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L-GROUPS AND THE LANGLANDS PROGRAM FOR COVERING GROUPS

Wee Teck Gan, Fan Gao & Martin H. Weissman

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L-GROUPS AND THE LANGLANDS PROGRAM FOR COVERING GROUPS

Wee Teck Gan, Fan Gao & Martin H. Weissman

Abstract. — This volume proposes an extension of the Langlands program to covers of quasisplit groups, where covers are those that arise from central extensions of reductive groups by K_2. By constructing an L-group for any such cover, one may conjecture a parameterization of genuine irreducible representations by Langlands parameters. Two constructions of the L-group are given, and related to each other in a final note. The proposed local Langlands conjecture for covers (LLCC) is proven for covers of split tori, spherical representations in the p-adic case, and discrete series for double-covers of real semisimple groups. The introduction of the L-group allows one to define partial L-functions and functoriality, including base change, for representations of covering groups.

Résumé **(L-groupes et le programme de Langlands pour les revêtements de groupes réductifs.)** — Ce volume propose une extension du programme de Langlands aux revêtements des groupes réductifs quasi-déployés qui proviennent des extensions centrales de ces groupes par K_2. On construit un L-groupe pour un tel revêtement, et on conjecture une paramétrisation de ses représentations irréductibles « spécifiques » (en anglais, « genuine ») par les paramètres de Langlands à valeurs dans ce L-groupe. En fait on donne deux constructions du L-groupe, qui sont reliées l'une à l'autre en fin d'article. La conjecture de Langlands locale proposée pour ces revêtements (LLCC) est prouvée pour les revêtements de tores déployés, les représentations sphériques dans le cas p-adique et les séries discrètes pour les revêtements doubles de groupes semi-simples réels. L'introduction du L-groupe permet de définir des fonctions L partielles et d'exprimer la fonctorialité, y compris le changement de base, pour ces représentations de revêtements.

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RÉSUMÉS DES ARTICLES

Dans cette introduction au présent volume de la série *Astérisque*, nous allons donner une brève discussion historique de l'étude des revêtements non linéaires des groupes réductifs, concernant leur structure, la théorie de leurs représentations et la théorie de leurs formes automorphes. Cela constitue une motivation historique et définit le cadre pour les contributions de ce volume.

Nous intégrons des revêtements de groupes réductifs quasi-déployés dans le programme de Langlands, en définissant un L-groupe associé à un tel revêtement. Nous travaillons avec tous les revêtements qui résultent d'extensions de groupes réductifs quasi-déployés par K_2 — la classe étudiée par Brylinski et Deligne. Nous utilisons ce L-groupe pour paramétrer des représentations irréductibles spécifiques dans de nombreux contextes, incluant les revêtements de tores déployés, les représentations sphériques, et les séries discrètes pour les revêtements doubles de groupes semi-simples réels. Une appendice étudie les torseurs et gerbes sur le site étale, puisqu'ils sont utilisés dans la construction du L-groupe.

Nous décrivons une extension conjecturale, en evolution, du programme de Langlands pour une classe de revêtements de groupes réductifs d'origine algébrique, étudiés par Brylinski et Deligne. Nous décrivons, en particulier, la construction, due à Weissman, d'une extension de L-groupe d'un tel revêtement de groupe (au-dessus d'un groupe réductif déployé). Nous étudions certaines de ses propriétés et discutons d'une variante de celui-ci. En utilisant cette extension de L-groupe, à l'aide du travail de Savin, McNamara, Weissman et W.-W. Li, nous décrivons une correspondance de Langlands locale pour des revêtements des tores (déployés) et pour les représentations non-ramifiées spécifiques. Nous définissons ensuite la notion de L-fonctions automorphes (partielles) attachées aux représentations automorphes spécifiques pour les groupes de Brylinski et Deligne. Enfin, nous verrons comment le formalisme de L-groupe explique certaines anomalies dans la théorie des représentations des revêtements de groupes réductifs et examinons quelques exemples de fonctorialité de Langlands tels que le changement de base.

Dans un article, l'auteur a défini un L-groupe associé à un revêtement de groupes réductifs quasi-déployés sur un corps local ou global. Dans un autre article, Wee Teck Gan et Fan Gao définissent (suite à une lettre inédite de l'auteur) un L-groupe associé à un revêtement de groupes réductifs quasi-déployés sur un corps local ou global. Dans cette courte note, nous donnons un isomorphisme entre ces L-groupes. De cette manière, les résultats et les conjectures discutés par Gan et Gao sont compatibles avec ceux de l'auteur. Les deux soutiennent les mêmes conjectures de type Langlands pour les revêtements des groupes.

ABSTRACTS

In this joint introduction to the present Astérisque volume, we shall give a short discussion of the historical developments in the study of nonlinear covering groups, touching on their structure theory, representation theory and the theory of automorphic forms. This serves as a historical motivation and sets the scene for the papers in this volume. Our discussion is necessarily subjective and will undoubtedly leave out the contributions of many authors, to whom we apologize in earnest.

We incorporate covers of quasisplit reductive groups into the Langlands program, defining an L-group associated to such a cover. We work with all covers that arise from extensions of quasisplit reductive groups by K_2—the class studied by Brylinski and Deligne. We use this L-group to parameterize genuine irreducible representations in many contexts, including covers of split tori, unramified representations, and discrete series for double covers of semisimple groups over $\mathbb{R}$. An appendix surveys torsors and gerbes on the étale site, as they are used in the construction of the L-group.

We describe an evolving and conjectural extension of the Langlands program for a class of nonlinear covering groups of algebraic origin studied by Brylinski and Deligne. In particular, we describe the construction of an L-group extension of such a covering group (over a split reductive group) due to Weissman, study some of its properties and discuss a variant of it. Using this L-group extension, we describe a local Langlands correspondence for covering (split) tori and unramified genuine representations, using work of Savin, McNamara, Weissman and W.-W. Li. We then define the notion of automorphic (partial) L-functions attached to genuine automorphic representations of the covering groups of Brylinski and

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Deligne. Finally, we see how the L-group formalism explains certain anomalies in the representation theory of covering groups and examine some examples of Langlands functoriality such as base change.

In one article, the author has defined an L-group associated to a cover of a quasisplit reductive group over a local or global field. In another article, Wee Teck Gan and Fan Gao define (following an unpublished letter of the author) an L-group associated to a cover of a pinned split reductive group over a local or global field. In this short note, we give an isomorphism between these L-groups. In this way, the results and conjectures discussed by Gan and Gao are compatible with those of the author. Both support the same Langlands-type conjectures for covering groups.

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L-GROUPS AND THE LANGLANDS PROGRAM FOR COVERING GROUPS: A HISTORICAL INTRODUCTION

by

Wee Teck Gan, Fan Gao & Martin H. Weissman

Abstract. — In this joint introduction to the present *Astérisque* volume, we shall give a short discussion of the historical developments in the study of nonlinear covering groups, touching on their structure theory, representation theory and the theory of automorphic forms. This serves as a historical motivation and sets the scene for the papers in this volume. Our discussion is necessarily subjective and will undoubtedly leave out the contributions of many authors, to whom we apologize in earnest.

Résumé **(L-groupes et le programme de Langlands pour les revêtements de groupes réductifs : une introduction historique)**

Dans cette introduction au présent volume de la série *Astérisque*, nous allons donner une brève discussion historique de l'étude des revêtements non linéaires des groupes réductifs, concernant leur structure, la théorie de leurs représentations et la théorie de leurs formes automorphes. Cela constitue une motivation historique et définit le cadre pour les contributions de ce volume.

1. Generalities

A *locally compact group* will mean a locally compact, Hausdorff, second countable topological group. Let G be a locally compact group and A a locally compact abelian group. We are interested in central extensions of G by A. Let us first define this notion; our treatment in this section follows the classic paper of Moore [**83**].

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1.1. Definition. — A *central extension of* G *by* A is a short exact sequence:

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

such that

- E is a locally compact group;
- i is continuous and $i(A)$ is a closed subgroup of the center of E;
- p is continuous and induces a topological isomorphism $E/i(A) \cong G$.

Equivalently, the third condition above can be replaced by the requirement that p is continuous and open (cf. [**80**, p.96]). We will ultimately be interested in the case when A is finite.

1.2. Definition. — Let E_1 and E_2 be two extensions of G by A. An *equivalence* from E_1 to E_2 is a continuous homomorphism $\phi \colon E_1 \to E_2$ inducing the identity maps on A and G:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "="'] & E_1 \arrow[r, "p_1"] \arrow[d, "\phi"'] & G \arrow[r] \arrow[d, "="'] & 1 \\ 1 \arrow[r] & A \arrow[r] & E_2 \arrow[r, "p_2"] & G \arrow[r] & 1. \end{tikzcd}$$

By the open mapping theorem, an equivalence is necessarily a topological isomorphism.

Let the set of equivalence classes of central extensions of G by A be denoted by $\mathrm{CExt}(G, A)$. The set $\mathrm{CExt}(G, A)$ has a natural abelian group structure, as we now explain.

Given two extensions E_1 and E_2 of G by A, we set

$$E = \{(h_1, h_2) \in E_1 \times E_2 : p_1(h_1) = p_2(h_2)\}/\delta(A)$$

where $\delta(a) = (a, a^{-1})$ is the skew diagonal embedding. This is the quotient of the fiber product $E_1 \times_G E_2$ by the skew diagonal embedding. Then E is a central extension of G by A,

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1,$$

by defining $i(a) = (a, 1) = (1, a) \in E$ and $p(h_1, h_2) = p_1(h_1) = p_2(h_2)$.

This E is the so-called Baer sum of E_1 and E_2, written $E_1 \dotplus E_2$ and this operation makes $\mathrm{CExt}(G, A)$ into an abelian group. In other words, the equivalence class of E depends only on the equivalence classes of E_1 and E_2.

In the context of abstract groups, the abelian group $\mathrm{CExt}(G, A)$ was first studied by Schur (1904) who introduced the notion of Schur multipliers. In modern language, Schur had introduced the cohomology group $H^2(G, A)$. We will however not go so far back in time in our historical discussion; a modern survey of the central extensions of finite groups of Lie type can be found in [**88**].

1.3. Categorical point of view. — If we fix G and A as before, define $\mathsf{CExt}(G, A)$ to be the category whose objects are central extensions of G by A, and whose morphisms are equivalences. Since all equivalences are isomorphisms, the category $\mathsf{CExt}(G, A)$ is a groupoid. The Baer sum is functorial,

$$\dotplus\colon \mathsf{CExt}(G, A) \times \mathsf{CExt}(G, A) \to \mathsf{CExt}(G, A),$$

making the category $\mathsf{CExt}(G, A)$ into a (strictly commutative) Picard category [**34**, Définition 1.4.2]

The neutral object in this category is the direct product $G \times A$. Given an object $E \in \mathsf{CExt}(G, A)$, a *splitting* of E is an equivalence (i.e., a morphism) from E to $G \times A$.

If $j\colon H \to G$ is a continuous homomorphism of locally compact groups, and $(E, i, p) \in \mathsf{CExt}(G, A)$, then we may *pull back* the extension E to define

$$j^*E = \{(h, e) \in H \times E : j(h) = p(e)\}.$$

Then $j^*E \in \mathsf{CExt}(H, A)$ by defining $i'\colon A \to j^*E$ by $i'(a) = (1, i(a))$ and $p'(h, e) = p(e)$.

If $f\colon A \to B$ is a continuous homomorphism of locally compact abelian groups, we may *push out* the extension (E, i, p) to define

$$f_*E = (B \times E)/\overline{\langle (f(a), i(a)^{-1}) : a \in A \rangle}.$$

Typically, f will be a closed map, and so it will not be necessary to take the closure in the quotient above. Then $f_*E \in \mathsf{CExt}(G, B)$ by defining $i''\colon B \to f_*E$ by $i''(b) = (b, 1)$ and $p''\colon f_*E \to G$ by $p''(b, e) = p(e)$.

Pullback and pushout define additive functors of Picard categories,

$$f_*\colon \mathsf{CExt}(G, A) \to \mathsf{CExt}(G, B), \quad f^*\colon \mathsf{CExt}(G, A) \to \mathsf{CExt}(H, A).$$

For isomorphism classes, these define homomorphisms of abelian groups,

$$f_*\colon \mathrm{CExt}(G, A) \to \mathrm{CExt}(G, B), \quad f^*\colon \mathrm{CExt}(G, A) \to \mathrm{CExt}(H, A).$$

1.4. Cohomological interpretation. — After the foundational work of Mackey [**73**], Moore wrote a series of papers [**82, 84, 85**] developing a cohomology theory for topological groups analogous to that for abstract groups. We summarize some of their results.

Moore defines for each $n \geq 0$ a cohomology group $H^n(G, A)$ using *measurable* cochains. These groups are functors which are covariant in A and contravariant in G. Note however that since the category of locally compact abelian groups is not an abelian category, this cohomology theory is not a derived functor cohomology theory. We describe the low degree cohomology groups concretely. Note that we are only interested in the case where A is trivial as a G-module. The 0-th cohomology group is $H^0(G, A) = A$. The first cohomology $H^1(G, A)$ is the group of continuous homomorphisms $G \to A$.

We describe $H^2(G, A)$ in more detail. Let $Z^2(G, A)$ be the group of measurable normalized 2-cocycles $z\colon G \times G \to A$; this means that $z(g, 1) = z(1, g) = 1$ for all

$g \in G$, and

$$z(g_1g_2, g_3)z(g_1, g_2) = z(g_1, g_2g_3)z(g_2, g_2) \text{ for all } g_1, g_2, g_3 \in G.$$

Let $C^1(G, A)$ be the group of normalized 1-cochains: measurable functions from G to A such that $f(1) = 1$. If $f \in C^1(G, A)$ is a measurable function, its coboundary $\partial f \in Z^2(G, A)$ is defined by

$$\partial f(g_1, g_2) = f(g_2) \cdot f(g_1g_2)^{-1} \cdot f(g_1)^{-1}.$$

The resulting cohomology group $H^2(G, A) = Z^2(G, A)/\partial C^1(G, A)$ is naturally isomorphic to $\mathrm{CExt}(G, A)$.

This can be understood categorically as follows. Consider the (small, strictly commutative Picard) category $\mathsf{H}^2(G, A)$, with objects set $Z^2(G, A)$, and where a morphism $z_1 \to z_2$ is defined to be an element $c \in C^1(G, A)$ such that $z_2 = z_1 + \partial c$. The Picard category structure arises from the abelian group structures on $Z^2(G, A)$ and $C^1(G, A)$. The isomorphism classes in $\mathsf{H}^2(G, A)$ form the cohomology group $H^2(G, A)$.

Describe a functor from $\mathsf{H}^2(G, A)$ to $\mathsf{CExt}(G, A)$ as follows: for an object $z \in Z^2(G, A)$, define an extension of G by A by $E = G \times A$, with multiplication

$$(g_1, a_1) \cdot (g_2, a_2) = (g_1g_2, a_1a_2 \cdot z(g_1, g_2)),$$

and maps

$$i : a \mapsto (1, a) \in E \quad \text{and} \quad p : (g, a) \mapsto g \in G.$$

A theorem of Mackey [**73**, Théorème 2] gives E a natural topology such that the above defines a locally compact group, and an extension of G by A. If $c \colon z_1 \to z_2$ is a morphism in $\mathsf{H}^2(G, A)$, i.e., $z_2 = z_1 + \partial c$, then c defines an equivalence of central extensions $E_1 \to E_2$ by the formula $f(g, a) = (g, c(g) \cdot a)$. The work of Mackey and Moore implies that this gives an equivalence of Picard categories, which we like to call "incarnation":

$$\mathsf{Inc}\colon \mathsf{H}^2(G, A) \to \mathsf{CExt}(G, A).$$

A consequence is the isomorphism of abelian groups, $H^2(G, A) \cong \mathrm{CExt}(G, A)$.

Surjectivity of this isomorphism is obtained as follows. Given a central extension $A \hookrightarrow E \twoheadrightarrow G$, Mackey proves that one can find a measurable section $s \colon G \to E$ (i.e., so that $p \circ s = id$). This is the best one can hope for: one cannot find a continuous section in general. From s, one defines a measurable 2-cocycle by:

$$z(g_1, g_2) = s(g_1g_2) \cdot s(g_1)^{-1}s(g_2)^{-1}.$$

The map $(g, a) \mapsto s(g) \cdot i(a)$ gives an isomorphism from $\mathsf{Inc}(z)$ to E.

1.5. $\mathrm{CExt}(G, -)$ **as a moduli functor.** — For another perspective, fix a locally compact group G. The assignment $A \mapsto \mathrm{CExt}(G, A)$ gives a functor,

$$\mathrm{CExt}(G, -)\colon \mathsf{LCA} \to \mathsf{Ab},$$

where LCA denotes the category of locally compact abelian groups and Ab denotes the category of (abstract) abelian groups. Indeed, we have seen above that $\mathrm{CExt}(G, A)$ has a natural abelian group structure and functoriality comes from pushout. Given $f\colon A \to B$ in LCA, pushout defines a group homomorphism $f_*\colon \mathrm{CExt}(G,A) \to \mathrm{CExt}(G,B)$.

Regarding $\mathrm{CExt}(G,-)$ as a functor $\mathsf{LCA} \to \mathsf{Ab}$, it is natural to ask:

Question. — Is the functor $\mathrm{CExt}(G,-)$ representable? If so, describe the representing object $\pi_1(G)$ of LCA explicitly.

As we shall see in the next section, this is the central motivating question behind the initial study of central extensions, as developed by Steinberg **[104]**, Moore **[83]**, Matsumoto **[74]**, Raghunathan-Prasad **[90, 91, 92]** and others.

If $\pi_1(G)$ exists, we call it the *fundamental group* of G, in which case we have isomorphisms, functorial in A:

$$H^2(G,A) \cong \mathrm{CExt}(G,A) \cong \mathrm{Hom}(\pi_1(G),A).$$

Note that this functorial isomorphism can only be unique up to automorphisms of $\pi_1(G)$. Also, observe that if $A = S^1$ (the unit circle), then

$$\pi_1(G) \cong \text{ the Pontryagin dual } H^2(G,S^1)^\vee \text{ of } H^2(G,S^1).$$

1.6. Universal extensions. — Suppose that $\pi_1(G)$ exists. Then since

$$\mathrm{CExt}(G,\pi_1(G)) \cong \mathrm{Hom}(\pi_1(G),\pi_1(G)),$$

there is an element $\tilde{G}$ of $\mathrm{CExt}(G,\pi_1(G))$ corresponding to the identity automorphism of $\pi_1(G)$. This extension

$$1 \longrightarrow \pi_1(G) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

is called a *universal central extension* because it has the following universal property: given any central extension E of G by A, there exists a *unique* continuous homomorphism $\phi\colon \tilde{G} \to E$ lying over the identity on G:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d, "f|_{\pi_1(G)}"'] & \tilde{G} \arrow[r] \arrow[d, "f"'] & G \arrow[r] \arrow[d, "="'] & 1 \\
1 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

The existence of such a universal central extension is equivalent to the representability of $\mathrm{CExt}(G,-)$.

1.7. Condition for representability. — Now let us consider the question about existence of $\pi_1(G)$. There is an obvious necessary condition for this existence. Indeed, consider the trivial extension $G \times S^1$ where S^1 is the unit circle. If $\pi_1(G)$ exists and $\tilde{G}$ is a universal central extension, then we have a unique map of extensions

$$f : \tilde{G} \longrightarrow G \times S^1.$$

However, if $\phi : G \longrightarrow S^1$ is any continuous homomorphism, then the map

$$f_\phi : \tilde{g} \mapsto f(\tilde{g}) \cdot (1, \phi(p(\tilde{g}))$$

is another morphism of extensions. Thus the uniqueness of f implies that $\mathrm{Hom}(G, S^1) = 0$. In other words, if $\pi_1(G)$ exists, then $H^1(G, S^1) = 0$, or equivalently, $[G, G]$ is dense in G, in which case we say that G is *topologically perfect.*

One may ask if the necessary condition above is sufficient for the existence of $\pi_1(G)$? Moore has given examples to show that it is not in general. We highlight some positive results in this direction due to Moore [**83**]:

Proposition 1.1. — *In the following cases, $\pi_1(G)$ exists:*

(i) *G is a discrete group which is perfect (equivalently, topologically perfect, since the topology is discrete);*
(ii) *G is topologically perfect and $H^2(G, S^1)$ is finite;*
(iii) *the component group G/G° of G is compact and $G = [G, G]$ (i.e., G is perfect).*

Recall that when $\pi_1(G)$ exists, $\pi_1(G) \cong H^2(G, S^1)^\vee$.

1.8. Relative fundamental groups. — One may also consider the problem of classifying the central extensions of G which are split over a subgroup $H \subset G$. Here, H need not be a closed subgroup; there are applications in which H is even dense in G. Thus we are interested in the representability of the functor $\mathsf{LCA} \to \mathsf{Ab}$ given by

$$A \mapsto \mathrm{Ker}(H^2(G, A) \longrightarrow H^2(H, A)).$$

One has the following result [**83**, Lemma 2.8]:

Proposition 1.2. — *Suppose that $\pi_1(G)$ and $\pi_1(H)$ both exist. The map $i : H \longrightarrow G$ induces $i_* : \pi_1(H) \longrightarrow \pi_1(G)$. Define*

$$\pi_1(G, H) = \pi_1(G)/\overline{i_*(\pi_1(H))}.$$

Then there is an isomorphism, functorial in A:

$$\mathrm{Ker}(H^2(G, A) \longrightarrow H^2(H, A)) \cong \mathrm{Hom}(\pi_1(G, H), A).$$

In other words, the above functor is represented by $\pi_1(G, H)$.

We call $\pi_1(G, H)$ the fundamental group of G relative to H.

1.9. Restricted direct product. — Given a countable collection (G_v, K_v), $v \in S$, where each G_v is a locally compact group and K_v is an open compact subgroup of G_v, one may form the restricted direct product

$$G = \prod_v{}' (G_v, K_v)$$

which is still a locally compact topological group.

The following proposition (based on [**83**, Theorem 12.1]) gives natural conditions under which $\pi_1(G)$ exists.

Proposition 1.3. — *Write $i_v \colon K_v \hookrightarrow G_v$ for the inclusions. Assume that*

- *$\pi_1(G_v)$ exists for all v;*
- *$\pi_1(K_v)$ exists for almost all v;*

– *$i_{v*}(\pi_1(K_v))$ is open in $\pi_1(G_v)$ for almost all v.*

Then $\pi_1(G)$ exists and is equal to the restricted direct product

$$\pi_1(G) = \prod_v{}' \left(\pi_1(G_v), i_{v*}(\pi_1(K_v))\right).$$

This concludes our discussion on generalities about central extensions and fundamental groups.

2. Abstract Chevalley Groups

In this section, we shall specialize to the case where $G = \mathbf{G}(k)$, with $\mathbf{G}$ a connected reductive group over a field k. In particular, when $\mathbf{G}$ is split, we give a summary of the beautiful work of Steinberg [**104, 105**], Moore [**83**] and Matsumoto [**74**] which describes the fundamental group of an abstract Chevalley group.

To begin, let $\mathbf{G}$ be a split, simple and simply connected linear algebraic group over an infinite field k. Set $G = \mathbf{G}(k)$ regarded as an abstract group (with discrete topology). In this case, G is known to be perfect. Thus $\pi_1(G)$ exists by Proposition 1.1(i) and there is a universal central extension

$$1 \longrightarrow \pi_1(G) \longrightarrow E_G \longrightarrow G \longrightarrow 1.$$

2.1. Steinberg's construction of E_G. — In [**104**], Steinberg gave an explicit construction of the universal central extension E_G using generators and relations. Fix

$$\mathbf{T} \subset \mathbf{B} = \mathbf{T} \cdot \mathbf{U}^+ \subset \mathbf{G},$$

a split maximal torus contained in a Borel subgroup of $\mathbf{G}$. This gives rise to a root system Φ with a set Δ of simple roots. For each $\alpha \in \Phi$, one has a root subgroup $\mathbf{U}_\alpha \cong \mathbf{G}_a$. If an isomorphism $x_\alpha \colon \mathbf{U}_\alpha \xrightarrow{\sim} \mathbf{G}_a$ is chosen for every $\alpha \in \Phi$, we define families of elements of G by

$$\begin{cases} w_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t), \\ h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1} \end{cases}$$

for $t \in k^\times$.

In [**105**, Chapter 6], Steinberg demonstrates that there exists a family of isomorphisms $\{x_\alpha : \alpha \in \Phi\}$ such that G is generated by $\{x_\alpha(t) : \alpha \in \Phi, t \in k\}$ modulo the relations

(A) $x_\alpha(s+t) = x_\alpha(s) x_\alpha(t)$.

(B) $[x_\alpha(t), x_\beta(s)] = \prod_{i,j} x_{i\alpha+j\beta}(n_{\alpha,\beta,i,j} t^i s^j)$ where the product is taken over (i,j) in lexicographic order and the $n_{\alpha,\beta,i,j}$'s are certain integers which we will not make precise here.

(B′) $w_\alpha(t) x_\alpha(s) w_\alpha(t)^{-1} = x_{-\alpha}(-st^{-2})$.

(C) $h_\alpha(st) = h_\alpha(s) h_\alpha(t)$.

The condition (B') is not necessary if $G \neq \mathrm{SL}_2$. The elements $h_\alpha(t)$ for $\alpha \in \Delta$ generate the group T, and the elements $x_\alpha(t)$ for $\alpha \in \Phi^+$, $t \in k$, generate the group U^+. Similarly, $\{x_\alpha(t) : \alpha \in \Phi^-, t \in k\}$ generates the opposite unipotent $U^- = \mathbf{U}^-(k)$.

Steinberg considered the group $\tilde{G}$ generated by elements $\tilde{x}_\alpha(t)$, for $\alpha \in \Phi$, modulo the (analogous) relations (A), (B) and (B') above (ignoring (C)). There is clearly a natural surjection

$$p_G : \tilde{G} \longrightarrow G$$

given by $\tilde{x}_\alpha(t) \mapsto x_\alpha(t)$.

From relations (A) and (B), it can be seen that $x_\alpha(t) \mapsto \tilde{x}_\alpha(t)$ extends to a homomorphism $\sigma^\pm \colon U^\pm \hookrightarrow \tilde{G}$ splitting p_G, i.e., $p_G \circ \sigma^\pm = \mathrm{Id}$.

Define elements $\tilde{w}_\alpha(t)$ and $\tilde{h}_\alpha(t)$ analogously to $w_\alpha(t)$ and $h_\alpha(t)$ above. We let $\tilde{T}$ denote the subgroup of $\tilde{G}$ generated by the $\tilde{h}_\alpha(t)$'s. Steinberg showed ([**105**, Chapter 7, Theorem 10]):

Theorem 2.1. — *The group* $\mathrm{Ker}(p_G)$ *is central in* $\tilde{G}$*, and*

$$1 \to \mathrm{Ker}(p_G) \to \tilde{G} \to G \to 1$$

is a universal central extension of G*. In particular* $\pi_1(G) = \mathrm{Ker}(p_G)$*. Furthermore,* $\pi_1(G) \subset \tilde{T}$*.*

2.2. Steinberg's cocycles. — In [**83**, p.194], Moore described a 2-cocycle which represents the universal extension $\tilde{G}$, depending on choices of Weyl representatives and an ordering of the simple roots. Let $\tilde{N}$ (respectively N) denote the subgroup of $\tilde{G}$ (resp. G) generated by the $\tilde{w}_\alpha(t)$'s (resp. $w_\alpha(t)$). Then

$$\tilde{N}/\tilde{T} \cong N/T = W$$

is the Weyl group of G. For each $w \in W$, we fix a representative $\tilde{w} \in \tilde{N}$ and denote its projection $p_G(\tilde{w}) \in N$ by $\dot{w}$.

Each element $g \in G$ lies in a unique Bruhat cell $B\dot{w}B$, and can be uniquely represented as:

$$g = u_w \cdot \dot{w} \cdot t \cdot u, \quad t = \prod_{\alpha \in \Delta} h_\alpha(t_\alpha) \in T,\, u \in U^+,\, u_w \in U_w = \prod_{\substack{\alpha>0 \\ w\alpha<0}} U_\alpha.$$

We define a section $s : G \longrightarrow \tilde{G}$ by setting

$$s(g) = \tilde{u}_w \cdot \tilde{w} \cdot \tilde{h} \cdot \tilde{u}$$

where $\tilde{u}_w = \sigma^+(u_w)$ and $\tilde{u} = \sigma^+(u)$ and $\tilde{h} = \prod_{\alpha \in \Delta} \tilde{h}_\alpha(t_\alpha)$. Here we must fix an ordering of the simple roots, in order for this product to make sense.

This gives a 2-cocycle $b_{\mathrm{univ}} \colon G \times G \longrightarrow \pi_1(G)$, given by

$$b_{\mathrm{univ}}(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}.$$

We call this the *universal Steinberg cocycle of* G.

Given any central extension $E \in \mathrm{CExt}(G, A)$, there is a unique homomorphism $f\colon \tilde{G} \to E$ lying over the identity map on G. Defining $s_E = f \circ s$ we have a section $s_E\colon G \to E$, and from this a cocycle

$$c_E(g_1, g_2) = f(b_{\mathrm{univ}}(g_1, g_2)) \in A$$

which incarnates E. We call c_E the *Steinberg cocycle of* E.

2.3. Moore's upper bound for $\pi_1(G)$. — After Steinberg's construction of the universal central extension by generators and relations, one can hope for an explicit presentation of $\pi_1(G)$. This question was taken up by Moore in [**83**]. His results are summarized in the following theorem (cf. [**83**, Lemma 8.1, Theorem 8.1, Lemma 8.2 and Lemma 8.4]).

Theorem 2.2. — (i) *$\pi_1(G)$ is generated by the elements $b_{\mathrm{univ}}(h_\alpha(s), h_\alpha(t))$ for $\alpha \in \Delta$ and $s, t \in k^\times$. In fact, if we fix a long root α_0, then $\pi_1(G)$ is generated by the elements $b_{\mathrm{univ}}(h_{\alpha_0}(s), h_{\alpha_0}(t))$ for $s, t \in k^\times$.*

(ii) *If $c = f \circ b_{\mathrm{univ}}$ is a Steinberg cocycle valued in A (with $f\colon \pi_1(G) \to A$), then c is completely determined by its restriction to $T \times T$. In fact, if α is a long root and T_α the 1-dimensional torus generated by $h_\alpha(t)$, then c is completely determined by its restriction to $T_\alpha \times T_\alpha$.*

Corollary 2.3. — *Let α be a long root in Φ and let $G_\alpha \cong \mathrm{SL}_2$ be the subgroup generated by U_α and $U_{-\alpha}$. Then for any A, the natural map*

$$H^2(G, A) \longrightarrow H^2(G_\alpha, A)$$

is an injection. Equivalently, the natural map

$$\pi_1(G_\alpha) \longrightarrow \pi_1(G)$$

is surjective.

This corollary is the key tool in the analysis of extensions of a split group.

The theorem significantly reduces the number of generators needed for $\pi_1(G)$. Indeed, in view of the theorem, we fix a long root α and set

$$b_{\mathrm{univ},\alpha}(s, t) = b_{\mathrm{univ}}(h_\alpha(s), h_\alpha(t))$$

so that

$$b_{\mathrm{univ},\alpha} : k^\times \times k^\times \longrightarrow \pi_1(G).$$

We know that $\pi_1(G)$ is generated by $b_{\mathrm{univ},\alpha}(s, t)$ for $s, t \in k^\times$. Moreover, Moore showed that under a simple condition, the function $b_{\mathrm{univ},\alpha}$ is bimultiplicative:

Proposition 2.4. — *If there exists a root β such that $\langle \beta^\vee, \alpha \rangle = 1$, then $b_{\mathrm{univ},\alpha}$ is bimultiplicative. This condition holds as long as $G \neq \mathrm{Sp}_{2n}$ ($n \geq 1$). If $G = \mathrm{Sp}_{2n}$ (e.g., if $G = \mathrm{SL}_2$), then $b_{\mathrm{univ},\alpha}$ may not be bimultiplicative.*

Now we want to know what are the relations satisfied by the $b_{\mathrm{univ},\alpha}(s, t)$. By working explicitly with the group SL_2, Moore was able to show (cf. [**83**, Theorem 9.2]):

Theorem 2.5. — *If $G = \mathrm{SL}_2$, then $\pi_1(G)$ is the group generated by the $b(s,t) := b_{\mathrm{univ},\alpha}(s,t)$ subject to the relations:*

(1) *(normalized cocycle identities)*

$$b(st,r)b(s,t) = b(s,tr)b(t,r), \quad b(s,1) = b(1,s) = 1.$$

(2) $b(s,t) = b(t^{-1},s)$.
(3) $b(s,t) = b(s,-st)$.
(4) $b(s,t) = b(s,(1-s)t)$ *if* $s \neq 1$.

There is in fact some redundancy in these relations: under (1) *and* (4), (2) *and* (3) *are equivalent.*

2.4. Definition. — We call functions $c : k^\times \times k^\times \longrightarrow A$ satisfying the above identities (1)-(4) *A-valued Steinberg cocycles on $k^\times$*. We denote the set of such functions by $\mathrm{St}(k^\times, A)$. Since the universal cocycle $b_{\mathrm{univ},\alpha}$ is bimultiplicative if $G \neq \mathrm{Sp}_{2n}(k)$, we define a subgroup $\mathrm{St}^\circ(k^\times, A) \subset \mathrm{St}(k^\times, A)$ consisting of those A-valued Steinberg cocycles which are bimultiplicative.

The elements of $\mathrm{St}^\circ(k^\times, A)$ can be more simply described as those maps $c\colon k^\times \times k^\times \to A$ satisfying just two conditions:

(1′) (bimultiplicative) $c(rs,t) = c(r,t)\cdot c(s,t)$ and $c(r,st) = c(r,s)\cdot c(s,t)$.
(2′) $c(s,1-s) = 1$ if $s \neq 1$.

The relations (1′) and (2′) are important in algebraic K-theory. Namely, they occur in the definition of the *Milnor-Quillen $\mathbf{K}_2$-group* **[80]**. This is the abelian group

$$\mathbf{K}_2(k) = \frac{k^\times \otimes_{\mathbb{Z}} k^\times}{\langle x \otimes (1-x) : x \neq 1\rangle}.$$

Thus $\mathrm{St}^\circ(k^\times, A) = \mathrm{Hom}(\mathbf{K}_2(k), A)$. A corollary of the above discussions and Pontryagin duality is:

Corollary 2.6. — (i) *For $G \neq \mathrm{Sp}_{2n}(k)$ (resp. $G = \mathrm{Sp}_{2n}(k)$) and any A, there is a natural inclusion*

$$H^2(G,A) \hookrightarrow \mathrm{St}^\circ(k^\times, A) \quad (\textit{resp. } \mathrm{St}(k^\times, A)).$$

(ii) *If $G = \mathrm{Sp}_{2n}(k)$, then $\pi_1(G)$ is a quotient of the group generated by $b_{\mathrm{univ},\alpha}(s,t)$ subject to the relations* (1)–(4) *of Theorem 2.5.*

(iii) *If $G \neq \mathrm{Sp}_{2n}(k)$, then $\pi_1(G)$ is a quotient of the group $\mathbf{K}_2(k)$.*

However, for a general Chevalley group G, Moore was unable to determine whether the relations in the corollary are enough to define $\pi_1(G)$, or whether more relations are necessary.

2.5. Matsumoto's determination of $\pi_1(G)$. — In [74], Matsumoto was able to complete Moore's results by showing:

Theorem 2.7. — (ii) *If $G = \mathrm{Sp}_{2n}(k)$, then $\pi_1(G)$ is isomorphic to the group generated by $b_{\mathrm{univ},\alpha}(s,t)$ subject to the relations* (1)–(4) *of Theorem 2.5. Thus*

$$H^2(G,A) \cong \mathrm{St}(k^\times, A).$$

(iii) *If $G \neq \mathrm{Sp}_{2n}(k)$, then $\pi_1(G)$ is isomorphic to* $\mathrm{K}_2(k)$. *Thus,*

$$H^2(G,A) \cong \mathrm{St}^\circ(k^\times, A).$$

We remark that since we have an upper bound $H^2(G,A) \hookrightarrow \mathrm{St}(k^\times, A)$ (or $\mathrm{St}^\circ(k^\times, A)$) from Moore, to show that this upper bound is attained is a question of construction of central extensions. Namely, given an element of $f \in \mathrm{St}(k^\times, A)$ or $\mathrm{St}^\circ(k^\times, A)$, one needs to construct a central extension of G by A whose associated Steinberg cocycle gives rise to f. This was what Matsumoto did.

3. Groups over Local Fields

In this section, we consider the main problem highlighted in the first section for groups over local fields. Let k be a local field and let $\mathbf{G}$ be a (algebraically) simply-connected semisimple group over k. We set $G = \mathbf{G}(k)$, so that G is a topological group and we are interested in its topological central extensions.

If $\mathbf{G}$ is k-isotropic, it is known that G is topologically perfect, so that there is a chance that $\pi_1(G)$ exists. The main result we want to highlight here is:

Theorem 3.1. — *Suppose that k is non-archimedean. Assume that $\mathbf{G}$ is absolutely simple and k-isotropic. Then*

$$H^2(G, S^1) \cong \mu(k)^\vee,$$

where $\mu(k)$ denotes the finite group of roots of unity contained in k. In particular, $\pi_1(G)$ exists and is equal to $\mu(k)$.

We make several remarks:

(1) The assumption that k is non-archimedean is for convenience: it allows us to give a simple statement. Over $\mathbb{R}$ or $\mathbb{C}$, the situation is completely understood.

(2) The condition that $\mathbf{G}$ be absolutely simple is not crucial. If $\mathbf{G}$ is just semisimple and simply-connected, then

$$\mathbf{G} \cong \prod_i \mathrm{Res}_{k_i/k} \mathbf{G}_i$$

with $\mathbf{G}_i$ absolutely simple. Thus if each $\mathbf{G}_i$ is k_i-isotropic, the theorem implies that

$$\pi_1(G) = \prod_i \mu(k_i).$$

(3) The theorem is the culmination of the work of Steinberg [**104**], Moore [**83**], Matsumoto [**74**], Deodhar [**37**], Prasad-Raghunathan [**90, 91, 92**], Prasad-Rapinchuk

[**93**], G. Prasad [**89**] and Deligne [**36**]. In the rest of the section, we will describe some ideas in its proof.

(4) If $\mathbf{G}$ is anisotropic, absolutely simple, and simply-connected, then $G = \mathrm{SL}_1(D)$ where D is a division algebra over k. In this case, one can still demand to compute $H^2(G, S^1)$, even though G is not perfect. Such a computation was done by Prasad-Rapinchuk. We will not discuss this here.

3.1. The case of split groups. — When $\mathbf{G}$ is split, the theorem was proved by the combined work of Moore and Matsumoto, which made decisive use of the analysis of the abstract universal central extension given in the last section. Let G_{abs} denote $\mathbf{G}(k)$ regarded as an abstract group (with discrete topology). Then since any topological central extension is an abstract extension, we have a natural map

$$H^2(G, A) \longrightarrow H^2(G_{\mathrm{abs}}, A_{\mathrm{abs}}).$$

It turns out that this natural map is always an inclusion (for any topological group G), so that there is a natural surjection

$$\pi_1(G_{\mathrm{abs}}) \longrightarrow \pi_1(G).$$

By Theorem 2.7, we know that

$$H^2(G_{\mathrm{abs}}, A_{\mathrm{abs}}) \cong \mathrm{St}(k^\times, A) \text{ or } \mathrm{St}^\circ(k^\times, A).$$

Thus it remains to determine which A-valued Steinberg cocycles correspond to topological extensions. The following result is both simple (to absorb) and natural:

Theorem 3.2. — *Let $E \in \mathrm{CExt}(G_{\mathrm{abs}}, A_{\mathrm{abs}})$. Then the following are equivalent:*

(i) *The Steinberg cocycle $c_E : G \times G \longrightarrow A$ of E is Borel measurable.*
(ii) *c_E is continuous on $T \times T$.*
(iii) *c_E is continuous on $T_\alpha \times T_\alpha$. (α a long root as before.)*
(iv) *E is a topological central extension.*

Thus, to classify topological central extension, we are reduced to classifying the set $\mathrm{St}_{\mathrm{cont}}(k^\times, A)$ of continuous A-valued Steinberg cocycles. This problem was solved by Moore (cf. [**83**, Chapter 2]). To describe his answer, we first recall that there is a natural supply of elements of $\mathrm{St}_{\mathrm{cont}}(k^\times, A)$ arising from local class field theory. Namely, if we let $\mu = \#\mu(k)$, there is a surjective μ-power residue symbol

$$(-,-) : k^\times \times k^\times \longrightarrow \mu(k).$$

Moore observes [**83**, Chapter II.(3)] that $(-,-)$ is an element of $\mathrm{St}^\circ_{\mathrm{cont}}(k^\times, \mu(k))$.

Now given any A and a homomorphism $f : \mu(k) \longrightarrow A$, we obtain an element

$$f \circ (-,-) \in \mathrm{St}^\circ_{\mathrm{cont}}(k^\times, A) \subset \mathrm{St}_{\mathrm{cont}}(k^\times, A).$$

Thus we have a map

$$\mathrm{Hom}(\mu(k), A) \longrightarrow \mathrm{St}_{\mathrm{cont}}(k^\times, A).$$

The result of Moore [**83**, Theorem 3.1] is:

Theorem 3.3. — *The natural map above is bijective:*

$$\mathrm{St}^{\circ}_{\mathrm{cont}}(k^{\times}, A) = \mathrm{St}_{\mathrm{cont}}(k^{\times}, A) = \mathrm{Hom}(\mu(k), A).$$

In particular, each element of $\mathrm{St}_{\mathrm{cont}}(k^{\times}, A)$ *is bimultiplicative (recall that* k *is non-archimedean here) and* $\pi_1(G) = \mu(k)$*.*

3.2. Deodhar's work for quasi-split groups. — Using a similar generators-relations approach based on a Chevalley-Steinberg system of épinglage, Deodhar [**37**] was able to extend Moore's results to the case when $\mathbf{G}$ is quasi-split. In particular, he obtained an upper bound for $\pi_1(G)$, namely that

$$\mu(k) \twoheadrightarrow \pi_1(G).$$

Once again, to establish that this is a bijection, one needs to construct topological central extensions. Thankfully, in this case, one does not need to give new constructions of central extensions. One can finesse the difficulty by using an observation of Deligne (unpublished) to reduce to the case of split groups.

Deligne's observation makes use of one consequence of Matsumoto's work which is useful to know. Suppose we have an embedding

$$i : \mathbf{SL}_2 \hookrightarrow \mathbf{G}.$$

Let $\mathbf{H}$ be a maximal split torus of $\mathbf{SL}_2$ and $\mathbf{T}$ a maximal split torus of $\mathbf{G}$ containing $i(\mathbf{H})$. There is then an embedding of $\mathbb{Z}$-modules:

$$X_*(\mathbf{H}) \hookrightarrow X_*(\mathbf{T}).$$

Fix a Weyl group invariant inner product $\langle -, - \rangle$ on $X_*(\mathbf{T}) \otimes \mathbb{R}$ such that for any long root α (so that $\alpha^{\vee}$ is short),

$$\langle \alpha^{\vee}, \alpha^{\vee} \rangle = 1.$$

Now take any generator μ of $X_*(\mathbf{H}) \cong \mathbb{Z}$ and set

$$n(i, G) = \langle \mu, \mu \rangle \geq 1.$$

Consider the induced map

$$i^* : H^2(G, A) \longrightarrow H^2(\mathrm{SL}_2, A).$$

Then the following lemma follows from [**74**, Lemma 5.4] and its proof.

Lemma 3.4. — *The image of* i^* *is* $n(i, G) \cdot H^2(\mathrm{SL}_2, A)$*.*

Now Deligne showed that for each quasi-split $\mathbf{G}$, one can find

- a split group $\mathbf{G}'$ containing $\mathbf{G}$,
- an embedding $i : \mathbf{SL}_2 \hookrightarrow \mathbf{G}$

such that $n(i, G') = 1$. Thus the composite

$$H^2(G', A) \longrightarrow H^2(G, A) \longrightarrow H^2(\mathrm{SL}_2, A)$$

is surjective. In particular, one can deduce that $H^2(G, A) \cong H^2(\mathrm{SL}_2, A) \cong \mathrm{Hom}(\mu(k), A)$.

3.3. The work of Prasad-Raghunathan for general k-isotropic groups. — When $\mathbf{G}$ is k-isotropic but not quasi-split, then the above strategy is not feasible because we do not have an explicit description of $\pi_1(G_{\mathrm{abs}})$ to begin with. In this case, Prasad and Raghunathan [**90, 91**] have to resort to more geometric ideas (using the Bruhat-Tits building of G) in order to compute $H^2(G, S^1)$. The details are too intricate to discuss here. In the end, they showed that $\pi_1(G)$ is a quotient of $\mu(k)$ with kernel at most of size 2. This was then strengthened to an isomorphism using the results of [**36**] and [**93**].

4. Adelic Groups

In this section, suppose that k is a global field and let $\mathbb{A}$ be its adele ring. For each place v of k, let k_v be the corresponding completion of k. Let $\mathbf{G}$ be a simply-connected semisimple group over k. We set

$$G_k = \mathbf{G}(k), \quad G_{\mathbb{A}} = \mathbf{G}(\mathbb{A}), \quad G_v = \mathbf{G}(k_v).$$

If S is a finite set of places of F, one may also work with the S-adeles $\mathbb{A}_S$; then $G_{\mathbb{A}_S}$ is the restricted product of the G_v for $v \notin S$. There is a natural diagonal map $i : G_k \hookrightarrow G_{\mathbb{A}_S}$, and one is interested in classifying topological central extensions of $G_{\mathbb{A}_S}$ which split over $i(G_k)$. These are classified by

$$M(S, G) = \mathrm{Ker}(H^2(G_{\mathbb{A}_S}, S^1) \longrightarrow H^2(G_k, S^1)).$$

This group is called the *S-metaplectic kernel*. If $S = \emptyset$, we call it the *absolute metaplectic kernel* and denote it simply by $M(G)$. The computation of $M(S, G)$ was achieved after a long series of papers by Prasad-Raghunathan [**90, 91, 92**] and Prasad-Rapinchuk [**93**].

One reason for focusing on central extensions of $G_{\mathbb{A}}$ which become split over G_k is that one is eventually interested in the theory of automorphic forms of coverings $\tilde{G}_{\mathbb{A}}$ of $G_{\mathbb{A}}$: these are functions on $i(G_k)\backslash\tilde{G}_{\mathbb{A}}$. Another reason is that the computation of $M(S, G)$ arises in the study of the congruence subgroup problem.

4.1. Local-to-global. — If $\mathbf{G}$ is k-isotropic, then $\pi_1(G_v)$ exists for all v by Theorem 3.1. Moreover, it is known that if K_v is a hyperspecial maximal compact subgroup of G_v, then K_v is perfect, so that $\pi_1(K_v)$ exists for almost all v by Proposition 1.1(iii). Moreover, for almost all v, the natural map

$$i_{v*} : \pi_1(K_v) \longrightarrow \pi_1(G_v)$$

is the zero map. Thus by Proposition 1.3, $\pi_1(G_{\mathbb{A}_S})$ exists and is equal to

$$\pi_1(G_{\mathbb{A}_S}) = \bigoplus_{v \notin S} \pi_1(G_v).$$

Further, the discrete group G_k is perfect so that $\pi_1(G_k)$ exists also. Thus, using Proposition 1.2, we have the relative fundamental group

$$\pi_1(G_{\mathbb{A}_S}, G_k) = \left(\bigoplus_{v \notin S} \pi_1(G_v)\right) / \overline{i_*(\pi_1(G_k))}.$$

Given all these, one deduces that the functor

$$A \mapsto M(S, G, A) = \mathrm{Ker}(H^2(G_{\mathbb{A}_S}, A) \longrightarrow H^2(G_k, A))$$

is represented by $\pi_1(G_{\mathbb{A}_S}, G_k)$, so that

$$M(S, G) = \mathrm{Hom}(\pi_1(G_{\mathbb{A}_S}, G_k), S^1).$$

Thus, if $\mathbf{G}$ is k-isotropic, the problem of computing $M(S, G)$ is the same as computing the relative fundamental group $\pi_1(G_{\mathbb{A}_S}, G_k)$. Since we know the local $\pi_1(G_v)$'s very explicitly, one approach to computing $\pi_1(G_{\mathbb{A}_S}, G_k)$ is to describe as explicitly as possible the closure of the image of $\pi_1(G_k)$. For this, one would need to know $\pi_1(G_k)$ very explicitly. As we noted in Section 2, we have this explicit description when $\mathbf{G}$ is quasi-split, thanks to the work of Steinberg, Moore, Matsumoto and Deodhar. When $\mathbf{G}$ is not quasi-split, such an approach to computing $\pi_1(G_{\mathbb{A}_S}, G_k)$ is not feasible. This is why for non-quasi-split groups, Prasad-Raghunathan and Prasad-Rapinchuk have to resort to completely different ideas to solve this problem.

In any case, the main global theorem of **[93]** is:

Theorem 4.1. — *Let $\mathbf{G}$ be an absolutely simple, simply connected semisimple group over k. If $\mathbf{G}$ is a special unitary group over a noncommutative division algebra, assume a certain conjecture (U). Let S be a finite (possibly empty) set of places of k. Then we have:*

(i) $M(S, G) \subset \mu(k)^\vee$.

(ii) *If S contains a non-archimedean place v where G_v is isotropic or a real place where G_v is not topologically simply-connected, then $M(S, G) = 0$.*

(iii) *If $S = \emptyset$, then $M(S, G) = \mu(k)^\vee$.*

As for the local theorem, this theorem is the culmination of the work of many people, culminating in the eventual work of Prasad-Rapinchuk **[93]**.

5. Brylinski-Deligne Theory

As our brief discussion of the historical development of the structure theory of covering groups shows, much of the earlier work is focused on determining the fundamental group or the universal central extension. This almost immediately restricts one to the case when $\mathbf{G}$ is a simply-connected linear algebraic group over a field k. The disadvantage of this is that it is a common strategy in Lie theory to prove results by induction through Levi subgroups of parabolic subgroups. However, the Levi subgroups are only reductive groups and not semisimple. Thus the structure theory of (topological) central extensions obtained in previous sections does not apply to the Levi subgroups of $\mathbf{G}$.

When $\mathbf{G}$ is not simply-connected, for example if $\mathbf{G}$ is a special orthogonal group, a typical way of obtaining central extensions of $G = \mathbf{G}(k)$ is to fix an embedding $\mathrm{G} \hookrightarrow \mathrm{SL}_r$ and then to pullback some known central extensions of SL_r. For example, one may embed GL_r into SL_{r+1} as an $r \times r$ block, and pullback a (topological) central extension of SL_{r+1}; this gives an extension which has been studied in some detail by Kazhdan and Patterson [**54**] and is one member of a family of covers of GL_r known as the Kazhdan-Patterson covers.

While such constructions give examples of covering groups, with some control on their structure through one's knowledge of the relevant 2-cocycles on SL_r, they do not amount to a systematic theory or classification.

In their 2001 IHÉS paper [**17**], Brylinski and Deligne approached the subject from a different angle. They returned to the very neat results obtained in the split simply-connected case by Steinberg, Moore and Matsumoto, where one has an extension of abstract groups

$$1 \longrightarrow \mathbf{K}_2(k) \longrightarrow \tilde{G} \longrightarrow G = \mathbf{G}(k) \longrightarrow 1,$$

which is universal if $\mathbf{G}$ is not of type C. Their idea (from our perspective) is to "remove the k" in the above short exact sequence. More precisely, regarding $\mathbf{K}_2$ and $\mathbf{G}$ as sheaves of groups on the big Zariski site of $\mathrm{Spec}(k)$, they consider the problem of understanding or classifying the central extensions of group sheaves

$$1 \longrightarrow \mathbf{K}_2 \longrightarrow \tilde{\mathbf{G}} \longrightarrow \mathbf{G} \longrightarrow 1.$$

Such a $\tilde{\mathbf{G}}$ is also called a multiplicative $\mathbf{K}_2$-torsor over $\mathbf{G}$ and the problem is to give a classification of the Picard category of such multiplicative $\mathbf{K}_2$-torsors with $\mathbf{G}$ fixed, i.e., to describe this category in simpler terms. Brylinski-Deligne managed to give a very reasonable answer to this classification problem which depends functorially on $\mathbf{G}$. Their results will be summarized and described in the papers in this volume.

Suppose one has a multiplicative $\mathbf{K}_2$-torsor $\tilde{\mathbf{G}}$ over a local field k. Then on taking k-points, one obtains a central extension of discrete groups

$$1 \longrightarrow \mathbf{K}_2(k) \longrightarrow \tilde{\mathbf{G}}(k) \longrightarrow \mathbf{G}(k) = G \longrightarrow 1.$$

Here the sequence remains exact on the right because $H^1_{\mathrm{Zar}}(k, \mathbf{K}_2) = 1$. If one pushes this sequence out via the norm residue symbol $\mathrm{K}_2(k) \longrightarrow \mu(k)$, then one obtains a topological central extension

$$1 \longrightarrow \mu(k) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

Thus, multiplicative $\mathbf{K}_2$-torsors over local fields give rise to topological central extensions. Such topological central extensions are thus of "algebraic origin".

Now suppose k is a global field with ring of adeles $\mathbb{A}$. Then the analog of the above construction shows that one inherits a central extension

$$1 \longrightarrow \mu(k) \longrightarrow \tilde{G}_{\mathbb{A}} \longrightarrow G_{\mathbb{A}} \longrightarrow 1$$

from a multiplicative $\mathbf{K}_2$-torsor over k. A key feature of the Brylinski-Deligne theory is that this central extension of $G_{\mathbb{A}}$ comes equipped with a canonical splitting $G_k \hookrightarrow \tilde{G}_{\mathbb{A}}$; this follows from reciprocity for norm residue symbols. In other words, a multiplicative $\mathbf{K}_2$-torsor over a global field gives rise to a topological central extension of the adelic group $G_{\mathbb{A}}$ together with a splitting over G_k. This means that one is immediately in a position to begin the study of automorphic forms of $\tilde{G}_{\mathbb{A}}$.

6. Representation Theory and Automorphic Forms

In this section, we shall give a brief discussion of several representative works on the representation theory, harmonic analysis and the theory of automorphic forms on covering groups.

Shortly after the middle of the last century, the classical theory of integer weight modular forms on the upper half plane was recast in the framework of automorphic representations on the group $\mathbf{SL}_2$. On the other hand, it has been known that fractional weights modular forms exist and play a significant role in classical modular form theory. One early example is the Jacobi theta function (a weight $1/2$ modular form), or more generally the theta function associated to an integer lattice of odd rank. It was then observed that such modular forms should correspond to automorphic representations on covering groups of $\mathbf{SL}_2(\mathbb{A})$. This gives a strong impetus for a systematic study of covering groups of adelic groups.

6.1. Segal-Shale-Weil representation. — One of the first systematic study of representations of a covering group is the work of Weil [**116**] on the so-called Segal-Shale-Weil representations (also called the oscillator representations) of the unique 2-fold cover of $\mathrm{Sp}_{2n}(k)$ (where k is a local field). This 2-fold cover is called the metaplectic group $\mathrm{Mp}_{2n}(k)$. Weil and others after him (such as Kubota [**56**] and Rao [**94**]) gave a comprehensive study of the 2-cocycles describing the metaplectic groups and its Weil representations. Weil's goal for developing this was to reformulate the theory of theta functions in the representation theoretic framework and to express previous results of Siegel (such as the Siegel mass formula and Siegel-Weil formula) in this language [**117**]. The Weil representations subsequently became a key ingredient in Howe's theory of dual pair correspondence (or theta correspondence).

6.2. The work of Kubota and Patterson. — In the late 1960s, almost concurrently as Moore and Matsumoto were doing their groundbreaking work, Kubota initiated a systematic study of the coverings of SL_2 or GL_2 (beyond the 2-fold cover), giving precise 2-cocycles for these covers [**56, 57**]. He was also interested in constructing analogs of Jacobi's theta function for these higher degree covers, using the residues of Eisenstein series. Patterson made a detailed study of the Fourier expansion of some of these theta functions on higher degree covers, noting that they contain interesting arithmetic information. In particular, for the 3-fold cover of GL_2, he showed in [**86, 87**] that the Fourier coefficients of the cubic theta function are cubic Gauss sums. Using this connection, Heath-Brown and Patterson [**51**] showed the equidistribution of the angular components of cubic Gauss sums. This suggests that one might find arithmetic applications by studying the Fourier expansion of interesting automorphic forms on covering groups. We will discuss some other of these arithmetic applications later on.

For higher degree covers, however, the structure of the Fourier coefficients of the generalized theta functions becomes much more complicated. This was subsequently explained by Deligne [**35**] as a consequence of the fact that Whittaker models are not unique for higher degree covers of $\mathbf{SL}_2$.

6.3. Shimura's correspondence. — One of the key milestones in the theory of automorphic forms on covering groups is Shimura's 1973 Annals paper [**102**], in which he developed a theory of Hecke operators for half integer weight modular forms and proved a correspondence between half integer weight modular forms and integer weight modular forms. Shimura proved the correspondence which bears his name by using the converse theorem of Weil. To do so, he introduced another innovation in his paper: a Rankin-Selberg integral for the standard L-function of a half-integer weight modular form. A slight variant of this Rankin-Selberg integral gives the symmetric square L-function of an integer weight modular form, which was used by Gelbart-Jacquet in their work on the symmetric square lifting from GL_2 to GL_3.

This influential paper of Shimura is the first to establish a lifting from Hecke eigenforms of a covering group to those of a linear group. It led to two independent lines of development, as we recall below. Both of these arise from the attempt to formulate Shimura's results in the setting of automorphic representations.

6.4. Kazhdan-Patterson covering and Flicker-Kazhdan lifting. — The first line of development from Shimura's paper is the work [**39**] of Flicker, who used the trace formula approach to prove the Shimura correspondence. More precisely, Flicker compared the trace formula of a particular degree n cover of GL_2 constructed by Kubota with that of GL_2 and proved a one-to-one correspondence between cuspidal automorphic representations of $\widetilde{\mathrm{GL}}_2$ and cuspidal representations of GL_2 whose central character is an n^{th} power. Thus, his work went beyond what Shimura did as he considered not just 2-fold covers of GL_2. In this adelic treatment of the Shimura correspondence, there is a local correspondence between genuine representations of the local covering group and those of the linear group GL_2. This local correspondence is expressed by a local character identity.

Following up on this work, Kazhdan-Patterson [**54**] considered degree n covers of GL_r which are obtained by pulling back from the degree n cover of SL_{r+1} (with GL_r embedded in SL_{r+1} in a standard way) and a standard twisting operation. Such covers are now called Kazhdan-Patterson covers and they generalize the Kubota covers of GL_2. In [**54**], Kazhdan and Patterson were largely interested in extending the results of Kubota noted above to the higher rank case of GL_r; in particular, they constructed generalizations of theta functions as residues of Eisenstein series. Further, their paper also laid the groundwork for an extension of Flicker's results from covers of GL_2 to the Kazhdan-Patterson covers of GL_r. This extension was initiated in their subsequent paper **[55]** and pursued further in the paper **[40]** of Flicker-Kazhdan. In **[40]**, the authors used a simple trace formula to prove a correspondence between cuspidal automorphic representations of the Kazhdan-Patterson covers of GL_r and cuspidal representations of GL_r under some simplifying local conditions (which allow the use of the simple trace formula). As in the GL_2 case, subordinate to this global correspondence is a local correspondence of representations based on a local character identity.

Somewhat unfortunately, it has been noted by several people that there are some errors in the papers **[54]** and **[40]**; given the subtlety of the structure theory of covering groups, this is quite understandable and certainly does not detract from the pioneering nature of these papers. The authors of **[40]** and **[54]** have however not provided an account and erratum for these errors. This is quite unfortunate, as there is no doubt that most of the results there must be true, at least if one imposes some conditions on the degree of the cover. Some further work in this direction, which cleared up some of the issues, were carried out by A. Kable **[53]**, P. Mezo **[79, 78]** and by Banks-Levy-Sepanski **[11]** among others.

6.5. The work of Waldspurger. — Another line of development originating from Shimura's 1973 paper is the work of Waldspurger which uses the technique of theta correspondence. In two papers **[112, 114]**, using the theta correspondence for $\mathrm{Mp}_2 \times \mathrm{SO}_3$, Waldspurger obtained a complete description of the automorphic discrete spectrum of the metaplectic group Mp_2 in terms of that of $\mathrm{PGL}_2 = \mathrm{SO}_3$. In particular, over local fields, Waldpsurger obtained a classification of the genuine representations of Mp_2 in the style of the local Langlands correspondence. Moreover, his description of the automorphic discrete spectrum of Mp_2 is in the style of the Arthur conjecture **[7]**, i.e., using local and global packets and having a global multiplicity formula.

What is especially intriguing is that the global multiplicity formula involves the global root number of a cuspidal representation of PGL_2. It should be noted, however, that at a critical point of his work, Waldspurger had needed to appeal to Flicker's results **[39]** obtained by the trace formula mentioned above. As a consequence, Waldspurger showed the existence of nonvanishing of central L-value of quadratic twists of automorphic L-functions of GL_2. Nowadays, however, one can avoid appealing to **[39]**, as Bump, Friedberg and Hoffstein have independently proven the necessary non-vanishing of central L-values. For a more detailed discussion of Waldspurger's work in light of Bump-Friedberg-Hoffstein **[19, 20, 44]**, the reader can look at **[47]**.

The work of Waldspurger has led to other significant arithmetic applications. For example, in **[113]**, he obtained a formula expressing the Fourier coefficients of half integral weight modular forms in terms of the central critical L-value of its Shimura correspondent. This result was applied by Tunnell **[111]** to provide a solution to the congruence number problem modulo the Birch-Swinnerton-Dyer conjecture.

It is natural to ask for an extension of Waldspurger's work to Mp_{2n} by using the theta correspondence for $\mathrm{Mp}_{2n} \times \mathrm{SO}_{2n+1}$. This has come slowly over the past 35 years. In the local setting, the local Shimura correspondence, giving a classification of the genuine representations of Mp_{2n} in terms of the representations of SO_{2n+1}, was shown by Adams-Barbasch [**3**] over $\mathbb{R}$ in the 1990's. The analogous result for p-adic fields was only shown fairly recently by Gan-Savin [**49**]. Over number fields, a conjectural extension of Waldspurger's results to Mp_{2n} was given in [**48**] and [**46**]. A recent preprint of Gan-Ichino gives a classification of the part of the automorphic discrete spectrum of Mp_{2n} associated to tempered A-parameters, thus proving the conjecture formulated in [**48**]. The reason for this long lapse in extending Waldspurger's results to Mp_{2n} is because that it requires recent advances in the theory of theta correspondences as well as the recent results of Arthur [**7**] on the automorphic discrete spectrum of classical groups. The result of Waldspurger on Fourier coefficients of half-integral weight modular form was extended to the setting of the Whittaker-Fourier coefficients of cuspidal representations of Mp_{2n} in a recent series of papers by Lapid-Mao [**63, 65, 64**].

6.6. Fourier coefficients of metaplectic Eisenstein series and generalized theta functions. — The work of Kubota and Patterson on the Fourier coefficients of generalized theta functions and metaplectic Eisenstein series was continued by several mathematicians in the 1980's and 1990's, most notably Bump, Friedberg, Hoffstein and their students or collaborators. An early work is the paper [**23**] of Bump-Hoffstein which shows that cubic L-functions occur in the Fourier expansion of Eisenstein series on a 3-fold Kazhdan-Patterson cover of GL_3. Several conjectures were highlighted and formulated in the paper [**26**] of Bump-Hoffstein and some further works in this direction include [**25, 10, 106, 107**]. There are complementary local results [**21**] on Whittaker functions of unramified genuine representations, analogous to the Casselman-Shalika formula in the linear case. These results on Fourier coefficients have found many stunning arithmetic applications, concerning nonvanishing of central values of twists of automorphic L-functions, such as [**19, 20, 14**].

These early works ultimately led Brubaker, Bump, Friedberg and Hoffstein to develop the theory of Weyl group multiple Dirichlet series in a series of papers (see for example, [**13, 15**]), together with important contributions from Chinta and Gunnells [**31, 32**]. This theory of Weyl group multiple Dirichlet series has found surprising connections with combinatorics, statistical physics and quantum groups. The local theory of the Casselman-Shalika formula culminated in the recent papers of Chinta-Offen and McNamara [**33, 75, 77**].

Automorphic descent in the covering setting is also explored by Friedberg and Ginzburg [**43, 42**]. Some other recent work on the Fourier coefficients of metaplectic Eisenstein series is contained in [**16, 41, 45**], and also the thesis work of Y.Q. Cai [**28, 27**].

6.7. Automorphic L-functions. — Another active area of research concerns the theory of automorphic L-functions for metaplectic forms. To have a definition of automorphic

L-functions, one would need to have an understanding of unramified representations and a notion of the dual group of a covering group. For Mp_{2n} and the Kazhdan-Patterson covers of GL_n (under some conditions on the degree of covering), one has a natural candidate for the dual group. For Mp_{2n} the natural dual group is $\mathrm{Sp}_{2n}(\mathbb{C})$ and for the Kazhdan-Patterson covers, it is $\mathrm{GL}_n(\mathbb{C})$, at least under some assumptions on the degree of cover. For these examples of covering groups, one can define Satake parameters for unramified representations and thus define the notion of partial automorphic L-functions. To show the usual analytic properties of these L-functions, one would try to find some Rankin-Selberg integrals for these automorphic L-functions. Some early work in this direction include **[24, 25, 18]**. On the other hand, the thesis work of D. Szpruch **[108, 109]** develops the Langlands-Shahidi theory for Mp_{2n}. A more recent preprint of Cai-Friedberg-Ginzburg-Kaplan **[29]** gives a sketch of a generalization of the doubling method of Piatetski-Shapiro-Rallis, which gives a Rankin-Selberg integral for the standard L-function of covers of classical groups.

Metaplectic forms have also proved useful in constructing Rankin-Selberg integrals for automorphic L-functions of linear groups. The prime example is the work of Bump-Ginzburg **[22]** which extended Shimura's original work to give a Rankin-Selberg integral for the symmetric square L-function of cuspidal representations of GL_n, using an Eisenstein series on a double cover of GL_n. Based on their work, the case of twisted symmetric square L-function is treated by Takeda **[110]**.

6.8. Savin's Hecke algebra correspondence. — In another direction, Savin studied and determined the structure of the Iwahori Hecke algebra for covers $\tilde{G}$ of simply-connected groups G **[100, 101]** and showed that they are isomorphic to the Iwahori Hecke algebra of an appropriate linear group. This gives a bijection between the irreducible genuine representations of $\tilde{G}$ with Iwahori-fixed vectors and those of the linear group. He did the same for the spherical Hecke algebra, thus obtaining a correspondence of unramified representations. These papers of Savin were the first to attempt a systematic development of the representation theory of general covering groups, going beyond treating families of examples. It gives strong suggestions for the dual groups of covers of simply-connected groups.

6.9. Character identitites. — The Flicker-Kazhdan local correspondence suggested that lifting of representations between covering and linear groups can be formulated in terms of local character identities. Such local character identities, in the context of Kazhdan-Patterson covers and other covering groups such as Mp_{2n}, were studied by J. Adams in a series of papers in the 1990's **[1, 2]**. Most of Adams' work is focused on coverings of real groups. It culminates in a long paper of Adams and Herb **[5]** which establishes such local character identities in a very general setting of coverings of real groups.

6.10. Real groups. — In Harish-Chandra's work on the invariant harmonic analysis of real Lie groups, he did not in fact limit himself to the case of linear algebraic groups, but allowed finite central covers of these. Hence, Harish-Chandra's classification of discrete series representations and his Plancherel theorem giving the decomposition of the regular representation $L^2(G)$ were shown for covers of real Lie groups. Likewise the technique of cohomological induction (the theory of Zuckerman functors) was also developed in this same setting.

Hence, one understands a lot more about the genuine representation theory of real covering groups. Indeed, there is a classification of such genuine representations called Vogan duality and some representative works in this direction are those of Renard-Trapa [**98, 99**] and Adams-Trapa [**6**], which led to a Kazhdan-Lusztig algorithm relating the irreducible characters of covering groups and those of standard modules. The recent paper [**4**] relates the unitary duals of covering groups and those of an appropriate linear group.

6.11. Invariant harmonic analysis, Eisenstein series and trace formula. — Our discussion above gives the impression that many results in the representation theory or the theory of automorphic forms on covering groups are based on the study of examples. While this is true to some extent, we would now like to highlight some general results which are necessary ingredients for a systematic theory.

We begin with invariant harmonic analysis as developed by Harish-Chandra. As mentioned above, Harish-Chandra's work on the invariant harmonic analysis of real Lie groups applies to finite covers of linear groups, such as his classification of discrete series representations and his Plancherel theorem giving the decomposition of the regular representation $L^2(G)$. His analogous results for p-adic groups were written up by Silberger [**103**] and also Waldspurger [**115**], but only in the context of linear reductive groups. Recently, many of these foundational results are extended to the covering case (with largely the same proof) by W.-W. Li [**67**] (character theory, orbital integrals and Plancherel theorem). Some other results, such as the theory of R-groups and the Howe's finiteness conjecture for invariant distributions, were extended to covering groups by C.H. Luo in his thesis work [**72**]. For smooth representation theory, many standard results developed in [**30**], such as the Langlands classification and the Casselman square-integrability or temperedness criterion, were extended to general covering groups by Ban-Jantzen [**8, 9**]. It is worth noting that the theory of Bernstein center also works in the same way as in the linear case, as noted by Deligne in his rendition [**12**] of the theory of Bernstein center.

Likewise, in the global setting, the Langlands theory of Eisenstein series was already developed in the setting of covering groups in Moeglin-Waldspurger's monograph [**81**]. In a striking series of recent papers [**69, 67, 68, 70**], W.-W. Li has developed the theory of the Arthur-Selberg trace formula for general covering groups, bringing it to the stage of the invariant trace formula.

7. A Langlands program for Brylinski-Deligne extensions

After the historical account of the previous sections, it is natural to ask if the framework of the Langlands program can be extended to the setting of covering groups. The classical Langlands program is built upon the rich and functorial structure theory of linear reductive groups. Such a structure theory of covering groups has now been developed in the work of Brylinski-Deligne and it is our belief that the Brylinski-Deligne theory serves as a good starting point for a systematic extension of the Langlands program to covering groups.

7.1. What constitutes a Langlands program? — Before one begins, it may be good to ask what exactly constitutes a Langlands program. For this, one can do no better than to turn to the starting point of the classical Langlands program, which is contained in the famous letter of Langlands to Weil. The key new ideas introduced in this letter are the notions of the dual group $G^\vee$ and the L-group ${}^LG$ of a connected reductive group $\mathbf{G}$. Langlands subsequently reworked in his monograph "Euler Products" [**59**] the theory of spherical functions and the Satake transform, reinterpreting Satake's results in the framework of the L-group. This allows him to classify the unramified representations of a quasi-split p-adic group in terms of unramified local Galois representations valued in the L-group, which immediately suggests (at least with hindsight) the local Langlands correspondence: classifying all irreducible representations of $\mathbf{G}(k)$ by local Galois representations valued in ${}^LG$. This unramified local Langlands correspondence also allows him to introduce the notion of "automorphic L-functions attached to a finite-dimensional representation of the L-group".

Thus, a key ingredient for a Langlands program is undoubtedly the notion of a dual group and an L-group, and a first test for any such candidate dual group or L-group is whether it gives a natural formulation of the Satake isomorphism, leading to a classification of unramified representations.

A second key realization of Langlands in the initial stage of the classical Langlands program is the difference between conjugacy and stable (or geometric) conjugacy in a reductive group [**60**]. More precisely, for a connected reductive group $\mathbf{G}$ defined over a local field k, say, one may consider a coarser equivalence relation than the usual notion of conjugacy in $\mathbf{G}(k)$. This coarser equivalence relation is conjugacy by elements of $\mathbf{G}(k^{\mathrm{sep}})$, where k^{sep} is a separable closure of k. This led him to develop the theory of endoscopy, including the definition of endoscopic groups [**58, 61**] and the definition of transfer factors with Shelstad [**62**].

To summarize, the two key ingredients for a Langlands program are, in our views:

- a definition of dual groups and L-groups;
- a theory of stable conjugacy and endoscopy.

We note that both these ingredients in the classical Langlands program require one to start with a reductive group $\mathbf{G}$ over k, and not just the topological group $\mathbf{G}(k)$. For example, suppose that k'/k is a separable finite extension of local fields, $\mathbf{G}$ a reductive group over k' and $\mathbf{H} := \mathrm{Res}_{k'/k}\mathbf{G}$, so that $\mathbf{G}(k') = \mathbf{H}(k)$ as topological groups, and there is no difference between the representation theories of $\mathbf{G}(k')$ and $\mathbf{H}(k)$. The dual groups and L-groups of $\mathbf{G}$ and $\mathbf{H}$ are however different, even if they can both be used to classify the irreducible representations of the same group. Similarly, the notion of stable conjugacy only makes sense because one has the notion of k^{sep}-points of a reductive group $\mathbf{G}$ over k, with an inclusion $\mathbf{G}(k) \hookrightarrow \mathbf{G}(k^{\mathrm{sep}})$. This suggests that to have these two ingredients in the setting of covering groups, one might need to work with covering groups of algebraic origin, such as those provided by the Brylinski-Deligne theory.

7.2. Dual Groups. — We now discuss some prior work on the two key ingredients of a Langlands program highlighted above. As we mentioned in the previous section, people knew what the dual groups of some examples of covering groups should be, such as for Mp_{2n}, some Kazhdan-Patterson covers and also covers of simply-connected groups. A systematic and general theory was developed in the work of Finkelberg-Lysenko [**38**] and Reich [**95**] in the context of the Geometric Langlands Program. This was followed in the classical context by the work of McNamara [**76**] and independently Weissman [**121**] who defined the modified dual root datum associated to a Brylinski-Deligne cover, using the invariants associated to such a multiplicative $\mathbf{K}_2$-torsor by [**17**].

7.3. Endoscopy. — The theory of endoscopy for covering groups was initiated by the work of Adams [**1**] and Renard [**96, 97**] in the context of $\mathrm{Mp}_{2n}(\mathbb{R})$. The thesis work of J. Schultz considered the case of Mp_2 over p-adic fields. The general case of Mp_{2n} over any local field was completed in the thesis work of W.-W. Li [**66**], with the endoscopic groups of Mp_{2n} being the groups $\mathrm{SO}_{2a+1} \times \mathrm{SO}_{2b+1}$, as (a, b) vary over ordered pairs of non-negative integers such that $a + b = n$. In particular, Li established the transfer of orbital integrals from Mp_{2n} to its endoscopic groups, the fundamental lemma for the unit element of the spherical Hecke algebra and the weighted fundamental lemma [**71**]. In his thesis work, C.H. Luo has shown the fundamental lemma for the whole spherical Hecke algebra, as well as established the expected local character identities for the local L-packets of Mp_{2n} defined by the local Shimura correspondence of [**49**]. Based on his theory of endoscopy, Li has begun the stabilization of the invariant trace formula for Mp_{2n}. It remains to see whether the case of Mp_{2n} is an anomaly or is an example of a theory of endoscopy which encompasses a large class of covering groups, such as the Brylinski-Deligne covers.

7.4. This volume. — This brings us to the current volume.

One of us (M.H.W.) has been thinking about using the Brylinski-Deligne theory as a starting point for the Langlands program for covering groups for some time. The paper [**119**] is an initial attempt to bring the Brylinski-Deligne structure theory to bear on the genuine representation theory of covering tori, whereas the paper [**120**] describes the interaction of the Brylinski-Deligne structure theory with the Bruhat-Tits theory of open compact subgroups, answering a question raised at the end of

[17], while the paper **[52]** applies this to the depth zero genuine representation theory of Brylinski-Deligne covers. The paper **[121]** gives a definition of the dual group and L-group of a Brylinski-Deligne cover of a split group, using the language of Hopf algebras. This turns out to be overly complicated, making the theory hard to use. Moreover, with hindsight, the candidate L-group there is not always the right one, as it does not make use of all the Brylinski-Deligne invariants. These initial attempts and ideas were communicated in a series of letters between M.H.W. and Deligne over the period 2007-2014 and Deligne's ideas and comments have been extremely helpful in every stage of the development. The interested reader can find this series of letters, which documents the evolution of some of the ideas discussed in this volume, in **[118]**.

These efforts culminate in the first paper of this volume (by M.H.W.) which defines the L-group of a Brylinski-Deligne cover of a quasi-split group using the language of étale gerbes, and tests this L-group for the purpose of representation theory, including the Satake isomorphism and classification of unramified representations as well as the classification of discrete series for covers of real groups. The second paper (by W.T.G. and F.G.) specializes to the case of covers of split groups and introduces another construction of the L-group also due to M.H.W, which is more down-to-earth, as it avoids the language of étale gerbes. There is some overlap between the second paper and the first, as the second paper also conducts the necessary tests for the legitimacy of the L-group, namely the Satake isomorphism and the representation theory of covers of split tori. This second paper then goes on to explore some cases of Langlands functoriality such as base change. As a consequence of these two papers, one can now define partial automorphic L-functions for automorphic representations of a Brylinski-Deligne cover. In a followup **[50]** to this work, one of us (F.G.) has extended the results of Langlands' "Euler Products" **[59]** to the covering setting, using the constant terms of Eisenstein series to show the meromorphic continuation of some of these automorphic L-functions (those of Langlands-Shahidi type). Finally, the third paper (by M.H.W.) of this volume shows that the two notions of L-groups used in the first two papers are in fact the same (for covers of split groups). Since the papers in the volume come with their own extended introductions, we shall refrain from giving a more detailed introduction here.

Finally, we note that this volume is simply a beginning, and we have only discussed one of the two key ingredients of a Langlands program. We have not addressed the issue of stable conjugacy and endoscopy, except for a brief speculative section in the second paper. We hope that this volume will stimulate further research in this area, leading one day to a fulfillment of the hope expressed by Deligne in his letter **[118]** to M.H.W. (Dated Dec. 14, 2007):

"For me, the aim is to understand "metaplectic" forms on semi-simple groups, the hope being that they are not "new" object, but rather correspond to usual automorphic forms on some other groups, on which they give new information. I would like to have precise conjectures on the hoped for correspondence, and I view my paper with Brylinski as setting a landscape in which conjectures should fit."

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L-GROUPS AND PARAMETERS FOR COVERING GROUPS

by

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Abstract. — We incorporate covers of quasisplit reductive groups into the Langlands program, defining an L-group associated to such a cover. We work with all covers that arise from extensions of quasisplit reductive groups by $\mathbf{K}_2$—the class studied by Brylinski and Deligne. We use this L-group to parameterize genuine irreducible representations in many contexts, including covers of split tori, unramified representations, and discrete series for double covers of semisimple groups over $\mathbb{R}$. An appendix surveys torsors and gerbes on the étale site, as they are used in the construction of the L-group.

***Résumé* (L-groupes et paramètres pour les revêtements de groupes).** — Nous intégrons des revêtements de groupes réductifs quasi-déployés dans le programme de Langlands, en définissant un L-groupe associé à un tel revêtement. Nous travaillons avec tous les revêtements qui résultent d'extensions de groupes réductifs quasi-déployés par $\mathbf{K}_2$ — la classe étudiée par Brylinski et Deligne. Nous utilisons ce L-groupe pour paramétrer des représentations irréductibles spécifiques dans de nombreux contextes, incluant les revêtements de tores déployés, les représentations sphériques, et les séries discrètes pour les revêtements doubles de groupes semi-simples réels. Une appendice étudie les torseurs et gerbes sur le site étale, puisqu'ils sont utilisés dans la construction du L-groupe.

Introduction

Constructions and conjectures. — Let $\mathbf{G}$ be a quasisplit reductive group over a local or global field F. In [**18**], Brylinski and Deligne introduce objects called *central extensions of* $\mathbf{G}$ *by* $\mathbf{K}_2$, and they express hope that for "a global field this will prove useful in the study of 'metaplectic' automorphic forms". We pursue their vision in this paper, and elaborate below.

Let n be a positive integer and let μ_n denote the group of nth roots of unity in F. Assume that μ_n has order n. Let $\mathbf{G}'$ be a central extension of $\mathbf{G}$ by $\mathbf{K}_2$, in the sense

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of [**18**]. We call the pair $\tilde{\mathbf{G}} := (\mathbf{G}', n)$ a "degree n cover" of $\mathbf{G}$. Fix a separable closure $\bar{F}/F$ and write $\mathrm{Gal}_F = \mathrm{Gal}(\bar{F}/F)$. Fix an injective character $\varepsilon \colon \mu_n \hookrightarrow \mathbb{C}^\times$.

Associated to $\mathbf{G}'$, Brylinski and Deligne associate three invariants, which we call Q, $\mathscr{D}$, and f. The first is a Weyl- and Galois-invariant quadratic form on the cocharacter lattice of a maximal torus in $\mathbf{G}$. The second is a central extension of this cocharacter lattice by $\mathscr{G}_m$ (in the category of sheaves of groups on $F_{\text{ét}}$). The third is difficult to describe here, but it reflects the rigidity of central extensions of simply-connected semisimple groups.

Part 1 of this article defines the L-group ${}^{\mathsf{L}}\tilde{G}$ of such a cover $\tilde{\mathbf{G}}$, from the three invariants Q, $\mathscr{D}$, and f (as well as n, ε, and $\bar{F}$). It is an extension of Gal_F by $\tilde{G}^\vee$, where $\tilde{G}^\vee$ is a pinned complex reductive group, on which Gal_F acts by pinned automorphisms. Unlike Langlands' L-group, ours does not come equipped with a distinguished splitting—although a noncanonical splitting often exists. Throughout the construction of ${}^{\mathsf{L}}\tilde{G}$, the arithmetically-inclined reader may replace $\mathbb{C}$ by any $\mathbb{Z}[1/n]$-algebra Ω endowed with $\varepsilon \colon \mu_n \hookrightarrow \Omega^\times$, without running into much difficulty. The dual group $\tilde{G}^\vee$ has been considered by other authors, and it appears in various forms in [**26**], [**51**], [**60**], and [**1**]. We tabulate the dual groups; each comes equipped with a 2-torsion element $\tau_Q(-1)$ in its center $\tilde{Z}^\vee$.

In our previous article [**78**], we limited our attention to split reductive groups, and constructed an L-group by bludgening Hopf algebras with two "twists". The construction here is more delicate, and more general. The "first twist" of [**78**] is encoded here in the following way. The quadratic Hilbert symbol may be used to define a canonical 2-cocycle, yielding an extension $\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_F \twoheadrightarrow \mathrm{Gal}_F$ which we call the metaGalois group. The metaGalois group may be of independent interest—one might look for its representations in nature, e.g., in the étale cohomology of a variety over $\mathbb{Q}(i)$ equipped with twisted descent data to $\mathbb{Q}$. Pushing out the metaGalois group via the central 2-torsion element in $\tilde{G}^\vee$ yields the first twist,

$$\tilde{Z}^\vee \hookrightarrow (\tau_Q)_* \widetilde{\mathrm{Gal}}_F \twoheadrightarrow \mathrm{Gal}_F\,. \tag{0.1}$$

The "second twist" of [**78**] provided the greatest challenge there and here. There, it was defined by twisting the multiplication in a Hopf algebra. After attempting many reformulations (e.g., a Tannakian approach), we found the gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ on $F_{\text{ét}}$, which applies to covers of quasisplit groups and serves as the second twist here. It is a bit different from gerbes that typically arise in the Langlands program, and so we include an appendix with relevant background on torsors and gerbes. The étale fundamental group of this gerbe provides the second twist,

$$\tilde{Z}^\vee \hookrightarrow \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})) \twoheadrightarrow \mathrm{Gal}_F\,. \tag{0.2}$$

The Baer sum of (0.1) and (0.2) gives an extension of Gal_F by $\tilde{Z}^\vee$. Pushing out to $\tilde{G}^\vee$ (respecting the Gal_F-action throughout) yields the L-group

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}_F,$$

of Part 1. In addition to its construction, we verify that this L-group behaves well with respect to Levi subgroups, passage between global fields, local fields, and rings of integers therein, and that the L-group construction is functorial for a class of "well-aligned" homomorphisms.

The construction of the L-group allows us to consider Weil parameters. When F is a local or global field, we consider the set $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/F)$ of $\tilde{G}^\vee$-orbits of Weil parameters $\mathcal{W}_F \to {}^{\mathsf{L}}\tilde{G}$. When $\mathcal{O}$ is the ring of integers in a nonarchimedean local field, we consider the set $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O})$ of $\tilde{G}^\vee$-orbits of unramified Weil parameters $\mathcal{W}_F \to {}^{\mathsf{L}}\tilde{G}$. One could similarly define Weil-Deligne parameters, (conjectural) global Langlands parameters, etc.

In a set of unpublished notes, we constructed an L-group for split groups without using the gerbe discussed above. This "E_1+E_2" construction has been studied further by Wee Teck Gan and Gao Fan in [**29**] and [**30**]. The construction of this paper agrees with the E_1+E_2 construction for split reductive groups; this is proven in a short note at the end of this volume.

With the construction of the L-group complete, we turn our attention to representation theory in Part 2. The cover $\tilde{\mathbf{G}}$ and character ε allow us to define ε-genuine irreducible representations of various sorts. The set $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\bullet)$ is defined in three contexts.

- F **a local field:** Brylinski and Deligne construct [**18**, §10.3] a central extension $\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G = \mathbf{G}(F)$, and we consider the set $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F)$ of equivalence classes of irreducible admissible ε-genuine representations of $\tilde{G}$.
- F **a global field:** Brylinski and Deligne construct [**18**, §10.4] a central extension $\mu_n \hookrightarrow \tilde{G}_{\mathbb{A}} \twoheadrightarrow G_{\mathbb{A}} = \mathbf{G}(\mathbb{A})$, canonically split over $\mathbf{G}(F)$, and we consider the set $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F)$ of equivalence classes of ε-genuine automorphic representations of $\tilde{G}_{\mathbb{A}}$.
- $\mathcal{O}$ **the integers in a nonarchimedean local field** F**:** Brylinski and Deligne construct [**18**, §10.3, 10.7] a central extension $\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G = \mathbf{G}(F)$, canonically split over $G^\circ = \mathbf{G}(\mathcal{O})$. We consider the set $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O})$ of equivalence classes of irreducible G°-spherical representations of $\tilde{G}$.

Part 2 introduces these classes of representations, reviewing or adapting foundational results as needed. These include old results, such as the basic theory of admissible, unitary, and tempered representations, and results which are recent for covering groups, such as the Satake isomorphism and Langlands classification. Some new features arise for covering groups: we introduce the notion of "central core character" which is a bit coarser than "central character". We place irreducible representations into "pouches" which should be subsets of L-packets in what follows later.

The remainder of the paper is devoted to supporting the following "Local Langlands Conjecture for Covers" (LLCC), an analog of the local Langlands conjectures (LLC). For the LLC, we refer to the excellent survey by Cogdell [**21**].

***Conjecture 0.1* (LLCC).** — When F is a local field, there is a *natural* finite-to-one parameterization,

$$\mathcal{L}_\varepsilon\colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/F).$$

Such a conjecture is nearly meaningless, without defining the adjective "natural". Naturality in the (traditional) local Langlands conjectures (LLC) includes compatibility with the bijective parameterizations for split tori (class field theory) and unramified representations (the Satake isomorphism), and with parabolic induction (the Langlands classification). In [**11**, §10], it is suggested that naturality includes desiderata which specify how central characters and twisting by characters should correspond to various properties of and operations on Weil parameters. One could add to these desiderata today, specifying for example how the formal degree (of discrete series) should correspond to adjoint γ-factors for Weil parameters (see [**37**]), or how the contragredient operation should correspond to the Chevalley involution (see [**3**]).

For covers, we can make a similar list of desiderata for the LLCC. We expect a finite-to-one parameterization for covering groups, compatible with our results for split tori (described in Part 3), with the unramified case (described in Part 4), with parabolic induction (via the Langlands classification for covers) and central core characters and twisting by characters (described in Part 2), and with formal degrees and adjoint γ-factors.

Unlike the LLC, we have not attempted to characterize the image of our conjectural parameterization in this paper, i.e., we have not identified the "relevant" parameters for covering groups. The cases of split tori and discrete series for real groups should suggest a characterization in the future.

Part 3 focuses on the case of "sharp" covers of split tori. For such a sharp cover (over a local or global field, or in the unramified setting), we define a *bijective* parameterization

$$\mathcal{L}_\varepsilon\colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/\bullet) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}/\bullet).$$

This parameterization is natural for pullbacks of covers via isomorphisms, for isomorphisms of covers of a given split torus, and for Baer sums of covers. This case constrains and guides many others, and it occupies the largest part of this article. The "sharp" case quickly leads to the general case of split tori, where the parameterization may no longer be surjective.

Part 4 includes three more cases where a precise parameterization is possible. First is the spherical/unramified case. When $\tilde{\mathbf{G}}$ is a cover of a quasisplit group over $\mathcal{O}$, we define a *bijective* parameterization

$$\mathcal{L}_\varepsilon\colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O}) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O}).$$

This parameterization follows from the Satake isomorphism (for covering groups), the parameterization for sharp covers of split tori above, and careful tracking of the Weyl group actions.

In [29], Gan and Gao have verified the LLCC for split tori and unramified representations of split reductive groups, in the context of the L-group given by the $E_1 + E_2$ construction.

Second, we consider an anisotropic torus $\mathbf{T}$ over $\mathbb{R}$ (i.e., $T = \mathbf{T}(\mathbb{R})$ is compact) and a sharp double cover $\tilde{\mathbf{T}}$ of $\mathbf{T}$. In this case, we define a *bijective* parameterization

$$\mathscr{L}_\varepsilon \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/\mathbb{R}) \rightarrow \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}/\mathbb{R}).$$

This case—when $\tilde{T}$ is a torus in the oldest-fashioned sense and the representation theory is dead-simple—is interesting from the standpoint of parameterization. It validates our choice of gerbe, since the equivalence class of the gerbe must be exactly right to correspond to the correct lattice of characters.

Third, when $\tilde{\mathbf{G}}$ is a double cover of a semisimple quasisplit group over $\mathbb{R}$, we define a finite-to-one parameterization of discrete series,

$$\mathscr{L}_\varepsilon \colon \mathbf{\Pi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}}/\mathbb{R}) \rightarrow \mathbf{\Phi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}}/\mathbb{R}).$$

This is based on the Harish-Chandra parameterization of discrete series, our parameterization for anisotropic tori, and careful tracking of involutions in the Weyl group.

Further questions. — As in the original Langlands conjectures, Conjecture 0.1 suggests directions for further investigation. Here are a few examples, within reach.

1. (Inspired by recent work of Wee Teck Gan and Fan Gao [29] on PGL_2). When $\tilde{\mathbf{G}}$ is a cover of a split group with trivial first invariant ($Q = 0$), one can find a z-extension (in the sense of Kottwitz [42]) $\mathbf{H} \rightarrow \mathbf{G}$ for which the pullback cover $\tilde{\mathbf{H}}$ is isomorphic to the trivial cover. This identifies genuine representations of $\tilde{G}$ with ordinary representations of H satisfying a constraint on the central character. This identification should be reflected on the side of Weil parameters, and the LLCC for such covers $\tilde{\mathbf{G}}$ should relate to the LLC for $\mathbf{H}$.
2. (Inspired by a conversation with Wee Teck Gan). When $\mathbf{G}$ is a group over $\mathbb{C}$, there are nontrivial covers $\tilde{\mathbf{G}}$, but the resulting covers of complex Lie groups split canonically. In this way, the genuine representations of $\tilde{G}$ correspond to ordinary representations of G. This should be reflected on the side of Weil parameters, and the LLCC for $\tilde{\mathbf{G}}$ should relate to the LLC for $\mathbf{G}$.
3. (Inspired by the work of Adams and Vogan [3]). When $\tilde{\mathbf{G}}$ is a cover of $\mathbf{G}$, a group over a local field F, one may define an inverse cover $\tilde{\mathbf{G}}^{\mathbf{op}}$ with respect to the Baer sum. If π is an ε-genuine representation of $\tilde{G}$, then the contragredient representation of π is naturally an ε-genuine representation of the inverse cover $\tilde{G}^{\mathrm{op}}$. The dual groups of $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{G}}^{\mathbf{op}}$ are the same, but the L-groups may not be. The contragredient should be reflected on the side of Weil parameters, extending the Chevalley involution of the dual group to a map connecting the L-group of $\tilde{\mathbf{G}}$ with the L-group of the inverse cover $\tilde{\mathbf{G}}^{\mathbf{op}}$.

Much broader investigations are possible as well, given the L-group constructed in this paper. Even restricting to the local case, examples include: the study of pure inner forms (strong inner forms for real groups) and "stability" for covering groups;

endoscopy for covering groups; the completion of the local Langlands conjectures for covers of reductive groups over $\mathbb{R}$; base change for covering groups; Langlands-Shahidi L-functions for covering groups (initiated by Fan Gao in [**30**] and Szpruch [**71**]); the parameterization of Iwahori-spherical and depth-zero representations; etc.

Philosophies. — A few principles are helpful when considering any putative Langlands program for covering groups.

1. There is no ε-genuine trivial (or Steinberg) representation for general covers, and so one should not expect a single distinguished splitting of the L-group.
2. If some set of things is parameterized by cohomology in degree 2, then that set of things should be viewed as the set of objects in a 2-category.
3. Things which "are trivial" (e.g., extensions, gerbes) can be isomorphic to trivial things in interesting ways.

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In [**29**] and [**30**], Wee Teck Gan and Fan Gao have tested some of the conjectures of this paper, and they have gone further in developing the Langlands program for covering groups. I have greatly appreciated our frequent conversations. Their results provided constraints which kept the constructions of this paper on track.

Pierre Deligne has kindly corresponded with me in a series of letters, and his generosity on these occasions has been incredibly helpful. His ideas led me to a deeper understanding of the crucial questions, and his correspondence motivated me to pursue this project further.

Notation

F: A field, typically local or global.

$\bar{F}$: A separable closure of F.

$\mathcal{O}$: The ring of integers in F, in the nonarchimedean local case.

$\mathbb{A}$: The ring of adeles of F, in the global case.

Fr: The geometric Frobenius automorphism.

S**:** A connected scheme, typically $\mathrm{Spec}(F)$ or $\mathrm{Spec}(\mathcal{O})$.

$\bar{s}$**:** The geometric point of S corresponding to $\bar{F}$.

Gal_S**:** The absolute Galois group $\pi_1^{\text{ét}}(S, \bar{s})$.

$\mathbf{X}$**:** An algebraic variety over S, or sheaf on S_{Zar}.

$\mathbf{G}_m$**:** The multiplicative group.

X **or** X_F**:** The F-points $\mathbf{X}(F)$ for such a variety.

$\underline{\mathbf{X}}$**:** A scheme over $\mathbb{Z}$.

$\underline{\boldsymbol{\mu}}_n$**:** The group scheme over $\mathbb{Z}$ of nth roots of unity.

μ_n**:** The group $\underline{\boldsymbol{\mu}}_n(S)$, assumed to be cyclic of order n.

$\mathscr{S}$**:** A sheaf on $S_{\text{ét}}$.

$\mathscr{S}[U]$**:** The sections of $\mathscr{S}$ over U ($U \rightarrow S$ étale).

$\mathscr{G}$**:** A local system on $S_{\text{ét}}$, of group schemes over $\mathbb{Z}$.

C**:** A category, with objects $\mathrm{Ob}(\mathsf{C})$.

E**:** A gerbe on $S_{\text{ét}}$.

$\mathsf{E}[U]$**:** The groupoid of sections of E over U.

A**:** An abelian group.

$A_{[n]}$**:** Its n-torsion subgroup.

$A_{/n}$**:** The quotient A/nA.

PART I
COVERING GROUPS AND THEIR L-GROUPS

1. Covering groups

Throughout this article, S will be a scheme in one of the following two classes: $S = \mathrm{Spec}(F)$ for a field F, or $S = \mathrm{Spec}(\mathcal{O})$ for a discrete valuation ring $\mathcal{O}$ with fraction field F. In the latter case, we assume that $\mathcal{O}$ contains a field, or that $\mathcal{O}$ has finite residue field. We will often fix a positive integer n, and we will assume that $\mu_n = \boldsymbol{\mu}_n(S)$ is a cyclic group of order n. In Section 4, we will place further restrictions on S.

1.1. Reductive groups. — Let $\mathbf{G}$ be a reductive group over S. We follow [**35**] in our conventions, so this means that $\mathbf{G}$ is a smooth group scheme over S such that $\mathbf{G}_{\bar{s}}$ is a connected reductive group for all geometric points $\bar{s}$ of S. Assume moreover that $\mathbf{G}$ is *quasisplit* over S.

Let $\mathbf{A}$ be a maximal S-split torus in $\mathbf{G}$, and let $\mathbf{T}$ be the centralizer of $\mathbf{A}$ in $\mathbf{G}$. Then $\mathbf{T}$ is a maximal torus in $\mathbf{G}$, and we say that $\mathbf{T}$ is a ***maximally split*** maximal torus. Let $\mathscr{X}$ and $\mathscr{Y}$ be the local systems (on $S_{\text{ét}}$) of characters and cocharacters of $\mathbf{T}$.

Let $\mathbf{N}$ be the normalizer of $\mathbf{T}$ in $\mathbf{G}$. Let $\mathscr{W}$ denote the Weyl group of the pair $(\mathbf{G}, \mathbf{T})$, viewed as a sheaf on $S_{\text{ét}}$ of finite groups. Then $\mathscr{W}[S] = \mathbf{N}(S)/\mathbf{T}(S)$ (see [**35**, Exposé XXVI, 7.1]). Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$, defined over S. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$.

Proposition 1.1. — *Assume as above that* $\mathbf{G}$ *is quasisplit, and* S *is the spectrum of a field or of a DVR. The group* $\mathbf{G}(S)$ *acts transitively, by conjugation, on the set of pairs* $(\mathbf{B}, \mathbf{T})$ *consisting of a Borel subgroup (defined over* S*) and a maximally split maximal torus therein.*

Proof. — As we work over a local base scheme S, [**35**, Exposé XXVI, Proposition 6.16] states that the group $\mathbf{G}(S)$ acts transitively on the set of maximal split subtori of $\mathbf{G}$ (defined over S).

Every maximally split maximal torus of $\mathbf{G}$ is the centralizer of such a maximal split torus, and thus $\mathbf{G}(S)$ acts transitively on the set of maximally split maximal tori in $\mathbf{G}$. The stabilizer of such a maximally split maximal torus $\mathbf{T}$ is the normalizer $\mathbf{N}(S)$. The Weyl group $\mathscr{W}[S] = \mathbf{N}(S)/\mathbf{T}(S)$ acts simply-transitively on the minimal parabolic subgroups containing $\mathbf{T}$ by [**35**, Exposé XXVI, Proposition 7.2]. This proves the proposition. $\square$

The roots and coroots (for the adjoint action of $\mathbf{T}$ on the Lie algebra of $\mathbf{G}$) form local systems Φ and $\Phi^\vee$ on $S_{\text{ét}}$, contained in $\mathscr{X}$ and $\mathscr{Y}$, respectively. The simple roots (with respect to the Borel subgroup $\mathbf{B}$) and their coroots form local systems of subsets $\Delta \subset \Phi$ and $\Delta^\vee \subset \Phi^\vee$, respectively. In this way we find a local system on $S_{\text{ét}}$ of based root data (cf. [**11**, §1.2]),

$$\Psi = (\mathscr{X}, \Phi, \Delta, \mathscr{Y}, \Phi^\vee, \Delta^\vee).$$

Write $\mathscr{Y}^{\text{sc}}$ for the subgroup of $\mathscr{Y}$ spanned by the coroots.

1.2. Covers. — In [**18**], Brylinski and Deligne study central extensions of $\mathbf{G}$ by $\mathbf{K}_2$, where $\mathbf{G}$ and $\mathbf{K}_2$ are viewed as sheaves of groups on the big Zariski site S_{Zar}. These extensions form a category we call $\mathsf{CExt}_S(\mathbf{G}, \mathbf{K}_2)$. Such a central extension will be written $\mathbf{K}_2 \hookrightarrow \mathbf{G}' \twoheadrightarrow \mathbf{G}$ in what follows. We add one more piece of data in the definition below.

Definition 1.2. — A degree n ***cover*** of $\mathbf{G}$ over S is a pair $\tilde{\mathbf{G}} = (\mathbf{G}', n)$, where

1. $\mathbf{K}_2 \hookrightarrow \mathbf{G}' \twoheadrightarrow \mathbf{G}$ is a central extension of $\mathbf{G}$ by $\mathbf{K}_2$ on S_{Zar};
2. n is a positive integer;
3. For all scheme-theoretic points $s \in S$, with residue field $\mathfrak{f}(s)$, $\#\boldsymbol{\mu}_n(\mathfrak{f}(s)) = n$.

Define $\mathsf{Cov}_n(\mathbf{G})$ (or $\mathsf{Cov}_{n/S}(\mathbf{G})$ to avoid confusion) to be the category of degree n covers of $\mathbf{G}$ over S. The objects are pairs $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ as above, and morphisms are those from $\mathsf{CExt}_S(\mathbf{G}, \mathbf{K}_2)$ (with n fixed).

If $\gamma\colon S_0 \to S$ is a morphism of schemes, then pulling back gives a functor $\gamma^*\colon \mathsf{Cov}_{n/S}(\mathbf{G}) \to \mathsf{Cov}_{n/S_0}(\mathbf{G}_{S_0})$. Indeed, a morphism of schemes gives inclusions of residue fields (in the opposite direction) and so Condition (3) is satisfied by the scheme S_0 when it is satisfied by the scheme S.

Central extensions $\mathbf{K}_2 \hookrightarrow \mathbf{G}' \twoheadrightarrow \mathbf{G}$ are classified by a triple of invariants $(Q, \mathscr{D}, f)$. For fields, this is carried out in [**18**], and the extension to DVRs (with finite residue field or containing a field) is found in [**79**]. The first invariant $Q\colon \mathscr{Y} \to \mathbb{Z}$ is a Galois-invariant Weyl-invariant quadratic form, i.e., $Q \in H^0_{\text{ét}}(S, \mathscr{S}ym^2(\mathscr{X})^{\mathscr{W}})$. The second invariant $\mathscr{D}$ is a central extension of sheaves of groups on $S_{\text{ét}}$, $\mathscr{G}_m \hookrightarrow \mathscr{D} \twoheadrightarrow \mathscr{Y}$. The third invariant f will be discussed later.

A cover $\tilde{\mathbf{G}}$ yields a symmetric $\mathbb{Z}$-bilinear form $\beta_Q\colon \mathscr{Y} \otimes_{\mathbb{Z}} \mathscr{Y} \to n^{-1}\mathbb{Z}$,

$$\beta_Q(y_1, y_2) := n^{-1} \cdot (Q(y_1 + y_2) - Q(y_1) - Q(y_2)).$$

This defines a local system $\mathscr{Y}_{Q,n} \subset \mathscr{Y}$,

$$\mathscr{Y}_{Q,n} = \{y \in \mathscr{Y} : \beta_Q(y, y') \in \mathbb{Z} \text{ for all } y' \in \mathscr{Y}\}.$$

The category of covers is equipped with the structure of a Picard groupoid; one may "add" covers via the Baer sum. If $\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_2$ are two covers of $\mathbf{G}$ of degree n, one obtains a cover $\tilde{\mathbf{G}}_1 \dotplus \tilde{\mathbf{G}}_2 = (\mathbf{G}_1' \dotplus \mathbf{G}_2', n)$.

When $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ is a degree n cover of $\mathbf{G}$, and $\mathbf{H} \to \mathbf{G}$ is a homomorphism of groups over S, write $\tilde{\mathbf{H}} = (\mathbf{H}', n)$ for the cover of $\mathbf{H}$ resulting from pulling back extensions by $\mathbf{K}_2$.

In three arithmetic contexts, a cover $\tilde{\mathbf{G}}$ yields a central extension of topological groups according to [**18**, §10.3, 10.4].

Global: If $S = \mathrm{Spec}(F)$ for a global field F, then $\tilde{\mathbf{G}}$ yields a central extension $\mu_n \hookrightarrow \tilde{G}_{\mathbb{A}} \twoheadrightarrow G_{\mathbb{A}}$, endowed with a splitting $\sigma_F\colon G_F \hookrightarrow \tilde{G}_{\mathbb{A}}$.

Local: If $S = \mathrm{Spec}(F)$ for a local field F, then $\tilde{\mathbf{G}}$ yields a central extension $\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G$, where $G = \mathbf{G}(F)$.

Local integral: If $S = \mathrm{Spec}(\mathscr{O})$, with $\mathscr{O}$ the ring of integers in a nonarchimedean local field F, then $\tilde{\mathbf{G}}$ yields a central extension $\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G$, where $G = \mathbf{G}(F)$, endowed with a splitting $\sigma^\circ\colon G^\circ \hookrightarrow G$.

Fix an injective character $\varepsilon\colon \mu_n \hookrightarrow \mathbb{C}^\times$. This allows one to define ε-genuine automorphic representations of $\tilde{G}_{\mathbb{A}}$ in the global context, ε-genuine admissible representations of $\tilde{G}$ in the local context, and ε-genuine G°-spherical representations of $\tilde{G}$ in the local integral context.

The purpose of this article is the construction an ***L-group*** associated to such a $\tilde{\mathbf{G}}$ and ε. We believe that this L-group will provide a parameterization of irreducible ε-genuine representations in the three contexts above.

1.3. Well-aligned homomorphisms. — Let $\mathbf{G}_1 \supset \mathbf{B}_1 \supset \mathbf{T}_1$ and $\mathbf{G}_2 \supset \mathbf{B}_2 \supset \mathbf{T}_2$ be quasisplit groups over S, endowed with Borel subgroups and maximally split maximal tori. Let $\tilde{\mathbf{G}}_1 = (\mathbf{G}_1', n)$ and $\tilde{\mathbf{G}}_2 = (\mathbf{G}_2', n)$ be covers (of the same degree) of $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively. Write $\mathscr{Y}_1$ and $\mathscr{Y}_2$ for the cocharacter lattices of $\mathbf{T}_1$ and $\mathbf{T}_2$, and Q_1, Q_2 for the quadratic forms arising from the covers. These quadratic forms yield sublattices $\mathscr{Y}_{1,Q_1,n}$ and $\mathscr{Y}_{2,Q_2,n}$.

Definition 1.3. — A ***well-aligned homomorphism*** $\tilde{\iota}$ from $(\tilde{\mathbf{G}}_1, \mathbf{B}_1, \mathbf{T}_1)$ to $(\tilde{\mathbf{G}}_2, \mathbf{B}_2, \mathbf{T}_2)$ is a pair $\tilde{\iota} = (\iota, \iota')$ of homomorphisms of sheaves of groups on S_{Zar}, making the following diagram commute,

$$\begin{array}{ccccc} \mathbf{K}_2 & \hookrightarrow & \mathbf{G}_1' & \twoheadrightarrow & \mathbf{G}_1 \\ \downarrow{=} & & \downarrow{\iota'} & & \downarrow{\iota} \\ \mathbf{K}_2 & \hookrightarrow & \mathbf{G}_2' & \twoheadrightarrow & \mathbf{G}_2 \end{array} \tag{1.1}$$

and satisfying the following additional axioms:

1. ι has normal image and smooth central kernel;
2. $\iota(\mathbf{B}_1) \subset \mathbf{B}_2$ and $\iota(\mathbf{T}_1) \subset \mathbf{T}_2$. Thus ι induces a map $\iota\colon \mathscr{Y}_1 \to \mathscr{Y}_2$;
3. (ι, ι') realizes $\mathbf{G}_1'$ as the pullback of $\mathbf{G}_2'$ via ι;
4. The homomorphism ι satisfies $\iota(\mathscr{Y}_{1,Q_1,n}) \subset \mathscr{Y}_{2,Q_2,n}$.

Remark 1.4. — Conditions (1) and (2) are inspired by [**11**, §1.4, 2.1,2.5], though more restrictive. By "normal image," we mean that for any geometric point $\bar{s} \to S$, the homomorphism $\iota\colon \mathbf{G}_{1,\bar{s}} \to \mathbf{G}_{2,\bar{s}}$ has normal image. Condition (3) implies that for all $y \in \mathscr{Y}_1$, $Q_1(y) = Q_2(\iota(y))$. In other words, Q_1 is the image of Q_2 via the map

$$\iota^*\colon H^0_{\text{ét}}(S, \mathscr{S}\!ym^2(\mathscr{X}_2)) \to H^0_{\text{ét}}(S, \mathscr{S}\!ym^2(\mathscr{X}_1)).$$

But Condition (3) does not imply Condition (4); one may cook up an example with $\mathbf{G}_1 = \mathbf{G}_{\mathrm{m}}$ and $\mathbf{G}_2 = \mathbf{G}_{\mathrm{m}}^2$ which satisfies (3) but not (4).

Proposition 1.5. — *The composition of well-aligned homomorphisms is well-aligned.*

Proof. — Suppose that (ι_1, ι_1') and (ι_2, ι_2') are well-aligned homomorphisms, with $\iota_1\colon \mathbf{G}_1 \to \mathbf{G}_2$ and $\iota_2\colon \mathbf{G}_2 \to \mathbf{G}_3$. Conditions (2), (3), and (4) are obviously satisfied by the composition $(\iota_2 \circ \iota_1, \iota_2' \circ \iota_1')$. For condition (1), notice that the kernel of $\iota_2 \circ \iota_1$ is contained in the kernel of ι_1, and hence is central. The only thing left is to verify that $\iota_2 \circ \iota_1$ has normal image. This may be checked by looking at geometric fibers; it seems well-known (cf. [**43**, §1.8]). □

Inner automorphisms are well-aligned homomorphisms.

Example 1.6. — *Suppose that $\tilde{\mathbf{G}}$ is a degree n cover of a quasisplit group $\mathbf{G}$. Suppose that $\mathbf{B}_0 \supset \mathbf{T}_0$ and $\mathbf{B} \supset \mathbf{T}$ are two Borel subgroups containing maximally split maximal tori. Suppose that $g \in \mathbf{G}(S)$, and write $\mathrm{Int}(g)$ for the resulting inner automorphism of $\mathbf{G}$. As noted in* [**18**, 0.N.4], *$\mathrm{Int}(g)$ lifts canonically to an automorphism $\mathrm{Int}(g)' \in \mathrm{Aut}(\mathbf{G}')$. If $\mathbf{B} = \mathrm{Int}(g)\mathbf{B}_0$ and $\mathbf{T} = \mathrm{Int}(g)\mathbf{T}_0$, then the pair $(\mathrm{Int}(g), \mathrm{Int}(g)')$ is a well-aligned homomorphism from $(\tilde{\mathbf{G}}, \mathbf{B}_0, \mathbf{T}_0)$ to $(\tilde{\mathbf{G}}, \mathbf{B}, \mathbf{T})$.*

While we focus on quasisplit groups in this article, the lifting of inner automorphisms allows one to consider "pure inner forms" of covers over a field.

Definition 1.7. — Let $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ be a degree n cover of a quasisplit group $\mathbf{G}$, over a field F. Let $\xi \in Z^1_{\text{ét}}(F, \mathbf{G})$ be a 1-cocycle. The image $\mathrm{Int}(\xi)$ in $Z^1_{\text{ét}}(F, \mathbf{Aut}(\mathbf{G}))$ defines an inner form $\mathbf{G}_\xi$ of $\mathbf{G}$. These are called the ***pure inner forms*** of $\mathbf{G}$. On the other hand, we may consider the image $\mathrm{Int}(\xi)'$ in $Z^1_{\text{ét}}(F, \mathbf{Aut}(\mathbf{G}'))$, which by [**18**, §7.1, 7.2] defines a central extension $\mathbf{G}'_\xi$ of $\mathbf{G}_\xi$ by $\mathbf{K}_2$. The cover $\tilde{\mathbf{G}}_\xi = (\mathbf{G}'_\xi, n)$ of $\mathbf{G}_\xi$ will be called a ***pure inner form*** of the cover $\tilde{\mathbf{G}}$.

We have not attempted to go further in the study of inner forms for covers, but presumably one should study something like strong real forms as in [**2**, Definition 1.12], and more general rigid forms as in [**39**], if one wishes to assemble L-packets for covering groups.

The next example of a well-aligned homomorphism is relevant for the study of central characters of genuine representations.

Example 1.8. — *Let $\tilde{\mathbf{G}}$ be a degree n cover of a quasisplit group $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$. Let $\mathbf{H}$ be the maximal torus in the center of $\mathbf{G}$, with cocharacter lattice $\mathscr{Y}_H \subset \mathscr{Y}$. Let $\mathbf{C}$ be the algebraic torus with cocharacter lattice $\mathscr{Y}_H \cap \mathscr{Y}_{Q,n}$, and $\iota \colon \mathbf{C} \to \mathbf{G}$ the resulting homomorphism (with central image). Let $\tilde{\mathbf{C}}$ denote the pullback of the cover $\tilde{\mathbf{G}}$ via ι. Then ι lifts to a well-aligned homomorphism from $\tilde{\mathbf{C}}$ to $\tilde{\mathbf{G}}$.*

The final example of a well-aligned homomorphism is relevant for the study of twisting genuine representations by one-dimensional representations of $\mathbf{G}$.

Example 1.9. — *Let $\mathbf{H}$ denote the maximal toral quotient of $\mathbf{G}$. In other words, $\mathbf{H}$ is the torus whose character lattice equals $\mathrm{Hom}(\mathbf{G}, \mathbf{G}_m)$. Let $p \colon \mathbf{G} \to \mathbf{H}$ denote the canonical homomorphism, and write $\iota \colon \mathbf{G} \to \mathbf{G} \times \mathbf{H}$ for the homomorphism $\mathrm{Id} \times p$. Write $\tilde{\mathbf{G}} \times \mathbf{H}$ for the cover $(\mathbf{G}' \times \mathbf{H}, n)$.*

The homomorphism ι realizes $\tilde{\mathbf{G}}$ as the pullback via ι of the cover $\tilde{\mathbf{G}} \times \mathbf{H}$. A Borel subgroup and torus in $\mathbf{G}$ determines a Borel subgroup and torus in $\mathbf{G} \times \mathbf{H}$. In this way, ι lifts to a well-aligned homomorphism of covers from $\tilde{\mathbf{G}}$ to $\tilde{\mathbf{G}} \times \mathbf{H}$.

2. The dual group

In this section, fix a degree n cover $\tilde{\mathbf{G}}$ of a quasisplit group $\mathbf{G}$ over S. Associated to $\tilde{\mathbf{G}}$, we define the "dual group," a local system on $S_{\text{ét}}$ of affine group schemes over $\mathbb{Z}$. We refer to Appendix A.1, for background on such local systems. We begin by reviewing the Langlands dual group of $\mathbf{G}$ in a framework suggested by Deligne (personal communication).

2.1. The Langlands dual group. — Choose, for now, a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ containing a maximally split maximal torus $\mathbf{T}$. The based root datum of $(\mathbf{G}, \mathbf{B}, \mathbf{T})$ was denoted Ψ, and the dual root datum,

$$\Psi^\vee = (\mathscr{Y}, \Phi^\vee, \Delta^\vee, \mathscr{X}, \Phi, \Delta),$$

is a local system of root data on $S_{\text{ét}}$.

This defines a unique (up to unique isomorphism) local system $\mathscr{G}^\vee$ on $S_{\text{ét}}$ of pinned reductive groups over $\mathbb{Z}$, called the ***Langlands dual group*** of $\mathbf{G}$. The center of $\mathscr{G}^\vee$ is a local system on $S_{\text{ét}}$ of groups of multiplicative type over $\mathbb{Z}$, given by

$$\mathscr{Z}^\vee = \operatorname{Spec}(\mathbb{Z}[\mathscr{Y}/\mathscr{Y}^{\text{sc}}]).$$

See Example A.2 for more on local systems and Spec in this context.

2.2. The dual group of a cover. — Now we adapt the definition of the dual group to covers. The ideas here are the same as those of [**78**]. The ideas for modifying root data originate in [**49**, §2.2] in the simply-connected case, in [**26**, Theorem 2.9] for the almost simple case, in [**51**, §11] and [**60**] in the reductive case. This dual group is also compatible with [**1**] and the Hecke algebra isomorphisms of Savin [**62**], and the most recent work of Lysenko [**50**].

Associated to the cover $\tilde{\mathbf{G}}$ of degree n, recall that $Q\colon \mathscr{Y} \to \mathbb{Z}$ is the first Brylinski-Deligne invariant, and $\beta_Q\colon \mathscr{Y} \otimes \mathscr{Y} \to n^{-1}\mathbb{Z}$ a symmetric bilinear form, and

$$\mathscr{Y}_{Q,n} = \{y \in \mathscr{Y} : \beta_Q(y, y') \in \mathbb{Z} \text{ for all } y' \in \mathscr{Y}\} \subset \mathscr{Y}.$$

Define $\mathscr{X}_{Q,n} = \{x \in n^{-1}\mathscr{X} : \langle x, y\rangle \in \mathbb{Z} \text{ for all } y \in \mathscr{Y}_{Q,n}\} \subset n^{-1}\mathscr{X}$. For each root $\phi \in \Phi$, define constants n_ϕ and m_ϕ,

$$n_\phi = \frac{n}{\operatorname{GCD}(n, Q(\phi^\vee))}, \quad m_\phi = \frac{Q(\phi^\vee)}{\operatorname{GCD}(n, Q(\phi^\vee))}. \tag{2.1}$$

Define ***modified roots*** and ***modified coroots*** by

$$\tilde{\phi} = n_\phi^{-1}\phi, \quad \tilde{\phi}^\vee = n_\phi \phi^\vee.$$

These define subsets $\tilde{\Phi} = \{\tilde{\phi} : \phi \in \Phi\} \subset \mathscr{X}_{Q,n}$ and $\tilde{\Phi}^\vee = \{\tilde{\phi}^\vee : \phi^\vee \in \Phi^\vee\} \subset \mathscr{Y}_{Q,n}$, as in [**78**]. Modifying the simple roots and their coroots, we have subsets $\tilde{\Delta} \subset \tilde{\Phi}$ and $\tilde{\Delta}^\vee \subset \tilde{\Phi}$.

By [**78**, Construction 1.3], this defines a local system of based root data $\tilde{\Psi}$ on $S_{\text{ét}}$. Write $\tilde{\Psi}^\vee$ for its dual,

$$\tilde{\Psi}^\vee = (\mathscr{Y}_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, \mathscr{X}_{Q,n}, \tilde{\Phi}, \tilde{\Delta}).$$

Write $\mathscr{Y}_{Q,n}^{\text{sc}}$ for the subgroup of $\mathscr{Y}_{Q,n}$ spanned by the modified coroots $\tilde{\Phi}^\vee$.

Define $\tilde{\mathscr{G}}^\vee$ to be the (unique up to unique isomorphism) local system on $S_{\text{ét}}$ of pinned reductive groups over $\mathbb{Z}$, associated to the local system of based root data $\tilde{\Psi}^\vee$. Its maximal torus is a local system on $S_{\text{ét}}$ of split tori over $\mathbb{Z}$,

$$\tilde{\mathscr{T}}^\vee = \operatorname{Spec}\left(\mathbb{Z}[\mathscr{Y}_{Q,n}]\right).$$

The center of $\tilde{\mathscr{G}}^\vee$ is a local system on $S_{\text{ét}}$ of groups of multiplicative type over $\mathbb{Z}$,

$$\tilde{\mathscr{Z}}^\vee = \operatorname{Spec}\left(\mathbb{Z}\left[\mathscr{Y}_{Q,n}/\mathscr{Y}_{Q,n}^{\text{sc}}\right]\right).$$

We call $\tilde{\mathscr{G}}^\vee$ (endowed with its pinning) the ***dual group*** of the cover $\tilde{\mathbf{G}}$.

Proposition 2.1. — *Suppose that $\tilde{\mathbf{G}}_0$ is another cover of $\mathbf{G}$ of degree n, with first Brylinski-Deligne invariant Q_0. If $Q \equiv Q_0$ modulo n, i.e., $Q(y) - Q_0(y) \in n\mathbb{Z}$ for all $y \in \mathscr{Y}$, then the resulting modified root data are equal: $\tilde{\Psi}^\vee = \tilde{\Psi}_0^\vee$. Thus the dual groups are equal, $\tilde{\mathscr{G}}^\vee = \tilde{\mathscr{G}}_0^\vee$.*

Proof. — One checks directly that $\beta_Q \equiv \beta_{Q_0}$ modulo $\mathbb{Z}$, from which it follows that

$$\mathscr{Y}_{Q,n} = \mathscr{Y}_{Q_0,n}, \quad \mathscr{X}_{Q,n} = \mathscr{X}_{Q_0,n}.$$

Similarly, one checks that the constants n_ϕ are equal,

$$\frac{n}{\operatorname{GCD}(n, Q_0(\phi^\vee))} = \frac{n}{\operatorname{GCD}(n, Q(\phi^\vee))}.$$

The result follows. □

The Weyl group of $\tilde{\mathscr{G}}^\vee$ with respect to $\tilde{\mathscr{T}}^\vee$ forms a local system $\tilde{\mathscr{W}}$ on $S_{\text{ét}}$ of finite groups, generated (locally on $S_{\text{ét}}$) by reflections $s_{\tilde{\phi}}$ for every $\tilde{\phi} \in \tilde{\Phi}$. The action of $\tilde{\mathscr{W}}$ on $\mathscr{Y}_{Q,n}$ is given by the formula

$$s_{\tilde{\phi}}(y) = y - \langle \tilde{\phi}, y \rangle \tilde{\phi}^\vee = y - \langle \phi, y \rangle \phi^\vee.$$

This identifies the root reflections $s_{\tilde{\phi}}$ with the root reflections s_ϕ, and hence identifies the Weyl group $\tilde{\mathscr{W}}$ with the Weyl group $\mathscr{W}$ of $\mathbf{G}$ with respect to $\mathbf{T}$ (where both are viewed as local systems on $S_{\text{ét}}$ of finite groups).

The dual group $\tilde{\mathscr{G}}^\vee$ comes with a distinguished 2-torsion element in its center, described here. From the quadratic form $Q\colon \mathscr{Y} \to \mathbb{Z}$, observe that $2Q(y) = n\beta_Q(y,y) \in n\mathbb{Z}$ for all $y \in \mathscr{Y}_{Q,n}$. Moreover, we have

$$Q(\tilde{\phi}^\vee) = n_\phi^2 Q(\phi^\vee) = n_\phi m_\phi n \in n\mathbb{Z},$$

for all $\phi \in \Phi$. Of course, $Q(ny) \in n\mathbb{Z}$ as well, for all $y \in \mathscr{Y}_{Q,n}$. We find a homomorphism of local systems of abelian groups on $S_{\text{ét}}$,

$$\overline{n^{-1}Q}\colon \frac{\mathscr{Y}_{Q,n}}{\mathscr{Y}_{Q,n}^{\text{sc}} + n\mathscr{Y}_{Q,n}} \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}, \quad y \mapsto n^{-1}Q(y) \bmod \mathbb{Z}.$$

Applying Spec yields a homomorphism of local systems on $S_{\text{ét}}$ of diagonalizable group schemes over $\mathbb{Z}$,

$$\tau_Q \in \operatorname{Hom}(\boldsymbol{\mu}_2, \tilde{\mathscr{Z}}_{[n]}^\vee).$$

Thus $\tau_Q(-1)$ is a Galois-invariant 2-torsion element in the center of $\tilde{\mathscr{G}}^\vee$. If n is odd, then $\tau_Q(-1) = 1$.

2.3. Well-aligned functoriality. — Consider a well-aligned homomorphism $\tilde{\iota}\colon \tilde{\mathbf{G}}_1 \to \tilde{\mathbf{G}}_2$ of covers, each endowed with Borel subgroup and maximally split maximal torus. Here we construct a corresponding homomorphism of dual groups,

$$\iota^\vee \colon \tilde{\mathscr{G}}_2^\vee \to \tilde{\mathscr{G}}_1^\vee.$$

These dual groups are constructed, locally on $S_{\text{ét}}$, from root data:

$$\tilde{\Psi}_1^\vee = \left(\mathscr{Y}_{1,Q_1,n}, \tilde{\Phi}_1^\vee, \tilde{\Delta}_1^\vee, \mathscr{X}_{1,Q_1,n}, \tilde{\Phi}_1, \tilde{\Delta}_1\right);$$
$$\tilde{\Psi}_2^\vee = \left(\mathscr{Y}_{2,Q_2,n}, \tilde{\Phi}_2^\vee, \tilde{\Delta}_2^\vee, \mathscr{X}_{2,Q_2,n}, \tilde{\Phi}_2, \tilde{\Delta}_2\right).$$

For the construction of $\iota^\vee$, it suffices to work locally on $S_{\text{ét}}$, on a finite étale cover over which $\mathbf{G}_1$ and $\mathbf{G}_2$ split. Condition (4) of Definition 1.3 gives a homomorphism $\iota\colon \mathscr{Y}_{1,Q_1,n} \to \mathscr{Y}_{2,Q_2,n}$, and its dual homomorphism $\iota^*\colon \mathscr{X}_{2,Q_2,n} \to \mathscr{X}_{1,Q_1,n}$. As ι has normal image, the coroots from $\Phi_1^\vee$ map to coroots from $\Phi_2^\vee$. Condition (3) implies that the scaled coroots in $\tilde{\Phi}_1^\vee \subset \mathscr{Y}_{1,Q_1,n}$ map to scaled coroots in $\tilde{\Phi}_2^\vee \subset \mathscr{Y}_{2,Q_2,n}$. Since the Borel subgroups are aligned, the simple scaled coroots in $\tilde{\Delta}_1^\vee$ map to simple scaled coroots in $\tilde{\Delta}_2^\vee$. Dually, the map $\iota^*\colon \mathscr{X}_{2,Q_2,n} \to \mathscr{X}_{1,Q_1,n}$ sends $\tilde{\Phi}_2$ to $\tilde{\Phi}_1$ and $\tilde{\Delta}_2$ to $\tilde{\Delta}_1$.

This allows us to assemble a homomorphism $\iota^\vee \colon \tilde{\mathscr{G}}_2^\vee \to \tilde{\mathscr{G}}_1^\vee$, (cf. [**11**, §2.1, 2.5]). On tori, let $\iota^\vee \colon \tilde{\mathscr{T}}_2^\vee \to \tilde{\mathscr{T}}_1^\vee$ be the homomorphism dual to the map of character lattices $\iota\colon \mathscr{Y}_{1,Q_1,n} \to \mathscr{Y}_{2,Q_2,n}$. Using the pinnings on $\tilde{\mathscr{G}}_2^\vee$ and $\tilde{\mathscr{G}}_1^\vee$, and the map of roots to roots, we obtain a homomorphism from the simply-connected cover $\tilde{\mathscr{G}}_{2,\text{sc}}^\vee$ of the derived subgroup, $\tilde{\mathscr{G}}_{2,\text{der}}^\vee$,

$$\iota_{\text{sc}}^\vee \colon \tilde{\mathscr{G}}_{2,\text{sc}}^\vee \to \tilde{\mathscr{G}}_1^\vee.$$

Let $\tilde{\mathscr{T}}_{2,\text{sc}}^\vee$ be the pullback of $\tilde{\mathscr{T}}_2^\vee$. The following diagram commutes.

$$\begin{tikzcd}
\tilde{\mathscr{T}}_{2,\text{sc}}^\vee \arrow[r] \arrow[d, hook] & \tilde{\mathscr{T}}_2^\vee \arrow[d, "\iota^\vee"] \\
\tilde{\mathscr{G}}_{2,\text{sc}}^\vee \arrow[r, "\iota_{\text{sc}}^\vee"] & \tilde{\mathscr{G}}_1^\vee.
\end{tikzcd}$$

The homomorphism $\iota_{\text{sc}}^\vee$ descends to the derived subgroup $\tilde{\mathscr{G}}_{2,\text{der}}^\vee$, since it is trivial on the kernel of $\tilde{\mathscr{T}}_{2,\text{sc}}^\vee \to \tilde{\mathscr{T}}_2^\vee$. In this way, we have a pair of homomorphisms of groups over $\mathbb{Z}$,

$$\iota_{\text{der}}^\vee \colon \tilde{\mathscr{G}}_{2,\text{der}}^\vee \to \tilde{\mathscr{G}}_1^\vee, \quad \iota^\vee \colon \tilde{\mathscr{T}}_2^\vee \to \tilde{\mathscr{T}}_1^\vee \subset \tilde{\mathscr{G}}_1^\vee.$$

These homomorphisms agree on their intersection, giving the desired homomorphism $\iota^\vee \colon \tilde{\mathscr{G}}_2^\vee \to \tilde{\mathscr{G}}_1^\vee$.

Since modified coroots are sent to modified coroots, we find that $\iota^\vee$ sends $\tilde{\mathscr{Z}}_2^\vee$ to $\tilde{\mathscr{Z}}_1^\vee$. The quadratic forms Q_1 and Q_2 induce two group homomorphisms:

$$\overline{n^{-1}Q_1}\colon \frac{\mathscr{Y}_{1,Q_1,n}}{\mathscr{Y}^{\mathrm{sc}}_{1,Q_1,n} + n\mathscr{Y}_{1,Q_1,n}} \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}, \quad \overline{n^{-1}Q_2}\colon \frac{\mathscr{Y}_{2,Q_2,n}}{\mathscr{Y}^{\mathrm{sc}}_{2,Q_2,n} + n\mathscr{Y}_{2,Q_2,n}} \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

These define two homomorphisms of group schemes over $\mathbb{Z}$,

$$\tau_{Q_1}\colon \boldsymbol{\mu}_2 \to \tilde{\mathscr{Z}}^\vee_{1,[n]}, \quad \tau_{Q_2}\colon \boldsymbol{\mu}_2 \to \tilde{\mathscr{Z}}^\vee_{2,[n]}.$$

As $Q_2(\iota(y)) = Q_1(y)$ for all $y \in \mathscr{Y}_1$, $\iota^\vee \circ \tau_{Q_1} = \tau_{Q_2}$. In other words, the homomorphism $\iota^\vee\colon \tilde{\mathscr{G}}_2^\vee \to \tilde{\mathscr{G}}_1^\vee$ sends the center to the center, and respects the 2-torsion elements therein, $\iota^\vee(\tau_{Q_2}(-1)) = \tau_{Q_1}(-1)$.

Given a pair of well-aligned homomorphisms,

$$\tilde{\mathbf{G}}_1 \xrightarrow{(\iota_1,\iota_1')} \tilde{\mathbf{G}}_2 \xrightarrow{(\iota_2,\iota_2')} \tilde{\mathbf{G}}_3,$$

their composition is a well-aligned homomorphism $(\iota_3, \iota_3') = (\iota_2\iota_1, \iota_2'\iota_1')$ from $\tilde{\mathbf{G}}_1$ to $\tilde{\mathbf{G}}_3$ by Proposition 1.5.

This gives a commutative diagram of sheaves of abelian groups.

$$\begin{tikzcd} \mathscr{Y}_{1,Q_1,n} \arrow[r, "\iota_1"'] \arrow[rr, bend left, "\iota_3"] & \mathscr{Y}_{2,Q_2,n} \arrow[r, "\iota_2"'] & \mathscr{Y}_{3,Q_3,n}. \end{tikzcd}$$

We find such a commutative diagram for dual groups, in the opposite direction.

$$\begin{tikzcd} \tilde{\mathscr{G}}_3^\vee \arrow[r, "\iota_2^\vee"'] \arrow[rr, bend left, "\iota_3^\vee"] & \tilde{\mathscr{G}}_2^\vee \arrow[r, "\iota_1^\vee"'] & \tilde{\mathscr{G}}_1^\vee. \end{tikzcd}$$

Let DGp_S^* denote the category whose objects are local systems $\mathscr{G}^\vee$ on $S_{\text{ét}}$ of group schemes over $\mathbb{Z}$, endowed with central morphisms $\boldsymbol{\mu}_2 \to \mathscr{G}^\vee$ (where $\boldsymbol{\mu}_2$ is viewed as the constant local system of group schemes). Morphisms in DGp_S^* are morphisms of local systems of group schemes over $\mathbb{Z}$, compatible with the central morphisms from $\boldsymbol{\mu}_2$.

Let WAC_S (Well-Aligned-Covers) denote the category whose objects are triples $(\tilde{\mathbf{G}}, \mathbf{B}, \mathbf{T})$ where $\tilde{\mathbf{G}}$ is a cover of a reductive group $\mathbf{G}$ over S, $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$, and $\mathbf{T}$ is a maximally split maximal torus of $\mathbf{G}$ contained in $\mathbf{B}$. Morphisms in WAC_S are well-aligned homomorphisms of covers.

We have proven the following result.

Proposition 2.2. — *The construction of the dual group defines a contravariant functor from* WAC_S *to* DGp_S^*.

Theorem 2.3. — *Let* $\tilde{\mathbf{G}}$ *be a degree* n *cover of a quasisplit group* $\mathbf{G}$ *over* S. *The dual group* $\tilde{\mathscr{G}}^\vee$ *is well-defined, up to unique isomorphism, by* $\tilde{\mathbf{G}}$ *alone.*

Proof. — The construction of the dual group depends on the choice of Borel and torus $\mathbf{B} \supset \mathbf{T}$. So it suffices to construct a canonical isomorphism of dual groups $\tilde{\mathscr{G}}^\vee \to \tilde{\mathscr{G}}_0^\vee$ for any pair of choices $\mathbf{B}_0 \supset \mathbf{T}_0$ and $\mathbf{B} \supset \mathbf{T}$. Such "well-definedness" is discussed in more detail in [**24**, §1.1].

By Proposition 1.1, there exists $g \in \mathbf{G}(S)$ such that $\mathrm{Int}(g)\mathbf{T}_0 = \mathbf{T}$ and $\mathrm{Int}(g)\mathbf{B}_0 = \mathbf{B}$. This automorphism $\mathrm{Int}(g)$ lifts to an automorphism of $\mathbf{G}'$ and defines a well-aligned isomorphism of covers,

$$\begin{tikzcd} \mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathbf{G}' \arrow[r, two heads] \arrow[d, "\mathrm{Int}(g)'"] & \mathbf{G} \arrow[d, "\mathrm{Int}(g)"] & \mathbf{B}_0 \arrow[d] & \mathbf{T}_0 \arrow[d] \\ \mathbf{K}_2 \arrow[r, hook] & \mathbf{G}' \arrow[r, two heads] & \mathbf{G} & \mathbf{B} & \mathbf{T}. \end{tikzcd}$$

As a well-aligned isomorphism of covers, this yields an isomorphism (of local systems on $S_{\text{ét}}$ of reductive groups over $\mathbb{Z}$)

$$\mathrm{Int}(g)^\vee : \tilde{\mathscr{G}}^\vee \xrightarrow{\sim} \tilde{\mathscr{G}}_0^\vee.$$

If $g' \in \mathbf{G}(S)$ also satisfies $\mathrm{Int}(g')\mathbf{T}_0 = \mathbf{T}$ and $\mathrm{Int}(g')\mathbf{B}_0 = \mathbf{B}$, then $g'g^{-1} \in \mathbf{N}(S) \cap \mathbf{B}(S) = \mathbf{T}(S)$. Thus $g' = tg$ for some $t \in \mathbf{T}(S)$. Hence $\mathrm{Int}(g') = \mathrm{Int}(t)\,\mathrm{Int}(g)$, and so by Proposition 2.2,

$$\mathrm{Int}(g')^\vee = \mathrm{Int}(g)^\vee\,\mathrm{Int}(t)^\vee : \tilde{\mathscr{G}}^\vee \to \tilde{\mathscr{G}}_0^\vee.$$

But $\mathrm{Int}(t)^\vee = \mathrm{Id}$, since $\mathrm{Int}(t)$ leaves all relevant data unchanged. Thus $\mathrm{Int}(g')^\vee = \mathrm{Int}(g)^\vee$. Hence we find a canonical isomorphism $\tilde{\mathscr{G}}^\vee \xrightarrow{\sim} \tilde{\mathscr{G}}_0^\vee$. □

2.4. Change of base scheme. — Let $\tilde{\mathbf{G}}$ be a degree n cover of a quasisplit group $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ over S, as before. Let $\gamma\colon S_0 \to S$ be a morphism of schemes, with $S_0 = \mathrm{Spec}(F_0)$ for some field F_0 or $S_0 = \mathrm{Spec}(\mathcal{O}_0)$ for some DVR $\mathcal{O}_0$ (with finite residue field or containing a field, as usual). Then γ gives rise to a pullback functor γ^* from sheaves on $S_{\text{ét}}$ to sheaves on $S_{0,\text{ét}}$. Pullback via γ defines a degree n cover $\tilde{\mathbf{G}}_0$ of a quasisplit group $\mathbf{G}_0 \supset \mathbf{B}_0 \supset \mathbf{T}_0$ over S_0.

The cocharacter lattice $\mathscr{Y}_0$ of $\mathbf{T}_0$ is a sheaf on $S_{0,\text{ét}}$. It is naturally isomorphic to the pullback $\gamma^*\mathscr{Y}$, with $\mathscr{Y}$ the cocharacter lattice of $\mathbf{T}$. Write $N\colon \gamma^*\mathscr{Y} \to \mathscr{Y}_0$ for the natural isomorphism. The quadratic form $Q\colon \mathscr{Y} \to \mathbb{Z}$ pulls back to a quadratic form $\gamma^*Q\colon \gamma^*\mathscr{Y} \to \mathbb{Z}$. The compatibility of Brylinski-Deligne invariants with pullbacks implies that $\gamma^*Q = N^*Q_0$, with Q_0 the first Brylinski-Deligne invariant of $\tilde{\mathbf{G}}_0$.

Remark 2.4. — This compatibility follows straightforwardly from [**18**, §3.10]; if $\tilde{\mathbf{T}}$ arises (after a finite étale $U \to S$) from the cocycle attached to $C \in \mathscr{X} \otimes \mathscr{X}$, then $\tilde{\mathbf{T}}_0$ arises from the pullback of this cocycle, i.e., from an element $C_0 \in \mathscr{X}_0 \otimes \mathscr{X}_0$ with $N^*C_0 = \gamma^*C$. The quadratic form Q is given by $Q(y) = C(y,y)$ and similarly $Q_0(y) = C_0(y,y)$. Since $N^*C_0 = \gamma^*C$, we find that $N^*Q_0 = \gamma^*Q$.

In this way, we find that N restricts to an isomorphism from $\gamma^*\mathscr{Y}_{Q,n}$ to $\mathscr{Y}_{0,Q_0,n}$, sending roots and coroots (the sheaves of sets $\gamma^*\tilde{\Phi}$, $\gamma^*\tilde{\Phi}^\vee$ on $S_{0,\text{ét}}$) to the corresponding roots and coroots in $\mathscr{Y}_{0,Q_0,n}$. As $\mathbf{B}_0 = \gamma^*\mathbf{B}$, simple roots and coroots are identified as well.

By our construction of the dual group, we find that N gives an isomorphism of local systems on $S_{0,\text{ét}}$ of pinned reductive groups over $\mathbb{Z}$,

$$N^\vee : \gamma^* \tilde{\mathscr{G}}^\vee \xrightarrow{\sim} \tilde{\mathscr{G}}_0^\vee.$$

2.5. Parabolic subgroups. — We keep the degree n cover $\tilde{\mathbf{G}}$ of the quasisplit group $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ over S. Let $\mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup defined over S, containing $\mathbf{B}$. Suppose that $\mathbf{P} = \mathbf{MN}$ is a Levi decomposition defined over S, with $\mathbf{N}$ the unipotent radical of $\mathbf{P}$ and $\mathbf{M}$ a Levi factor containing $\mathbf{T}$. Let $\mathbf{B_M}$ denote the Borel subgroup $\mathbf{B} \cap \mathbf{M}$ of $\mathbf{M}$. Write $\tilde{\mathbf{M}} = (\mathbf{M}', n)$ for the cover of $\mathbf{M}$ arising from pulling back $\tilde{\mathbf{G}}$.

The first Brylinski-Deligne invariant Q is the same for $\tilde{\mathbf{G}}$ as for $\tilde{\mathbf{M}}$, as it depends only upon the cover $\tilde{\mathbf{T}}$ of their common maximal torus. Write $\Phi_{\mathbf{M}}$ and $\Phi_{\mathbf{M}}^\vee$ for the roots and coroots of $\mathbf{M}$; these are subsets of Φ and $\Phi^\vee$, respectively, and by agreement of the first Brylinski-Deligne invariant,

$$\tilde{\Phi}_{\mathbf{M}} \subset \tilde{\Phi}, \quad \tilde{\Phi}_{\mathbf{M}}^\vee \subset \tilde{\Phi}^\vee.$$

As $\mathbf{B_M} = \mathbf{B} \cap \mathbf{M}$, we have $\tilde{\Delta}_{\mathbf{M}} \subset \tilde{\Delta}$ and $\tilde{\Delta}_{\mathbf{M}}^\vee \subset \tilde{\Delta}^\vee$. We find a pair of local systems on $S_{\text{ét}}$ of based root data,

$$\left(\mathscr{Y}_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, \mathscr{X}_{Q,n}, \tilde{\Phi}, \tilde{\Delta}\right), \quad \left(\mathscr{Y}_{Q,n}, \tilde{\Phi}_{\mathbf{M}}^\vee, \tilde{\Delta}_{\mathbf{M}}^\vee, \mathscr{X}_{Q,n}, \tilde{\Phi}_{\mathbf{M}}, \tilde{\Delta}_{\mathbf{M}}\right).$$

The first root datum defines a local system $\tilde{\mathscr{G}}^\vee$ on $S_{\text{ét}}$ of pinned reductive groups over $\mathbb{Z}$. The second root datum defines a local system $\tilde{\mathscr{M}}^\vee$ on $S_{\text{ét}}$ of pinned reductive Levi subgroups of $\tilde{\mathscr{G}}^\vee$.

Thus the dual group of $\tilde{\mathbf{M}}$ is naturally a Levi subgroup of the dual group $\tilde{\mathscr{G}}^\vee$. Moreover, by agreement of the first Brylinski-Deligne invariants, the central 2-torsion element of $\tilde{\mathscr{M}}^\vee$ coincides with the central 2-torsion element $\tau_Q(-1) \in \tilde{\mathscr{G}}^\vee$.

2.6. Weyl action on the dual torus. — As before, keep the degree n cover $\tilde{\mathbf{G}}$ of $\mathbf{G}$. Write $\tilde{\mathbf{T}} = (\mathbf{T}', n)$ for the resulting cover of a maximally split maximal torus $\mathbf{T}$. Assume here that $\mathbf{T}$ splits over a cyclic Galois cover of S. Suppose that $w \in \mathscr{W}[S]$ is an element of the Weyl group, represented by an element $\dot{w} \in \mathscr{N}[S]$ (here $\mathscr{N}$ is the sheaf on $S_{\text{ét}}$ represented by the normalizer of $\mathbf{T}$).

Then $\mathrm{Int}(\dot{w}) : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ defines a well-aligned homomorphism from $\tilde{\mathbf{T}}$ to itself:

$$\begin{tikzcd}
\mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathbf{T}' \arrow[r, two heads] \arrow[d, "\mathrm{Int}(\dot{w})"] & \mathbf{T} \arrow[d, "\mathrm{Int}(w)"] \\
\mathbf{K}_2 \arrow[r, hook] & \mathbf{T}' \arrow[r, two heads] & \mathbf{T}.
\end{tikzcd}$$

As such, $\mathrm{Int}(\dot{w})$ defines a map of dual groups,

$$\mathrm{Int}(\dot{w})^\vee : \tilde{\mathscr{T}}^\vee \to \tilde{\mathscr{T}}^\vee.$$

As $\mathrm{Int}(t)^\vee = \mathrm{Id}$ for all $t \in \mathbf{T}(S)$, this homomorphism of dual groups depends only on the element w of the Weyl group, not on the chosen representative $\dot{w}$. Thus we write

$$\mathrm{Int}(w)^\vee : \tilde{\mathscr{T}}^\vee \to \tilde{\mathscr{T}}^\vee.$$

On the other hand the element $w \in \mathscr{W}[S]$ corresponds to a Galois-invariant element of the Weyl group of the dual group. From [**11**, Lemma 6.2], there exists an element $n^\vee \in \tilde{\mathscr{G}}^\vee[S]$ such that $n^\vee$ normalizes $\tilde{\mathscr{T}}$ and the resulting action of $n^\vee$ on the character lattice $\mathscr{Y}_{Q,n}$ of $\tilde{\mathscr{T}}$ coincides with the action of w on $\mathscr{Y}_{Q,n}$. Such an element $n^\vee$ is unique up to multiplication by elements of $\tilde{\mathscr{T}}^\vee[S]$. Since $\mathrm{Int}(w)^\vee$ and $\mathrm{Int}(n^\vee)$ define the same action on $\mathscr{Y}_{Q,n}$, we find that

$$\mathrm{Int}(w)^\vee = \mathrm{Int}(n^\vee), \text{ as elements of } \mathrm{Aut}(\tilde{\mathscr{T}}^\vee).$$

2.7. Specific cases. — Here we give many specific cases of covers, associated dual groups, and 2-torsion elements in their centers corresponding to $\tau_Q(-1)$.

2.7.1. *Method for simply-connected groups.* — Let $\mathbf{G}$ be a simply-connected semisimple group over S, with Borel subgroup $\mathbf{B}$ containing a maximally split maximal torus $\mathbf{T}$ over S. Let S'/S be a Galois cover over which $\mathbf{T}$ splits. If t is an integer, there exists a unique Weyl-invariant quadratic form Q_t with value t on all short coroots. This Q_t is a multiple of the Killing form.

Corresponding to Q_t, there is a unique, up to unique isomorphism, object $\mathbf{G}^{(t)} \in \mathsf{CExt}_S(\mathbf{G}, \mathbf{K}_2)$ with first Brylinski-Deligne invariant Q_t (by [**18**, §7.3(i)] when working over a field or [**79**, §3.3.3] over a DVR). Write β_t for the resulting $n^{-1}\mathbb{Z}$-valued bilinear form. Weyl-invariance of Q_t implies that

$$\beta_t(\phi^\vee, y) = n^{-1} Q_t(\phi^\vee)\langle \phi, y\rangle, \text{ for all } \phi \in \Phi, \text{ and all } y \in \mathscr{Y}. \tag{2.2}$$

(See the proof of Lemma 3.13 for a derivation.) It follows that, for any $y \in \mathscr{Y}$,

$$\beta_t(\phi^\vee, y) \in \mathbb{Z} \text{ if and only if } \langle \phi, y \rangle \in n_\phi \mathbb{Z}. \tag{2.3}$$

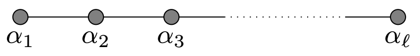
Let $Y = \mathscr{Y}[S']$, $X = \mathscr{X}[S']$, and $\Delta = \Delta[S'] = \{\alpha_1, \ldots, \alpha_\ell\}$ a basis of simple roots corresponding to $\mathbf{B}$. Since we assume $\mathbf{G}$ is simply connected, $Y = \bigoplus_{i=1}^{\ell} \alpha_i^\vee \mathbb{Z}$. Write $n_i = n_{\alpha_i}$. From (2.3), we find a characterization of $Y_{Q,n} = \mathscr{Y}_{Q,n}[S']$:

$$y \in Y_{Q,n} \text{ if and only if } \langle \alpha_i, y \rangle \in n_i \mathbb{Z} \text{ for all } 1 \leq i \leq \ell. \tag{2.4}$$

This, in turn, can be used to tabulate dual groups. We provide tables here reference, noting that such information can also be found in the examples of [**26**, §2.4]. But we also include data on the central 2-torsion elements that we have not found in the literature. Our tabulation was greatly assisted by using SAGE [**69**], especially the recently updated package which deftly handles root data.

The following tables only include split groups, and we write $\tilde{\mathbf{G}}^\vee = \tilde{\mathscr{G}}^\vee[S]$ for the dual group over S (a pinned reductive group scheme over $\mathbb{Z}$). For quasisplit groups, one can view the dual group as a pinned reductive group scheme over $\mathbb{Z}$ endowed with Galois action by pinned automorphisms.

2.7.2. $\mathbf{SL}_{\ell+1}$. — For $\mathbf{G} = \mathbf{SL}_{\ell+1}$, the standard Borel subgroup and maximal torus, and system of roots $\alpha_1, \ldots, \alpha_\ell$, the Dynkin diagram is



Consider the cover $\tilde{\mathbf{G}}$ of degree n arising from the quadratic form satisfying $Q(\phi^\vee) = 1$ for all $\phi \in \Phi$. Then $n_i = n$ and $\tilde{\alpha}_i^\vee = n\alpha_i^\vee$ for all $1 \leq i \leq \ell$. Hence

$$Y_{Q,n}^{\mathrm{sc}} = nY = n\alpha_1^\vee \mathbb{Z} + \cdots + n\alpha_\ell^\vee \mathbb{Z}.$$

The Cartan matrix of $\tilde{\mathbf{G}}^\vee$ has entries $\tilde{C}_{ij} = \langle \tilde{\alpha}_i, \tilde{\alpha}_j^\vee \rangle = \langle \alpha_i, \alpha_j^\vee \rangle$, and so $\tilde{\mathbf{G}}^\vee$ is isogenous to $\underline{\mathbf{SL}}_{\ell+1}$. To determine the dual group up to isomorphism, it suffices to compute the order of the center, since the center of $\underline{\mathbf{SL}}_{\ell+1}$ is $\underline{\boldsymbol{\mu}}_{\ell+1}$. The order of the center is equal to the index $\# Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}$, and this is computable from (2.4). The results are given in Table 1.

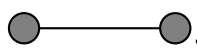
	Group $\mathbf{G}$				
n	$\mathbf{SL}_2$	$\mathbf{SL}_3$	$\mathbf{SL}_4$	$\mathbf{SL}_5$	$\mathbf{SL}_6$
1	$\underline{\mathbf{PGL}}_2$	$\underline{\mathbf{PGL}}_3$	$\underline{\mathbf{PGL}}_4$	$\underline{\mathbf{PGL}}_5$	$\underline{\mathbf{PGL}}_6$
2	${}^*\underline{\mathbf{SL}}_2$	$\underline{\mathbf{PGL}}_3$	$\underline{\mathbf{SL}}_4/\underline{\boldsymbol{\mu}}_2$	$\underline{\mathbf{PGL}}_5$	${}^*\underline{\mathbf{SL}}_6/\underline{\boldsymbol{\mu}}_3$
3	$\underline{\mathbf{PGL}}_2$	$\underline{\mathbf{SL}}_3$	$\underline{\mathbf{PGL}}_4$	$\underline{\mathbf{PGL}}_5$	$\underline{\mathbf{SL}}_6/\underline{\boldsymbol{\mu}}_2$
4	$\underline{\mathbf{SL}}_2$	$\underline{\mathbf{PGL}}_3$	${}^*\underline{\mathbf{SL}}_4$	$\underline{\mathbf{PGL}}_5$	$\underline{\mathbf{SL}}_6/\underline{\boldsymbol{\mu}}_3$
5	$\underline{\mathbf{PGL}}_2$	$\underline{\mathbf{PGL}}_3$	$\underline{\mathbf{PGL}}_4$	$\underline{\mathbf{SL}}_5$	$\underline{\mathbf{PGL}}_6$
6	${}^*\underline{\mathbf{SL}}_2$	$\underline{\mathbf{SL}}_3$	$\underline{\mathbf{SL}}_4/\underline{\boldsymbol{\mu}}_2$	$\underline{\mathbf{PGL}}_5$	${}^*\underline{\mathbf{SL}}_6$

TABLE 1. Table of dual groups for degree n covers of $\mathbf{SL}_{\ell+1}$. Groups marked with $*$ have $\tau_Q(-1)$ nontrivial.

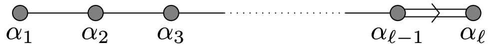
The dual groups $\tilde{\mathbf{G}}^\vee$ are consistent with the Iwahori-Hecke algebra isomorphisms found by Savin in [**62**, Theorem 7.8]. In other words, the dual group $\tilde{\mathbf{G}}^\vee$ coincides with the Langlands dual group of $\mathbf{SL}_{\ell+1}/\mathbf{Z}_{[n]}$, where $\mathbf{Z}_{[n]}$ is the n-torsion subgroup of the center of $\mathbf{SL}_{\ell+1}$.

The central 2-torsion elements $\tau_Q(-1)$ follow a somewhat predictable pattern. For covering degree 2, $\tau_Q(-1)$ is nontrivial for $\mathbf{SL}_2$, $\mathbf{SL}_6$, $\mathbf{SL}_{10}$, $\mathbf{SL}_{14}$, etc. In covering degree 4, $\tau_Q(-1)$ is nontrivial for $\mathbf{SL}_4$, $\mathbf{SL}_{12}$, $\mathbf{SL}_{20}$, etc. In covering degree 6, $\tau_Q(-1)$ is nontrivial for $\mathbf{SL}_2$, $\mathbf{SL}_6$, $\mathbf{SL}_{10}$, $\mathbf{SL}_{14}$, etc. In covering degree 8, $\tau_Q(-1)$ is nontrivial for $\mathbf{SL}_8$, $\mathbf{SL}_{24}$, etc. In general, we suspect the following:

$$\tau_Q(-1) \text{ is nontrivial for a degree } 2^e \cdot k \ (k \text{ odd}) \text{ cover of } \mathbf{SL}_m \\ \text{if and only if } m = 2^e \cdot j \text{ for } j \text{ odd}.$$

To illustrate the quasisplit case, consider $\mathbf{G} = \mathbf{SU}_3$, a quasisplit special unitary group associated to a degree 2 Galois cover S'/S. There is a unique degree n cover of $\mathbf{G}$ arising from the quadratic form taking values 1 at all coroots. The dual group of the degree 2 cover of $\mathbf{G} = \mathbf{SU}_3$ is identified with $\underline{\mathbf{PGL}}_3$, a pinned reductive group over $\mathbb{Z}$, endowed with $\mathrm{Gal}(S'/S)$-action by outer automorphism corresponding to the nontrivial automorphism of the Dynkin diagram . Thus the dual group of the double-cover of $\mathbf{SU}_3$ coincides with the Langlands dual group of the linear group $\mathbf{SU}_3$.

2.7.3. $\mathbf{Spin}_{2\ell+1}$. — For $\mathbf{G} = \mathbf{Spin}_{2\ell+1}$, the Dynkin diagram has type B.



Let $\tilde{\mathbf{G}}$ be the cover of degree n, associated to the quadratic form taking the value 1 at all short coroots. Thus $Q(\alpha_i^\vee) = 2$ for all long coroots $1 \leq i \leq \ell - 1$. If n is odd, then $n_\alpha = n$ for all coroots $\alpha^\vee$. If n is even, then $n_i = n/2$ for $1 \leq i \leq \ell - 1$ and $n_\ell = n$. When n is even, short coroots become long and long become short, after modification. We find that the dual group is isogenous to $\underline{\mathbf{Sp}}_{2\ell}$ if n is odd, and is isogenous to $\underline{\mathbf{Spin}}_{2\ell+1}$ if n is even.

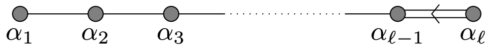
The centers of $\underline{\mathbf{Sp}}_{2\ell}$ and $\underline{\mathbf{Spin}}_{2\ell+1}$ are cyclic of order two. Thus the dual group can be identified by the order of its center.

	Group $\mathbf{G}$					
n	$\underline{\mathbf{Spin}}_7$	$\underline{\mathbf{Spin}}_9$	$\underline{\mathbf{Spin}}_{11}$	$\underline{\mathbf{Spin}}_{13}$	$\underline{\mathbf{Spin}}_{15}$	$\underline{\mathbf{Spin}}_{17}$
1	$\underline{\mathbf{PGSp}}_6$	$\underline{\mathbf{PGSp}}_8$	$\underline{\mathbf{PGSp}}_{10}$	$\underline{\mathbf{PGSp}}_{12}$	$\underline{\mathbf{PGSp}}_{14}$	$\underline{\mathbf{PGSp}}_{16}$
2	$\underline{\mathbf{SO}}_7$	$\underline{\mathbf{Spin}}_9$	$\underline{\mathbf{SO}}_{11}$	${}^*\underline{\mathbf{Spin}}_{13}$	$\underline{\mathbf{SO}}_{15}$	$\underline{\mathbf{Spin}}_{17}$
3	$\underline{\mathbf{PGSp}}_6$	$\underline{\mathbf{PGSp}}_8$	$\underline{\mathbf{PGSp}}_{10}$	$\underline{\mathbf{PGSp}}_{12}$	$\underline{\mathbf{PGSp}}_{14}$	$\underline{\mathbf{PGSp}}_{16}$
4	${}^*\underline{\mathbf{Spin}}_7$	$\underline{\mathbf{Spin}}_9$	${}^*\underline{\mathbf{Spin}}_{11}$	$\underline{\mathbf{Spin}}_{13}$	${}^*\underline{\mathbf{Spin}}_{15}$	$\underline{\mathbf{Spin}}_{17}$
5	$\underline{\mathbf{PGSp}}_6$	$\underline{\mathbf{PGSp}}_8$	$\underline{\mathbf{PGSp}}_{10}$	$\underline{\mathbf{PGSp}}_{12}$	$\underline{\mathbf{PGSp}}_{14}$	$\underline{\mathbf{PGSp}}_{16}$
6	$\underline{\mathbf{SO}}_7$	$\underline{\mathbf{Spin}}_9$	$\underline{\mathbf{SO}}_{11}$	${}^*\underline{\mathbf{Spin}}_{13}$	$\underline{\mathbf{SO}}_{15}$	$\underline{\mathbf{Spin}}_{17}$

TABLE 2. Table of dual groups for degree n covers of $\mathbf{Spin}_{2\ell+1}$. Groups marked with $*$ have $\tau_Q(-1)$ nontrivial.

Table 2 describes the dual groups. Note that, in this case, the isogeny class of the dual group depends on the covering degree modulo 4. In covering degree $4k+2$, we find that $\tau_Q(-1)$ is nontrivial for $\mathbf{Spin}_{8j+5}$ (corresponding to rank $4j+2$) for all positive integers j. In covering degree $4k$, we find that $\tau_Q(-1)$ is nontrivial for $\mathbf{Spin}_{4j+3}$ for all positive integers j.

2.7.4. $\mathbf{Sp}_{2\ell}$. — For $\mathbf{G} = \mathbf{Sp}_{2\ell}$, the Dynkin diagram has type C.



Let $\tilde{\mathbf{G}}$ be the cover of degree n, associated to the quadratic form taking the value 1 at all short coroots. As in type B, we find that short coroots become long, and long become short, after modification when n is even. We find that the dual group is isogenous to $\underline{\mathbf{Sp}}_{2\ell}$ if n is even, and is isogenous to $\underline{\mathbf{Spin}}_{2\ell+1}$ if n is odd. As before, the dual group can be identified by the order of its center.

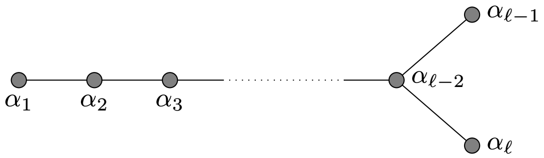
As Table 3 illustrates, the dual group of the degree n cover of $\mathbf{Sp}_{2\ell}$ is the simply-connected Chevalley group $\underline{\mathbf{Sp}}_{2\ell}$ when n is even, and the dual group is the adjoint

	Group $\mathbf{G}$		
n	$\mathbf{Sp}_6$	$\mathbf{Sp}_8$	$\mathbf{Sp}_{10}$
1	$\underline{\mathbf{SO}}_7$	$\underline{\mathbf{SO}}_9$	$\underline{\mathbf{SO}}_{11}$
2	${}^*\underline{\mathbf{Sp}}_6$	${}^*\underline{\mathbf{Sp}}_8$	${}^*\underline{\mathbf{Sp}}_{10}$
3	$\underline{\mathbf{SO}}_7$	$\underline{\mathbf{SO}}_9$	$\underline{\mathbf{SO}}_{11}$
4	$\underline{\mathbf{Sp}}_6$	$\underline{\mathbf{Sp}}_8$	$\underline{\mathbf{Sp}}_{10}$
5	$\underline{\mathbf{SO}}_7$	$\underline{\mathbf{SO}}_9$	$\underline{\mathbf{SO}}_{11}$
6	${}^*\underline{\mathbf{Sp}}_6$	${}^*\underline{\mathbf{Sp}}_8$	${}^*\underline{\mathbf{Sp}}_{10}$

TABLE 3. Table of dual groups for degree n covers of $\mathbf{Sp}_{2\ell}$. Groups marked with $*$ have $\tau_Q(-1)$ nontrivial.

group $\underline{\mathbf{SO}}_{2\ell+1} = \underline{\mathbf{Spin}}_{2\ell+1}/\boldsymbol{\mu}_2$ when n is odd. The central 2-torsion element $\tau_Q(-1)$ is nontrivial when the covering degree is $4k+2$ for some non-negative integer k. This is consistent (in covering degree 2) with expectations from the classical theta correspondence for metaplectic groups.

2.7.5. $\mathbf{Spin}_{2\ell}$. — For $\mathbf{G} = \mathbf{Spin}_{2\ell}$, $\ell \geq 4$, the Dynkin diagram has type D.



Let $\tilde{\mathbf{G}}$ be the cover of degree n, associated to the quadratic form taking the value 1 at all coroots. By the same methods as in type A, we find that the dual group is isogenous to $\underline{\mathbf{Spin}}_{2\ell}$.

If ℓ is odd, then the center of $\underline{\mathbf{Spin}}_{2\ell}$ is a cyclic group of order 4. In this case, the dual group is determined by the order of its center.

If ℓ is even, then the center of $\underline{\mathbf{Spin}}_{2\ell}$ is isomorphic to $\underline{\boldsymbol{\mu}}_2 \times \underline{\boldsymbol{\mu}}_2$, and so the dual group is not a priori determined by the order of its center. But fortunately, the order of the center of the dual group always equals 1 or 4 when ℓ is even, and this suffices to identify the dual group.

As Table 4 illustrates, the dual group of an odd-degree cover of $\mathbf{Spin}_{2\ell}$ coincides with the Langlands dual group of the linear group $\mathbf{Spin}_{2\ell}$; this dual group is the adjoint group $\underline{\mathbf{PGO}}_{2\ell}$. But the dual group of an even-degree cover of $\mathbf{Spin}_{2\ell}$ depends on the parity of ℓ and the covering degree modulo 4. As in type A, these dual groups agree with expectations from the Hecke algebra isomorphisms of Savin [**62**].

When the covering degree is a multiple of 4, the element $\tau_Q(-1)$ is nontrivial for $\mathbf{Spin}_{4j+2}$ for all $j \geq 2$. Since $\mathbf{Spin}_{4j+2}$ has a unique central element of order two, this suffices to describe τ_Q. When the covering degree has the form $4k+2$, the element

	Group $\mathbf{G}$					
n	$\mathbf{Spin}_8$	$\mathbf{Spin}_{10}$	$\mathbf{Spin}_{12}$	$\mathbf{Spin}_{14}$	$\mathbf{Spin}_{16}$	$\mathbf{Spin}_{18}$
1	$\underline{\mathbf{PGO}}_8$	$\underline{\mathbf{PGO}}_{10}$	$\underline{\mathbf{PGO}}_{12}$	$\underline{\mathbf{PGO}}_{14}$	$\underline{\mathbf{PGO}}_{16}$	$\underline{\mathbf{PGO}}_{18}$
2	$\underline{\mathbf{Spin}}_8$	$\underline{\mathbf{SO}}_{10}$	$^*\underline{\mathbf{Spin}}_{12}$	$\underline{\mathbf{SO}}_{14}$	$\underline{\mathbf{Spin}}_{16}$	$\underline{\mathbf{SO}}_{18}$
3	$\underline{\mathbf{PGO}}_8$	$\underline{\mathbf{PGO}}_{10}$	$\underline{\mathbf{PGO}}_{12}$	$\underline{\mathbf{PGO}}_{14}$	$\underline{\mathbf{PGO}}_{16}$	$\underline{\mathbf{PGO}}_{18}$
4	$\underline{\mathbf{Spin}}_8$	$^*\underline{\mathbf{Spin}}_{10}$	$\underline{\mathbf{Spin}}_{12}$	$^*\underline{\mathbf{Spin}}_{14}$	$\underline{\mathbf{Spin}}_{16}$	$^*\underline{\mathbf{Spin}}_{18}$
5	$\underline{\mathbf{PGO}}_8$	$\underline{\mathbf{PGO}}_{10}$	$\underline{\mathbf{PGO}}_{12}$	$\underline{\mathbf{PGO}}_{14}$	$\underline{\mathbf{PGO}}_{16}$	$\underline{\mathbf{PGO}}_{18}$
6	$\underline{\mathbf{Spin}}_8$	$\underline{\mathbf{SO}}_{10}$	$^*\underline{\mathbf{Spin}}_{12}$	$\underline{\mathbf{SO}}_{14}$	$\underline{\mathbf{Spin}}_{16}$	$\underline{\mathbf{SO}}_{18}$

TABLE 4. Table of dual groups for degree n covers of $\mathbf{Spin}_{2\ell}$. Groups marked with $*$ have $\tau_Q(-1)$ nontrivial.

$\tau_Q(-1)$ is nontrivial for $\mathbf{Spin}_{8j+4}$ for all $j \geq 1$. The center of the group $\underline{\mathbf{Spin}}_{8j+4}$ is isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, which has three distinct 2-torsion elements. However, only one of these is invariant under the nontrivial outer automorphism of the pinned Chevalley group $\underline{\mathbf{Spin}}_{8j+4}$. This one must be $\tau_Q(-1)$, since Q is invariant under this outer automorphism.

2.7.6. *Exceptional groups.* — Let $\mathbf{G}$ be a simply-connected split simple group of type E_ℓ (with $\ell = 6, 7, 8$), F_4, or G_2. Let $\tilde{\mathbf{G}}$ be the cover of degree n, associated to the quadratic form taking the value 1 at all short coroots (all coroots in type E). As in types A and D, we find that the dual group is semisimple and isogenous to the Chevalley group of the same type as $\mathbf{G}$. In types E_8, F_4, and G_2, the simply-connected group is centerless, and so $\tilde{\mathbf{G}}^\vee$ coincides with the simply-connected Chevalley group of the same type.

The center of $\underline{\mathbf{E}}_6$ has order 3, and the center of $\underline{\mathbf{E}}_7$ has order 2. Hence the dual group $\tilde{\mathbf{G}}^\vee$ is determined by the order of its center. The dual groups are listed in Table 5. In type E, these dual groups agree with expectations from [**62**]. The central

	Group $\mathbf{G}$				
n	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$	$\mathbf{F}_4$	$\mathbf{G}_2$
1	$\underline{\mathbf{E}}_6/\boldsymbol{\mu}_3$	$\underline{\mathbf{E}}_7/\boldsymbol{\mu}_2$	$\underline{\mathbf{E}}_8$	$\underline{\mathbf{F}}_4$	$\underline{\mathbf{G}}_2$
2	$\underline{\mathbf{E}}_6/\boldsymbol{\mu}_3$	$^*\underline{\mathbf{E}}_7$	$\underline{\mathbf{E}}_8$	$\underline{\mathbf{F}}_4$	$\underline{\mathbf{G}}_2$
3	$\underline{\mathbf{E}}_6$	$\underline{\mathbf{E}}_7/\boldsymbol{\mu}_2$	$\underline{\mathbf{E}}_8$	$\underline{\mathbf{F}}_4$	$\underline{\mathbf{G}}_2$
4	$\underline{\mathbf{E}}_6/\boldsymbol{\mu}_3$	$\underline{\mathbf{E}}_7$	$\underline{\mathbf{E}}_8$	$\underline{\mathbf{F}}_4$	$\underline{\mathbf{G}}_2$
5	$\underline{\mathbf{E}}_6/\boldsymbol{\mu}_3$	$\underline{\mathbf{E}}_7/\boldsymbol{\mu}_2$	$\underline{\mathbf{E}}_8$	$\underline{\mathbf{F}}_4$	$\underline{\mathbf{G}}_2$
6	$\underline{\mathbf{E}}_6$	$^*\underline{\mathbf{E}}_7$	$\underline{\mathbf{E}}_8$	$\underline{\mathbf{F}}_4$	$\underline{\mathbf{G}}_2$

TABLE 5. Table of dual groups for degree n covers of exceptional groups. Groups marked with $*$ have $\tau_Q(-1)$ nontrivial.

2-torsion element is nontrivial for $\mathbf{E}_7$, when the covering degree equals $4j+2$ for some $j \geq 0$.

2.7.7. $\mathbf{GL}_r$. — Suppose that $\mathbf{G}$ is split reductive, and the derived subgroup of $\mathbf{G}$ is simply-connected. Let $\mathbf{T}$ be a split maximal torus in $\mathbf{G}$ with cocharacter lattice Y. Then, for any Weyl-invariant quadratic form $Q \colon Y \to \mathbb{Z}$, there exists a cover $\tilde{\mathbf{G}}$ with first Brylinski-Deligne invariant Q.

For example, when $\mathbf{G} = \mathbf{GL}_r$, there is a two-parameter family of such Weyl-invariant quadratic forms. Write $\mathbf{T}$ for the standard maximal torus of diagonal matrices, and identify $Y = \mathbb{Z}^r$ in the usual way. For any pair of integers q, c, there exists a unique Weyl-invariant quadratic form $Q_{q,c}$ satisfying

$$Q(1, -1, 0, \dots, 0) = q \text{ and } Q(1, 0, \dots, 0) = 1 + c.$$

The n-fold covers $\widetilde{GL}_r^{(c)}$ studied by Kazhdan and Patterson [**40**, §0.1] can be constructed from Brylinski-Deligne extensions with first invariant $Q_{1,c}$. The proof of following result is left to the reader.

Proposition 2.5. — *Let $\tilde{\mathbf{G}}$ be a degree n cover of $\mathbf{G} = \mathbf{GL}_r$ with first Brylinski-Deligne invariant $Q_{1,c}$. If $\mathrm{GCD}(n, 1 + r + 2rc) = 1$, then $\tilde{\mathbf{G}}^\vee$ is isomorphic to $\underline{\mathbf{GL}}_r$. If $\mathrm{GCD}(n, r) = 1$, then the derived subgroup of $\tilde{\mathbf{G}}^\vee$ is isomorphic to $\underline{\mathbf{SL}}_r$ and thus there exists an isogeny $\tilde{\mathbf{G}}^\vee \to \underline{\mathbf{GL}}_r$.*

This may place the work of Kazhdan and Flicker [**28**] in a functorial context.

2.7.8. $\mathbf{GSp}_{2r}$. — For $\mathbf{G} = \mathbf{GSp}_{2r}$, and a standard choice of split maximal torus and Borel subgroup, we write $e_0, \dots, e_r$ for a basis of Y, $f_0, \dots, f_r$ for the dual basis of X, and the simple roots and coroots are

$$\begin{aligned} &\alpha_1 = f_1 - f_2, \dots, \alpha_{r-1} = f_{r-1} - f_r, && \alpha_r = 2f_r - f_0; \\ &\alpha_1^\vee = e_1 - e_2, \dots, \alpha_{r-1}^\vee = e_{r-1} - e_r, && \alpha_r^\vee = e_r. \end{aligned}$$

The Weyl group is $S_r \ltimes \mu_2^r$, with S_r acting by permutation of indices $1, \dots, r$ (fixing e_0 and f_0), and elements w_j (for $1 \leq j \leq r$) of order two which satisfy

$$w_j(e_j) = -e_j, \quad w_j(e_i) = e_i \text{ for } i \neq j, 0, \quad w_j(e_0) = e_0 + e_1.$$

Weyl-invariant quadratic forms on Y are in bijection with pairs (κ, ν) of integers. For any such pair, there is a unique Weyl-invariant quadratic form $Q_{\kappa,\nu}$ satisfying

$$Q_{\kappa,\nu}(e_0) = \kappa, \quad Q_{\kappa,\nu}(e_i) = \nu \text{ for } 1 \leq i \leq r.$$

The proof of following result is left to the reader.

Proposition 2.6. — *Let $\tilde{\mathbf{G}}$ be a degree 2 cover of $\mathbf{G} = \mathbf{GSp}_{2r}$, with first Brylinski-Deligne invariant $Q_{0,1}$. Then the dual group $\tilde{\mathbf{G}}^\vee$ is isomorphic to $\underline{\mathbf{GSp}}_{2r}$ if r is odd, and to $\underline{\mathbf{PGSp}}_{2r} \times \underline{\mathbf{G}}_m$ if r is even.*

We find that double-covers of $\mathbf{GSp}_{2r}$ behave differently depending on the parity of r; this phenomenon is consistent with the work of Szpruch [**71**] on principal series.

3. The gerbe associated to a cover

In this section, we construct a gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ on $S_{\text{ét}}$ associated to a degree n cover $\tilde{\mathbf{G}}$ of a quasisplit group $\mathbf{G}$ and an injective character $\varepsilon\colon \mu_n \hookrightarrow \mathbb{C}^\times$. Fix $\tilde{\mathbf{G}}$, $\mathbf{G}$, and ε throughout. Also, choose a Borel subgroup containing a maximally split maximal torus $\mathbf{B} \supset \mathbf{T}$; we will see that our construction is independent of this choice (in a 2-categorical sense).

We make one assumption about our cover $\tilde{\mathbf{G}}$, which enables our construction and is essentially nonrestrictive.

***Assumption 3.1* (Odd n implies even Q).** — If n is odd, then we assume $Q\colon \mathscr{Y} \to \mathbb{Z}$ takes only even values.

If $\tilde{\mathbf{G}}$ does not satisfy this assumption, i.e., n is odd and $Q(y)$ is odd for some $y \in \mathscr{Y}$, then replace $\tilde{\mathbf{G}}$ by $(n+1)\dot{\times}\tilde{\mathbf{G}}$ (its Baer sum with itself $n+1$ times). The first Brylinski-Deligne invariant becomes $(n+1)Q$, which is even-valued. By Proposition 2.1, the dual group $\tilde{\mathscr{G}}^\vee$ does not change since $Q \equiv (n+1)Q$ modulo n. Moreover, the resulting extensions of groups over local or global fields, e.g., $\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G$, remain the same (up to natural isomorphism). Indeed, the Baer sum of $\tilde{G}$ with itself $n+1$ times is naturally isomorphic to the pushout via the $(n+1)$th power map $\mu_n \to \mu_n$, which equals the identity map.

We work with sheaves of abelian groups on $S_{\text{ét}}$, and great care is required to avoid confusion between those in the left column and the right column below. Define

$$
\begin{aligned}
\hat{\mathscr{T}} &= \mathscr{H}om(\mathscr{Y}_{Q,n}, \mathscr{G}_m), & \tilde{\mathscr{T}}^\vee &= \mathscr{H}om(\mathscr{Y}_{Q,n}, \mathbb{C}^\times);\\
\hat{\mathscr{T}}_{\text{sc}} &= \mathscr{H}om(\mathscr{Y}^{\text{sc}}_{Q,n}, \mathscr{G}_m), & \tilde{\mathscr{T}}^\vee_{\text{sc}} &= \mathscr{H}om(\mathscr{Y}^{\text{sc}}_{Q,n}, \mathbb{C}^\times);\\
\hat{\mathscr{Z}} &= \mathscr{H}om(\mathscr{Y}_{Q,n}/\mathscr{Y}^{\text{sc}}_{Q,n}, \mathscr{G}_m), & \tilde{\mathscr{Z}}^\vee &= \mathscr{H}om(\mathscr{Y}_{Q,n}/\mathscr{Y}^{\text{sc}}_{Q,n}, \mathbb{C}^\times).
\end{aligned}
$$

Here $\mathbb{C}^\times$ denotes the constant sheaf on $S_{\text{ét}}$. Thus, in the right column, we find the complex points of the dual groups,

$$\tilde{\mathscr{T}}^\vee = \tilde{\mathscr{T}}^\vee(\mathbb{C}), \quad \tilde{\mathscr{T}}^\vee_{\text{sc}} = \tilde{\mathscr{T}}^\vee_{\text{sc}}(\mathbb{C}), \quad \tilde{\mathscr{Z}}^\vee = \tilde{\mathscr{Z}}^\vee(\mathbb{C}).$$

Composing with ε defines homomorphisms of local systems of abelian groups,

$$
\begin{aligned}
\hat{\mathscr{T}}_{[n]} &= \mathscr{H}om(\mathscr{Y}_{Q,n}, \mu_n) \xrightarrow{\varepsilon} \tilde{\mathscr{T}}^\vee;\\
\hat{\mathscr{T}}_{\text{sc},[n]} &= \mathscr{H}om(\mathscr{Y}^{\text{sc}}_{Q,n}, \mu_n) \xrightarrow{\varepsilon} \tilde{\mathscr{T}}^\vee_{\text{sc}};\\
\hat{\mathscr{Z}}_{[n]} &= \mathscr{H}om(\mathscr{Y}_{Q,n}/\mathscr{Y}^{\text{sc}}_{Q,n}, \mu_n) \xrightarrow{\varepsilon} \tilde{\mathscr{Z}}^\vee.
\end{aligned}
$$

3.1. The gerbe associated to a cover of a torus. — Associated to the cover $\tilde{\mathbf{T}} = (\mathbf{T}', n)$, the second Brylinski-Deligne invariant is a central extension of sheaves of groups on $S_{\text{ét}}$,

$$\mathscr{G}_m \hookrightarrow \mathscr{D} \twoheadrightarrow \mathscr{Y}.$$

The commutator of this extension is given in [**18**, Proposition 3.11],

$$\text{Comm}(y_1, y_2) = (-1)^{n\beta_Q(y_1,y_2)}, \text{ for all } y_1, y_2 \in \mathscr{Y}. \tag{3.1}$$

Pulling back via $\mathscr{Y}_{Q,n} \hookrightarrow \mathscr{Y}$, we find an extension of sheaves of groups,

$$\mathscr{G}_m \hookrightarrow \mathscr{D}_{Q,n} \twoheadrightarrow \mathscr{Y}_{Q,n}. \tag{3.2}$$

Proposition 3.2. — *$\mathscr{D}_{Q,n}$ is a commutative extension.*

Proof. — If n is even and $y_1, y_2 \in \mathscr{Y}_{Q,n}$, then $\beta_Q(y_1, y_2) \in \mathbb{Z}$ and $n\beta_Q(y_1, y_2) \in 2\mathbb{Z}$. On the other hand, if n is odd, Assumption 3.1 implies that

$$n\beta_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2) \in 2\mathbb{Z}.$$

The commutator Formula (3.1) finishes the proof. □

Let $\mathscr{Spl}(\mathscr{D}_{Q,n})$ denote the sheaf of *splittings* of the commutative extension (3.2). In other words, $\mathscr{Spl}(\mathscr{D}_{Q,n})$ is the subsheaf of $\mathscr{Hom}(\mathscr{Y}_{Q,n}, \mathscr{D}_{Q,n})$ consisting of homomorphisms which split (3.2).

Over any finite étale $U \to S$ splitting $\mathbf{T}$, $\mathscr{Y}_{Q,n}$ restricts to a constant sheaf of free abelian groups. Thus $\mathscr{Spl}(\mathscr{D}_{Q,n})$ is a $\hat{\mathscr{T}}$-torsor on $S_{\text{ét}}$, which obtains a point over any such U. The equivalence class of this torsor is determined by its cohomology class $\left[\mathscr{Spl}(\mathscr{D}_{Q,n})\right] \in H^1_{\text{ét}}(S, \hat{\mathscr{T}})$.

Consider the Kummer sequence, $\hat{\mathscr{T}}_{[n]} \hookrightarrow \hat{\mathscr{T}} \xrightarrow{n} \hat{\mathscr{T}}$. Write $\varkappa$ (for Kummer) for the connecting map in cohomology, $\varkappa\colon H^1_{\text{ét}}(S, \hat{\mathscr{T}}) \to H^2_{\text{ét}}(S, \hat{\mathscr{T}}_{[n]})$. This map in cohomology corresponds to the functor which sends a $\hat{\mathscr{T}}$-torsor to its gerbe of nth roots (see A.3.4 for details). We write $\sqrt[n]{\mathscr{Spl}(\mathscr{D}_{Q,n})}$ for the gerbe of nth roots of the $\hat{\mathscr{T}}$-torsor $\mathscr{Spl}(\mathscr{D}_{Q,n})$. It is banded by the local system $\hat{\mathscr{T}}_{[n]}$ and its equivalence class satisfies

$$\left[\sqrt[n]{\mathscr{Spl}(\mathscr{D}_{Q,n})}\right] = \varkappa[\mathscr{Spl}(\mathscr{D}_{Q,n})].$$

Finally we push out via the homomorphism of local systems,

$$\varepsilon\colon \hat{\mathscr{T}}_{[n]} = \mathscr{Hom}(\mathscr{Y}_{Q,n}, \mu_n) \to \mathscr{Hom}(\mathscr{Y}_{Q,n}, \mathbb{C}^\times) = \tilde{\mathscr{T}}^\vee.$$

Definition 3.3. — The ***gerbe associated to the cover $\tilde{\mathbf{T}}$*** is defined by

$$\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}) := \varepsilon_* \sqrt[n]{\mathscr{Spl}(\mathscr{D}_{Q,n})}.$$

It is a gerbe on $S_{\text{ét}}$ banded by the local system of abelian groups $\tilde{\mathscr{T}}^\vee$.

Example 3.4. — *Suppose that $\mathbf{T}$ is a split torus. Then the exact sequence of sheaves $\mathscr{G}_m \hookrightarrow \mathscr{D}_{Q,n} \twoheadrightarrow \mathscr{Y}_{Q,n}$ splits. Indeed, $\mathscr{Y}_{Q,n}$ is a constant sheaf of free abelian groups, and Hilbert's Theorem 90 gives a short exact sequence*

$$\mathscr{G}_m[S] \hookrightarrow \mathscr{D}_{Q,n}[S] \twoheadrightarrow \mathscr{Y}_{Q,n}[S]. \tag{3.3}$$

Since $\mathscr{Y}_{Q,n}[S]$ is a free abelian group, this exact sequence splits, and any such splitting defines an S-point of the torsor $\mathscr{Spl}(\mathscr{D}_{Q,n})$. An S-point of $\mathscr{Spl}(\mathscr{D}_{Q,n})$, in turn, neutralizes of the gerbe $\sqrt[n]{\mathscr{Spl}(\mathscr{D}_{Q,n})}$.

Thus when $\mathbf{T}$ is a split torus, the gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{T}})$ is trivial. Any splitting of the sequence (3.3) *defines a neutralization of $\mathsf{E}_\varepsilon(\tilde{\mathbf{T}})$.*

3.2. The gerbe of liftings. — Recall that $\mathscr{Y}_{Q,n}^{\mathrm{sc}}$ denotes the subgroup of $\mathscr{Y}_{Q,n}$ spanned by the modified coroots $\tilde{\Phi}^\vee$, and $\hat{\mathscr{T}}_{\mathrm{sc}} = \mathscr{H}\!om(\mathscr{Y}_{Q,n}^{\mathrm{sc}}, \mathscr{G}_m)$. The inclusion $\mathscr{Y}_{Q,n}^{\mathrm{sc}} \hookrightarrow \mathscr{Y}_{Q,n}$ corresponds to a surjective homomorphism,

$$p\colon \hat{\mathscr{T}} \to \hat{\mathscr{T}}_{\mathrm{sc}}.$$

The extension $\mathscr{G}_m \hookrightarrow \mathscr{D}_{Q,n} \twoheadrightarrow \mathscr{Y}_{Q,n}$ pulls back via $\mathscr{Y}_{Q,n}^{\mathrm{sc}} \hookrightarrow \mathscr{Y}_{Q,n}$ to an extension,

$$\mathscr{G}_m \hookrightarrow \mathscr{D}_{Q,n}^{\mathrm{sc}} \twoheadrightarrow \mathscr{Y}_{Q,n}^{\mathrm{sc}}.$$

A splitting of $\mathscr{D}_{Q,n}$ pulls back to a splitting of $\mathscr{D}_{Q,n}^{\mathrm{sc}}$, providing a map of torsors,

$$p^*\colon \mathscr{S}\!p\ell(\mathscr{D}_{Q,n}) \to \mathscr{S}\!p\ell(\mathscr{D}_{Q,n}^{\mathrm{sc}}),$$

lying over $p\colon \hat{\mathscr{T}} \to \hat{\mathscr{T}}_{\mathrm{sc}}$. Taking nth roots of torsors gives a functor of gerbes,

$$\sqrt[n]{p^*}\colon \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D}_{Q,n})} \to \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D}_{Q,n}^{\mathrm{sc}})},$$

lying over $p\colon \hat{\mathscr{T}}_{[n]} \to \hat{\mathscr{T}}_{\mathrm{sc},[n]}$ (see Appendix A.3.4).

Recall that $\tilde{\mathscr{T}}^\vee = \mathscr{H}\!om(\mathscr{Y}_{Q,n}, \mathbb{C}^\times)$ and $\tilde{\mathscr{T}}^\vee_{\mathrm{sc}} = \mathscr{H}\!om(\mathscr{Y}_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times)$. Define $\mathbf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}}) := \varepsilon_* \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D}_{Q,n}^{\mathrm{sc}})}$ by analogy to $\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}) = \varepsilon_* \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D}_{Q,n})}$. Pushing out via ε, the functor $\sqrt[n]{p^*}$ yields a functor of gerbes

$$\mathbf{p} = \varepsilon_* \sqrt[n]{p^*}\colon \mathbf{E}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}}),$$

lying over the homomorphism $p\colon \tilde{\mathscr{T}}^\vee \to \tilde{\mathscr{T}}^\vee_{\mathrm{sc}}$.

In the next section, we define the *Whittaker torsor*, which gives an object $\boldsymbol{w}$ neutralizing the gerbe $\mathbf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}})$. We take this construction of $\boldsymbol{w}$ for granted at the moment.

Definition 3.5. — Define $\mathbf{E}_\varepsilon(\tilde{\mathbf{G}})$ to be the gerbe $\mathbf{p}^{-1}(\boldsymbol{w})$ of liftings of $\boldsymbol{w}$ via $\mathbf{p}$ (see A.3.3). In other words, $\mathbf{E}_\varepsilon(\tilde{\mathbf{G}})$ is the category of pairs $(\mathbf{e}, j)$ where $\mathbf{e}$ is an object of $\mathbf{E}_\varepsilon(\tilde{\mathbf{T}})$ and $j\colon \mathbf{p}(\mathbf{e}) \to \boldsymbol{w}$ is an isomorphism in $\mathbf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}})$. This is a gerbe on $S_{\text{ét}}$ banded by $\tilde{\mathscr{Z}}^\vee = \mathrm{Ker}(\tilde{\mathscr{T}}^\vee \xrightarrow{p} \tilde{\mathscr{T}}^\vee_{\mathrm{sc}})$.

The cohomology classes of our gerbes now fit into a sequence

$$\begin{array}{ccccc}
\left[\mathbf{E}_\varepsilon(\tilde{\mathbf{G}})\right] & \longmapsto & \left[\mathbf{E}_\varepsilon(\tilde{\mathbf{T}})\right] & \longmapsto & \left[\mathbf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}})\right] = 0 \\
\pitchfork & & \pitchfork & & \pitchfork \\
H^2_{\text{ét}}(S, \tilde{\mathscr{Z}}^\vee) & \longrightarrow & H^2_{\text{ét}}(S, \tilde{\mathscr{T}}^\vee) & \longrightarrow & H^2_{\text{ét}}(S, \tilde{\mathscr{T}}^\vee_{\mathrm{sc}}).
\end{array}$$

Remark 3.6. — The construction of this gerbe relies on the (soon-to-be-defined) Whittaker torsor in a crucial way. We view this as a good thing, since any putative Langlands correspondence should also connect the existence of Whittaker models to properties of the Langlands parameter (cf. [**74**]).

3.3. The Whittaker torsor. — Now we construct the object $\boldsymbol{w}$ neutralizing the gerbe $\mathsf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}})$ over S. Let $\mathbf{U}$ denote the unipotent radical of the Borel subgroup $\mathbf{B} \subset \mathbf{G}$, and let $\mathscr{U}$ be the sheaf of groups on $S_{\mathrm{\acute{e}t}}$ that it represents. Let $\mathbf{G}_a$ denote the additive group scheme over S, and $\mathscr{G}_a$ the sheaf of groups on $S_{\mathrm{\acute{e}t}}$ that it represents. Recall that $\Delta \subset \Phi$ denotes the subset of simple roots corresponding to the Borel subgroup $\mathbf{B}$.

For $S' \to S$ finite étale and splitting $\mathbf{T}$, and $\alpha \in \Delta[S']$, write $\mathbf{U}_\alpha$ for the one-dimensional root subgroup of $\mathbf{U}_{S'}$ associated to ϕ. Let $\mathscr{U}_\alpha$ be the associated sheaf of abelian groups on $S'_{\mathrm{\acute{e}t}}$. Write $\mathscr{H}om^*(\mathscr{U}_\alpha, \mathscr{G}_a)$ for the sheaf (on $S'_{\mathrm{\acute{e}t}}$) of isomorphisms from $\mathscr{U}_\alpha$ to $\mathscr{G}_{a,S'}$. The sheaf $\mathscr{H}om^*(\mathscr{U}_\alpha, \mathscr{G}_a)$ naturally forms a $\mathscr{G}_m$-torsor on $S'_{\mathrm{\acute{e}t}}$, by the formula

$$[h * \xi](u) = h^{-1} \cdot \xi(u) \text{ for all } h \in \mathscr{G}_m, \xi \in \mathscr{H}om^*(\mathscr{U}_\alpha, \mathscr{G}_a).$$

Definition 3.7. — The ***Whittaker torsor*** is the subsheaf $\mathscr{W}hit \subset \mathscr{H}om(\mathscr{U}, \mathscr{G}_a)$ consisting of those homomorphisms which (locally on $S_{\mathrm{\acute{e}t}}$) restrict to an isomorphism on every simple root subgroup. The sheaf $\mathscr{W}hit$ is given the structure of a $\hat{\mathscr{T}}_{\mathrm{sc}}$-torsor as follows: for a Galois cover $S' \to S$ splitting $\mathbf{T}$, we have

$$\hat{\mathscr{T}}_{\mathrm{sc}}[S'] = \mathscr{H}om(\mathscr{Y}_{Q,n}^{\mathrm{sc}}, \mathscr{G}_m)[S'] = \mathscr{H}om\left(\bigoplus_{\alpha \in \Delta[S']} \mathbb{Z}\tilde{\alpha}^\vee, \mathscr{G}_m \right)[S'] \equiv \prod_{\alpha \in \Delta[S']} \mathscr{G}_m[S'].$$

Similarly, we can decompose the Whittaker sheaf

$$\mathscr{W}hit[S'] \equiv \bigoplus_{\alpha \in \Delta[S']} \mathscr{H}om^*(\mathscr{U}_\alpha, \mathscr{G}_a)[S'].$$

The $\mathscr{G}_m$-torsor structure on $\mathscr{H}om^*(\mathscr{U}_\alpha, \mathscr{G}_a)$ yields (simple root by simple root) a $\hat{\mathscr{T}}_{\mathrm{sc}}$-torsor structure on $\mathscr{W}hit$. Although we have defined the torsor structure locally on $S_{\mathrm{\acute{e}t}}$, the action descends since the $\mathrm{Gal}(S'/S)$-actions are compatible throughout.

The third Brylinski-Deligne invariant of $\tilde{\mathbf{G}}$ is a homomorphism $f\colon \mathscr{D}_Q \to \mathscr{D}$ of groups on $S_{\mathrm{\acute{e}t}}$.

$$\begin{tikzcd}
\mathscr{G}_m \arrow[r, hook] \arrow[d, "="] & \mathscr{D}_Q \arrow[r, two heads] \arrow[d, hook, "f"] & \mathscr{Y}^{\mathrm{sc}} \arrow[d, hook] \\
\mathscr{G}_m \arrow[r, hook] & \mathscr{D} \arrow[r, two heads] & \mathscr{Y}.
\end{tikzcd}$$

Here $\mathscr{D}_Q$ is a sheaf on $S_{\mathrm{\acute{e}t}}$ which depends (up to unique isomorphism) only on the Weyl- and Galois-invariant quadratic form $Q\colon \mathscr{Y}^{\mathrm{sc}} \to \mathbb{Z}$. This is reviewed in [**79**, §1.3, 3.3], and characterized in [**18**, §11] when working over a field.

Consider a Galois cover $S' \to S$ splitting $\mathbf{T}$ as before. For any $\eta \in \mathscr{W}hit[S']$, and any simple root $\alpha \in \Delta[S']$, there exists a unique element $e_{\eta,\alpha} \in \mathscr{U}_\alpha[S']$ such that $\eta(e_{\eta,\alpha}) = 1$. From these, [**18**, §11.2] gives elements $[e_{\eta,\alpha}] \in \mathscr{D}_Q[S']$ lying over the simple coroots $\alpha^\vee \in \mathscr{Y}^{\mathrm{sc}}[S']$.

Remark 3.8. — When $S = \mathrm{Spec}(F)$ this follows directly from [**18**, §11.2]. When $S = \mathrm{Spec}(\mathscr{O})$, $S' = \mathrm{Spec}(\mathscr{O}')$, $\eta \in \mathscr{W}hit[S']$, and F' is the fraction field of $\mathscr{O}'$, we find elements

$e_{\eta,\alpha} \in \mathscr{U}_\alpha[F']$; as η gives an isomorphism from $\mathscr{U}_\alpha$ to $\mathscr{G}_a$ (as sheaves of groups on $\mathscr{O}'_{\text{ét}}$), it follows that $e_{\eta,\alpha} \in \mathscr{U}_\alpha[\mathscr{O}']$ as well. The map $e \mapsto [e]$ of [**18**, §11.1] similarly makes sense over $\mathscr{O}'$ as well as it does over a field; since we assume $\mathbf{G}$ is a reductive group over $\mathscr{O}$, split over $\mathscr{O}'$, every root SL_2 over F' arises from one over $\mathscr{O}'$. Thus the results of [**18**, §11.2] apply in the setting of $S = \operatorname{Spec}(\mathscr{O})$ as well as in the setting of a field.

Using the elements $[e_{\eta,\alpha}] \in \mathscr{D}_Q[S']$ lying over the simple coroots $\alpha^\vee$, define

$$\omega(\eta)(\tilde{\alpha}^\vee) := r_\alpha f([e_{\eta,\alpha}])^{n_\alpha} \in \mathscr{D}^{\text{sc}}_{Q,n}[S'], \text{ lying over } \tilde{\alpha}^\vee = n_\alpha \alpha^\vee \in \mathscr{Y}^{\text{sc}}_{Q,n}[S'],$$

where the sign r_α is defined by

$$r_\alpha := (-1)^{\frac{Q(\alpha^\vee) n_\alpha (n_\alpha - 1)}{2}}.$$

The map $\omega(\eta)\colon \tilde{\alpha}^\vee \mapsto r_\alpha f([e_{\eta,\alpha}])^{n_\alpha}$ extends uniquely to a splitting of the sequence

$$\mathscr{G}_m[S'] \hookrightarrow \mathscr{D}^{\text{sc}}_{Q,n}[S'] \twoheadrightarrow \mathscr{Y}^{\text{sc}}_{Q,n}[S']. \tag{3.4}$$

As $(\mathscr{Y}^{\text{sc}}_{Q,n})_{S'}$ is a constant sheaf, this gives an element $\omega(\eta) \in \mathscr{S}\!\mathit{pl}(\mathscr{D}^{\text{sc}}_{Q,n})[S']$. Allowing η to vary, and appyling Galois descent (cf. [**18**, Proposition 11.7]), we find a map of sheaves on $S_{\text{ét}}$,

$$\omega\colon \mathscr{W}\!\mathit{hit} \to \mathscr{S}\!\mathit{pl}(\mathscr{D}^{\text{sc}}_{Q,n}).$$

To summarize, ω is the map that sends a nondegenerate character η of $\mathbf{U}$ to the splitting $\omega(\eta)$, which (locally on $S_{\text{ét}}$) sends each modified simple coroot $\tilde{\alpha}^\vee$ to the element $r_\alpha f([e_{\eta,\alpha}])^{n_\alpha}$ of $\mathscr{D}_{Q,n}$.

Remark 3.9. — For the purposes of this paper, there is some flexibility in the choice of signs r_α. The signs here are defined in such a way that our hypothesized local Langlands correspondence for covers matches what is known for covers of $\mathbf{SL}_2$, e.g., metaplectic correspondences of Shimura and Waldspurger.

Both $\mathscr{W}\!\mathit{hit}$ and $\mathscr{S}\!\mathit{pl}(\mathscr{D}^{\text{sc}}_{Q,n})$ are $\hat{\mathscr{T}}_{\text{sc}}$-torsors, and the following proposition describes how ω interacts with the torsor structure.

Proposition 3.10. — *Let $\nu\colon \hat{\mathscr{T}}_{\text{sc}} \to \hat{\mathscr{T}}_{\text{sc}}$ be the homomorphism corresponding to the unique homomorphism $\mathscr{Y}^{\text{sc}}_{Q,n} \to \mathscr{Y}^{\text{sc}}_{Q,n}$ which sends $\tilde{\alpha}^\vee$ to $-n_\alpha Q(\alpha^\vee)\tilde{\alpha}^\vee$ for all simple roots α. Then ω lies over ν, i.e., the following diagram commutes.*

$$\begin{array}{ccc} \hat{\mathscr{T}}_{\text{sc}} \times \mathscr{W}\!\mathit{hit} & \xrightarrow{\ *\ } & \mathscr{W}\!\mathit{hit} \\ \downarrow{\scriptstyle \nu \times \omega} & & \downarrow{\scriptstyle \omega} \\ \hat{\mathscr{T}}_{\text{sc}} \times \mathscr{S}\!\mathit{pl}(\mathscr{D}^{\text{sc}}_{Q,n}) & \xrightarrow{\ *\ } & \mathscr{S}\!\mathit{pl}(\mathscr{D}^{\text{sc}}_{Q,n}). \end{array}$$

Proof. — We must trace through the action of $\hat{\mathscr{T}}_{\text{sc}} = \mathscr{H}\!\mathit{om}(\mathscr{Y}^{\text{sc}}_{Q,n}, \mathscr{G}_m)$; we work over a finite étale cover of S over which $\mathbf{T}$ splits in what follows. Then, for any simple root

$\alpha \in \Delta$, and any element $h \in \mathscr{G}_m$, there exists a unique element $h_\alpha \in \hat{\mathscr{T}}_{\mathrm{sc}}$ such that for all $\beta \in \Delta$,

$$h_\alpha(\tilde{\beta}^\vee) = \begin{cases} 1 & \text{if } \beta \neq \alpha; \\ h & \text{if } \beta = \alpha. \end{cases}$$

If $\eta \in \mathscr{W}\!\mathit{hit}$ then $[h_\alpha * \eta](e_{h_\alpha * \eta, \alpha}) = 1$ and so $\eta(e_{h_\alpha * \eta, \alpha}) = h$. Therefore,

$$e_{h_\alpha * \eta, \beta} = \begin{cases} e_{\eta, \beta} & \text{if } \beta \neq \alpha; \\ h * e_{\eta, \alpha} & \text{if } \beta = \alpha. \end{cases}$$

If $\beta \neq \alpha$, then $\omega(h_\alpha * \eta)(\tilde{\beta}^\vee) = r_\beta f([e_{h_\alpha * \eta, \beta}])^{n_\beta} = r_\beta f([e_{\eta,\beta}])^{n_\beta} = \omega(\eta)(\tilde{\beta}^\vee)$. On the other hand, in the case $\beta = \alpha$ we compute using [**18**, Equation (11.2.1)],

$$\begin{aligned} \omega(h_\alpha * \eta)(\tilde{\alpha}^\vee) = r_\alpha f([e_{h_\alpha * \eta, \alpha}])^{n_\alpha} &= r_\alpha f\left([h * e_{\eta,\alpha}]\right)^{n_\alpha} \\ &= r_\alpha f\left(h^{-Q(\alpha^\vee)} \cdot [e_{\eta,\alpha}]\right)^{n_\alpha} \\ &= r_\alpha h^{-n_\alpha Q(\phi_\alpha^\vee)} \cdot f\left([e_{\eta,\alpha}]\right)^{n_\alpha} \\ &= h^{-n_\alpha Q(\phi_\alpha^\vee)} \cdot \omega(\eta)(\tilde{\alpha}^\vee). \end{aligned}$$

This computation demonstrates that the diagram commutes as desired. □

Now let $\mu\colon \hat{\mathscr{T}}_{\mathrm{sc}} \twoheadrightarrow \hat{\mathscr{T}}_{\mathrm{sc}}$ be the homomorphism corresponding to the unique homomorphism $\mathscr{Y}_{Q,n}^{\mathrm{sc}} \hookrightarrow \mathscr{Y}_{Q,n}^{\mathrm{sc}}$ which sends $\tilde{\alpha}^\vee$ to $-m_\alpha \tilde{\alpha}^\vee$ for all $\alpha \in \Delta$. As $Q(\alpha^\vee) n_\alpha = m_\alpha \cdot n$, we find that $\nu = n \circ \mu$, where n denotes the nth-power map.

Let $\mu_* \mathscr{W}\!\mathit{hit}$ denote the pushout of the $\hat{\mathscr{T}}_{\mathrm{sc}}$-torsor $\mathscr{W}\!\mathit{hit}$, via μ. Since ν factors through μ, we find that $\omega\colon \mathscr{W}\!\mathit{hit} \to \mathscr{S}\!\mathit{pl}(\mathscr{D}_{Q,n}^{\mathrm{sc}})$ factors uniquely through $\bar{\omega}\colon \mu_* \mathscr{W}\!\mathit{hit} \to \mathscr{S}\!\mathit{pl}(\mathscr{D}_{Q,n}^{\mathrm{sc}})$, making the following diagram commute.

$$\begin{array}{ccc} \hat{\mathscr{T}}_{\mathrm{sc}} \times (\mu_* \mathscr{W}\!\mathit{hit}) & \xrightarrow{\ *\ } & \mu_* \mathscr{W}\!\mathit{hit} \\ \big\downarrow{\scriptstyle n \times \bar{\omega}} & & \big\downarrow{\scriptstyle \bar{\omega}} \\ \hat{\mathscr{T}}_{\mathrm{sc}} \times \mathscr{S}\!\mathit{pl}(\mathscr{D}_{Q,n}^{\mathrm{sc}}) & \xrightarrow{\ *\ } & \mathscr{S}\!\mathit{pl}(\mathscr{D}_{Q,n}^{\mathrm{sc}}). \end{array}$$

The pair $(\mu_* \mathscr{W}\!\mathit{hit}, \bar{\omega})$ is therefore an object of the category $\sqrt[n]{\mathscr{S}\!\mathit{pl}(\mathscr{D}_{Q,n}^{\mathrm{sc}})}[S]$; it *neutralizes* the gerbe $\sqrt[n]{\mathscr{S}\!\mathit{pl}(\mathscr{D}_{Q,n}^{\mathrm{sc}})}$. In particular,

$$\left[\sqrt[n]{\mathscr{S}\!\mathit{pl}(\mathscr{D}_{Q,n}^{\mathrm{sc}})}\right] = 0.$$

Write $\boldsymbol{w} = (\mu_* \mathscr{W}\!\mathit{hit}, \bar{\omega})$ for this object. Pushing out via ε, we view $\boldsymbol{w}$ as an S-object of $\mathsf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}})$. This completes the construction of the gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}) = \mathsf{p}^{-1}(\mathsf{w})$ associated to the cover $\tilde{\mathbf{G}}$ and character ε.

Example 3.11. — *Suppose that $\mathscr{Y}_{Q,n}/\mathscr{Y}_{Q,n}^{\mathrm{sc}}$ is torsion-free and a constant sheaf (equivalently, the center of $\tilde{\mathscr{G}}^\vee$ is connected and constant as a sheaf on $S_{\text{ét}}$). Then the* *following short exact sequence splits:*

$$\mathscr{Y}_{Q,n}^{\mathrm{sc}} \hookrightarrow \mathscr{Y}_{Q,n} \twoheadrightarrow \mathscr{Y}_{Q,n}/\mathscr{Y}_{Q,n}^{\mathrm{sc}}.$$

Given such a splitting, write $\mathscr{Y}_{Q,n}^{\mathrm{cent}} \subset \mathscr{Y}_{Q,n}$ *for the image of* $\mathscr{Y}_{Q,n}/\mathscr{Y}_{Q,n}^{\mathrm{sc}}$ *via the splitting. The identification* $\mathscr{Y}_{Q,n} = \mathscr{Y}_{Q,n}^{\mathrm{sc}} \oplus \mathscr{Y}_{Q,n}^{\mathrm{cent}}$ *corresponds to an isomorphism* $\hat{\mathscr{T}} \xrightarrow{\sim} \hat{\mathscr{T}}_{\mathrm{sc}} \times \hat{\mathscr{Z}}$. *Let* $\mathscr{D}_{Q,n}^{\mathrm{cent}}$ *be the pullback of* $\mathscr{D}_{Q,n}$ *to* $\mathscr{Y}_{Q,n}^{\mathrm{cent}}$. *From Example 3.4, the short exact sequence* $\mathscr{G}_m \hookrightarrow \mathscr{D}_{Q,n}^{\mathrm{cent}} \twoheadrightarrow \mathscr{Y}_{Q,n}^{\mathrm{cent}}$ *splits, providing an object of* $\sqrt[n]{\mathcal{S}p\ell(\mathscr{D}_{Q,n}^{\mathrm{cent}})}$.

Chasing diagrams gives a map of object sets,

$$\sqrt[n]{\mathcal{S}p\ell(\mathscr{D}_{Q,n}^{\mathrm{sc}})}[S] \times \sqrt[n]{\mathcal{S}p\ell(\mathscr{D}_{Q,n}^{\mathrm{cent}})}[S] \to \sqrt[n]{\mathcal{S}p\ell(\mathscr{D}_{Q,n})}[S].$$

A splitting of $\mathscr{D}_{Q,n}^{\mathrm{cent}}$ *gives an object of* $\sqrt[n]{\mathcal{S}p\ell(\mathscr{D}_{Q,n}^{\mathrm{cent}})}[S]$ *and* $\boldsymbol{w}$ *provides an object of* $\sqrt[n]{\mathcal{S}p\ell(\mathscr{D}_{Q,n}^{\mathrm{sc}})}[S]$. *Hence the gerbe* $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ *is neutral when* $\mathscr{Y}_{Q,n}/\mathscr{Y}_{Q,n}^{\mathrm{sc}}$ *is torsion-free and a constant sheaf.*

3.4. Well-aligned functoriality. — Consider a well-aligned homomorphism $\tilde{\iota}\colon \tilde{\mathbf{G}}_1 \to \tilde{\mathbf{G}}_2$ of covers, each endowed with Borel subgroup and maximally split maximal torus, i.e., a morphism in the category WAC_S. Fix ε as before. We have constructed gerbes $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1)$ and $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_2)$ associated to $\tilde{\mathbf{G}}_1$ and $\tilde{\mathbf{G}}_2$, banded by $\tilde{\mathscr{Z}}_1^\vee$ and $\tilde{\mathscr{Z}}_2^\vee$, respectively. We have constructed a homomorphism of dual groups $\iota^\vee\colon \tilde{\mathscr{G}}_2^\vee \to \tilde{\mathscr{G}}_1^\vee$ in Section 2.3, which (after taking $\mathbb{C}$-points) restricts to $\iota^\vee\colon \tilde{\mathscr{Z}}_2^\vee \to \tilde{\mathscr{Z}}_1^\vee$. Here we construct a functor of gerbes $\mathsf{i}\colon \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_2) \to \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1)$, lying over $\iota^\vee\colon \tilde{\mathscr{Z}}_2^\vee \to \tilde{\mathscr{Z}}_1^\vee$.

Well-alignedness give a commutative diagram in which the first row is the pullback of the second.

$$\begin{tikzcd} \mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathbf{T}_1' \arrow[r, two heads] \arrow[d, "\iota'"] & \mathbf{T}_1 \arrow[d, "\iota"] \\ \mathbf{K}_2 \arrow[r, hook] & \mathbf{T}_2' \arrow[r, two heads] & \mathbf{T}_2. \end{tikzcd}$$

This gives a commutative diagram for the second Brylinski-Deligne invariants. After pulling back to $\mathscr{Y}_{1,Q_1,n}$ and $\mathscr{Y}_{2,Q_2,n}$, we get a commutative diagram of sheaves of abelian groups on $S_{\text{ét}}$, in which the top row is the pullback of the bottom:

$$\begin{tikzcd} \mathscr{G}_m \arrow[r, hook] \arrow[d, "="] & \mathscr{D}_{1,Q_1,n} \arrow[r, two heads] \arrow[d, "\iota'"] & \mathscr{Y}_{1,Q_1,n} \arrow[d, "\iota"] \\ \mathscr{G}_m \arrow[r, hook] & \mathscr{D}_{2,Q_2,n} \arrow[r, two heads] & \mathscr{Y}_{2,Q_2,n}. \end{tikzcd}$$

(Assumption 3.1 is in effect, so Q_1 and Q_2 are even-valued if n is odd.)

We have homomorphisms of sheaves of abelian groups,

$$\iota\colon \mathscr{Y}_{1,Q_1,n} \to \mathscr{Y}_{2,Q_2,n}, \quad \mathscr{Y}_{1,Q_1,n}^{\mathrm{sc}} \to \mathscr{Y}_{2,Q_2,n}^{\mathrm{sc}}, \quad \frac{\mathscr{Y}_{1,Q_1,n}}{\mathscr{Y}_{1,Q_1,n}^{\mathrm{sc}}} \to \frac{\mathscr{Y}_{2,Q_2,n}}{\mathscr{Y}_{2,Q_2,n}^{\mathrm{sc}}}.$$

Applying $\mathscr{H}om(\bullet, \mathscr{G}_m)$ yields homomorphisms of sheaves of abelian groups,

$$\hat{\iota}\colon \hat{\mathscr{T}}_2 \to \hat{\mathscr{T}}_1, \quad \hat{\mathscr{T}}_{\mathrm{sc},2} \to \hat{\mathscr{T}}_{\mathrm{sc},1}, \quad \hat{\mathscr{Z}}_2 \to \hat{\mathscr{Z}}_1.$$

A splitting of $\mathscr{D}_{2,Q_2,n}$ pulls back to a splitting of $\mathscr{D}_{1,Q_1,n}$ (call this pullback map ι^*), giving a commutative diagram of sheaves on $S_{\text{ét}}$.

$$\begin{array}{ccc} \hat{\mathscr{T}}_2 \times \mathscr{S}\!p\ell(\mathscr{D}_{2,Q_2,n}) & \xrightarrow{\ *\ } & \mathscr{S}\!p\ell(\mathscr{D}_{2,Q_2,n}) \\ \downarrow{\scriptstyle \hat{\iota}\times\iota^*} & & \downarrow{\scriptstyle \iota^*} \\ \hat{\mathscr{T}}_1 \times \mathscr{S}\!p\ell(\mathscr{D}_{1,Q_1,n}) & \xrightarrow{\ *\ } & \mathscr{S}\!p\ell(\mathscr{D}_{1,Q_1,n}). \end{array}$$

This defines a functor of gerbes

$$\sqrt[n]{\iota^*}\colon \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D}_{2,Q_2,n})} \to \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D}_{1,Q_1,n})},$$

lying over $\hat{\iota}\colon \hat{\mathscr{T}}_{2,[n]} \to \hat{\mathscr{T}}_{1,[n]}$. Pushing out via ε yields a functor of gerbes,

$$\mathbf{i}\colon \mathbf{E}_\varepsilon(\tilde{\mathbf{T}}_2) \to \mathbf{E}_\varepsilon(\tilde{\mathbf{T}}_1).$$

The same process applies to $\iota\colon \mathscr{Y}^{\mathrm{sc}}_{1,Q_1,n} \to \mathscr{Y}^{\mathrm{sc}}_{2,Q_2,n}$, giving a functor of gerbes, $\mathbf{i}^{\mathrm{sc}}\colon \mathbf{E}^{\mathrm{sc}}_\varepsilon(\tilde{\mathbf{T}}_2) \to \mathbf{E}^{\mathrm{sc}}_\varepsilon(\tilde{\mathbf{T}}_1)$. By pulling back in stages, we find a square of gerbes and functors, and a natural isomorphism $\mathsf{S}\colon \mathbf{p}_1 \circ \mathbf{i} \overset{\sim}{\Rightarrow} \mathbf{i}^{\mathrm{sc}} \circ \mathbf{p}_2$ making the diagram 2-commute.

$$\begin{array}{ccc} \mathbf{E}_\varepsilon(\tilde{\mathbf{T}}_2) & \xrightarrow{\ \mathbf{i}\ } & \mathbf{E}_\varepsilon(\tilde{\mathbf{T}}_1) \\ \downarrow{\scriptstyle \mathbf{p}_2} & & \downarrow{\scriptstyle \mathbf{p}_1} \\ \mathbf{E}^{\mathrm{sc}}_\varepsilon(\tilde{\mathbf{T}}_2) & \xrightarrow{\ \mathbf{i}^{\mathrm{sc}}\ } & \mathbf{E}^{\mathrm{sc}}_\varepsilon(\tilde{\mathbf{T}}_1). \end{array}$$

If $\mathbf{U}_1$ and $\mathbf{U}_2$ are the unipotent radicals of $\mathbf{B}_1$ and $\mathbf{B}_2$, respectively, then pulling back gives a map $\iota^*\colon \mathscr{W}\!hit_2 \to \mathscr{W}\!hit_1$. Condition (1) of well-alignedness states that $\mathrm{Ker}(\iota)$ is contained in the center of $\mathbf{G}_1$, and so simple root subgroups in $\mathbf{U}_1$ map isomorphically to simple root subgroups in $\mathbf{U}_2$. Compatibility of quadratic forms Q_1 and Q_2 implies compatibility in the constants n_α, m_α, and r_α for simple roots, and so we find a commutative diagram

$$\begin{array}{ccc} (\mu_2)_* \mathscr{W}\!hit_2 & \xrightarrow{\ \bar{\omega}_2\ } & \mathscr{S}\!p\ell(\mathscr{D}^{\mathrm{sc}}_{2,Q_2,n}) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ (\mu_1)_* \mathscr{W}\!hit_1 & \xrightarrow{\ \bar{\omega}_1\ } & \mathscr{S}\!p\ell(\mathscr{D}^{\mathrm{sc}}_{1,Q_1,n}). \end{array}$$

Write $\boldsymbol{w}_1$ for the object $((\mu_1)_* \mathscr{W}\!hit_1, \bar{\omega}_1)$ of $\mathbf{E}^{\mathrm{sc}}_\varepsilon(\tilde{\mathbf{T}}_1)$, and similarly $\boldsymbol{w}_2$ for the object $((\mu_2)_* \mathscr{W}\!hit_2, \bar{\omega}_2)$ of $\mathbf{E}^{\mathrm{sc}}_\varepsilon(\tilde{\mathbf{T}}_2)$. The commutative diagram above gives an isomorphism $f\colon \mathbf{i}^{\mathrm{sc}}(\boldsymbol{w}_2) \xrightarrow{\sim} \boldsymbol{w}_1$ in $\mathbf{E}^{\mathrm{sc}}_\varepsilon(\tilde{\mathbf{T}}_1)$.

If $(\boldsymbol{e}, j)$ is an object of $\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}_2) = \mathbf{p}_2^{-1}(\boldsymbol{w}_2)$, i.e., $\boldsymbol{e}$ is an object of $\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}_2)$ and $j\colon \mathbf{p}_2(\boldsymbol{e}) \to \boldsymbol{w}_2$ is an isomorphism, then we find a sequence of isomorphisms,

$$\mathbf{i}(j)\colon \mathbf{p}_1(\mathbf{i}(\boldsymbol{e})) \xrightarrow{\mathsf{S}} \mathbf{i}^{\mathrm{sc}}(\mathbf{p}_2(\boldsymbol{e})) \xrightarrow{j} \mathbf{i}^{\mathrm{sc}}(\boldsymbol{w}_2) \xrightarrow{f} \boldsymbol{w}_1.$$

In this way $(\mathbf{i}(e), \mathbf{i}(j))$ becomes an object of $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1) = \mathbf{p}_1^{-1}(w_1)$. This extends to a functor of gerbes, $\mathbf{i}\colon \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_2) \to \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1)$, lying over $\iota^\vee\colon \tilde{\mathscr{Z}}_2^\vee \to \tilde{\mathscr{Z}}_1^\vee$.

Given a pair of well-aligned homomorphisms,

$$\tilde{\mathbf{G}}_1 \xrightarrow{\tilde{\iota}_1} \tilde{\mathbf{G}}_2 \xrightarrow{\tilde{\iota}_2} \tilde{\mathbf{G}}_2,$$

with $\tilde{\iota}_3 = \tilde{\iota}_2 \circ \tilde{\iota}_1$, we find three functors of gerbes

$$\begin{tikzcd} \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_3) \arrow[r, "\mathbf{i}_2"'] \arrow[rr, bend left, "\mathbf{i}_3"] & \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_2) \arrow[r, "\mathbf{i}_1"'] & \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1), \end{tikzcd}$$

lying over three homomorphisms of sheaves,

$$\begin{tikzcd} \tilde{\mathscr{Z}}_3^\vee \arrow[r, "\iota_2^\vee"'] \arrow[rr, bend left, "\iota_3^\vee"] & \tilde{\mathscr{Z}}_2^\vee \arrow[r, "\iota_1^\vee"'] & \tilde{\mathscr{Z}}_1^\vee. \end{tikzcd}$$

The functoriality of pullbacks, pushouts, taking nth roots of torsors, etc., yields a *natural isomorphism* of functors:

$$N(\iota_1, \iota_2)\colon \mathbf{i}_3 \overset{\sim}{\Rightarrow} \mathbf{i}_1 \circ \mathbf{i}_2. \tag{3.5}$$

We can summarize these results using the language of "weak functors" [**68**, Tag 003G]. We have given all the data for such a weak functor, and a reader who wishes to check commutativity of a few diagrams may verify the following.

Proposition 3.12. — *The construction of the associated gerbe defines a weak functor from* WAC_S *(the category of triples* $(\tilde{\mathbf{G}}, \mathbf{B}, \mathbf{T})$ *and well-aligned homomorphisms), to the 2-category of gerbes on* $S_{\text{ét}}$*, functors of gerbes, and natural isomorphisms thereof.*

3.5. Well-definedness. — The construction of the associated gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ depended on the choice of Borel subgroup $\mathbf{B}$ and maximally split maximal torus $\mathbf{T}$. Now we demonstrate that $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ is well-defined independently of these choices, in a suitable 2-categorical sense.

Consider another choice $\mathbf{B}_0 \supset \mathbf{T}_0$. Our constructions, with these two choices of tori and Borel subgroups, yield two dual groups $\tilde{\mathscr{G}}_0^\vee$ and $\tilde{\mathscr{G}}^\vee$ and two gerbes $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ and $\mathsf{E}_{0,\varepsilon}(\tilde{\mathbf{G}})$.

Let $\mathscr{Y}_0$ and $\mathscr{Y}$ be the cocharacter lattices of $\mathbf{T}_0$ and $\mathbf{T}$, respectively, and $\Phi_0^\vee, \Phi^\vee$ the coroots therein. The Borel subgroups provide systems of simple coroots $\Delta_0^\vee$ and $\Delta^\vee$, respectively (sheaves of sets on $S_{\text{ét}}$). Similarly we have character lattices $\mathscr{X}_0, \mathscr{X}$, roots Φ_0, Φ, and simple roots Δ_0, Δ. The cover $\tilde{\mathbf{G}}$ yields quadratic forms $Q\colon \mathscr{Y} \to \mathbb{Z}$ and $Q_0\colon \mathscr{Y}_0 \to \mathbb{Z}$. The second Brylinski-Deligne invariant gives extensions $\mathscr{G}_m \hookrightarrow \mathscr{D} \twoheadrightarrow \mathscr{Y}$ and $\mathscr{G}_m \hookrightarrow \mathscr{D}_0 \twoheadrightarrow \mathscr{Y}_0$.

By Proposition 1.1, there exists $g \in \mathbf{G}(S)$ such that $\mathrm{Int}(g)\mathbf{T}_0 = \mathbf{T}$ and $\mathrm{Int}(g)\mathbf{B}_0 = \mathbf{B}$. This automorphism $\mathrm{Int}(g)$ lifts to an automorphism of $\mathbf{G}'$. This defines a well-aligned isomorphism of covers.

$$\begin{array}{ccccccc}
\mathbf{K}_2 & \hookrightarrow & \mathbf{G}' & \twoheadrightarrow & \mathbf{G} & \qquad \mathbf{B}_0 & \qquad \mathbf{T}_0 \\
\downarrow{=} & & \downarrow{\mathrm{Int}(g)'} & & \downarrow{\mathrm{Int}(g)} & \qquad \downarrow & \qquad \downarrow \\
\mathbf{K}_2 & \hookrightarrow & \mathbf{G}' & \twoheadrightarrow & \mathbf{G} & \qquad \mathbf{B} & \qquad \mathbf{T}.
\end{array}$$

This well-aligned isomorphism of covers yields an equivalence of gerbes,

$$\mathsf{Int}(g) \colon \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}) \to \mathsf{E}_{0,\varepsilon}(\tilde{\mathbf{G}}),$$

lying over $\mathrm{Int}(g)^\vee \colon \tilde{\mathscr{Z}}^\vee \to \tilde{\mathscr{Z}}_0^\vee$.

Suppose that $g' \in \mathbf{G}(S)$ also satisfies $\mathrm{Int}(g')\mathbf{T}_0 = \mathbf{T}$ and $\mathrm{Int}(g')\mathbf{B}_0 = \mathbf{B}$. As in the proof of Theorem 2.3, $g' = tg$ for a unique $t \in \mathbf{T}(S)$ and $\mathrm{Int}(g') = \mathrm{Int}(t)\,\mathrm{Int}(g)$.

This gives a natural isomorphism of functors,

$$N(g', g) \colon \mathsf{Int}(g') \overset{\sim}{\Rightarrow} \mathsf{Int}(g) \circ \mathsf{Int}(t).$$

Our upcoming Proposition 3.15 will provide a natural isomorphism $\mathsf{Int}(t) \overset{\sim}{\Rightarrow} \mathsf{Id}$. Assuming this for the moment, we find a natural isomorphism $\mathsf{Int}(g') \overset{\sim}{\Rightarrow} \mathsf{Int}(g)$. This demonstrates that the gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ is well-defined in the following 2-categorical sense:

1. for each pair $i = (\mathbf{B}, \mathbf{T})$ consisting of a Borel subgroup of $\mathbf{G}$ and a maximally split maximal torus therein, we have constructed a gerbe $\mathsf{E}_\varepsilon^i(\tilde{\mathbf{G}})$;
2. for each pair $i = (\mathbf{B}, \mathbf{T})$, $j = (\mathbf{B}_0, \mathbf{T}_0)$, we have constructed a family $P(i,j)$ of gerbe equivalences $\mathsf{Int}(g) \colon \mathsf{E}_\varepsilon^i(\tilde{\mathbf{G}}) \to \mathsf{E}_\varepsilon^j(\tilde{\mathbf{G}})$, indexed by those g which conjugate i to j;
3. for any two elements g, g' which conjugate i to j, there is a distinguished natural isomorphism of gerbe equivalences from $\mathsf{Int}(g')$ to $\mathsf{Int}(g)$.

Once we define the natural isomorphism $\mathsf{Int}(t) \overset{\sim}{\Rightarrow} \mathsf{Id}$, we will have defined $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ uniquely up to equivalence, the equivalences being defined uniquely up to unique natural isomorphism (we learned this notion of well-definedness from reading various works of James Milne). Defining the natural isomorphism requires some computation and is the subject of the section below.

3.5.1. *The isomorphism* $\mathsf{Int}(t) \overset{\sim}{\Rightarrow} \mathsf{Id}$. — For what follows, define $\delta_Q \colon \mathscr{Y} \to n^{-1}\mathscr{X}$ to be the unique homomorphism satisfying

$$\langle \delta_Q(y_1), y_2 \rangle = \beta_Q(y_1, y_2) \text{ for all } y_1, y_2 \in \mathscr{Y}. \tag{3.6}$$

The constants m_ϕ and n_ϕ arise in the following useful result.

Lemma 3.13. — *For all $\phi \in \Phi$, we have $\delta_Q(n_\phi \phi^\vee) = \delta_Q(\tilde{\phi}^\vee) = m_\phi \phi$.*

Proof. — For all $\phi \in \Phi$, and all $y \in \mathscr{Y}$, we have

$$\langle \delta_Q(\tilde{\phi}^\vee), y \rangle = \beta_Q(n_\phi \phi^\vee, y). \tag{3.7}$$

Weyl-invariance of the quadratic form (applying the root reflection for ϕ) implies

$$\beta_Q(\phi^\vee, y) = \beta_Q(-\phi^\vee, y - \langle \phi, y\rangle \phi^\vee) = -\beta_Q(\phi^\vee, y) + \beta_Q(\phi^\vee, \phi^\vee)\langle \phi, y\rangle.$$

Adding $\beta_Q(\phi^\vee, y)$ to both sides and dividing by two, we find that

$$\beta_Q(\phi^\vee, y) = n^{-1} Q(\phi^\vee)\langle \phi, y\rangle.$$

Substituting into (3.7) yields

$$\langle \delta_Q(\tilde{\phi}^\vee), y\rangle = n_\phi \beta_Q(\phi^\vee, y) = \langle n_\phi n^{-1} Q(\phi^\vee)\cdot \phi, y\rangle.$$

Since this holds for all $y \in \mathscr{Y}$, we have

$$\delta_Q(\tilde{\phi}^\vee) = n_\phi n^{-1} Q(\phi^\vee)\cdot \phi = m_\phi \cdot \phi. \qquad \square$$

Consider the homomorphisms of sheaves of abelian groups on $S_{\text{ét}}$,

$$\mathscr{T} \xrightarrow{\delta_Q} \hat{\mathscr{T}} \overset{p}{\twoheadrightarrow} \hat{\mathscr{T}}_{\text{sc}},$$

obtained by applying $\mathscr{H}om(\bullet, \mathscr{G}_m)$ to the homomorphisms

$$\mathscr{Y}_{Q,n}^{\text{sc}} \hookrightarrow \mathscr{Y}_{Q,n} \xrightarrow{\delta_Q} \mathscr{X}.$$

An object of $\mathbf{E}_\varepsilon(\tilde{\mathbf{G}})$ is a triple $(\mathscr{H}, h, j)$ where

- $\mathscr{H} \xrightarrow{h} \mathscr{S}pl(\mathscr{D}_{Q,n})$ is an nth root of the $\hat{\mathscr{T}}$-torsor $\mathscr{S}pl(\mathscr{D}_{Q,n})$.
- $j\colon p_*\mathscr{H} \to \mu_*\mathscr{W}hit$ is an isomorphism in the gerbe $\mathbf{E}^{\text{sc}}_\varepsilon(\tilde{\mathbf{T}}) = \varepsilon_* \sqrt[n]{\mathscr{S}pl(\mathscr{D}^{\text{sc}}_{Q,n})}$. Thus $j = \tau^\vee \wedge j_0$ where $j_0\colon p_*\mathscr{H} \to \mu_*\mathscr{W}hit$ is an isomorphism in the gerbe $\sqrt[n]{\mathscr{S}pl(\mathscr{D}^{\text{sc}}_{Q,n})}$ and $\tau^\vee \in \tilde{\mathscr{T}}^\vee_{\text{sc}}$. See A.3.2 for a general description of morphisms in gerbes obtained by pushing out.

Given $\mathscr{H} \xrightarrow{h} \mathscr{S}pl(\mathscr{D}_{Q,n})$ and $\hat{t} \in \hat{\mathscr{T}}$, write $\hat{t}\circ h\colon \mathscr{H} \to \mathscr{S}pl(\mathscr{D}_{Q,n})$ for the map obtained by composing h with the automorphism $\hat{t}$ of the $\hat{\mathscr{T}}$-torsor $\mathscr{S}pl(\mathscr{D}_{Q,n})$.

Similarly, given $j = \tau^\vee \wedge j_0$, with $p_*\mathscr{H} \xrightarrow{j_0} \mu_*\mathscr{W}hit$, $\tau^\vee \in \tilde{\mathscr{T}}^\vee_{\text{sc}}$, and given $\hat{t}_{\text{sc}} \in \hat{\mathscr{T}}_{\text{sc}}$, write $\hat{t}_{\text{sc}} \circ j$ for $\tau^\vee \wedge (\hat{t}_{\text{sc}} \circ j_0)$. Here $\hat{t}_{\text{sc}} \circ j_0\colon p_*\mathscr{H} \to \mu_*\mathscr{W}hit$ is the map obtained by composing j with the automorphism $\hat{t}_{\text{sc}}$ of the $\hat{\mathscr{T}}_{\text{sc}}$-torsor $\mu_*\mathscr{W}hit$. Similarly, write $j\circ\hat{t}_{\text{sc}}$ for $\tau^\vee \wedge (j_0 \circ \hat{t}_{\text{sc}})$. Since j_0 is an isomorphism of $\hat{\mathscr{T}}_{\text{sc}}$-torsors, we have $j\circ\hat{t}_{\text{sc}} = \hat{t}_{\text{sc}}\circ j$.

The following result describes the functor $\mathbf{Int}(t)$ explicitly.

Lemma 3.14. — *For all $t \in \mathbf{T}(S)$, the equivalence of gerbes $\mathbf{Int}(t)\colon \mathbf{E}_\varepsilon(\tilde{\mathbf{G}}) \to \mathbf{E}_\varepsilon(\tilde{\mathbf{G}})$ sends an object $(\mathscr{H}, h, j)$ to the object $(\mathscr{H}, \delta_Q(t)^{-n}\circ h, p(\delta_Q(t))^{-1}\circ j)$.*

Proof. — We work locally on $S_{\text{ét}}$ throughout the proof. The action $\mathrm{Int}(t)$ of t on the extension $\mathscr{G}_m \hookrightarrow \mathscr{D}_{Q,n} \twoheadrightarrow \mathscr{Y}_{Q,n}$ is given by [**18**, Equation 11.11.1], which states (in different notation) that

$$\mathrm{Int}(t)d = d\cdot \delta_Q(y)(t)^{-n} = d\cdot y\left(\delta_Q(t)^{-n}\right),$$

for all $d \in \mathscr{D}_{Q,n}$ lying over $y \in \mathscr{Y}_{Q,n}$. We find that $\mathrm{Int}(t)$ is the automorphism of the extension $\mathscr{D}_{Q,n}$ determined by the element $\delta_Q(t)^{-n} \in \hat{\mathscr{T}} = \mathscr{H}om(\mathscr{Y}_{Q,n}, \mathscr{G}_m)$. Hence the functor $\mathsf{Int}(t)\colon \mathsf{E}_\varepsilon(\tilde{\mathbf{T}}) \to \mathsf{E}_\varepsilon(\tilde{\mathbf{T}})$ sends $(\mathscr{H}, h)$ to $(\mathscr{H}, \delta_Q(t)^{-n} \circ h)$. It remains to see how $\mathsf{Int}(t)$ affects the third term in a triple $(\mathscr{H}, h, j)$.

Conjugation by t gives a map of torsors $\mathscr{W}\!hit \to \mathscr{W}\!hit$. Let j_0' be the map which makes the following triangle commute:

$$\begin{tikzcd} p_*\mathscr{H} \arrow[r, "j_0"] \arrow[dr, "j_0'"'] & \mu_*\mathscr{W}\!hit \arrow[d, "\mathrm{Int}(t)"] \\ & \mu_*\mathscr{W}\!hit. \end{tikzcd}$$

We have $\mathsf{Int}(t)(\mathscr{H}, h, \tau^\vee \wedge j_0) = (\mathscr{H}, \delta_Q(t)^{-n} \circ h, \tau^\vee \wedge j_0')$ and must describe j_0'.

Take an element $\eta = (\eta_\alpha : \alpha \in \Delta) \in \mathscr{W}\!hit$ (decomposed with respect to the basis of simple roots Δ). Conjugation by t yields a new element $\mathrm{Int}(t)\eta$ satisfying

$$\mathrm{Int}(t)\eta(u) = \eta(t^{-1}ut), \quad \text{for all } u \in \mathscr{U}.$$

Decomposing along the simple root subgroups,

$$\mathrm{Int}(t)\eta = (\alpha(t) * \eta_\alpha : \alpha \in \Delta) \in \mathscr{W}\!hit,$$

where $[a * \eta_\alpha](e) = \eta_\alpha(a^{-1}e)$ for any $a \in \mathscr{G}_m$.

Comparing to the action of $\hat{\mathscr{T}}_{\mathrm{sc}}$ on $\mathscr{W}\!hit$, we find that

$$\mathrm{Int}(t)\eta = \theta(t) * \eta,$$

where $\theta\colon \mathscr{T} \to \hat{\mathscr{T}}_{\mathrm{sc}}$ denotes the homomorphism dual to the homomorphism of character lattices sending $\tilde{\alpha}^\vee \in \mathscr{Y}_{Q,n}^{\mathrm{sc}}$ to $\alpha \in \mathscr{X}$.

In the quotient $\hat{\mathscr{T}}_{\mathrm{sc}}$-torsor $\mu_*\mathscr{W}\!hit$, the action is given by $\mathrm{Int}(t)\bar{\eta} = \mu(\theta(t)) \cdot \bar{\eta}$, for all $\bar{\eta} \in \mu_*\mathscr{W}\!hit$. More explicitly, $\mu \circ \theta\colon \hat{\mathscr{T}}_{\mathrm{sc}} \to \mathscr{T}$ is the homomorphism dual to the map of character lattices sending $\tilde{\alpha}^\vee$ to $-m_\alpha \alpha$. By Lemma 3.13, we have $\mu(\theta(t)) = p(\delta_Q(t))^{-1}$ (recall that p corresponds to the inclusion $\mathscr{Y}_{Q,n}^{\mathrm{sc}} \hookrightarrow \mathscr{Y}_{Q,n}$). It follows that $j_0' = \mathrm{Int}(t) \circ j_0 = p(\delta_Q(t))^{-1} \circ j_0$. This yields the result:

$$\mathsf{Int}(t)(\mathscr{H}, h, j) = (\mathscr{H}, \delta_Q(t)^{-n} \circ h, p(\delta_Q(t))^{-1} \circ j). \qquad \square$$

Proposition 3.15. — *Let $(\mathscr{H}, h, j)$ be an object of the category $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$. Then, for all $t \in \mathbf{T}(S)$, the morphism $\delta_Q(t)^{-1}\colon \mathscr{H} \to \mathscr{H}$ defines an isomorphism from $\mathsf{Int}(t)(\mathscr{H}, h, j)$ to $(\mathscr{H}, h, j)$. As objects vary, this defines a natural isomorphism $\mathsf{A}(t)\colon \mathsf{Int}(t) \Rightarrow \mathsf{Id}$ of functors from $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ to itself. For a pair of elements $t_1, t_2 \in \mathbf{T}(S)$, we have a commutative diagram of functors and natural isomorphisms.*

$$\begin{tikzcd} \mathsf{Int}(t_1t_2) \arrow[r, Rightarrow, "="] \arrow[d, Rightarrow, "\mathsf{A}(t_1t_2)"] & \mathsf{Int}(t_1) \circ \mathsf{Int}(t_2) \arrow[d, Rightarrow, "\mathsf{A}(t_1)\circ\mathsf{A}(t_2)"] \\ \mathsf{Id} \arrow[r, Rightarrow, "="] & \mathsf{Id}. \end{tikzcd}$$

Proof. — As $\mathscr{H} \xrightarrow{h} \mathscr{S}\!p\ell(\mathscr{D}_{Q,n})$ is an nth root of the torsor $\mathscr{S}\!p\ell(\mathscr{D}_{Q,n})$, we have $\delta_Q(t)^{-n} \circ h = h \circ \delta_Q(t)^{-1}$. As $j = \tau^\vee \wedge j_0$, with j_0 an isomorphism of $\hat{\mathscr{T}}_{\mathrm{sc}}$-torsors, we have $p(\delta_Q(t)^{-1}) \circ j = j \circ p(\delta_Q(t)^{-1})$.

Hence the isomorphism $\delta_Q(t)^{-1} \colon \mathscr{H} \to \mathscr{H}$ defines an isomorphism

$$\begin{aligned}\mathsf{Int}(t)(\mathscr{H}, h, j) &= (\mathscr{H}, \delta_Q(t)^{-n} \circ h, p(\delta_Q(t)^{-1}) \circ j) \\ &= (\mathscr{H}, h \circ \delta_Q(t)^{-1}, j \circ p(\delta_Q(t)^{-1})) \\ &\xrightarrow{\delta_Q(t)^{-1}} (\mathscr{H}, h, j).\end{aligned}$$

As their definition depends only on t, these isomorphisms $\delta_Q(t)^{-1}$ form a natural isomorphism $\mathsf{A}(t) \colon \mathsf{Int}(t) \Rightarrow \mathsf{Id}$. As $\delta_Q(t_1 t_2)^{-1} = \delta_Q(t_1)^{-1}\delta_Q(t_2)^{-1}$, we find an *equality* of functors $\mathsf{Int}(t_1 t_2) = \mathsf{Int}(t_1) \circ \mathsf{Int}(t_2)$, and the commutative diagram of the proposition. □

3.6. Change of base scheme. — Let $\tilde{\mathbf{G}}$ be a degree n cover of a quasisplit group $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ over S, as before. Let $\gamma \colon S_0 \to S$ be a morphism of schemes as in Section 2.4. Then γ gives rise to a pullback functor γ^* from sheaves on $S_{\text{ét}}$ to sheaves on $S_{0,\text{ét}}$, and from gerbes on $S_{\text{ét}}$ to gerbes on $S_{0,\text{ét}}$.

Assuming that $\boldsymbol{\mu}_n(S)$ and $\boldsymbol{\mu}_n(S_0)$ are cyclic groups of order n, we may identify these groups via $\gamma \colon \boldsymbol{\mu}_n(S) \to \boldsymbol{\mu}_n(S_0)$, and a character $\varepsilon \colon \mu_n \hookrightarrow \mathbb{C}^\times$ corresponds to a character $\varepsilon_0 \colon \boldsymbol{\mu}_n(S_0) \hookrightarrow \mathbb{C}^\times$.

Pullback via γ defines a degree n cover $\tilde{\mathbf{G}}_0$ of a quasisplit group $\mathbf{G}_0 \supset \mathbf{B}_0 \supset \mathbf{T}_0$ over S_0. We have constructed gerbes $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ and $\mathsf{E}_{\varepsilon_0}(\tilde{\mathbf{G}}_0)$ associated to this data, banded by $\tilde{\mathscr{Z}}^\vee$ and $\tilde{\mathscr{Z}}_0^\vee$, respectively. We also consider the pullback gerbe $\gamma^*\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$, banded by $\gamma^*\tilde{\mathscr{Z}}^\vee$, and recall from Section 2.4 that there is a natural isomorphism $N^\vee \colon \gamma^*\tilde{\mathscr{Z}}^\vee \to \tilde{\mathscr{Z}}_0^\vee$.

Similarly, we find isomorphisms of sheaves on $S_{0,\text{ét}}$,

1. $\hat{N} \colon \gamma^*\hat{\mathscr{T}} \xrightarrow{\sim} \hat{\mathscr{T}}_0$ (from $N \colon \gamma^*\mathscr{Y}_{Q,n} \xrightarrow{\sim} \mathscr{Y}_{0,Q_0,n}$);
2. $N^\vee \colon \gamma^*\tilde{\mathscr{T}}_{\text{sc}}^\vee \xrightarrow{\sim} \tilde{\mathscr{T}}_{0,\text{sc}}^\vee$;
3. $N \colon \gamma^*\mathscr{D}_{Q,n} \xrightarrow{\sim} \mathscr{D}_{0,Q_0,n}$ (the construction of the second Brylinski-Deligne invariant is compatible with pullbacks);
4. $N \colon \gamma^*\mathscr{W}\!\mathit{hit} \xrightarrow{\sim} \mathscr{W}\!\mathit{hit}_0$ (since $\mathbf{B}_0 = \gamma^*\mathbf{B}$).

From these observed isomorphisms, we find that the pullback of étale sheaves from $S_{\text{ét}}$ to $S_{0,\text{ét}}$ defines a functor

$$\mathsf{N}' \colon \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}) \to \gamma_*\mathsf{E}_{\varepsilon_0}(\tilde{\mathbf{G}}_0)),$$

given on objects by $\mathsf{N}'(\mathscr{H}, h, j) = (\gamma^*\mathscr{H}, \gamma^* h, \gamma^* j)$. For example, if $\mathscr{H}$ is a $\hat{\mathscr{T}}$-torsor, then $\gamma^*\mathscr{H}$, a priori a $\gamma^*\hat{\mathscr{T}}$-torsor, becomes a $\hat{\mathscr{T}}_0$-torsor via $\hat{N}$.

From [**32**, Chapitre V, Proposition 3.1.8], such an equivalence of gerbes N' determines a unique, up to unique natural isomorphism, equivalence of gerbes,

$$\mathsf{N} \colon \gamma^*\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}) \to \mathsf{E}_{\varepsilon_0}(\tilde{\mathbf{G}}_0),$$

lying over the natural isomorphism of bands $N^\vee \colon \gamma^*\tilde{\mathscr{Z}}^\vee \to \tilde{\mathscr{Z}}_0^\vee$. In this way, the construction of the gerbe associated to a cover is compatible with change of base scheme.

3.7. Parabolic subgroups. — Return to the degree n cover $\tilde{\mathbf{G}}$ of a quasisplit group $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ over S. Let $\mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup defined over S, containing $\mathbf{B}$. As in Section 2.5, we consider a Levi decomposition $\mathbf{P} = \mathbf{MN}$ and resulting cover $\tilde{\mathbf{M}}$. Fix $\varepsilon \colon \mu_n \hookrightarrow \mathbb{C}^\times$ as before.

Consider the gerbes $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ and $\mathsf{E}_\varepsilon(\tilde{\mathbf{M}})$. An object of $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ is a triple $(\mathscr{H}, h, j)$, where $(\mathscr{H}, h)$ is an nth root of the torsor $\mathscr{Spl}(\mathscr{D}_{Q,n})$, $j = \tau^\vee \wedge j_0$, $\tau^\vee \in \tilde{\mathscr{T}}^\vee_{\mathrm{sc}}$, and $j_0 \colon p_* \mathscr{H} \to \mu_* \mathscr{Whit}$ is an isomorphism of $\hat{\mathscr{T}}_{\mathrm{sc}}$-torsors.

Restriction of characters via $\mathbf{U} \supset \mathbf{U_M} = \mathbf{U} \cap \mathbf{M}$ provides a homomorphism of sheaves from $\mathscr{Whit}$ to $\mathscr{Whit}_{\mathbf{M}}$ (the Whittaker torsor for $\mathbf{M}$). The inclusion $\mathscr{Y}^{\mathrm{sc}}_{\mathbf{M}} \subset \mathscr{Y}^{\mathrm{sc}}$ of coroot lattices provides a homomorphism of sheaves of abelian groups $\hat{\mathscr{T}}_{\mathrm{sc}} \twoheadrightarrow \hat{\mathscr{T}}_{\mathbf{M},\mathrm{sc}}$, where

$$\hat{\mathscr{T}}_{\mathrm{sc}} = \mathscr{Hom}(\mathscr{Y}^{\mathrm{sc}}, \mathscr{G}_m), \quad \hat{\mathscr{T}}_{\mathbf{M},\mathrm{sc}} = \mathscr{Hom}(\mathscr{Y}^{\mathrm{sc}}_{\mathbf{M}}, \mathscr{G}_m).$$

Similarly, it provides a homomorphism $\tilde{\mathscr{T}}^\vee_{\mathrm{sc}} \twoheadrightarrow \tilde{\mathscr{T}}^\vee_{\mathbf{M},\mathrm{sc}}$, where

$$\tilde{\mathscr{T}}^\vee_{\mathrm{sc}} = \mathscr{Hom}(\mathscr{Y}^{\mathrm{sc}}, \mathbb{C}^\times), \quad \tilde{\mathscr{T}}^\vee_{\mathbf{M},\mathrm{sc}} = \mathscr{Hom}(\mathscr{Y}^{\mathrm{sc}}_{\mathbf{M}}, \mathbb{C}^\times).$$

Define $\tau^\vee_{\mathbf{M}}$ to be the image of $\tau^\vee$ under this homomorphism.

The constants defining $\mu \colon \hat{\mathscr{T}}_{\mathrm{sc}} \to \hat{\mathscr{T}}_{\mathrm{sc}}$ are the same as those defining the corresponding map, $\mu_{\mathbf{M}} \colon \hat{\mathscr{T}}_{\mathbf{M},\mathrm{sc}} \to \hat{\mathscr{T}}_{\mathbf{M},\mathrm{sc}}$. We find a commutative diagram

$$\begin{tikzcd}
\hat{\mathscr{T}}_{\mathrm{sc}} \times \mu_* \mathscr{Whit} \arrow[r, "*"] \arrow[d] & \mu_* \mathscr{Whit} \arrow[d] \\
\hat{\mathscr{T}}_{\mathbf{M},\mathrm{sc}} \times (\mu_{\mathbf{M}})_* \mathscr{Whit}_{\mathbf{M}} \arrow[r, "*"] & (\mu_{\mathbf{M}})_* \mathscr{Whit}_{\mathbf{M}}.
\end{tikzcd}$$

Composing $j_0 \colon p_* \mathscr{H} \to \mu_* \mathscr{Whit}$ with the map to $(\mu_{\mathbf{M}})_* \mathscr{Whit}_{\mathbf{M}}$ defines a map

$$j_{\mathbf{M},0} \colon \mathscr{H} \to (\mu_{\mathbf{M}})_* \mathscr{Whit}_{\mathbf{M}}.$$

This defines a functor of gerbes,

$$\mathsf{i} \colon \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}) \to \mathsf{E}_\varepsilon(\tilde{\mathbf{M}}), \quad (\mathscr{H}, h, \tau^\vee \wedge j_0) \mapsto (\mathscr{H}, h, \tau^\vee_{\mathbf{M}} \wedge j_{\mathbf{M},0}),$$

lying over $\iota^\vee \colon \tilde{\mathscr{Z}}^\vee \hookrightarrow \tilde{\mathscr{Z}}^\vee_{\mathbf{M}}$.

3.8. Weyl action on the gerbe associated to the cover of the torus. — We keep the degree n cover $\tilde{\mathbf{G}}$ of a quasisplit group $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ over S. Let $\dot{w} \in \mathscr{N}[S]$ represent an element of the Weyl group $w \in \mathscr{W}[S]$. As in Section 2.6, $\mathrm{Int}(\dot{w})$ defines a well-aligned homomorphism from $\tilde{\mathbf{T}}$ to itself.

$$\begin{tikzcd}
\mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathbf{T}' \arrow[r, two heads] \arrow[d, "\mathrm{Int}(\dot{w})"] & \mathbf{T} \arrow[d, "\mathrm{Int}(w)"] \\
\mathbf{K}_2 \arrow[r, hook] & \mathbf{T}' \arrow[r, two heads] & \mathbf{T}.
\end{tikzcd}$$

This defines an equivalence of gerbes on $S_{\text{ét}}$,

$$\mathsf{Int}(\dot{w}) \colon \mathsf{E}_\varepsilon(\tilde{\mathbf{T}}) \to \mathsf{E}_\varepsilon(\tilde{\mathbf{T}}),$$

lying over the homomorphism $\mathrm{Int}(w)^\vee \colon \tilde{\mathscr{T}}^\vee \to \tilde{\mathscr{T}}^\vee$.

Given two such representatives $\dot{w}, \ddot{w} \in \mathscr{N}[S]$ for w, there exists a unique $t \in \mathbf{T}(S)$ such that $\ddot{w} = \dot{w} \cdot t$. The natural isomorphism of functors $\mathsf{A}(\mathsf{t}) \colon \mathbf{Int}(t) \stackrel{\sim}{\Rightarrow} \mathbf{Id}$, defined in Proposition 3.15, yields a natural isomorphism

$$\mathbf{Int}(\ddot{w}) \stackrel{\sim}{\Rightarrow} \mathbf{Int}(\dot{w}).$$

In this way, there is an equivalence of gerbes $\mathbf{Int}(w) \colon \mathbf{E}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{E}_\varepsilon(\tilde{\mathbf{T}})$, defined uniquely up to unique natural isomorphism.

For what follows later, it will be useful to describe the functor $\mathbf{Int}(\dot{w})$ explicitly. So consider an object $\mathscr{H} \xrightarrow{h} \mathscr{S}\!pl(\mathscr{D}_{Q,n})$ of $\mathbf{E}_\varepsilon(\tilde{\mathbf{T}})$. Thus $\mathscr{H}$ is a $\hat{\mathscr{T}}$-torsor on $S_{\text{ét}}$, and h is a map of sheaves on $S_{\text{ét}}$ satisfying

$$h(\hat{\tau} * a) = \hat{\tau}^n * h(a), \text{ for all } a \in \mathscr{H}, \hat{\tau} \in \hat{\mathscr{T}}.$$

The well-aligned homomorphism $\mathrm{Int}(\dot{w}) \colon \tilde{\mathbf{T}} \to \tilde{\mathbf{T}}$ yields a commutative diagram when we take the second Brylinski-Deligne invariant.

$$\begin{tikzcd}
\mathscr{G}_m \arrow[r, hook] \arrow[d, "="] & \mathscr{D}_{Q,n} \arrow[r, two heads] \arrow[d, "\mathrm{Int}(\dot{w})"] & \mathscr{Y}_{Q,n} \arrow[d, "\mathrm{Int}(w)"] \\
\mathscr{G}_m \arrow[r, hook] & \mathscr{D}_{Q,n} \arrow[r, two heads] & \mathscr{Y}_{Q,n}.
\end{tikzcd}$$

If $s \in \mathscr{S}\!pl(\mathscr{D}_{Q,n})$, define $\dot{w}(s) \in \mathscr{S}\!pl(\mathscr{D}_{Q,n})$ by

$$\dot{w}(s) = \mathrm{Int}(\dot{w}) \circ s \circ \mathrm{Int}(w)^{-1}.$$

Allowing s to vary, this gives a map of sheaves on $S_{\text{ét}}$,

$$\dot{w} \colon \mathscr{S}\!pl(\mathscr{D}_{Q,n}) \to \mathscr{S}\!pl(\mathscr{D}_{Q,n})$$

which satisfies

$$\dot{w}(\hat{\tau} * s) = w(\hat{\tau}) * \dot{w}(s).$$

The functor $\mathbf{Int}(\dot{w}) \colon \mathbf{E}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{E}_\varepsilon(\tilde{\mathbf{T}})$ sends $(\mathscr{H}, h)$ to $({}^w\mathscr{H}, \dot{w} \circ h)$, where ${}^w\mathscr{H}$ is the $\hat{\mathscr{T}}$-torsor on $S_{\text{ét}}$ which coincides with $\mathscr{H}$ as sheaves of sets on $S_{\text{ét}}$, and in which the torsor structure is twisted:

$$\hat{\tau} *_w x := w^{-1}\hat{\tau} * x.$$

An explicit description of $\dot{w} \colon \mathscr{S}\!pl(\mathscr{D}_{Q,n}) \to \mathscr{S}\!pl(\mathscr{D}_{Q,n})$ seems unwieldy, in general. But in a special case, we may describe $\dot{w}(s)$ explicitly. For this special case, define $\mathbf{T}_d$ to be the maximal S-split torus in $\mathbf{T}$; its cocharacter lattice Y_d coincides with $\mathscr{Y}[S]$. Write X_d for the character lattice of this split torus, and Φ_d for the subset of X_d consisting of the roots of $\mathbf{G}$ relative to $\mathbf{T}_d$. Write W_d for the Weyl group of the relative root system; it can be identified with $\mathscr{W}[S]$. If $\beta \in \Phi_d$ is a relative root, then define w_β to be the corresponding reflection in W_d. We refer to the seminal paper of Borel and Tits for the results stated here about relative root systems; for example, the following is a result of [**13**, Théorème 5.3]:

$$W_d = \langle w_\beta : \beta \in \Phi_d \text{ and } 2\beta \notin \Phi_d \rangle.$$

Suppose that $\beta \in \Phi_d$ and $2\beta \notin \Phi_d$ hereafter, i.e., β is non-multipliable. Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be the absolute roots (in $\Phi[U]$ for some Galois cover $U \to S$) whose restrictions to $\mathbf{T}_d$ coincide with β. These roots are pairwise orthogonal, and form a single Galois orbit. Let S'/S be the étale cover corresponding to the finite Galois module $\{\alpha_1, \ldots, \alpha_\ell\}$. As the roots $\alpha_1, \ldots, \alpha_\ell$ lie in the same Galois orbit, the associated constants $n_{\alpha_1}, \ldots, n_{\alpha_\ell}$ are equal, and we write n_β for their common value. Write q_β for the common value of $Q(\alpha_1^\vee), \ldots, Q(\alpha_\ell^\vee)$.

Define $\beta^\vee = \sum_{i=1}^{\ell} \alpha_i^\vee$ for the resulting cocharacter of $\mathbf{T}_d$, and define $\tilde{\beta}^\vee = n_\beta \beta^\vee$. The structure theory of quasisplit groups provides a homomorphism defined over S, with finite kernel, $\mathbf{Res}_{S'/S}\mathbf{SL}_2 \to \mathbf{G}$. Composing with the natural embedding $\mathbf{SL}_2 \hookrightarrow \mathbf{Res}_{S'/S}\mathbf{SL}_2$, we find a homomorphism with finite kernel,

$$r_\beta \colon \mathbf{SL}_2 \to \mathbf{G}.$$

The standard coroot $t \mapsto diag(t, t^{-1})$ of $\mathbf{SL}_2$ maps to the cocharacter $\beta^\vee \in Y_d$.

The cover $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ pulls back via r_β to a cover $(\mathbf{SL}_2', n)$ of $\mathbf{SL}_2$, defined uniquely up to unique isomorphism by the integer $Q(\beta^\vee) = \ell q_\beta$. The second Brylinski-Deligne invariant for $\mathbf{SL}_2'$, an extension $\mathscr{D}_{\ell q_\beta}$, fits into a commutative diagram

$$\begin{tikzcd}
\mathscr{G}_m \arrow[r, hook] \arrow[d, "="] & \mathscr{D}_{\ell q_\beta} \arrow[r, two heads] \arrow[d, hook] & \mathbb{Z} \arrow[d, hook, "a \mapsto a\cdot\beta^\vee"] \\
\mathscr{G}_m \arrow[r, hook] & \mathscr{D} \arrow[r, two heads] & \mathscr{Y}.
\end{tikzcd}$$

Let $\mathbf{U}_*$ be the locally closed subscheme of $\mathbf{SL}_2$ obtained from the closed subgroup of upper-triangular unipotent matrices, by deleting the identity section. Let $\mathbf{U}_*^-$ be the analogous scheme of lower-triangular unipotent matrices. If $e \in \mathbf{U}_*(S)$, then there exists $e^- \in \mathbf{U}_*^-(S)$ such that $\dot{w}_\beta := r_\beta(ee^-e)$ represents the relative root reflection w_β (cf. [**25**, §1.8]). This provides an element $[e] \in \mathscr{D}$ lying over $\beta^\vee$ by the construction of [**18**, §11.2] (the construction for $\mathbf{SL}_2$ is in [**18**], and apply the inclusion $\mathscr{D}_{\ell q_\beta} \hookrightarrow \mathscr{D}$). Raising to the n_β power yields an element $[e]^{n_\beta} \in \mathscr{D}_{Q,n}$ lying over $\tilde{\beta}^\vee \in \mathscr{Y}_{Q,n}$.

The following lemma is a slight adaptation of [**18**, Equation 11.11.2].

Lemma 3.16. — *Suppose that $\dot{w} = r_\beta(ee^-e)$ and $d \in \mathscr{D}_{Q,n}$ lies over $y \in \mathscr{Y}_{Q,n}$. Then*

$$\mathrm{Int}(\dot{w})d = d \cdot \prod_{i=1}^{\ell} \left([e_i]^{-n_\beta \langle \tilde{\alpha}_i, y \rangle} \cdot (-1)^{q_\beta \varepsilon(-n_\beta \langle \tilde{\alpha}_i, y \rangle)} \right).$$

Here $\varepsilon(N) = N(N+1)/2$. The elements $[e_i] \in \mathscr{D}$ are described in the proof.

Proof. — Extending scalars to the étale cover S'/S, we have an isomorphism

$$\left(\mathbf{Res}_{S'/S}\mathbf{SL}_2\right)_{S'} \cong \prod_{i=1}^{\ell} \mathbf{SL}_{2,S'}.$$

The natural embedding $\mathbf{SL}_2 \hookrightarrow \mathbf{Res}_{S'/S}\mathbf{SL}_2$ is identified with the diagonal embedding, after extending scalars to S'.

In this way, we may write $e = \prod_{i=1}^{\ell} e_i$ and $e^- = \prod_{i=1}^{\ell} e_i^-$, where e_i and e_i^- commute with e_j and e_j^- for $i \neq j$. In this way, $w_\beta = \prod w_{\alpha_i}$ and the latter product consists of commuting reflections in $\mathscr{W}[S']$. Write $\dot{w}_i = e_i e_i^- e_i$, and note that $\dot{w}_\beta = \prod \dot{w}_i$ (a product of commuting elements of $\mathbf{G}(S')$). Tracing through the construction of [**18**, §11.1], we find that $[e] = \prod_{i=1}^{\ell} [e_i]$, where the latter product consists of commuting elements of $\mathscr{D}$, with $[e_i]$ lying over $\alpha_i^\vee$. Define $\tilde{\beta} = \sum_{i=1}^{\ell} \tilde{\alpha}_i$ in what follows.

[**18**, Equation 11.11.2] gives a formula for each $1 \leq i \leq \ell$,

$$\mathrm{Int}(\dot{w}_i) d = d \cdot [e_i]^{-\langle \alpha_i, y \rangle} \cdot (-1)^{q_\beta \varepsilon(-\langle \alpha_i, y \rangle)}.$$

Since $\alpha_i = n_\beta \tilde{\alpha}_i$ for all i, we have

$$\mathrm{Int}(\dot{w}_i) d = d \cdot [e_i]^{-n_\beta \langle \tilde{\alpha}_i, y \rangle} \cdot (-1)^{q_\beta \varepsilon(-n_\beta \langle \tilde{\alpha}_i, y \rangle))}. \tag{3.8}$$

Since $\langle \alpha_i, \alpha_j^\vee \rangle = 0$ for $i \neq j$, repeated application of this equation yields

$$\mathrm{Int}(\dot{w}) d = d \cdot \prod_{i=1}^{\ell} \left([e_i]^{-n_\beta \langle \tilde{\alpha}_i, y \rangle} \cdot (-1)^{q_\beta \varepsilon(-n_\beta \langle \tilde{\alpha}_i, y \rangle))} \right). \qquad \square$$

Now, if $s \in \mathscr{S\!p\ell}(\mathscr{D}_{Q,n})$, and $\dot{w}_\beta = r_\beta(e e^- e)$ as above, then we say that s is ***aligned with*** $\dot{w}_\beta$ if $s(\tilde{\alpha}_i^\vee) = [e_i]^{n_\beta}$ for all $1 \leq i \leq \ell$.

Theorem 3.17. — *For $\dot{w}_\beta = r_\beta(e e^- e)$ as before, and $s \in \mathscr{S\!p\ell}(\mathscr{D}_{Q,n})$ aligned with $\dot{w}_\beta$, we have*

$$\dot{w}_\beta(s) = \begin{cases} \prod_{i=1}^{\ell} \tilde{\alpha}_i(-1)^{n/2} * s & \text{if } q_\beta \text{ is odd;} \\ s & \text{if } q_\beta \text{ is even.} \end{cases}$$

(By Assumption 3.1, q_β odd implies n is even. Each $\tilde{\alpha}_i \in \mathscr{X}_{Q,n}$ is viewed as a cocharacter of $\hat{\mathscr{T}}$ here.)

Proof. — From the Formula (3.8), we compute

$$\begin{aligned}
[\dot{w}_\beta(s)](y) &= \mathrm{Int}(\dot{w}_\beta) s (\mathrm{Int}(w_\beta)^{-1} y) \\
&= \mathrm{Int}(\dot{w}_\beta) s \left(y - \sum_j \langle \tilde{\alpha}_j, y \rangle \tilde{\alpha}_j^\vee \right) \\
&= \mathrm{Int}(\dot{w}_\beta) s(y) \cdot \prod_j \mathrm{Int}(\dot{w}_\beta) s(\tilde{\alpha}_j^\vee)^{-\langle \tilde{\alpha}_j, y \rangle} \\
&= s(y) \cdot \prod_{i=1}^{\ell} [e_i]^{-n_\beta \langle \tilde{\alpha}_i, y \rangle} \cdot (-1)^{q_\beta \varepsilon(-n_\beta \langle \tilde{\alpha}_i, y \rangle)} \\
&\quad \cdot \prod_{j=1}^{\ell} \left(s(\tilde{\alpha}_j^\vee)^{-\langle \tilde{\alpha}_j, y \rangle} \cdot \prod_{i=1}^{\ell} \left([e_i]^{n_\beta \langle \tilde{\alpha}_j, y \rangle \langle \tilde{\alpha}_i, \tilde{\alpha}_j^\vee \rangle} \cdot (-1)^{-q_\beta \langle \tilde{\alpha}_j, y \rangle \varepsilon(-n_\beta \langle \tilde{\alpha}_i, \tilde{\alpha}_j^\vee \rangle)} \right) \right) \\
&= s(y) \cdot \prod_{i=1}^{\ell} (-1)^{q_\beta \varepsilon(-n_\beta \langle \tilde{\alpha}_i, y \rangle) - q_\beta \langle \tilde{\alpha}_i, y \rangle \varepsilon(-2 n_\beta)}.
\end{aligned}$$

In the last line, we use the fact that $s(\tilde{\alpha}_i^\vee) = [e_i]^{n_\beta}$ for all $1 \leq i \leq \ell$, and we use the orthogonality relation $\langle \tilde{\alpha}_j, \tilde{\alpha}_i^\vee \rangle = 0$ for $i \neq j$, and $\langle \tilde{\alpha}_j, \tilde{\alpha}_j^\vee \rangle = 2$.

For all $N \in \mathbb{Z}$, $\varepsilon(2N) = N$ modulo 2, and $\varepsilon(-N) - N = \varepsilon(N)$ modulo 2. Hence

$$\varepsilon(-n_\beta \langle \tilde{\alpha}_i, y \rangle) - \langle \tilde{\alpha}_i, y \rangle \varepsilon(-2n_\beta) = \varepsilon(-n_\beta \langle \tilde{\alpha}_i, y \rangle) - n_\beta \langle \tilde{\alpha}_i, y \rangle = \varepsilon(n_\beta \langle \tilde{\alpha}_i, y \rangle), \text{ modulo } 2.$$

We find that

$$[\dot{w}_\beta(s)](y) = s(y) \cdot \prod_{i=1}^{\ell} (-1)^{q_\beta \cdot \varepsilon(n_\beta \langle \tilde{\alpha}_i, y \rangle)}.$$

If q_β is even, then the exponent of (-1) is even and $\dot{w}_\beta(s) = s$ as claimed.

If q_β is odd, then n is even and n_β is even. Moreover, n is a multiple of 4 if and only if n_β is a multiple of 4. We find that

$$\prod_{i=1}^{\ell} (-1)^{q_\beta \varepsilon(n_\beta \langle \tilde{\alpha}_i, y \rangle)} = \prod_{i=1}^{\ell} (-1)^{n_\beta \langle \tilde{\alpha}_i, y \rangle / 2} = \prod_{i=1}^{\ell} (-1)^{n \langle \tilde{\alpha}_i, y \rangle / 2}.$$

This is precisely the evaluation of y at $\prod_i \tilde{\alpha}_i(-1)^{n/2} \in \hat{\mathscr{T}}_{[2]} = \mathscr{H}om(\mathscr{Y}_{Q,n}, \mu_2)$. □

4. The metaGalois group

We now specialize to three classes of base scheme of arithmetic interest.

Global: $S = \mathrm{Spec}(F)$ for a global field F;

Local: $S = \mathrm{Spec}(F)$ for a local field F;

Local integral: $S = \mathrm{Spec}(\mathscr{O})$ for the ring of integers $\mathscr{O}$ in a nonarchimedean local field F.

Choose a separable closure $\bar{F}/F$ in all three cases, and write $\mathrm{Gal}_F = \mathrm{Gal}(\bar{F}/F)$ for the resulting absolute Galois group. In the local integral case, the separable closure $\bar{F}/F$ provides a geometric base point $\bar{s}$ for $\mathrm{Spec}(\mathscr{O})$ as well, and define $\mathrm{Gal}_{\mathscr{O}} = \pi_1^{\text{ét}}(\mathrm{Spec}(\mathscr{O}), \bar{s})$. This is a profinite group, topologically generated by a geometric Frobenius element Fr; thus we write $\mathrm{Gal}_{\mathscr{O}} = \langle \mathrm{Fr} \rangle_{\mathrm{prof}}$.

When $S = \mathrm{Spec}(F)$ for a local or global field, or $S = \mathrm{Spec}(\mathscr{O})$ for the ring of integers in a nonarchimedean local field, write Gal_S for Gal_F or $\mathrm{Gal}_{\mathscr{O}}$ accordingly.

4.1. Construction of the metaGalois group. — The metaGalois group will be a profinite group fitting into a central extension,

$$\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_S \twoheadrightarrow \mathrm{Gal}_S .$$

When F has characteristic two, we *define* the metaGalois group $\widetilde{\mathrm{Gal}}_F$ to be the trivial extension $\mathrm{Gal}_F \times \mu_2$. The metaGalois group $\widetilde{\mathrm{Gal}}_{\mathscr{O}}$ will *not be defined* when $\mathscr{O}$ has residual characteristic 2, reflecting the idea that metaGalois representations cannot be "unramified at 2" (though one might propose an alternative notion of "minimally ramified").

4.1.1. *Local fields.* — When F is a local field (with $2 \neq 0$) the quadratic Hilbert symbol defines a symmetric nondegenerate $\mathbb{Z}$-bilinear form

$$\mathrm{Hilb}_2 \colon F_{/2}^{\times} \times F_{/2}^{\times} \to \mu_2.$$

The abelianized Galois group $\mathrm{Gal}_F^{\mathrm{ab}}$ is defined, up to unique isomorphism, from F alone. When F is nonarchimedean, we normalize the valuation so that $\mathrm{val}(F^{\times}) = \mathbb{Z}$, and we normalize the reciprocity map of local class field theory to send a geometric Frobenius element to an element of valuation 1. The reciprocity map gives a surjective homomorphism

$$\mathrm{rec}_{F/2} \colon \mathrm{Gal}_F^{\mathrm{ab}} \twoheadrightarrow F_{/2}^{\times}.$$

Composing the Hilbert symbol with $\mathrm{rec}_{F/2}$ defines a function

$$h \colon \mathrm{Gal}_F^{\mathrm{ab}} \times \mathrm{Gal}_F^{\mathrm{ab}} \to \mu_2,$$

and it is straightforward to verify that h is a (bimultiplicative) continuous symmetric 2-cocycle. This incarnates a commutative extension

$$\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_F^{\mathrm{ab}} \twoheadrightarrow \mathrm{Gal}_F^{\mathrm{ab}}$$

of profinite groups. Concretely, $\widetilde{\mathrm{Gal}}_F^{\mathrm{ab}} = \mathrm{Gal}_F^{\mathrm{ab}} \times \mu_2$ as sets, and

$$(\gamma_1, \varepsilon_1) \cdot (\gamma_2, \varepsilon_2) := (\gamma_1\gamma_2, \varepsilon_1\varepsilon_2 \cdot h(\gamma_1, \gamma_2)).$$

The pullback of this extension to Gal_F will be called the ***metaGalois group of*** F, written $\widetilde{\mathrm{Gal}}_F$. It is a central extension of Gal_F by μ_2,

$$\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_F \twoheadrightarrow \mathrm{Gal}_F.$$

4.1.2. *The local integral case.* — Suppose that F is a nonarchimedean *nondyadic* (i.e., $\mathrm{val}(2) = 0$) local field. Then the quadratic Hilbert symbol satisfies

$$\mathrm{Hilb}_2(u, v) = 1 \text{ for all } u, v \in \mathcal{O}^{\times}.$$

Write $\mathcal{I} \subset \mathrm{Gal}_F$ for the inertial subgroup, so that $\mathrm{rec}_F(\mathcal{I}) = \mathcal{O}^{\times}$. The cocycle h is trivial when restricted to $\mathcal{I} \times \mathcal{I}$. Thus $\gamma \mapsto (\gamma, 1)$ gives a canonical splitting $\sigma^\circ \colon \mathcal{I} \hookrightarrow \widetilde{\mathrm{Gal}}_F$. The natural map $\mathrm{Gal}_F \twoheadrightarrow \mathrm{Gal}_{\mathcal{O}}$ identifies $\mathrm{Gal}_{\mathcal{O}}$ with $\mathrm{Gal}_F / \mathcal{I}$. Define $\widetilde{\mathrm{Gal}}_{\mathcal{O}} = \widetilde{\mathrm{Gal}}_F / \sigma^\circ(\mathcal{I})$ to obtain a commutative diagram with exact rows.

$$\begin{tikzcd}
\mu_2 \arrow[r, hook] \arrow[d, "="] & \widetilde{\mathrm{Gal}}_F \arrow[r] \arrow[d, two heads] & \mathrm{Gal}_F \arrow[d, two heads] \\
\mu_2 \arrow[r, hook] & \widetilde{\mathrm{Gal}}_{\mathcal{O}} \arrow[r] & \mathrm{Gal}_{\mathcal{O}}.
\end{tikzcd}$$

We call $\widetilde{\mathrm{Gal}}_{\mathcal{O}}$ the ***metaGalois group of*** $\mathcal{O}$. If $\gamma \in \mathrm{Gal}_F$ lifts Fr, then every element of $\widetilde{\mathrm{Gal}}_{\mathcal{O}}$ is equal to $(\gamma^{\hat{n}}, \pm 1) \pmod{\sigma^\circ(\mathcal{I})}$ for some $\hat{n} \in \hat{\mathbb{Z}}$. In this way, a Frobenius lift provides an isomorphism from $\widetilde{\mathrm{Gal}}_{\mathcal{O}}$ to the group with underlying set $\langle \gamma \rangle_{\mathrm{prof}} \times \mu_2$ and multiplication given by

$$(\gamma^{\hat{n}_1}, \varepsilon_1) \cdot (\gamma^{\hat{n}_2}, \varepsilon_2) = \left(\gamma^{\hat{n}_1 + \hat{n}_2}, \varepsilon_1\varepsilon_2 \cdot (-1)^{\hat{n}_1\hat{n}_2(q-1)/2}\right),$$

where q is the cardinality of the residue field of $\mathcal{O}$.

4.1.3. *Global fields.* — When F is a global field (with $2 \neq 0$ as before), the Hilbert symbol defines a symmetric $\mathbb{Z}$-bilinear form,

$$\mathrm{Hilb}_2 \colon \mathbb{A}^\times_{/2} \times \mathbb{A}^\times_{/2} \to \mu_2,$$

obtained as the product of local Hilbert symbols. This defines a continuous symmetric 2-cocycle, from which we get a commutative extension,

$$\mu_2 \hookrightarrow \widetilde{\mathbb{A}^\times_{/2}} \twoheadrightarrow \mathbb{A}^\times_{/2}. \tag{4.1}$$

Global quadratic reciprocity for the Hilbert symbol ($\mathrm{Hilb}_2(u,v) = 1$ for all $u, v \in F^\times$) provides a canonical splitting $\sigma_F \colon F^\times_{/2} \hookrightarrow \widetilde{\mathbb{A}^\times_{/2}}$. Taking the quotient yields a commutative extension,

$$\mu_2 \hookrightarrow \frac{\widetilde{\mathbb{A}^\times_{/2}}}{\sigma_F(F^\times_{/2})} \twoheadrightarrow \frac{\mathbb{A}^\times_{/2}}{F^\times_{/2}}. \tag{4.2}$$

The global reciprocity map of class field theory gives an surjective homomorphism,

$$\mathrm{rec}_{F/2} \colon \mathrm{Gal}_F^{\mathrm{ab}} \twoheadrightarrow \mathbb{A}^\times_{/2}/F^\times_{/2} \equiv (\mathbb{A}^\times/F^\times)_{/2}.$$

Pulling back (4.2) via $\mathrm{rec}_{F/2}$ yields a commutative extension,

$$\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_F^{\mathrm{ab}} \twoheadrightarrow \mathrm{Gal}_F^{\mathrm{ab}}.$$

Pulling back via $\mathrm{Gal}_F \twoheadrightarrow \mathrm{Gal}_F^{\mathrm{ab}}$ defines the ***metaGalois group of*** F,

$$\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_F \twoheadrightarrow \mathrm{Gal}_F.$$

4.1.4. *Compatibilities.* — If $v \in \mathcal{V}$ is a place of a global field F, then an embedding $\bar{F} \hookrightarrow \bar{F}_v$ of separable closures determines an injective homomorphism $\iota_v \colon \mathrm{Gal}_{F_v} \hookrightarrow \mathrm{Gal}_F$. As the global Hilbert symbol is the product of local ones, we find a homomorphism $\tilde{\iota}_v$ realizing $\widetilde{\mathrm{Gal}}_{F_v}$ as the pullback of the extension $\widetilde{\mathrm{Gal}}_F$

$$\begin{tikzcd}
\mu_2 \arrow[r, hook] \arrow[d, "="] & \widetilde{\mathrm{Gal}}_{F_v} \arrow[r] \arrow[d, hook, "\tilde{\iota}_v"] & \mathrm{Gal}_{F_v} \arrow[d, hook, "\iota_v"] \\
\mu_2 \arrow[r, hook] & \widetilde{\mathrm{Gal}}_F \arrow[r] & \mathrm{Gal}_F .
\end{tikzcd}$$

For local and global fields F, a choice of separable closure $\bar{F}/F$ entered the construction of the metaGalois group. Suppose that $\bar{F}_0$ is another separable closure of F. Every F-algebra isomorphism $\iota \colon \bar{F}_0 \xrightarrow{\sim} \bar{F}$ yields an isomorphism $\iota \colon \mathrm{Gal}(\bar{F}_0/F) \xrightarrow{\sim} \mathrm{Gal}(\bar{F}/F)$. The resulting isomorphism $\mathrm{Gal}(\bar{F}_0/F)^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$ does not depend on ι.

The separable closure $\bar{F}_0$ yields a cocycle $h_0 \colon \mathrm{Gal}(\bar{F}_0/F)^{\mathrm{ab}} \times \mathrm{Gal}(\bar{F}_0/F)^{\mathrm{ab}} \to \mu_2$, and thus a metaGalois group $\widetilde{\mathrm{Gal}}(\bar{F}_0/F)$. Since the defining cocycles h_0 and h factor through abelianized Galois groups, the isomorphism $\iota\colon \mathrm{Gal}(\bar{F}_0/F) \to \mathrm{Gal}(\bar{F}/F)$ lifts canonically to an isomorphism of metaGalois groups

$$\begin{tikzcd} \mu_2 \arrow[r, hook] \arrow[d, "="] & \widetilde{\mathrm{Gal}}(\bar{F}_0/F) \arrow[r, two heads] \arrow[d, "\tilde{\iota}"] & \mathrm{Gal}(\bar{F}_0/F) \arrow[d, "\iota"] \\ \mu_2 \arrow[r, hook] & \widetilde{\mathrm{Gal}}(\bar{F}/F) \arrow[r, two heads] & \mathrm{Gal}(\bar{F}/F). \end{tikzcd}$$

4.2. The Brauer class. — When F is a local or global field, the metaGalois group is an extension $\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_F \twoheadrightarrow \mathrm{Gal}_F$. As such, it has a cohomology class in the Brauer group

$$\left[\widetilde{\mathrm{Gal}}_F\right] \in H^2_{\text{ét}}(F, \mu_2) = \mathrm{Br}(F)_{[2]}.$$

This Brauer class is often trivial—the metaGalois group often splits, though it rarely has a canonical splitting.

Proposition 4.1. — *Suppose that F is a* nondyadic *(i.e.,* $\mathrm{val}(2) = 0$*) nonarchimedean local field. Then $\left[\widetilde{\mathrm{Gal}}_F\right]$ is the trivial class.*

Proof. — The projection $\widetilde{\mathrm{Gal}}_F \to \widetilde{\mathrm{Gal}}_{\mathcal{O}}$ identifies the metaGalois group of F with the pullback of the metaGalois group of $\mathcal{O}$

$$\begin{tikzcd} \mu_2 \arrow[r] \arrow[d, "="] & \widetilde{\mathrm{Gal}}_F \arrow[r] \arrow[d] & \mathrm{Gal}_F \arrow[d] \\ \mu_2 \arrow[r] & \widetilde{\mathrm{Gal}}_{\mathcal{O}} \arrow[r] & \mathrm{Gal}_{\mathcal{O}} = \langle \mathrm{Fr} \rangle_{\mathrm{prof}}. \end{tikzcd}$$

But every extension of $\hat{\mathbb{Z}}$ by μ_2 splits (though not canonically); hence the metaGalois group splits and its Brauer class is trivial. $\square$

Proposition 4.2. — *Over $\mathbb{R}$, the metaGalois group is a nonsplit extension, so $\left[\widetilde{\mathrm{Gal}}_{\mathbb{R}}\right]$ is the unique nontrivial class in the Brauer group* $\mathrm{Br}(\mathbb{R})$.

Proof. — Let σ denote complex conjugation, $\mathrm{Gal}_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{\mathrm{Id}, \sigma\}$. The metaGalois group is a cyclic group of order 4 sitting in an extension

$$\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_{\mathbb{R}} \twoheadrightarrow \mathrm{Gal}_{\mathbb{R}}.$$

Indeed, the cocycle h satisfies $h(\sigma, \sigma) = \mathrm{Hilb}_2(-1, -1) = -1$. Thus

$$(\sigma, 1) \cdot (\sigma, 1) = (\mathrm{Id}, -1) \in \widetilde{\mathrm{Gal}}_{\mathbb{R}}.$$

Hence $(\sigma, 1)$ is an element of order 4 and $\left[\widetilde{\mathrm{Gal}}_{\mathbb{R}}\right]$ is nontrivial. $\square$

Proposition 4.3. — *Let F_2 be a* dyadic *nonarchimedean local field of characteristic zero. Then $\left[\widetilde{\mathrm{Gal}}_{F_2}\right]$ is trivial if $[F_2 : \mathbb{Q}_2]$ is even, and is nontrivial if $[F_2 : \mathbb{Q}_2]$ is odd.*

Proof. — Let $d = [F_2 : \mathbb{Q}_2]$. By approximation, there exists a global field F such that $F \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is isomorphic to F_2 as an F-algebra. Indeed, the primitive element theorem allows us to write $F_2 = \mathbb{Q}_2(z)$ for $z \in F_2$ a root of a monic irreducible polynomial $P \in \mathbb{Q}_2[X]$. This gives an isomorphism of $\mathbb{Q}_2$-algebras from $\mathbb{Q}_2[X]/(P)$ to F_2. A small change in the coefficients of P will not change the isomorphism class of the field $\mathbb{Q}_2[X]/(P)$, by Krasner's lemma. Hence, by density of $\mathbb{Q}$ in $\mathbb{Q}_2$, we may assume that $P \in \mathbb{Q}[X]$. As P is monic irreducible over $\mathbb{Q}_2$, it is monic irreducible over $\mathbb{Q}$. The field $F = \mathbb{Q}[X]/(P)$ satisfies the condition that $F \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is isomorphic to F_2 as an F-algebra.

The global metaGalois group $\widetilde{\mathrm{Gal}}_F$ has a Brauer class β_F with local components $\beta_{F,v}$ satisfying

- $\beta_{F,v}$ is trivial when F_v has odd residual characteristic (by Proposition 4.1);
- $\beta_{F,2} = \left[\widetilde{\mathrm{Gal}}_{F_2}\right]$ at the unique place of even residual characteristic;
- $\beta_{F,v}$ is nontrivial at all real places (by Proposition 4.2);
- (Parity condition) $\beta_{F,v}$ is nontrivial at a set of places of even cardinality.

We have $d = [F_2 : \mathbb{Q}_2] = [F : \mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of real places, and r_2 the number of complex places. It follows that d is even if and only if r_1 is even. The parity condition on the global Brauer class implies that r_1 is even if and only if $\beta_{F,2}$ is the trivial class. □

Corollary 4.4. — *Let F be a global field, with $2 \neq 0$ in F. Then the Brauer class of $\widetilde{\mathrm{Gal}}_F$ is that of the unique quaternion algebra which is ramified at all real places and all dyadic places of odd degree over $\mathbb{Q}_2$.*

Proof. — This follows directly from the previous three propositions, and the local-global compatibility of the metaGalois group. □

In particular, the Brauer class $\left[\widetilde{\mathrm{Gal}}_{\mathbb{Q}}\right]$ is that of the quaternion algebra $\frac{(-1,-1)}{\mathbb{Q}}$ ramified only at 2 and ∞. If F is a global field of characteristic $p \neq 2$, then $\left[\widetilde{\mathrm{Gal}}_F\right]$ is the trivial class.

4.3. Splitting by additive characters. — The metaGalois group may be a nonsplit extension of Gal_S by μ_2, and even when it splits, it rarely splits canonically. However, an additive character suffices to split the metaGalois group after pushing out via $\mu_2 \hookrightarrow \mu_4$. In the three cases of interest, define a $\mathscr{G}_m[S]$-torsor Ψ_S as follows.

- When F is local, let Ψ_F be the set of nontrivial continuous homomorphisms from F to $\mathbb{C}^\times$. If $u \in F^\times$, $\psi \in \Psi_F$, write $[u * \psi](x) = \psi(u^{-1}x)$. In this way, Ψ_F is a $F^\times$-torsor.
- When F is global, let Ψ_F be the set of nontrivial continuous homomorphisms from $\mathbb{A}/F$ to $\mathbb{C}^\times$. If $u \in F^\times$, $\psi \in \Psi_F$, write $[u * \psi](x) = \psi(u^{-1}x)$. In this way, Ψ_F is a $F^\times$-torsor.

- When F is local nonarchimedean, with ring of integers $\mathcal{O}$, let $\Psi_{\mathcal{O}}$ be the set of nontrivial continuous homomorphisms from $F/\mathcal{O}$ to $\mathbb{C}^\times$. If $u \in \mathcal{O}^\times$, $\psi \in \Psi_{\mathcal{O}}$, write $[u * \psi](x) = \psi(u^{-1}x)$. In this way, $\Psi_{\mathcal{O}}$ is a $\mathcal{O}^\times$-torsor.

Define here $\mu_4 = \boldsymbol{\mu}_4(\mathbb{C}) = \{1, -1, i, -i\}$. When F is a local field (with $2 \neq 0$), and $\psi \in \Psi_F$, the ***Weil index*** is a function $\mathbf{w}_F(\bullet, \psi) \colon F^\times_{/2} \to \mu_4$ which satisfies

$$\frac{\mathbf{w}_F(uv, \psi)}{\mathbf{w}_F(u, \psi)\mathbf{w}_F(v, \psi)} = \mathrm{Hilb}_2(u, v). \tag{4.3}$$

Our $\mathbf{w}_F(u, \psi)$ is defined to be $\gamma(ux^2)/\gamma(x^2)$ in Weil's notation from [**76**, §29] and is written $\gamma_F(u, \psi)$ in [**59**, §A.3] and elsewhere. From [**59**, Proposition A.11], the local Weil indices are trivial on $\mathcal{O}^\times$ at all nondyadic places.

When F is a global field and $\psi \in \Psi_F$, the ***Weil index*** is the function

$$\mathbf{w}_F(\bullet, \psi) \colon \mathbb{A}^\times_{/2} \to \mu_4$$

defined as the product of local Weil indices. The global Weil index is trivial on $F^\times$ by [**76**, §II.30, Proposition 5]. As the global Hilbert symbol $\mathrm{Hilb}_2 \colon \mathbb{A}^\times \times \mathbb{A}^\times \to \mu_2$ is defined as the product of local Hilbert symbols, the Formula (4.3) holds in the global setting too.

Write $\widetilde{\mathrm{Gal}}_S^{(4)}$ for the pushout of $\widetilde{\mathrm{Gal}}_S$ via the inclusion $\mu_2 \hookrightarrow \mu_4$ (when $S = \mathrm{Spec}(F)$ or $S = \mathrm{Spec}(\mathcal{O})$ as usual)

$$\begin{tikzcd} \mu_2 \arrow[r, hook] \arrow[d, hook, "\iota"] & \widetilde{\mathrm{Gal}}_S \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\ \mu_4 \arrow[r, hook] & \widetilde{\mathrm{Gal}}_S^{(4)} \arrow[r, two heads] & \mathrm{Gal}_S . \arrow[l, dashed, bend right, "s(\psi)?"'] \end{tikzcd}$$

The splittings of $\widetilde{\mathrm{Gal}}_S^{(4)}$, if they exist, form a $\mathrm{Hom}(\mathrm{Gal}_S, \mu_4)$-torsor. In what follows, if $u \in \mathscr{G}_m[S]$, define $\chi_u \colon \mathrm{Gal}_S \to \mu_2$ to be the quadratic character associated to the étale extension $F[\sqrt{u}]$ (in the local or global case) or $\mathcal{O}[\sqrt{u}]$ (in the nondyadic local integral case).

Proposition 4.5. — *For each additive character $\psi \in \Psi_S$, the Weil index provides a splitting $s(\psi) \colon \mathrm{Gal}_S \to \widetilde{\mathrm{Gal}}_S^{(4)}$. Moreover, this system of splittings satisfies*

$$s(u * \psi) = \chi_u * s(\psi) \text{ for all } u \in \mathscr{G}_m[S].$$

The splittings $s(\psi)$ are described in three cases below.

4.3.1. *Local fields.* — When F is a local field, the pushout $\widetilde{\mathrm{Gal}}_S^{(4)}$ can be identified with the product $\mathrm{Gal}_F \times \mu_4$ as a set, with multiplication given by

$$(\gamma_1, \zeta_1) \cdot (\gamma_2, \zeta_2) = \left(\gamma_1\gamma_2, \zeta_1\zeta_2 \cdot \mathrm{Hilb}_2(\mathrm{rec}_{F/2}(\gamma_1), \mathrm{rec}_{F/2}(\gamma_2))\right).$$

For $\psi \in \Psi_F$, (4.3) provides a splitting $s(\psi) \colon \mathrm{Gal}_F \to \widetilde{\mathrm{Gal}}_F^{(4)}$,

$$s(\psi)(\gamma) = \left(\gamma, \mathbf{w}_F(\mathrm{rec}_{F/2}(\gamma), \psi)\right), \text{ for all } \gamma \in \mathrm{Gal}_F .$$

If $u \in F^\times$, then [**59**, Corollary A.5] states that $\mathbf{w}_F(a, u*\psi) = \mathrm{Hilb}_2(a,u) \cdot \mathbf{w}_F(a,\psi)$. Since $\mathrm{Hilb}_2(\mathrm{rec}_{F/2}\gamma, u) = \chi_u(\gamma)$, we find $s(u*\psi) = \chi_u * s(\psi)$.

4.3.2. *The local integral case.* — When F is a nonarchimedean, nondyadic local field, the local Weil index is trivial on $\mathcal{O}^\times$. Given a character $\psi \in \Psi_{\mathcal{O}} \subset \Psi_F$, the splitting $s(\psi) \colon \mathrm{Gal}_F \to \widetilde{\mathrm{Gal}}_F^{(4)}$ coincides with the canonical splitting σ° on inertia,

$$s(\psi)(\gamma) = \sigma^\circ(\gamma) = (\gamma, 1), \text{ for all } \gamma \in \mathcal{J}.$$

It follows that $s(\psi)$ descends to a splitting of $\widetilde{\mathrm{Gal}}_{\mathcal{O}}^{(4)}$ at nondyadic places.

$$\begin{tikzcd} \mu_4 \arrow[r, hook] & \widetilde{\mathrm{Gal}}_{\mathcal{O}}^{(4)} \arrow[r, two heads] & \mathrm{Gal}_{\mathcal{O}} \arrow[l, bend right, "s(\psi)"'] . \end{tikzcd}$$

If $u \in \mathcal{O}^\times$, write $\bar{u}$ for its image in the residue field $\mathbb{F}_q$. As before, we have $s(u*\psi) = \chi_u \cdot s(\psi)$. But now, the quadratic character χ_u is restricted to $\mathrm{Gal}_{\mathcal{O}} = \langle \mathrm{Fr} \rangle_{\mathrm{prof}}$; we have

$$\chi_u(\mathrm{Fr}) = \bar{u}^{(q-1)/2} \in \mu_2.$$

In other words, χ_u is the character of $\mathrm{Gal}_{\mathcal{O}}$ which sends Fr to the Legendre symbol of the reduction of u.

4.3.3. *Global fields.* — In the global setting, pushing out via $\mu_2 \hookrightarrow \mu_4$ gives a short exact sequence

$$\mu_4 \hookrightarrow \widetilde{\mathbb{A}_{/2}^\times}^{(4)} \twoheadrightarrow \mathbb{A}_{/2}^\times. \tag{4.4}$$

The middle term is given by $\widetilde{\mathbb{A}_{/2}^\times}^{(4)} = \mathbb{A}_{/2}^\times \times \mu_4$ as a set, with multiplication given by

$$(u_1, \zeta_1) \cdot (u_2, \zeta_2) = (u_1 u_2, \zeta_1 \zeta_2 \cdot \mathrm{Hilb}_2(u_1, u_2)).$$

A character $\psi \in \Psi_F$ provides a splitting of the extension (4.4),

$$s_{\mathbb{A}}(\psi)(u) = (u, \mathbf{w}_F(u, \psi)) \text{ for all } u \in \mathbb{A}_{/2}^\times.$$

Since $\mathbf{w}_F(u,\psi) = 1$ for all $u \in F^\times$, this splitting restricts to the canonical splitting $\sigma_F \colon F_{/2}^\times \to \widetilde{\mathbb{A}_{/2}^\times}$. Thus $s_{\mathbb{A}}(\psi)$ descends and pulls back to a splitting $s(\psi) \colon \mathrm{Gal}_F \to \widetilde{\mathrm{Gal}}_F^{(4)}$. If $u \in F^\times$, then our local results and local-global compatibility imply that $s(u*\psi) = \chi_u \cdot s(\psi)$.

4.4. Restriction. — Suppose that F'/F is a finite separable extension with $F' \subset \bar{F}$. In the local integral case, suppose that F'/F is unramified and let $\mathcal{O}'$ be the ring of integers in F'. Write $S' = \mathrm{Spec}(F')$ in the cases of local or global fields, and write $S' = \mathrm{Spec}(\mathcal{O}')$ in the local integral case. We have defined metaGalois groups for S

and S'

$$
\begin{tikzcd}
\mu_2 \arrow[r, hook] \arrow[d, "="] & \widetilde{\mathrm{Gal}}_{S'} \arrow[r, two heads] \arrow[d, dashed, "?"] & \mathrm{Gal}_{S'} \arrow[d, hook] \\
\mu_2 \arrow[r, hook] & \widetilde{\mathrm{Gal}}_S \arrow[r, two heads] & \mathrm{Gal}_S .
\end{tikzcd}
\tag{4.5}
$$

The inclusion $F' \subset \bar{F}$ gives an inclusion of Galois groups $\mathrm{Gal}_{S'} \hookrightarrow \mathrm{Gal}_S$, but a natural inclusion of metaGalois groups is not obvious. In particular, the cocycle defining $\widetilde{\mathrm{Gal}}_S$ does not restrict to the cocycle defining $\widetilde{\mathrm{Gal}}_{S'}$.

Fortunately, a beautiful insight of Wee Teck Gan gives such an inclusion of meta-Galois groups, using a "lifting theorem" of Edward Bender [**8**]. We explain this insight here.

In the case of local fields, consider a nonzero element $u \in F'$, and the "trace form" (cf. [**63**]) $F' \to F$ given by $x \mapsto \mathrm{Tr}_{F'/F}(ux^2)$. Viewing this as a quadratic form on a finite-dimensional F-vector space F', it has a Hasse-Witt invariant (an element of $\{\pm 1\}$). Define

$$\mathrm{HW}(u) = \frac{\text{Hasse-Witt invariant of } x \mapsto \mathrm{Tr}_{F'/F}(ux^2)}{\text{Hasse-Witt invariant of } x \mapsto \mathrm{Tr}_{F'/F}(x^2)}.$$

This function depends only on the square class of u.

Bender's theorem [**8**, Theorem 1] states that

$$\mathrm{Hilb}_{F',2}(u,v) = \frac{\mathrm{HW}(u)\,\mathrm{HW}(v)}{\mathrm{HW}(uv)} \cdot \mathrm{Hilb}_{F,2}(\mathrm{N}_{F'/F}\, u, \mathrm{N}_{F'/F}\, v).$$

Let $\iota\colon \mathrm{Gal}_{F'} \hookrightarrow \mathrm{Gal}_F$ be the canonical inclusion, so that $\mathrm{rec}_F(\iota(\gamma)) = \mathrm{N}_{F'/F}\,\mathrm{rec}_F(\gamma)$ for all $\gamma \in \mathrm{Gal}_{F'}$.

Proposition 4.6. — *Let F be a local field (with $2 \neq 0$ as usual). Then the function $\tilde{\iota}\colon \widetilde{\mathrm{Gal}}_{F'} \hookrightarrow \widetilde{\mathrm{Gal}}_F$, given by*

$$\tilde{\iota}(\gamma, \pm 1) = (\iota(\gamma), \pm\,\mathrm{HW}(\mathrm{rec}_{F'}\,\gamma))$$

is a group homomorphism completing the commutative diagram (4.5).

Proof. — Consider any $\gamma_1, \gamma_2 \in \mathrm{Gal}_{F'}$ and define $u_1 := \mathrm{rec}_{F'}(\gamma_1)$, $u_2 := \mathrm{rec}_{F'}(\gamma_2)$. Thus $\mathrm{rec}_F(\iota(\gamma_1)) = \mathrm{N}_{F'/F}\, u_1$ and $\mathrm{rec}_F(\iota(\gamma_2)) = \mathrm{N}_{F'/F}\, u_2$. For all $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, we compute

$$
\begin{aligned}
\tilde{\iota}((\gamma_1,\varepsilon_1)\cdot(\gamma_2,\varepsilon_2)) &= \tilde{\iota}\,(\gamma_1\gamma_2, \varepsilon_1\varepsilon_2\,\mathrm{Hilb}_{F',2}\,(\mathrm{rec}_{F'}(\gamma_1), \mathrm{rec}_{F'}(\gamma_2))) \\
&= (\iota(\gamma_1\gamma_2), \varepsilon_1\varepsilon_2\,\mathrm{Hilb}_{F',2}(u_1,u_2)\cdot\mathrm{HW}(u_1u_2)) \\
&= \big(\iota(\gamma_1)\iota(\gamma_2), \varepsilon_1\varepsilon_2\,\mathrm{Hilb}_{F,2}(\mathrm{N}_{F'/F}\,u_1, \mathrm{N}_{F'/F}\,u_2)\,\mathrm{HW}(u_1)\,\mathrm{HW}(u_2)\big) \\
&= (\iota(\gamma_1), \varepsilon_1\,\mathrm{HW}(u_1))\cdot(\iota(\gamma_2), \varepsilon_2\,\mathrm{HW}(u_2)) \\
&= \tilde{\iota}(\gamma_1,\varepsilon_1)\cdot\tilde{\iota}(\gamma_2,\varepsilon_2).
\end{aligned}
$$

□

In the local integral case, when $\mathcal{O}$ is the ring of integers in a nondyadic nonarchimedean field, $\mathrm{HW}(u) = 1$ for all $u \in \mathcal{O}^\times$. From this it follows that $\tilde{\iota}\colon \widetilde{\mathrm{Gal}}_{F'} \hookrightarrow \widetilde{\mathrm{Gal}}_F$ descends to an injective homomorphism

$$\begin{tikzcd} \mu_2 \arrow[r, hook] \arrow[d, "="] & \widetilde{\mathrm{Gal}}_{\mathcal{O}'} \arrow[r, two heads] \arrow[d, "\tilde{\iota}"] & \mathrm{Gal}_{\mathcal{O}'} \arrow[d, hook, "\iota"] \\ \mu_2 \arrow[r, hook] & \widetilde{\mathrm{Gal}}_{\mathcal{O}} \arrow[r, two heads] & \mathrm{Gal}_{\mathcal{O}}. \end{tikzcd}$$

In the global case, when F is a number field, we note that $\prod_v \mathrm{HW}_v(u) = 1$ for all $u \in F^\times$ (here HW_v denotes the invariant as the place v). From this it follows that the injective homomorphisms $\tilde{\iota}_v\colon \widetilde{\mathrm{Gal}}_{F'_v} \hookrightarrow \widetilde{\mathrm{Gal}}_{F_v}$ yield a injective homomorphism globally

$$\begin{tikzcd} \mu_2 \arrow[r, hook] \arrow[d, "="] & \widetilde{\mathrm{Gal}}_{F'} \arrow[r, two heads] \arrow[d, "\tilde{\iota}"] & \mathrm{Gal}_{F'} \arrow[d, hook, "\iota"] \\ \mu_2 \arrow[r, hook] & \widetilde{\mathrm{Gal}}_{F} \arrow[r, two heads] & \mathrm{Gal}_{F}. \end{tikzcd}$$

Taken together, these inclusions $\tilde{\iota}\colon \widetilde{\mathrm{Gal}}_{S'} \hookrightarrow \widetilde{\mathrm{Gal}}_S$ allow one to canonically "restrict" metaGalois representations (representations of $\widetilde{\mathrm{Gal}}_S$).

5. L-groups, parameters, L-functions

5.1. L-groups. — We use the term "L-group" to refer to a broad class of extensions of Galois groups by complex reductive groups. Unlike Langlands, Vogan, and others, we do not assume that our L-groups are endowed with a conjugacy class of splittings. Our L-groups are more closely related to the "weak E-groups" of [**74**, Definition 3.24]. But we maintain the letter "L" since our L-groups are still connected to L-functions.

The other difference between our L-groups and those in the literature is that (for reasons which will become clear) we consider our L-groups as objects of a 2-category. A base scheme S and geometric point $\bar{s} \to S$ will be fixed as in the previous section.

Definition 5.1. — An ***L-group*** is a pair $(G^\vee, {}^{\mathsf{L}}G)$, where $G^\vee$ is a complex linear algebraic group (not necessarily connected) and ${}^{\mathsf{L}}G$ is an extension of locally compact groups

$$G^\vee \hookrightarrow {}^{\mathsf{L}}G \twoheadrightarrow \mathrm{Gal}_S,$$

for which the conjugation action of any element of ${}^{\mathsf{L}}G$ on $G^\vee$ is complex-algebraic.

Remark 5.2. — For complex linear algebraic groups, we do not distinguish between the underlying variety and its $\mathbb{C}$-points. Thus we say $G^\vee$ is a complex linear algebraic group, and also view $G^\vee$ as a locally compact group.

Of course, Langlands' L-group ${}^{\mathsf{L}}G = \mathrm{Gal}_F \ltimes G^\vee$ (associated to a reductive group $\mathbf{G}$ over a field F) is an example. When V is a finite-dimensional complex vector space, the direct product $\mathrm{Gal}_S \times \mathrm{GL}(V)$ is an L-group. Since we don't assume $G^\vee$ to be connected, our metaGalois group $\widetilde{\mathrm{Gal}}_S$ is an L-group.

Definition 5.3. — Given two L-groups,

$$G_1^\vee \hookrightarrow {}^{\mathsf{L}}G_1 \twoheadrightarrow \mathrm{Gal}_S, \quad G_2^\vee \hookrightarrow {}^{\mathsf{L}}G_2 \twoheadrightarrow \mathrm{Gal}_S,$$

an ***L-morphism*** ${}^{\mathsf{L}}\rho\colon {}^{\mathsf{L}}G_1 \to {}^{\mathsf{L}}G_2$ will mean a continuous group homomorphism lying over $\mathrm{Id}\colon \mathrm{Gal}_S \to \mathrm{Gal}_S$, which restricts to a complex algebraic homomorphism $\rho^\vee\colon G_1^\vee \to G_2^\vee$. An ***L-equivalence*** will mean an invertible L-morphism.

In other words, an L-morphism fits into a commutative diagram, with the middle column continuous and the left column complex-algebraic

$$\begin{tikzcd}
G_1^\vee \arrow[r, hook] \arrow[d, "\rho^\vee"] & {}^{\mathsf{L}}G_1 \arrow[r, two heads] \arrow[d, "{}^{\mathsf{L}}\rho"] & \mathrm{Gal}_S \arrow[d, "="] \\
G_2^\vee \arrow[r, hook] & {}^{\mathsf{L}}G_2 \arrow[r, two heads] & \mathrm{Gal}_S .
\end{tikzcd}$$

Definition 5.4. — Given two L-morphisms ${}^{\mathsf{L}}\rho, {}^{\mathsf{L}}\rho'\colon {}^{\mathsf{L}}G_1 \to {}^{\mathsf{L}}G_2$, a natural isomorphism ${}^{\mathsf{L}}\rho \overset{\sim}{\Rightarrow} {}^{\mathsf{L}}\rho'$ will mean an element $a \in Z_2^\vee$ (the center of $G_2^\vee$) such that

$${}^{\mathsf{L}}\rho'(g) = a \cdot {}^{\mathsf{L}}\rho(g) \cdot a^{-1} \text{ for all } g \in {}^{\mathsf{L}}G_1.$$

In particular, note that ${}^{\mathsf{L}}\rho$ and ${}^{\mathsf{L}}\rho'$ coincide on $G_1^\vee$ when they are naturally isomorphic.

While the axioms for a 2-category are not satisfied if one looks at *all* L-groups, L-morphisms, and natural isomorphisms, this does define a 2-category of L-groups, L-*equivalences*, and natural isomorphisms.

In many cases of interest (e.g., when ${}^{\mathsf{L}}G_2$ arises as the L-group of a split reductive group), the only natural isomorphism is the identity. However, in some nonsplit cases, e.g., ${}^{\mathsf{L}}G_2 = \mathrm{Gal}_S \ltimes SL_3(\mathbb{C})$, the Langlands L-group of a quasisplit $\mathbf{G} = \mathbf{PGU}_3$, a nontrivial element $a \in Z_2^\vee$ does not lie in the center of ${}^{\mathsf{L}}G_2$. Such an element a may determine a nonidentity natural isomorphism.

An ***L-representation*** of an L-group ${}^{\mathsf{L}}G$ will mean a pair (ρ, V), where V is a finite-dimensional complex vector space, and $\rho\colon {}^{\mathsf{L}}G \to \mathrm{GL}(V)$ is a continuous homomorphism whose restriction to $G^\vee$ is complex algebraic. Giving an L-representation of ${}^{\mathsf{L}}G$ is the same as giving an L-morphism ${}^{\mathsf{L}}\rho\colon {}^{\mathsf{L}}G \to \mathrm{Gal}_S \times \mathrm{GL}(V)$.

5.2. Parameters. — Write $\mathcal{W}_S$ for the ***Weil group***. When $S = \mathrm{Spec}(F)$ for a local or global field, this Weil group $\mathcal{W}_S$ is $\mathcal{W}_F$ defined as in [**5**]; when $S = \mathrm{Spec}(\mathcal{O})$, we define $\mathcal{W}_S$ to be the free cyclic group $\langle \mathrm{Fr} \rangle \cong \mathbb{Z}$ generated by a geometric Frobenius Fr. In all cases, the Weil group is endowed with a continuous homomorphism $\mathcal{W}_S \to \mathrm{Gal}_S$ with dense image.

Let $G^\vee \hookrightarrow {}^{\mathsf{L}}G \twoheadrightarrow \mathrm{Gal}_S$ be an L-group. A ***Weil parameter*** is a continuous homomorphism $\phi\colon \mathcal{W}_S \to {}^{\mathsf{L}}G$ lying over $\mathcal{W}_S \to \mathrm{Gal}_S$, such that $\phi(w)$ is semisimple for all $w \in \mathcal{W}_S$ (see [**11**, §8.2]). The reader may follow [**11**] and [**33**] to define Weil-Deligne parameters in this general context, when working over a local field.

Write $\mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G)$ for the set of ${}^{\mathsf{L}}G$-valued Weil parameters. It is endowed with an action of $G^\vee$ by conjugation: if $g \in G^\vee$ and ϕ is a parameter, then define

$$ {}^g\phi(w) = \phi(g^{-1}wg). $$

Two parameters are called ***equivalent*** if they are in the same $G^\vee$-orbit.

Composition with an L-morphism ${}^{\mathsf{L}}\rho \colon {}^{\mathsf{L}}G_1 \to {}^{\mathsf{L}}G_2$ defines a map,

$$ {}^{\mathsf{L}}\rho\colon\ \mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_1) \to \mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_2). $$

Moreover, this map is equivariant, in the sense that for all $g_1 \in G_1^\vee$ and all parameters $\phi \in \mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_1)$, we have

$$ {}^{\mathsf{L}}\rho\left({}^{g_1}\phi\right) = {}^{\rho^\vee(g_1)}\left({}^{\mathsf{L}}\rho(\phi)\right). $$

Thus the L-morphism ρ descends to a well-defined map of equivalence classes

$$ {}^{\mathsf{L}}\rho\colon \frac{\mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_1)}{G_1^\vee - \text{conjugation}} \to \frac{\mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_2)}{G_2^\vee - \text{conjugation}}. $$

Next, consider a natural isomorphism of L-morphisms $\rho \overset{\sim}{\Rightarrow} \rho'$, with $\rho, \rho' \colon {}^{\mathsf{L}}G_1 \to {}^{\mathsf{L}}G_2$. We find two maps of parameter spaces,

$$ {}^{\mathsf{L}}\rho, {}^{\mathsf{L}}\rho'\colon\ \mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_1) \to \mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_2), $$

and an element $a \in Z_2^\vee$ such that ${}^{\mathsf{L}}\rho'$ is obtained from ${}^{\mathsf{L}}\rho$ by conjugation by a.

It follows that ${}^{\mathsf{L}}\rho$ and ${}^{\mathsf{L}}\rho'$ induce the *same* map on equivalence classes,

$$ {}^{\mathsf{L}}\rho = {}^{\mathsf{L}}\rho'\colon \frac{\mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_1)}{G_1^\vee - \text{conjugation}} \to \frac{\mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G_2)}{G_2^\vee - \text{conjugation}}. $$

Suppose that an L-group ${}^{\mathsf{L}}G$ is defined up to L-equivalence, and the L-equivalence defined up to unique natural isomorphism. Then the set of equivalence classes of parameters

$$ \frac{\mathrm{Par}(\mathcal{W}_S, {}^{\mathsf{L}}G)}{G^\vee - \text{conjugation}} $$

is uniquely defined up to unique isomorphism.

Remark 5.5. — Refinements of the Langlands parameterization for quasisplit groups suggest that one should look not only at equivalence classes of (Weil or Weil-Deligne) parameters, but also irreducible representations of the component group of the centralizer of a parameter. Or, following Vogan [**74**], one can look at $G^\vee$-equivariant perverse sheaves on a suitable variety of parameters. The fact that conjugation by $a \in Z_2^\vee$ commutes with the conjugation action of $G_2^\vee$ implies that conjugation by a preserves not only the equivalence class of a Weil parameter for ${}^{\mathsf{L}}G_2$, but also the equivalence class of such a refined parameter. If an L-group is defined up to L-equivalence, and the L-equivalence defined up to unique natural isomorphism, then the set of equivalence classes of refined parameters is uniquely defined up to unique isomorphism.

5.3. L-functions. — Let $G^\vee \hookrightarrow {}^{\mathsf{L}}G \twoheadrightarrow \mathrm{Gal}_S$ be an L-group, and $\phi\colon \mathcal{W}_S \to {}^{\mathsf{L}}G$ a Weil parameter (or we may take ϕ to be a Weil-Deligne parameter in the local case). Let (ρ, V) be an L-representation of ${}^{\mathsf{L}}G$. Then

$$\rho \circ \phi\colon \mathcal{W}_S \to \mathrm{GL}(V)$$

is a Weil representation (or Weil-Deligne representation in the local case). As such we obtain an L-function (as defined by Weil and discussed in [**72**, §3.3]),

$$L(\phi, \rho, s) := L(\rho \circ \phi, s).$$

Choosing an additive character ψ as well gives an ε-factor (see [**72**, §3.4], based on work of Langlands and Deligne),

$$\varepsilon(\phi, \rho, \psi, s) := \varepsilon(\rho \circ \phi, \psi, s).$$

In the local integral case $S = \mathrm{Spec}(\mathcal{O})$, we have $\mathcal{W}_S = \langle \mathrm{Fr} \rangle$, and we define the L-functions and ε-factors to be those coming from the unramified representation of $\mathcal{W}_F$ by pullback.

In the setting of Langlands L-groups, a zoo of L-representations arises from complex algebraic representations of $G^\vee$, yielding well-known "standard" L-functions, symmetric power and exterior power L-functions, etc.

But in our very broad setting, we limit our discussion to *adjoint* L-functions, as these play an important role in representation theory and their definition is "internal". Consider any L-morphism $\rho\colon {}^{\mathsf{L}}H \to {}^{\mathsf{L}}G$ of L-groups

$$\begin{tikzcd} H^\vee \arrow[r, hook] \arrow[d, "\rho^\vee"] & {}^{\mathsf{L}}H \arrow[r, two heads] \arrow[d, "{}^{\mathsf{L}}\rho"] & \mathrm{Gal}_S \arrow[d, "="] \\ G^\vee \arrow[r, hook] & {}^{\mathsf{L}}G \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd}$$

For example, we might consider the case where $H^\vee$ is a Levi subgroup of $G^\vee$ (as arises in the Langlands-Shahidi method, [**64**]).

Let $\mathfrak{g}^\vee$ be the complex Lie algebra of $G^\vee$. The homomorphism ${}^{\mathsf{L}}\rho$ followed by conjugation gives an adjoint representation:

$$\mathrm{Ad}_\rho\colon {}^{\mathsf{L}}H \to \mathrm{GL}\left(\mathfrak{g}^\vee\right).$$

Suppose we have a decomposition of $\mathfrak{g}^\vee$ as a representation of ${}^{\mathsf{L}}H$,

$$\mathfrak{g}^\vee = \bigoplus_{i=0}^{h} \mathfrak{g}_i^\vee. \tag{5.1}$$

For example, when $H^\vee$ is a Levi subgroup of a parabolic $P^\vee \subset G^\vee$, we may decompose $\mathfrak{g}^\vee$ into $\mathfrak{h}^\vee$ and the steps in the nilradical of the Lie algebra of $P^\vee$ and its opposite.

A decomposition (5.1) gives representations $\mathrm{Ad}_i\colon {}^{\mathsf{L}}H \to \mathrm{GL}\left(\mathfrak{g}_i^\vee\right)$. When $\phi\colon \mathcal{W}_S \to {}^{\mathsf{L}}H$ is a Weil parameter, we obtain L-functions

$$L(\phi, \mathrm{Ad}_i, s) := L(\mathrm{Ad}_i \circ \phi, s).$$

In particular, when ${}^{\mathsf{L}}H = {}^{\mathsf{L}}G$, and $\rho = \mathrm{Id}$, we write Ad for the adjoint representation of ${}^{\mathsf{L}}G$ on $\mathfrak{g}^\vee$. This yields ***the adjoint L-function*** $L(\phi, \mathrm{Ad}, s)$ for any Weil parameter $\phi\colon \mathscr{W}_S \to {}^{\mathsf{L}}G$. When $H^\vee$ is a Levi subgroup of a parabolic in $G^\vee$, and Ad_i arises from a step in the nilradical of the parabolic, we call $L(\phi, \mathrm{Ad}_i, s)$ a ***Langlands-Shahidi L-function***.

Remark 5.6. — The importance of such L-functions for covering groups is suggested by recent work of D. Szpruch [**70**], who demonstrates that the Langlands-Shahidi construction of L-functions carries over to the metaplectic group. But it is not clear how to extend the Langlands-Shahidi method to other covering groups, where uniqueness of Whittaker models often fails. The thesis work of Gao Fan [**30**] takes some promising steps in this direction. The general machinery of adjoint L-functions also suggests an analog, for covering groups, of the Hiraga-Ichino-Ikeda conjecture [**37**, Conjecture 1.4] on formal degrees (see Ichino-Lapid-Mao [**38**]). It is also supported by the simpler observation that theta correspondence for the metaplectic group $\widetilde{Sp}_{2n}$ provides a definition of adjoint L-functions independently of choices of additive characters.

5.4. The L-group of a cover. — Now we define the L-group of a cover. Let $\tilde{\mathbf{G}}$ be a degree n cover of a quasisplit group $\mathbf{G}$ over S. Fix an injective character $\varepsilon\colon \mu_n \hookrightarrow \mathbb{C}^\times$. Choose a separable closure $\bar{F}/F$, yielding a geometric base point $\bar{s} \to S$ and the absolute Galois group $\mathrm{Gal}_S = \pi_1^{\text{ét}}(S, \bar{s})$.

Recall the constructions of the previous three sections.

- $\tilde{\mathscr{G}}^\vee$ denotes the dual group of $\tilde{\mathbf{G}}$, a local system on $S_{\text{ét}}$ of pinned reductive groups over $\mathbb{Z}$, with center $\tilde{\mathscr{Z}}^\vee$. It is endowed with a homomorphism $\tau_Q\colon \boldsymbol{\mu}_2 \to \tilde{\mathscr{Z}}^\vee$.
- $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ is the gerbe associated to $\tilde{\mathbf{G}}$, a gerbe on $S_{\text{ét}}$ banded by $\tilde{\mathscr{Z}}^\vee = \tilde{\mathscr{Z}}^\vee(\mathbb{C})$.
- $\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_S \twoheadrightarrow \mathrm{Gal}_S$ is the metaGalois group.

Define $\tilde{Z}^\vee = \tilde{\mathscr{Z}}^\vee_{\bar{s}} = \tilde{\mathscr{Z}}^\vee_{\bar{s}}(\mathbb{C})$. This is the center of the complex pinned reductive group $\tilde{G}^\vee = \tilde{\mathscr{G}}^\vee_{\bar{s}} = \tilde{\mathscr{G}}^\vee_{\bar{s}}(\mathbb{C})$. Note that Gal_S acts by pinned automorphisms on $\tilde{G}^\vee$.

Pushing out $\widetilde{\mathrm{Gal}}_S$ via $\tau_Q\colon \mu_2 \to \tilde{Z}^\vee$ defines an L-group,

$$\tilde{Z}^\vee \hookrightarrow (\tau_Q)_* \widetilde{\mathrm{Gal}}_S \twoheadrightarrow \mathrm{Gal}_S\,. \tag{5.2}$$

From Theorem A.7, the fundamental group of the gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ at the base point $\bar{s}$ is an L-group, well-defined up to L-equivalence, and the L-equivalence well-defined up to unique natural isomorphism,

$$\tilde{Z}^\vee \hookrightarrow \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{s}) \twoheadrightarrow \mathrm{Gal}_S\,. \tag{5.3}$$

Remark 5.7. — The extensions (5.2) and (5.3) play the role of the first and second twist in [**78**]. In fact (5.2) is canonically isomorphic to the first twist in the split case; the extension (5.3) may not coincide with the second twist under some circumstances, and the construction here is more general than [**78**] in both cases.

The Baer sum of (5.2) and (5.3) is an L-group which will be called ${}^{\mathsf{L}}\tilde{Z}$,

$$\tilde{Z}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{Z} \twoheadrightarrow \mathrm{Gal}_S\,.$$

The L-group of $\tilde{\mathbf{G}}$ is defined to be the pushout of ${}^{\mathsf{L}}\tilde{Z}$ via the inclusion $\tilde{Z}^\vee \hookrightarrow \tilde{G}^\vee$,

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}_S .$$

More explicitly, this pushout is in the Gal_S-equivariant sense. In other words,

$${}^{\mathsf{L}}\tilde{G} = \frac{\tilde{G}^\vee \rtimes {}^{\mathsf{L}}\tilde{Z}}{\langle (z, z^{-1}) : z \in \tilde{Z}^\vee \rangle},$$

where the semidirect product action ${}^{\mathsf{L}}\tilde{Z} \to \mathrm{Aut}(\tilde{G}^\vee)$ is given by projection ${}^{\mathsf{L}}\tilde{Z} \to \mathrm{Gal}_S$ followed by the action $\mathrm{Gal}_S \to \mathrm{Aut}(\tilde{G}^\vee)$.

By construction, ${}^{\mathsf{L}}\tilde{G}$ is well-defined by $\tilde{\mathbf{G}}$ and ε up to L-equivalence, and the equivalence defined uniquely up to unique natural isomorphism. To describe ${}^{\mathsf{L}}\tilde{G}$ on the nose (not "up to L-equivalence"), one must choose a geometric base point $\bar{z}$ for the gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})$ over $\bar{s}$.

5.5. Well-aligned functoriality. — Our notation here follows that of Section 3.4: consider a well-aligned homomorphism $\tilde{\iota}\colon \tilde{\mathbf{G}}_1 \to \tilde{\mathbf{G}}_2$ of covers, each endowed with Borel subgroup and maximally split maximal torus, i.e., a morphism in the category WAC_S. Fix ε as before. We have constructed gerbes $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1)$ and $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_2)$ associated to $\tilde{\mathbf{G}}_1$ and $\tilde{\mathbf{G}}_2$, banded by $\tilde{\mathscr{Z}}_1^\vee$ and $\tilde{\mathscr{Z}}_2^\vee$, respectively. We have constructed a homomorphism of dual groups $\iota^\vee\colon \tilde{\mathscr{G}}_2^\vee \to \tilde{\mathscr{G}}_1^\vee$ in Section 2.3, which (after taking $\mathbb{C}$-points) restricts to $\iota^\vee\colon \tilde{\mathscr{Z}}_2^\vee \to \tilde{\mathscr{Z}}_1^\vee$. This homomorphism $\iota^\vee$ is compatible with the 2-torsion elements, i.e., $\iota^\vee \circ \tau_{Q_2} = \tau_{Q_1}$. In Section 3.4, we constructed a functor of gerbes $\mathsf{i}\colon \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_2) \to \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1)$, lying over $\iota^\vee\colon \tilde{\mathscr{Z}}_2^\vee \to \tilde{\mathscr{Z}}_1^\vee$.

Define $\tilde{Z}_1^\vee = \tilde{\mathscr{Z}}_{1,\bar{s}}^\vee(\mathbb{C})$ and $\tilde{Z}_2^\vee = \tilde{\mathscr{Z}}_{2,\bar{s}}^\vee(\mathbb{C})$. These are the centers of the complex pinned reductive groups $\tilde{G}_1^\vee = \tilde{\mathscr{G}}_{1,\bar{s}}^\vee(\mathbb{C})$ and $\tilde{G}_2^\vee = \tilde{\mathscr{G}}_{2,\bar{s}}^\vee(\mathbb{C})$.

The compatibility $\iota^\vee \circ \tau_{Q_2} = \tau_{Q_1}$ defines an L-morphism,

$$\begin{tikzcd} \tilde{Z}_2^\vee \arrow[r, hook] \arrow[d] & (\tau_{Q_2})_* \widetilde{\mathrm{Gal}}_S \arrow[r, two heads] \arrow[d] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{Z}_1^\vee \arrow[r, hook] & (\tau_{Q_1})_* \widetilde{\mathrm{Gal}}_S \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd} \tag{5.4}$$

The functor of gerbes $\mathsf{i}\colon \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_2) \to \mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1)$ defines an L-morphism, well-defined up to unique natural isomorphism

$$\begin{tikzcd} \tilde{Z}_2^\vee \arrow[r, hook] \arrow[d] & \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_2), \bar{s}) \arrow[r, two heads] \arrow[d] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{Z}_1^\vee \arrow[r, hook] & \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}_1), \bar{s}) \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd} \tag{5.5}$$

Applying Baer sums to (5.4) and (5.5) yields an L-morphism,

$$\begin{tikzcd} \tilde{Z}_2^\vee \arrow[r, hook] \arrow[d] & {}^{\mathsf{L}}\tilde{Z}_2 \arrow[r, two heads] \arrow[d] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{Z}_1^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{Z}_1 \arrow[r, two heads] & \mathrm{Gal}_S\,. \end{tikzcd}$$

Pushing out yields an L-morphism, well-defined up to natural isomorphism,

$$\begin{tikzcd} \tilde{G}_2^\vee \arrow[r, hook] \arrow[d, "\iota^\vee"] & {}^{\mathsf{L}}\tilde{G}_2 \arrow[r, two heads] \arrow[d, "{}^{\mathsf{L}}\iota"] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{G}_1^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{G}_1 \arrow[r, two heads] & \mathrm{Gal}_S\,. \end{tikzcd}$$

In this way, the construction of the L-group is contravariantly functorial for well-aligned homomorphisms.

5.6. Local-global compatibility. — Suppose that $\gamma\colon F \hookrightarrow F_v$ is the inclusion of a global field F into its localization at a place. Let $\tilde{\mathbf{G}}$ be a degree n cover of a quasisplit group $\mathbf{G}$ over F, and let $\varepsilon\colon \mu_n \hookrightarrow \mathbb{C}^\times$ be an injective character. Let $\bar{F} \hookrightarrow \bar{F}_v$ be an inclusion of separable closures, inducing an inclusion $\mathrm{Gal}_{F_v} \hookrightarrow \mathrm{Gal}_F$ of absolute Galois groups. Write $S = \mathrm{Spec}(F)$ and $S_v = \mathrm{Spec}(F_v)$, and $\bar{s} \to S$ and $\bar{s}_v \to S_v$ for the geometric base points arising from $\bar{F} \hookrightarrow \bar{F}_v$.

Write $\tilde{\mathbf{G}}_v$ for the pullback of $\tilde{\mathbf{G}}$ via $\mathrm{Spec}(F_v) \to \mathrm{Spec}(F)$. Similarly, write Q_v for its Brylinski-Deligne invariant. The results of Section 2.4 identify $\tilde{G}^\vee = \tilde{\mathscr{G}}_{\bar{s}}^\vee(\mathbb{C})$ with the corresponding dual group for $\tilde{\mathbf{G}}_v$ (relative to the separable closure $\bar{F}_v$). Thus we simply write $\tilde{G}^\vee$ for their dual groups and $\tilde{Z}^\vee$ for the centers thereof.

The results of Sections 2.4 and 4.1.4 together provide an L-morphism, unique up to unique natural isomorphism

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, "="] & (\tau_{Q_v})_* {}^{\mathsf{L}}\widetilde{\mathrm{Gal}}_{F_v} \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_{F_v} \arrow[d, hook] \\ \tilde{Z}^\vee \arrow[r, hook] & (\tau_Q)_* {}^{\mathsf{L}}\widetilde{\mathrm{Gal}}_F \arrow[r, two heads] & \mathrm{Gal}_F\,. \end{tikzcd} \tag{5.6}$$

The results of Section 3.6 and following Theorem A.7 give an L-morphism, unique up to unique natural isomorphism

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, "="] & \pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}_v), \bar{s}_v) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_{F_v} \arrow[d, hook] \\ \tilde{Z}^\vee \arrow[r, hook] & \pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{s}) \arrow[r, two heads] & \mathrm{Gal}_F\,. \end{tikzcd} \tag{5.7}$$

Applying Baer sums to (5.6) and (5.7), and pushing out via $\tilde{Z}^\vee \hookrightarrow \tilde{G}^\vee$, yields an L-morphism, unique up to unique natural isomorphism

$$\begin{tikzcd} \tilde{G}^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{L}}\tilde{G}_v \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_{F_v} \arrow[d, hook] \\ \tilde{G}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{G} \arrow[r, two heads] & \mathrm{Gal}_F . \end{tikzcd}$$

In this way, we identify the L-group of $\tilde{\mathbf{G}}_v$ with the pullback of the L-group of $\tilde{\mathbf{G}}$, via the inclusion of Galois groups $\mathrm{Gal}_{F_v} \hookrightarrow \mathrm{Gal}_F$.

5.7. Parabolic subgroups. — Return to a degree n cover $\tilde{\mathbf{G}}$ of a quasisplit group $\mathbf{G}$ over S, and let $\mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup defined over S. As before, consider a Levi decomposition $\mathbf{P} = \mathbf{MN}$ and the resulting cover $\tilde{\mathbf{M}}$. Fix $\varepsilon \colon \mu_n \hookrightarrow \mathbb{C}^\times$.

Compatibility of the dual groups from Section 2.5 gives inclusions

$$\tilde{Z}^\vee \hookrightarrow \tilde{Z}_M^\vee \hookrightarrow \tilde{M}^\vee \hookrightarrow \tilde{G}^\vee,$$

where $\tilde{Z}^\vee$ denotes the center of $\tilde{G}^\vee$, and $\tilde{Z}_M^\vee$ denotes the center of $\tilde{M}^\vee$. As these inclusions are compatible with the 2-torsion elements in $\tilde{Z}^\vee$ and $\tilde{Z}_M^\vee$, we find an L-morphism

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, hook] & (\tau_Q)_* {}^{\mathsf{L}}\widetilde{\mathrm{Gal}}_F \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{Z}_M^\vee \arrow[r, hook] & (\tau_{Q_{\mathbf{M}}})_* {}^{\mathsf{L}}\widetilde{\mathrm{Gal}}_F \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd} \tag{5.8}$$

Section 3.7 provided a functor of gerbes $\mathbf{i} \colon \mathbf{E}_\varepsilon(\tilde{\mathbf{G}}) \to \mathbf{E}_\varepsilon(\tilde{\mathbf{M}})$, lying over $\mathscr{Z}^\vee \hookrightarrow \mathscr{Z}_{\mathbf{M}}^\vee$. This defines an L-morphism of étale fundamental groups, up to unique natural isomorphism

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, hook] & \pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{s}) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{Z}_M^\vee \arrow[r, hook] & \pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{M}}), \bar{s}) \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd} \tag{5.9}$$

Applying Baer sums to (5.8) and (5.9) yields an L-morphism, uniquely defined up to unique natural isomorphism

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, hook] & {}^{\mathsf{L}}\tilde{Z} \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{Z}_M^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{Z}_M \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd}$$

The universal property of pushouts yields an L-morphism

$$\begin{tikzcd} \tilde{M}^\vee \arrow[r, hook] \arrow[d, hook] & {}^{\mathsf{L}}\tilde{M} \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{G}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{G} \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd}$$

This L-morphism is well-defined up to conjugation by $\tilde{Z}_M^\vee$.

As a special case, when $\mathbf{P} = \mathbf{B} = \mathbf{TU}$, we find an L-morphism

$$\begin{tikzcd} \tilde{T}^\vee \arrow[r, hook] \arrow[d, hook] & {}^{\mathsf{L}}\tilde{T} \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{G}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{G} \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd}$$

A ***principal series*** parameter for $\tilde{\mathbf{G}}$ is a Weil parameter $\phi\colon \mathscr{W}_S \to {}^{\mathsf{L}}\tilde{G}$, which factors through a Weil parameter $\mathscr{W}_S \to {}^{\mathsf{L}}\tilde{T}$ via the L-morphism above. By [**58**, Theorem 2], principal series parameters exist, i.e., there exist Weil parameters $\mathscr{W}_S \to {}^{\mathsf{L}}\tilde{T}$.

5.8. The Weyl-group action on the L-group of a cover of torus. — We keep the degree n cover $\tilde{\mathbf{G}}$ of a quasisplit group $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ over S. As before, $\tilde{Z}^\vee = \tilde{\mathscr{Z}}_{\bar{s}}(\mathbb{C})$ and similarly $\tilde{T}^\vee = \tilde{\mathscr{T}}_{\bar{s}}(\mathbb{C})$ and $\tilde{G}^\vee = \tilde{\mathscr{G}}_{\bar{s}}(\mathbb{C})$. Write $\tilde{N}^\vee$ for the normalizer of $\tilde{T}^\vee$ in $\tilde{G}^\vee$. Define $W = \mathscr{W}_{\bar{s}}$, a finite group endowed with Gal_S-action. As a group endowed with Gal_S-action, W is identified with $\tilde{W}^\vee = \tilde{N}^\vee / \tilde{T}^\vee$. Assume in this section (as in Section 2.6) that $\mathbf{T}$ splits over a *cyclic* Galois cover of S.

For convenience, write $\mathscr{O}_{\bar{s}}^\times = \mathscr{G}_{m,\bar{s}}$; thus $\mathscr{O}_{\bar{s}}^\times = \bar{F}^\times$ if F is a local or global field, and $\mathscr{O}_{\bar{s}}^\times$ is endowed with an action of Gal_S.

Choose a geometric base point $\bar{z} = (\mathscr{H}, h, j) \in \mathsf{E}_\varepsilon(\tilde{\mathbf{G}})_{\bar{s}}$. Such a geometric base point defines a geometric base point $\bar{x} = (\mathscr{H}, h) \in \mathsf{E}_\varepsilon(\tilde{\mathbf{T}})_{\bar{s}}$, and a commutative diagram

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, hook] & \pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z}) \arrow[r, two heads] \arrow[d] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{T}^\vee \arrow[r, hook] & \pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x}) \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd}$$

Taking the Baer sum with $(\tau_Q)_* \widetilde{\mathrm{Gal}}_S$, we find an L-morphism

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, hook] & {}^{\mathsf{L}}\tilde{Z}_{\bar{z}} \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{T}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd}$$

Here we place $\bar{z}$ and $\bar{x}$ in subscripts to emphasize the dependence on geometric base point. We must be careful here to discuss L-morphisms "on the nose" and not just up to natural isomorphism.

This L-morphism identifies ${}^{\mathsf{L}}\tilde{T}_{\bar{x}}$ with $({}^{\mathsf{L}}\tilde{Z}_{\bar{z}} \ltimes \tilde{T}^\vee)/\langle (u, u^{-1}) : u \in \tilde{Z}^\vee \rangle$. As described in the previous section, the universal property of pushouts yields an L-morphism,

$$\begin{tikzcd} \tilde{T}^\vee \arrow[r, hook] \arrow[d, hook] & {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{G}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{G}_{\bar{z}} \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd}$$

In this way, we find a chain of subgroups, ${}^{\mathsf{L}}\tilde{Z}_{\bar{z}} \hookrightarrow {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \hookrightarrow {}^{\mathsf{L}}\tilde{G}_{\bar{z}}$ lying over Gal_S. The inclusion ${}^{\mathsf{L}}\tilde{T}_{\bar{x}} \hookrightarrow {}^{\mathsf{L}}\tilde{G}_{\bar{z}}$ is an inclusion of semidirect products,

$${}^{\mathsf{L}}\tilde{T}_{\bar{x}} = \frac{{}^{\mathsf{L}}\tilde{Z}_{\bar{z}} \ltimes \tilde{T}^\vee}{\langle (u, u^{-1}) : u \in \tilde{Z}^\vee \rangle} \hookrightarrow \frac{{}^{\mathsf{L}}\tilde{Z}_{\bar{z}} \ltimes \tilde{G}^\vee}{\langle (u, u^{-1}) : u \in \tilde{Z}^\vee \rangle} = {}^{\mathsf{L}}\tilde{G}_{\bar{z}}.$$

For all $\zeta \in {}^{\mathsf{L}}\tilde{Z}_{\bar{z}}$, $g^\vee \in \tilde{G}^\vee$, we write $[\zeta, g^\vee]$ for the corresponding element of ${}^{\mathsf{L}}\tilde{G}_{\bar{z}}$.

Let $\dot{w} \in \mathscr{N}[S]$ represent an element of the Weyl group $w \in \mathscr{W}[S] = W^{\mathrm{Gal}_S}$. Let $w^\vee$ denote the element of $\tilde{W}^\vee$ corresponding to the element w. There exists a Gal_S-invariant representative $n^\vee \in \tilde{N}^\vee$ for $w^\vee$ (by [**11**, Lemma 6.2]). Conjugation by $n^\vee$ gives an L-morphism $\mathrm{Int}(n^\vee)\colon {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \to {}^{\mathsf{L}}\tilde{T}_{\bar{x}}$.

The Gal_S-invariant representative $n^\vee$ of $w^\vee$ is unique up to multiplication by $(\tilde{T}^\vee)^{\mathrm{Gal}_S}$. Hence the L-morphism $\mathrm{Int}(n^\vee)$ depends only on $w^\vee$, which in turn depends only on w. Therefore we write $\mathrm{Int}(w^\vee)$ instead of $\mathrm{Int}(n^\vee)$, as we describe the L-morphism below

$$\begin{tikzcd} \tilde{T}^\vee \arrow[r, hook] \arrow[d, "\mathrm{Int}(w^\vee)"] & {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \arrow[r, two heads] \arrow[d, "\mathrm{Int}(w^\vee)"] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{T}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \arrow[r, two heads] & \mathrm{Gal}_S . \end{tikzcd} \tag{5.10}$$

Proposition 5.8. — *For all $w \in \mathscr{W}[S]$, and all $\zeta \in {}^{\mathsf{L}}\tilde{Z}_{\bar{z}} \subset {}^{\mathsf{L}}\tilde{T}_{\bar{x}}$, we have* $\mathrm{Int}(w^\vee)\zeta = \zeta$.

Proof. — We compute directly, writing γ for the image of ζ in Gal_S, and $n^\vee$ for a Gal_S-invariant representative of $w^\vee$ in $\tilde{N}^\vee$.

$$\begin{aligned} \mathrm{Int}(w^\vee)\zeta &= \mathrm{Int}(n^\vee)[\zeta, 1] \quad (\text{in the group } {}^{\mathsf{L}}\tilde{G}_{\bar{z}}), \\ &= [1, n^\vee] \cdot [\zeta, 1] \cdot [1, (n^\vee)^{-1}], \\ &= [1, n^\vee] \cdot [\zeta, 1] \cdot [1, (n^\vee)^{-1}] \cdot [\zeta, 1]^{-1}[\zeta, 1], \\ &= [1, n^\vee] \cdot [1, {}^\gamma(n^\vee)^{-1}] \cdot [\zeta, 1], \\ &= [\zeta, 1] = \zeta \quad (\text{since } n^\vee \text{ is } \mathrm{Gal}_S\text{-invariant}). \end{aligned}$$

□

The previous proposition describes the automorphism $\mathrm{Int}(w^\vee)$ of ${}^{\mathsf{L}}\tilde{T}_{\bar{x}}$. For any element $[\zeta, t^\vee] \in {}^{\mathsf{L}}\tilde{T}_{\bar{x}}$, we find that

$$\mathrm{Int}(w^\vee)[\zeta, t^\vee] = [\zeta, \mathrm{Int}(w^\vee)t^\vee].$$

There is another action of the Weyl group on ${}^{\mathsf{L}}\tilde{T}_{\bar{x}}$, arising as in Sections 2.6 and 3.8 from the well-aligned homomorphism $\mathrm{Int}(\dot{w})$

$$\begin{tikzcd} \mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathbf{T}' \arrow[r, two heads] \arrow[d, "\mathrm{Int}(\dot{w})"] & \mathbf{T} \arrow[d, "\mathrm{Int}(w)"] \\ \mathbf{K}_2 \arrow[r, hook] & \mathbf{T}' \arrow[r, two heads] & \mathbf{T}. \end{tikzcd}$$

From Section 5.5, the well-aligned homomorphism defines an L-equivalence lying over $\mathrm{Int}(w)^\vee$, up to natural isomorphism

$$\begin{tikzcd} \tilde{T}^\vee \arrow[r, hook] \arrow[d, "\mathrm{Int}(w)^\vee"] & {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \arrow[r, two heads] \arrow[d, "{}^{\mathsf{L}}\,\mathrm{Int}(\dot{w})"] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{T}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \arrow[r, two heads] & \mathrm{Gal}_S\,. \end{tikzcd} \tag{5.11}$$

From Section 2.6, we know that $\mathrm{Int}(w^\vee) = \mathrm{Int}(w)^\vee$, as automorphisms of the dual torus $\tilde{T}^\vee$. The rest of this section will be devoted to demonstrating that the L-morphism $\mathrm{Int}(w^\vee)$ in (5.10) is naturally isomorphic to the L-morphism ${}^{\mathsf{L}}\,\mathrm{Int}(\dot{w})$ in (5.11). For this, we must describe ${}^{\mathsf{L}}\,\mathrm{Int}(\dot{w})$ in much more detail.

The well-aligned homomorphism $\mathrm{Int}(\dot{w})$ gives an equivalence of gerbes,

$$\mathsf{Int}(\dot{w}) \colon \mathsf{E}_\varepsilon(\tilde{\mathbf{T}}) \to \mathsf{E}_\varepsilon(\tilde{\mathbf{T}}),$$

lying over the homomorphism $\mathrm{Int}(w)^\vee \colon \tilde{\mathscr{T}}^\vee \to \tilde{\mathscr{T}}^\vee$. This equivalence sends the object $\bar{x} = (\mathscr{H}, h)$ to the object $\mathsf{Int}(\dot{w})\bar{x} = ({}^w\mathscr{H}, \dot{w} \circ h)$. The latter object is described in Section 3.8.

Suppose that $\dot{\rho} \colon \bar{x} \to \mathsf{Int}(\dot{w})\bar{x}$ is an isomorphism in the gerbe $\mathsf{E}_\varepsilon(\tilde{\mathbf{T}})$. Then $\mathsf{Int}(\dot{w})$ and the isomorphism $\dot{\rho}$ define an L-equivalence we call $I(\dot{w}, \dot{\rho})$,

$$\begin{tikzcd} \tilde{T}^\vee \arrow[r, hook] \arrow[d, "\mathrm{Int}(w)^\vee"] & \pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x}) \arrow[r, two heads] \arrow[d, "I(\dot{w},\dot{\rho})"] & \mathrm{Gal}_S \arrow[d, "="] \\ \tilde{T}^\vee \arrow[r, hook] & \pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x}) \arrow[r, two heads] & \mathrm{Gal}_S\,. \end{tikzcd}$$

Explicitly, if $\phi \colon \bar{x} \to {}^\gamma\bar{x}$ is an element of $\pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x})$ lying over $\gamma \in \mathrm{Gal}_S$,

$$I(\dot{w}, \dot{\rho})\phi = {}^\gamma\dot{\rho}^{-1} \circ \mathsf{Int}(\dot{w})\phi \circ \dot{\rho}.$$

$I(\dot{w}, \dot{\rho})\phi$ fits into the commutative diagram below

$$\begin{tikzcd} \bar{x} \arrow[r, "I(\dot{w},\dot{\rho})\phi"] \arrow[d, "\dot{\rho}"] & {}^\gamma\bar{x} \arrow[d, "{}^\gamma\dot{\rho}"] \\ \mathsf{Int}(\dot{w})\bar{x} \arrow[r, "\mathsf{Int}(\dot{w})\phi"] & \mathsf{Int}(\dot{w})^\gamma\bar{x}. \end{tikzcd}$$

On the right side of the diagram, we use the fact that $\gamma(\dot{w}) = \dot{w}$, and therefore

$$\mathsf{Int}(\dot{w})^\gamma\bar{x} = {}^\gamma\left(\mathsf{Int}(\dot{w})\bar{x}\right).$$

Proposition 5.9. — *Allowing $\dot{w}$ to vary over representatives for w in $\mathscr{N}[S]$, and allowing $\dot{\rho}$ to vary over isomorphisms from $\bar{x}$ to $\mathsf{Int}(\dot{w})\bar{x}$, the family of L-equivalences $I(\dot{w},\dot{\rho})$ defines an L-equivalence $I(w)$ up to unique natural isomorphism*

$$\begin{tikzcd}
\tilde{T}^\vee \arrow[r, hook] \arrow[d, "\mathrm{Int}(w)^\vee"] & \pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x}) \arrow[r, two heads] \arrow[d, "I(w)"] & \mathrm{Gal}_S \arrow[d, "="] \\
\tilde{T}^\vee \arrow[r, hook] & \pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x}) \arrow[r, two heads] & \mathrm{Gal}_S .
\end{tikzcd}$$

Proof. — We first vary representatives of w. If $\dot{w}, \ddot{w} \in \mathscr{N}[S]$ are two representatives of $w \in \mathscr{W}[S]$, then the unique element $t \in \mathscr{T}[S]$ satisfying $\ddot{w} = \dot{w}t$ determines a natural isomorphism of functors $\mathsf{A}(t)\colon \mathsf{Int}(\ddot{w}) \Rightarrow \mathsf{Int}(\dot{w})$, according to Proposition 3.15.

If $\dot{\rho}\colon \bar{x} \to \mathsf{Int}(\dot{w})\bar{x}$ is an isomorphism, define $\ddot{\rho}\colon \bar{x} \to \mathsf{Int}(\ddot{w})\bar{x}$ by $\ddot{\rho} = \mathsf{A}(t)_{\bar{x}}^{-1} \circ \dot{\rho}$, as in the commutative triangle below

$$\begin{tikzcd}
 & \mathsf{Int}(\ddot{w})\bar{x} \arrow[d, "\mathsf{A}(t)_{\bar{x}}"] \\
\bar{x} \arrow[ur, "\ddot{\rho}"] \arrow[r, "\dot{\rho}"'] & \mathsf{Int}(\dot{w})\bar{x}.
\end{tikzcd}$$

Naturality of $\mathsf{A}(t)$ implies that the following diagram commutes

$$\begin{tikzcd}
\mathsf{Int}(\ddot{w})\bar{x} \arrow[r, "\mathsf{Int}(\ddot{w})\phi"] \arrow[d, "\mathsf{A}(t)_{\bar{x}}"] & \mathsf{Int}(\ddot{w})^\gamma\bar{x} \arrow[d, "\mathsf{A}(t)_{^\gamma\bar{x}} = {^\gamma}\mathsf{A}(t)_{\bar{x}}"] \\
\mathsf{Int}(\dot{w})\bar{x} \arrow[r, "\mathsf{Int}(\dot{w})\phi"] & \mathsf{Int}(\dot{w})^\gamma\bar{x}.
\end{tikzcd}$$

It follows that

$$\begin{aligned}
I(\ddot{w},\ddot{\rho})\phi &= {^\gamma}\ddot{\rho}^{-1} \circ \mathsf{Int}(\ddot{w})\phi \circ \ddot{\rho} \\
&= {^\gamma}\dot{\rho}^{-1} \circ {^\gamma}\mathsf{A}(t)_{\bar{x}} \circ \mathsf{Int}(\ddot{w})\phi \circ \mathsf{A}(t)_{\bar{x}}^{-1} \circ \dot{\rho} \\
&= {^\gamma}\dot{\rho}^{-1} \circ \mathsf{Int}(\dot{w})\phi \circ \dot{\rho} = I(\dot{w},\dot{\rho})\phi.
\end{aligned}$$

Hence $I(\ddot{w},\ddot{\rho}) = I(\dot{w},\dot{\rho})$.

Next, we consider varying the isomorphism $\dot{\rho}$ while keeping $\dot{w}$ fixed. Consider another isomorphism $\breve{\rho}\colon \bar{x} \to \mathsf{Int}(\dot{w})\bar{x}$. There exists a unique $\tau^\vee \in \tilde{T}^\vee$ such that $\breve{\rho} = \dot{\rho} \circ \tau^\vee$. It follows that ${^\gamma}\breve{\rho} = \dot{\rho} \circ \gamma(\tau^\vee)$. Therefore,

$$\begin{aligned}
I(\dot{w},\breve{\rho})\phi &= {^\gamma}\breve{\rho}^{-1} \circ \mathsf{Int}(\dot{w})\phi \circ \breve{\rho} \\
&= \frac{1}{\gamma(\tau^\vee)} \circ {^\gamma}\dot{\rho}^{-1} \circ \mathsf{Int}(\dot{w})\phi \circ \dot{\rho} \circ \tau^\vee \\
&= \frac{\tau^\vee}{\gamma(\tau^\vee)} \circ \left({^\gamma}\dot{\rho}^{-1} \circ \mathsf{Int}(\dot{w})\phi \circ \dot{\rho}\right) \\
&= \frac{\tau^\vee}{\gamma(\tau^\vee)} \circ I(\dot{w},\dot{\rho})\phi.
\end{aligned}$$

Therefore, $\tau^\vee$ defines a natural isomorphism from the L-equivalence $I(\dot{w}, \dot{\rho})$ to the L-equivalence $I(\dot{w}, \breve{\rho})$.

Hence, as $\dot{w}$ and $\dot{\rho}$ vary, the L-equivalence $I(\dot{w}, \dot{\rho})$ varies by uniquely-determined natural isomorphism. Thus we find an L-equivalence $I(w)$ defined uniquely up to unique natural isomorphism. □

The following lemma will allow us to study $I(\dot{w}, \dot{\rho})$, in an important special case. As we work in the geometric fiber throughout, recall that $\tilde{T}^\vee = \tilde{\mathscr{T}}^\vee_{\bar{s}}$, and define $H = \mathscr{H}_{\bar{s}}$, $\mathrm{Spl}(\mathscr{D}_{Q,n}) = \mathscr{S}\!pl(\mathscr{D}_{Q,n})_{\bar{s}}$, and $\hat{T} = \hat{\mathscr{T}}_{\bar{s}}$. Define $\Phi = \Phi_{\bar{s}}$ for the set of absolute roots. Write $Y_{Q,n} = \mathscr{Y}_{Q,n,\bar{s}}$, so that

$$\hat{T} = \mathrm{Hom}(Y_{Q,n}, \mathscr{O}_{\bar{s}}^\times), \quad \tilde{T}^\vee = \mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times).$$

Lemma 5.10. — *Suppose that $\dot{w} = r_\beta(ee^-e)$ arises from a relative root β, for which 2β is not a root, as in Theorem 3.17. Then there exists a pair $(\dot{\rho}, \dot{a})$, with $\dot{\rho}$ an isomorphism $\bar{x} \xrightarrow{\sim} \mathbf{Int}(\dot{w})\bar{x}$ in the gerbe $\sqrt[n]{\mathscr{S}\!pl(\mathscr{D}_{Q,n})}$, and $\dot{a} \in H = \mathscr{H}_{\bar{s}}$, such that $\dot{\rho}(\dot{a}) = \dot{a}$.*

Proof. — Since we work in the geometric fiber, there exists a splitting $\sigma \in \mathrm{Spl}(\mathscr{D}_{Q,n})$ such that σ is aligned with $\dot{w}$ (aligned as in Theorem 3.17). Moreover, $h\colon H \to \mathrm{Spl}(\mathscr{D}_{Q,n})$ is surjective (Definition 1.2(3) implies that $\hat{T} \xrightarrow{n} \hat{T}$ is surjective), so there exists $a \in H$ such that $h(a) = \sigma$.

Let $\rho\colon \bar{x} \to \mathbf{Int}(\dot{w})\bar{x}$ be any isomorphism. We find that $\rho\colon H \to {}^wH$ satisfies

$$\sigma = h(a) = \dot{w}(h(\rho(a))).$$

There exists a unique $r \in \hat{T}$ such that $\rho(a) = r * a$. From this it follows that

$$\sigma = \dot{w}(h(r * a)) = \dot{w}(r^n * \sigma) = w^{-1}(r)^n * \dot{w}(\sigma).$$

Let $\{\alpha_1, \ldots, \alpha_\ell\}$ denote the roots in Φ which restrict to the relative root β. Define $\tilde{\beta}\colon \mathscr{O}_{\bar{s}}^\times \to \hat{T} = \mathrm{Hom}(Y_{Q,n}, \mathscr{O}_{\bar{s}}^\times)$ by

$$\tilde{\beta}(u)(y) = \prod_{i=1}^{\ell} u^{\langle \tilde{\alpha}_i, y \rangle}.$$

Since $w = \prod w_{\alpha_i}$ (orthogonal root reflections), $w(\tilde{\beta}(u)) = \tilde{\beta}(u)^{-1}$.

Theorem 3.17 gives the formula $\dot{w}(\sigma) = \tilde{\beta}(\pm 1) * \sigma$, where the sign is -1 if and only if q_β and $n/2$ are both odd. Hence

$$\sigma = w^{-1}(r^n) * \tilde{\beta}(\pm 1) * \sigma.$$

Since $w(\tilde{\beta}(\pm 1)) = \tilde{\beta}(\pm 1)$, we have

$$r^n = \tilde{\beta}(\pm 1).$$

Let $\xi \in \mathscr{O}_{\bar{s}}^\times$ be an element satisfying $\xi^n = \pm 1$ (the same sign as above). Here we use the fact that n is coprime to the characteristics of all residue fields of prime ideals in S. Thus $\tilde{\beta}(\xi)/r \in \hat{T}_{[n]}$. Choose a square root $\sqrt{\xi}$ of ξ in $\mathscr{O}_{\bar{s}}^\times$. (In characteristic 2, we have $-1 = 1$, and we may take $\xi = 1$ and $\sqrt{\xi} = 1$.)

View $\tilde{\beta}(\xi)/r$ as an automorphism of $\mathbf{Int}(\dot{w})\bar{x}$ in $\sqrt[n]{\mathscr{Spl}(\mathscr{D}_{Q,n})}$ and define

$$\dot{\rho} = (\tilde{\beta}(\xi)/r) \circ \rho, \quad \dot{a} = \tilde{\beta}(\sqrt{\xi}) * a.$$

Then we find

$$\begin{aligned}
\dot{\rho}(\dot{a}) &= \dot{\rho}\left(\tilde{\beta}(\sqrt{\xi}) * a\right) \\
&= \tilde{\beta}(\sqrt{\xi})^{-1} * \dot{\rho}(a) \\
&= \tilde{\beta}(\sqrt{\xi})^{-1} * \frac{\tilde{\beta}(\xi)}{r} * \rho(a) \\
&= \tilde{\beta}(\sqrt{\xi})^{-1} \cdot \tilde{\beta}(\xi) * a \\
&= \tilde{\beta}(\sqrt{\xi})^{-1} \cdot \tilde{\beta}(\xi) \cdot \tilde{\beta}(\sqrt{\xi})^{-1} * \dot{a} = \dot{a}.
\end{aligned}$$

□

The functor of gerbes $\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}) \to \mathbf{E}_\varepsilon(\tilde{\mathbf{T}})$ sends $\bar{z}$ to $\bar{x}$ and induces an L-morphism,

$$\begin{tikzcd}
\tilde{Z}^\vee \arrow[r, hook] \arrow[d, hook] & \pi_1(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z}) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_S \arrow[d, "="] \\
\tilde{T}^\vee \arrow[r, hook] & \pi_1(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x}) \arrow[r, two heads] & \mathrm{Gal}_S .
\end{tikzcd}$$

In this way we view $\pi_1(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z})$ as a subset of $\pi_1(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x})$.

Proposition 5.11. — *Suppose that $\phi\colon \bar{z} \to {}^\gamma\bar{z}$ is an element of $\pi_1(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z})$ lying over $\gamma \in \mathrm{Gal}_S$. Suppose that $\dot{w} = r_\beta(ee^-e)$ arises from a relative root β as in the previous lemma, and choose $\dot{\rho}\colon \bar{x} \to \mathbf{Int}(\dot{w})\bar{x}$ and $\dot{a} \in \mathscr{H}_{\bar{s}}$ as in that lemma. Then*

$$I(\dot{w}, \dot{\rho})\phi = \phi.$$

Proof. — We continue to work in the geometric fiber throughout this proof. Note that $\dot{a} \in H$ and every point of H can be expressed as $\hat{\tau} * \dot{a}$ for some $\hat{\tau} \in \hat{T}$. Since $\dot{\rho}(\dot{a}) = \dot{a}$, we have for all $\hat{\tau} \in \hat{T}$,

$$\dot{\rho}(\hat{\tau} * \dot{a}) = w^{-1}(\hat{\tau}) * \dot{a}.$$

Since $\dot{\rho}$ intertwines h and $\dot{w} \circ h$, we find that

$$h(\dot{a}) = \dot{w}(h(\dot{\rho}(\dot{a}))) = \dot{w}(h(\dot{a})).$$

Now we consider an isomorphism $\phi\colon \bar{z} \to {}^\gamma\bar{z}$ in the gerbe $\mathbf{E}_\varepsilon(\tilde{\mathbf{G}})$. Such an isomorphism can be written as a contraction $\phi = t^\vee \wedge \phi_0$, where $\phi_0\colon \bar{x} \to {}^\gamma\bar{x}$ is an isomorphism in $\sqrt[n]{\mathscr{Spl}(\mathscr{D}_{Q,n})}$ and $t^\vee \in \tilde{T}^\vee = \mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times)$.

The object ${}^\gamma\bar{z}$ is equal to $({}^\gamma\mathscr{H}, {}^\gamma h, {}^\gamma j)$, where ${}^\gamma\mathscr{H}$ is the torsor with the same underlying sheaf of sets $\mathscr{H}$, and torsor structure given by

$$\hat{\tau} *_\gamma a = \gamma^{-1}(\hat{\tau}) * a \text{ for all } a \in H.$$

The isomorphism ϕ_0, viewed as a morphism of $\hat{T}$-torsors from H to ${}^\gamma H$ lying over $\gamma\colon \hat{T} \to \hat{T}$, satisfies $\phi_0(\dot{a}) = \hat{f} * \dot{a}$ for some $\hat{f} \in \hat{T}$. For all $\hat{\tau} \in \hat{T}$,

$$\phi_0(\hat{\tau} * \dot{a}) = \gamma^{-1}(\hat{\tau}) * \hat{f} * \dot{a}.$$

The isomorphism $\phi = t^\vee \wedge \phi_0$ is therefore determined by the element

$$t^\vee \wedge \hat{f} \in \tilde{T}^\vee \wedge_{\hat{T}_{[n]}} \hat{T} = \frac{\tilde{T}^\vee \times \hat{T}}{\langle (\varepsilon(\zeta), \zeta^{-1}) : \zeta \in \hat{T}_{[n]} \rangle}.$$

Recall that $j \colon p_* \mathscr{H} \to \mu_* \mathscr{W}\!hit$ is an isomorphism in the gerbe $\mathbf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}})$. Since ϕ is an isomorphism in $\mathbf{E}_\varepsilon(\tilde{\mathbf{G}})$, the following diagram commutes in $\mathbf{E}_\varepsilon^{\mathrm{sc}}(\tilde{\mathbf{T}})$

$$\begin{array}{ccc} p_* \mathscr{H} & \xrightarrow{j} & \mu_* \mathscr{W}\!hit \\ \downarrow{\scriptstyle p_*\phi} & & \downarrow{\scriptstyle \gamma} \\ p_* {}^\gamma \mathscr{H} & \xrightarrow{{}^\gamma j} & \mu_* \mathscr{W}\!hit. \end{array}$$

The commutativity of this diagram is equivalent to the fact that

$$t^\vee \wedge \hat{f} \in \operatorname{Ker}\left(p \colon \tilde{T}^\vee \wedge_{\hat{T}_{[n]}} \hat{T} \to \tilde{T}^\vee_{\mathrm{sc}} \wedge_{\hat{T}_{\mathrm{sc},[n]}} \hat{T}_{\mathrm{sc}} \right).$$

Note that $\hat{T} = \operatorname{Hom}(Y_{Q,n}, \bar{F}^\times)$ and $\tilde{T}^\vee = \operatorname{Hom}(Y_{Q,n}, \mathbb{C}^\times)$, and they are contracted over $\hat{T}_{[n]} = \operatorname{Hom}(Y_{Q,n}, \mu_n)$ via ε. The map p is given by restricting homomorphisms from $Y_{Q,n}$ to $Y_{Q,n}^{\mathrm{sc}}$. Recall that $\{\alpha_1, \dots, \alpha_\ell\}$ are the roots restricting to the given relative root β.

Since $t^\vee \wedge \hat{f} \in \operatorname{Ker}(p)$, there exist $\zeta_i \in \mu_n$ for all $1 \leq i \leq \ell$, such that

$$t^\vee(\tilde{\alpha}_i^\vee) = \varepsilon(\zeta_i)^{-1}, \quad \hat{f}(\tilde{\alpha}_i^\vee) = \zeta_i.$$

For each $y \in Y_{Q,n}$ the simple root reflection acts by $w_{\alpha_i}(y) = y - \langle \tilde{\alpha}_i, y \rangle \tilde{\alpha}_i^\vee$. Therefore, for all $1 \leq i \leq \ell$, we have

$$w_{\alpha_i}(t^\vee) = t^\vee \cdot \tilde{\alpha}_i(\varepsilon(\zeta_i))^{-1}, \quad w_{\alpha_i}(\hat{f}) = \hat{f} \cdot \tilde{\alpha}_i(\zeta_i).$$

Here $\tilde{\alpha}_i(\zeta_i)$ denotes the element of $\hat{T}_{[n]} = \operatorname{Hom}(Y_{Q,n}, \mu_n)$ given by

$$\tilde{\alpha}_i(\zeta_i)(y) = \zeta_i^{\langle \tilde{\alpha}_i, y \rangle},$$

and similarly $\tilde{\alpha}_i(\varepsilon(\zeta_i))$ is an element of $\tilde{T}^\vee_{[n]}$.

Hence, for all $1 \leq i \leq \ell$, we have

$$w_{\alpha_i}(t^\vee \wedge \hat{f}) = t^\vee \cdot \tilde{\alpha}_i(\varepsilon(\zeta_i))^{-1} \wedge \hat{f} \cdot \tilde{\alpha}_i(\zeta_i) = t^\vee \wedge \hat{f}.$$

As $w = \prod_{i=1}^\ell w_{\alpha_i}$, we have $w(t^\vee \wedge \hat{f}) = t^\vee \wedge \hat{f}$.

Now we compute

$$\begin{aligned} I(\dot{w}, \dot{\rho})\phi_0(\dot{a}) &= \left[{}^\gamma \dot{\rho}^{-1} \circ \mathbf{Int}(\dot{w}) \phi_0 \circ \dot{\rho} \right](\dot{a}) \\ &= \dot{\rho}^{-1}\left(\phi_0\left(\dot{\rho}(\dot{a}) \right) \right) \\ &= \dot{\rho}^{-1}\left(\phi_0(\dot{a}) \right) \\ &= \dot{\rho}^{-1}\left(\hat{f} * \dot{a} \right) \\ &= w(\hat{f}) * \dot{a} \\ &= w(\hat{f}) \cdot \hat{f}^{-1} * \phi_0(\dot{a}). \end{aligned}$$

Therefore,

$$I(\dot{w}, \dot{\rho})\phi_0 = \frac{w(\hat{f})}{\hat{f}} * \phi_0.$$

Since $\phi = t^\vee \wedge \phi_0$, we compute

$$\begin{aligned} I(\dot{w}, \dot{\rho})\phi &= I(\dot{w}, \dot{\rho})(t^\vee \wedge \phi_0) \\ &= w(t^\vee) \wedge I(\dot{w}, \dot{\rho})\phi_0 \\ &= w(t^\vee) \wedge \left(\frac{w(\hat{f})}{\hat{f}} * \phi_0 \right) \\ &= t^\vee \frac{w(t^\vee)}{t^\vee} \wedge \left(\frac{w(\hat{f})}{\hat{f}} * \phi_0 \right) \\ &= t^\vee \wedge \phi_0 = \phi. \end{aligned}$$

The last step follows from the identity $w(t^\vee) \wedge w(\hat{f}) = t^\vee \wedge \hat{f}$. □

Remark 5.12. — The notation may be difficult to follow in the above argument. In the contraction of torsors $\tilde{T}^\vee \wedge_{\hat{T}_{[n]}} \mathrm{Hom}(\bar{x}, {}^\gamma\bar{x})$, we have

$$\tau_1^\vee \wedge (\hat{u}_1 * \phi_0) = \tau_2^\vee \wedge (\hat{u}_2 * \phi_0)$$

whenever $\tau_1^\vee \wedge u_1 = \tau_2^\vee \wedge u_2$ in $\tilde{T}^\vee \wedge_{\hat{T}_{[n]}} \hat{T}$.

Theorem 5.13. — *For any $w \in W^{\mathrm{Gal}_S}$, the L-morphism ${}^{\mathsf{L}}\mathrm{Int}(\dot{w}) \colon {}^{\mathsf{L}}\tilde{T} \to {}^{\mathsf{L}}\tilde{T}$ is naturally isomorphic to the L-morphism $\mathrm{Int}(w^\vee)$.*

Proof. — Suppose first that $\dot{w} = r_\beta(ee^-e)$, and $\dot{\rho}$ is chosen as in the previous proposition. Then the L-morphism

$$I(\dot{w}, \dot{\rho}) \colon \pi_1(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x}) \to \pi_1(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{x})$$

restricts to the identity on $\pi_1(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z})$.

The Weyl action on $(\tau_Q)_* \widetilde{\mathrm{Gal}}_F$ is also trivial, and so we find that the L-morphism

$${}^{\mathsf{L}}\mathrm{Int}(\dot{w}) \colon {}^{\mathsf{L}}\tilde{T}_{\bar{x}} \to {}^{\mathsf{L}}\tilde{T}_{\bar{x}}$$

restricts to the identity on ${}^{\mathsf{L}}\tilde{Z}_{\bar{z}}$. (Strictly speaking, ${}^{\mathsf{L}}\mathrm{Int}(\dot{w})$ is only defined up to natural isomorphism; but fixing $\dot{\rho}$ defines ${}^{\mathsf{L}}\mathrm{Int}(\dot{w})$ on the nose.)

The semidirect product decomposition,

$${}^{\mathsf{L}}\tilde{T}_{\bar{x}} = \frac{{}^{\mathsf{L}}\tilde{Z}_{\bar{z}} \ltimes \tilde{T}^\vee}{\langle (u, u^{-1}) : u \in \tilde{z}^\vee \rangle},$$

implies that there is a unique L-morphism from ${}^{\mathsf{L}}\tilde{T}_{\bar{x}}$ to itself which acts as the identity on ${}^{\mathsf{L}}\tilde{Z}_{\bar{z}}$ and acts via $\mathrm{Int}(w^\vee) = \mathrm{Int}(w)^\vee$ on $\tilde{T}^\vee$. Proposition 5.8 implies that

$${}^{\mathsf{L}}\mathrm{Int}(\dot{w}) \text{ is naturally isomorphic to } \mathrm{Int}(w^\vee).$$

Since W^{Gal_S} is generated by relative root reflections (for relative roots β such that 2β is not a relative root), we find that ${}^{\mathsf{L}}\,\mathrm{Int}(\dot{w})$ is naturally isomorphic to $\mathrm{Int}(w^\vee)$ for all $w \in W^{\mathrm{Gal}_S}$. □

PART II
GENUINE REPRESENTATIONS

In this part, we review basic facts about the ε-genuine representations (admissible and automorphic) of covering groups $\tilde{G}$. Most of these facts are corollaries of previous works by many authors, consolidated and organized for convenience. However, a new feature of representation theory is the organization of representations into pouches based on their character. As we review features of the ε-genuine representation theory, we introduce corresponding features of L-groups.

6. Local fields

Let F be a local field, and let $\mathbf{G}$ be a quasisplit reductive group over F. Let n be a positive integer and let $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ be a degree n cover of $\mathbf{G}$ over F (Definition 1.2). Fix an injective character $\varepsilon\colon \mu_n \hookrightarrow \mathbb{C}^\times$.

The cover $\tilde{\mathbf{G}}$ yields a short exact sequence

$$\mathbf{K}_2(F) \hookrightarrow \mathbf{G}'(F) \twoheadrightarrow \mathbf{G}(F). \tag{6.1}$$

Define $G = \mathbf{G}(F)$. The Hilbert symbol provides a homomorphism $\mathrm{Hilb}_n\colon \mathbf{K}_2(F) \to \mu_n$. (When $F \cong \mathbb{C}$, the Hilbert symbol is trivial; otherwise the Hilbert symbol is surjective). Pushing out (6.1) via Hilb_n yields a central extension of groups

$$\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G. \tag{6.2}$$

As a topological group, $\tilde{G}$ is described in [**18**, Construction 10.3]. When F is nonarchimedean, $\tilde{G}$ is a totally disconnected locally compact group. When F is archimedean, $\tilde{G}$ is a real-reductive group in the sense of Harish-Chandra. When $F \cong \mathbb{C}$, the Hilbert symbol is trivial, and the extension $\tilde{G} \to G$ splits canonically.

Let K be a maximal compact subgroup of G, and let $\tilde{K}$ denote its preimage in $\tilde{G}$. Then $\tilde{K}$ is a maximal compact subgroup of $\tilde{G}$. Define $\mathbf{\Pi}(\tilde{K})$ to be the set of equivalence classes of continuous irreducible finite-dimensional representations of $\tilde{K}$ on complex vector spaces (the ***irreps*** of $\tilde{K}$).

Definition 6.1. — Suppose that F is nonarchimedean. An ***admissible representation*** of $\tilde{G}$ is a pair (π, V) where V is a complex vector space and $\pi\colon \tilde{G} \to \mathrm{GL}(V)$ is a group homomorphism, such that

1. For all $[\chi] \in \mathbf{\Pi}(\tilde{K})$, the $(\tilde{K}, [\chi])$-isotypic subspace $V_{[\chi]}$ is finite-dimensional;

2. As a representation of $\tilde{K}$, $V = \bigoplus_{[\chi] \in \mathbf{\Pi}(\tilde{K})} V_{[\chi]}$.

When F is archimedean, write $\mathfrak{g}$ for the complexified Lie algebra of $\tilde{G}$; this is naturally identified with the complexified Lie algebra of G.

Definition 6.2. — Suppose that F is archimedean. An ***admissible representation*** of $\tilde{G}$ will mean an admissible $(\mathfrak{g}, \tilde{K})$-module V. In particular, V decomposes as a direct sum of finite-dimensional $\tilde{K}$-isotypic representations, $V = \bigoplus_{[\chi] \in \mathbf{\Pi}(\tilde{K})} V_{[\chi]}$ as in the nonarchimedean case.

In both nonarchimedean and archimedean cases, an admissible representation V of $\tilde{G}$ is called ε-***genuine*** if for all $\zeta \in \mu_n$ and all $v \in V$, we have $\pi(\zeta)v = \varepsilon(\zeta) \cdot v$. Note that $\mu_n \subset \tilde{K}$, so an admissible representation is ε-genuine if and only if all of its $\tilde{K}$-isotypic components are ε-genuine. Define $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$ (or $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F)$ in case of confusion) to be the set of equivalence classes of irreducible admissible ε-genuine representations of $\tilde{G}$.

In the nonarchimedean case, write $\mathscr{H}_\varepsilon(\tilde{G})$ for the convolution algebra

$$\mathscr{H}_\varepsilon(\tilde{G}) = \{f \in C_c^\infty(\tilde{G}) : f(\zeta \tilde{g}) = \varepsilon(\zeta) \cdot f(\tilde{g}) \text{ for all } \zeta \in \mu_n, \tilde{g} \in \tilde{G}\}.$$

Here we fix a Haar measure on $\tilde{G}$, and convolution is given by

$$[f_1 * f_2](g) = \int_G f_1(x) \cdot f_2(x^{-1}g) dx.$$

As f_1 and f_2 are "ε-genuine functions," the integrand is a well-defined function on $G = \tilde{G}/\mu_n$.

In the archimedean case, write $\mathscr{H}_\varepsilon(\tilde{G})$ for the convolution algebra of ε-genuine, left and right $\tilde{K}$-finite distributions on $\tilde{G}$ supported on $\tilde{K}$. In both cases, we have a faithful functor from the category of admissible ε-genuine representations of $\tilde{G}$ to the category of (nondegenerate) $\mathscr{H}_\varepsilon(\tilde{G})$-modules.

6.1. Unitary, discrete series, and tempered representations

Definition 6.3. — A ***unitary representation*** of $\tilde{G}$ is a pair $(\pi, \hat{V})$, where $\hat{V}$ is a separable Hilbert space, and $\pi \colon \tilde{G} \to U(\hat{V})$ is an action of $\tilde{G}$ on $\hat{V}$ by unitary transformations, such that the resulting map $\tilde{G} \times \hat{V} \to \hat{V}$ is continuous.

A unitary representation $(\pi, \hat{V})$ of $\tilde{G}$ is called ***irreducible*** if the only closed $\tilde{G}$-invariant subspaces of $\hat{V}$ are 0 and $\hat{V}$. Write $\mathbf{\Pi}_\varepsilon^{\text{unit}}(\tilde{\mathbf{G}})$ for the set of equivalence classes of irreducible ε-genuine unitary representations of $\tilde{G}$.

The following theorem combines a few fundamental results, essentially due to Harish-Chandra. See **[75]** for proofs in the (more difficult) archimedean case.

Theorem 6.4. — *If $(\pi, \hat{V})$ is an irreducible unitary representation of $\tilde{G}$, then the subspace $V \subset \hat{V}$ of $\tilde{K}$-finite vectors inherits the structure of an irreducible admissible representation of $\tilde{G}$. This defines an injective function,*

$$\mathbf{\Pi}_\varepsilon^{\text{unit}}(\tilde{\mathbf{G}}) \hookrightarrow \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}).$$

Definition 6.5. — Suppose that $(\pi, \hat{V})$ is an irreducible unitary ε-genuine representation of $\tilde{G}$. For all $v_1, v_2 \in \hat{V}$, the ***matrix coefficient*** $m_{v_1,v_2} \colon \tilde{G} \to \mathbb{C}$ is the function

$$m_{v_1,v_2}(g) = \langle v_1, \pi(g) v_2 \rangle.$$

We say that $(\pi, \hat{V})$ is a ***discrete series*** (respectively, ***tempered***) representation if for all $\tilde{K}$-finite vectors $v_1, v_2 \in \hat{V}$,

$$m_{v_1,v_2} \in L^2(\tilde{G}/Z(\tilde{G})), \quad (\text{resp., } L^{2+\varepsilon}(\tilde{G}/Z(\tilde{G})) \text{ for all } \varepsilon > 0).$$

Write $\mathbf{\Pi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}})$ (respectively $\mathbf{\Pi}_\varepsilon^{\mathrm{temp}}(\tilde{\mathbf{G}})$) for the set of equivalence classes of discrete series (resp. tempered) ε-genuine unitary representations of $\tilde{G}$.

In this way, we organize the irreducible admissible ε-genuine representations of $\tilde{G}$ in a nested fashion.

$$\mathbf{\Pi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}}) \subset \mathbf{\Pi}_\varepsilon^{\mathrm{temp}}(\tilde{\mathbf{G}}) \subset \mathbf{\Pi}_\varepsilon^{\mathrm{unit}}(\tilde{\mathbf{G}}) \subset \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}).$$

This mirrors the familiar "uncovered" case, where there are inclusions,

$$\mathbf{\Pi}^{\mathrm{disc}}(\mathbf{G}) \subset \mathbf{\Pi}^{\mathrm{temp}}(\mathbf{G}) \subset \mathbf{\Pi}^{\mathrm{unit}}(\mathbf{G}) \subset \mathbf{\Pi}(\mathbf{G}).$$

6.2. Tori. — In previous papers [**77**] [**80**], we have studied the ε-genuine representations of covers of tori over local and global fields. Here we review the main results over local fields. Let $\mathbf{T}$ be a torus over a local field F, and let $\tilde{\mathbf{T}}$ be a degree n cover of $\mathbf{T}$. The resulting central extension

$$\mu_n \hookrightarrow \tilde{T} \twoheadrightarrow T$$

is a "Heisenberg" type group, and its irreducible representations are therefore determined by their central character. Let $Z(\tilde{T})$ denote the center of $\tilde{T}$. For any $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$, let $\chi_{[\pi]}$ be its central character. Then we have

$$\chi_{[\pi]} \in \mathrm{Hom}_\varepsilon(Z(\tilde{T}), \mathbb{C}^\times),$$

the set of ε-genuine continuous homomorphisms.

Theorem 6.6. — *The map $[\pi] \to \chi_{[\pi]}$ gives a bijection $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}) \xrightarrow{\sim} \mathrm{Hom}_\varepsilon(Z(\tilde{T}), \mathbb{C}^\times)$.*

Proof. — A proof of this analog of the Stone von-Neumann theorem can be found in [**77**, Theorem 3.1] and [**1**, Proposition 2.2] (for the archimedean case). □

Let Y be the cocharacter lattice of $\mathbf{T}$. Let Q be the first Brylinski-Deligne invariant of the cover $\tilde{\mathbf{T}}$, and define

$$Y_{Q,n} = \{y \in Y : n^{-1} B_Q(y, y') \in \mathbb{Z} \text{ for all } y' \in Y\},$$

as in [**77**]. Write $\mathbf{T}_{Q,n}$ for the F-torus with cocharacter lattice $Y_{Q,n}$. The inclusion $Y_{Q,n} \hookrightarrow Y$ corresponds to an isogeny $\mathbf{T}_{Q,n} \to \mathbf{T}$, and we define $C^\dagger(T) = \mathrm{Im}(\mathbf{T}_{Q,n}(F) \to \mathbf{T}(F))$. Define $C(\tilde{T})$ for the preimage of $C^\dagger(T)$ in $\tilde{T}$. The group $C(\tilde{T})$ is called the ***central core*** of $\tilde{T}$.

The following result is contained in [**77**, Theorem 1.3].

Theorem 6.7. — *$C(\tilde{T})$ is a finite-index subgroup of $Z(\tilde{T})$. If $\mathbf{T}$ is split then $C(\tilde{T}) = Z(\tilde{T})$.*

Let $\tilde{\mathbf{T}}_{Q,n}$ be the pullback of the cover $\tilde{\mathbf{T}}$ to $\mathbf{T}_{Q,n}$. Then $\tilde{T}_{Q,n}$ is abelian and $C(\tilde{T}) = \operatorname{Im}(\tilde{T}_{Q,n} \to \tilde{T})$. If $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$, then define $\omega_\pi \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_{Q,n})$ to be the pullback of the central character of π to $\tilde{T}_{Q,n}$. The character ω_π is called the ***central core character*** of $[\pi]$.

Definition 6.8. — If $[\pi_1], [\pi_2] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$, we say that $[\pi_1]$ and $[\pi_2]$ belong to the same ***pouch*** if they share the same central core character. In other words, pouches for covers of tori are the fibers of the central core character map

$$\omega \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_{Q,n}).$$

If $\mathbf{T}$ is split, then the pouches are singletons. The image of ω is a subject for another paper.

On the L-group side, note that the map $\tilde{\mathbf{T}}_{Q,n} \to \tilde{\mathbf{T}}$ is well-aligned. Hence it corresponds to an L-morphism,

$$\begin{array}{ccccc} \tilde{T}^\vee & \hookrightarrow & {}^{\mathsf{L}}\tilde{T} & \twoheadrightarrow & \mathrm{Gal}_F \\ \downarrow & & \downarrow & & \downarrow{=} \\ \tilde{T}^\vee_{Q,n} & \hookrightarrow & {}^{\mathsf{L}}\tilde{T}_{Q,n} & \twoheadrightarrow & \mathrm{Gal}_F . \end{array}$$

Here the dual groups $\tilde{T}^\vee$ and $\tilde{T}^\vee_{Q,n}$ are both equal to $\operatorname{Hom}(Y_{Q,n}, \mathbb{C}^\times)$, the homomorphism $\tilde{T}^\vee \to \tilde{T}^\vee_{Q,n}$ is the identity, and the above diagram is an L-isomorphism. Hence there is a natural bijection of Weil parameters,

$${}^{\mathsf{L}}\omega \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_{Q,n}).$$

6.3. Central core character. — An element $g \in G$ is called ***regular semisimple*** if it is semisimple and the neutral component of its centralizer $\mathbf{Z}^\circ_{\mathbf{G}}(g)$ is a maximal torus in $\mathbf{G}$. In this case, g is certainly an element of this maximal torus since it centralizes the maximal torus. Write G^{reg} for the locus of regular semisimple elements of $G = \mathbf{G}(F)$; this is a dense open subset of G. Define $\tilde{G}^{\mathrm{reg}}$ to be the preimage of G^{reg} in $\tilde{G}$.

Let $\mathbf{S} = \mathbf{Z}^\circ(\mathbf{G})$ be the neutral component of the center of $\mathbf{G}$, i.e., the maximal torus contained in the center of $\mathbf{G}$. Let $\tilde{\mathbf{S}}$ the degree n cover of $\mathbf{S}$ obtained by pulling back $\tilde{\mathbf{G}}$. If $\mathbf{T}$ is any maximal torus of $\mathbf{G}$, then $\mathbf{S} \subset \mathbf{T}$. The cocharacter lattice of $\mathbf{S}$ is naturally identified with the sublattice Y^W of Weyl-invariants in Y. Define $Y^W_{Q,n} = Y_{Q,n} \cap Y^W$, and let $\mathbf{S}_{Q,n}$ be the F-torus with cocharacter lattice $Y^W_{Q,n}$. The inclusion $Y^W_{Q,n} \hookrightarrow Y^W$ corresponds to an F-isogeny $\mathbf{S}_{Q,n} \to \mathbf{S}$, and we define

$$C^\dagger(G) = \operatorname{Im}(\mathbf{S}_{Q,n}(F) \to \mathbf{S}(F)).$$

Definition 6.9. — The ***central core*** of $\tilde{G}$ is the preimage $C(\tilde{G})$ of $C^\dagger(G)$ in $\tilde{G}$.

Alternatively, pulling back $\tilde{\mathbf{G}}$ yields a cover $\tilde{\mathbf{S}}_{Q,n}$ and a continuous map $\tilde{S}_{Q,n} \to \tilde{S}$. The central core of $\tilde{G}$ is the image:

$$C(\tilde{G}) = \operatorname{Im}\left(\tilde{S}_{Q,n} \to \tilde{S}\right).$$

Note that

$$Y_{Q,n}^W \subset (Y^W)_{Q,n} = \{y_1 \in Y^W : B_Q(y_1, y_2) \in n\mathbb{Z} \text{ for all } y_2 \in Y^W\}.$$

It follows that $C^\dagger(G) \subset C^\dagger(S)$, and so $C(\tilde{G}) \subset C(\tilde{S})$.

Proposition 6.10. — *The central core $C(\tilde{G})$ is a finite-index subgroup of $Z(\tilde{G})$.*

Proof. — Suppose that $g \in G^{\mathrm{reg}}$, and let $\mathbf{T} = \mathbf{Z}_{\mathbf{G}}^\circ(g)$ denote the neutral component of its centralizer (a maximal F-torus in $\mathbf{G}$). Since $Y_{Q,n}^W \subset Y_{Q,n} \subset Y$, we find that

$$C(\tilde{G}) \subset Z(\tilde{T}),$$

by Theorem 6.7. Hence if $\tilde{g} \in \tilde{G}^{\mathrm{reg}}$ is a lift of g, then $\tilde{g} \in \tilde{T}$, and so $C(\tilde{G})$ commutes with $\tilde{g}$. Since $C(\tilde{G})$ commutes with every regular semisimple element $\tilde{g} \in \tilde{G}^{\mathrm{reg}}$, the density of $\tilde{G}^{\mathrm{reg}}$ in $\tilde{G}$ implies that $C(\tilde{G}) \subset Z(\tilde{G})$.

Since the isogeny $\mathbf{S}_{Q,n} \to \mathbf{S}$ has degree coprime to the characteristic of F, the image $C^\dagger(G)$ has finite index in S. We find a chain of subgroups of G and finite-index inclusions $C^\dagger(G) \subset S \subset Z(G)$. Writing $Z^\dagger(G)$ for the image of $Z(\tilde{G})$ in G, we have $C^\dagger(G) \subset Z^\dagger(G) \subset Z(G)$ and therefore $C^\dagger(G)$ has finite index in $Z^\dagger(G)$. Thus $C(\tilde{G})$ has finite index in $Z(\tilde{G})$. □

Definition 6.11. — For any $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$, let ω_π be the pullback of the central character of $[\pi]$ to $\tilde{S}_{Q,n}$. The character $\omega_\pi \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{S}}_{Q,n})$ is called the ***central core character*** of $[\pi]$.

In this way, the central core character provides a map,

$$\omega \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}) \to \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{S}}_{Q,n}).$$

Observe that the map $\tilde{\mathbf{S}}_{Q,n} \to \tilde{\mathbf{G}}$ is well-aligned. Therefore, it induces an L-morphism,

$$\begin{tikzcd}
\tilde{G}^\vee \arrow[r, hook] \arrow[d] & {}^{\mathsf{L}}\tilde{G} \arrow[r, two heads] \arrow[d] & \mathrm{Gal}_F \arrow[d, "="] \\
\tilde{S}_{Q,n}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{S}_{Q,n} \arrow[r, two heads] & \mathrm{Gal}_F .
\end{tikzcd}$$

Hence there is a natural map of Weil parameters, generalizing the case of tori,

$${}^{\mathsf{L}}\omega \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}) \to \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{S}}_{Q,n}).$$

6.4. Characters and pouches. — Just as we partitioned $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$ into pouches, we can partition $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$ into pouches using the character distribution (occasionally assuming $\mathrm{char}(F) = 0$). When $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$, the character of $[\pi]$ is a conjugation-invariant distribution $\mathrm{Tr}[\pi]$ on $\tilde{G}$. Let Θ_π denote the restriction of $[\pi]$ to $\tilde{G}^{\mathrm{reg}}$, i.e., to test functions supported in the regular locus. The following theorem in harmonic analysis is a result of many extensions to a deep result of Harish-Chandra.

Theorem 6.12. — *Θ_π coincides, after choice of Haar measure, with a smooth (real analytic in the archimedean case, locally constant in the nonarchimedean case) function on $\tilde{G}^{\mathrm{reg}}$. In other words, there exists a smooth function $\Theta_\pi \colon \tilde{G}^{\mathrm{reg}} \to \mathbb{C}$ such that for every test function $f \in C_c^\infty(\tilde{G}^{\mathrm{reg}})$ we have*

$$\mathrm{Tr}[\pi](f) = \int_{\tilde{G}^{\mathrm{reg}}} \Theta_\pi(x) f(x) dx.$$

Proof. — The key ideas are in the work of Harish-Chandra for semisimple groups at first; see [**36**]. The idea extends in a straightforward way for real-reductive groups in the Harish-Chandra class (including our covering groups); see [**75**, §8.4.1] for a treatment. For a large class of p-adic groups, see Clozel [**20**]; the extension to covering groups requires no new ideas. In characteristic p, we refer to Gopal Prasad's appendix to [**4**] and the treatment of Bushnell and Henniart [**19**, Corollary A.11]. □

Theorem 6.13. — *Suppose that F has characteristic zero. If $[\pi_1], [\pi_2] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$ and $\Theta_{\pi_1} = \Theta_{\pi_2}$, then $[\pi_1] = [\pi_2]$. In other words, the character distribution, restricted to the regular locus, determines the isomorphism class of a genuine irreducible representation.*

Proof. — The character $\mathrm{Tr}[\pi]$ determines the equivalence class $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$. This result follows from [**41**, §X.1, Theorem 10.6] when F is archimedean, and from [**9**, Corollary 2.20] when F is nonarchimedean. In characteristic zero, it is known [**47**, Corollaire 4.3.3] that the character distribution $\mathrm{Tr}[\pi]$ is locally integrable, and determined by the function Θ_π: for all test functions $f \in C_c^\infty(\tilde{G})$ (not necessarily supported on the regular locus!),

$$\mathrm{Tr}[\pi](f) = \int_{\tilde{G}^{\mathrm{reg}}} \Theta_\pi(x) f(x) dx. \qquad \square$$

Remark 6.14. — If F has characteristic $p \neq 0$, the local integrability of characters is an open question (see Rodier [**61**] and Lemaire [**46**] for two cases where the characters are proven locally integrable). For this reason, in characteristic p, we do not know whether each equivalence class $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$ is determined by Θ_π.

Definition 6.15. — Suppose that $\tilde{x} \in \tilde{G}^{\mathrm{reg}}$, mapping to $x \in G^{\mathrm{reg}}$. Let $\mathbf{T} = \mathbf{Z}_{\mathbf{G}}^\circ(x)$ be the maximal torus centralizing x. We say that $\tilde{x}$ is ***genuinely supportive*** if $\tilde{x} \in Z(\tilde{T})$. Let $\tilde{G}^{\mathrm{greg}}$ denote the set of genuinely supportive elements of $\tilde{G}^{\mathrm{reg}}$.

Proposition 6.16. — *If $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$ then Θ_π is supported on $\tilde{G}^{\mathrm{greg}}$.*

Proof. — Θ_π is a conjugation-invariant function on $\tilde{G}^{\mathrm{reg}}$. If $\tilde{x} \in \tilde{G}^{\mathrm{reg}}$ and $\tilde{x}$ is not genuinely supportive, then there exists $t \in T \subset Z_G(x)$ and $\zeta \in \mu_n$ such that

$$\mathrm{Int}(t)\tilde{x} = \zeta \cdot \tilde{x}, \text{ and } 1 \neq \zeta.$$

It follows that $\Theta_\pi(\tilde{x}) = \Theta_\pi(\mathrm{Int}(t)\tilde{x}) = \varepsilon(\zeta) \cdot \Theta_\pi(\tilde{x})$. Hence $\Theta_\pi(x) = 0$. □

For purposes of parameterization, it seems appropriate to place irreducible representations in the same "pouch" if their characters agree on a locus called the "regular core".

Definition 6.17. — Suppose that $\tilde{x} \in G^{\mathrm{reg}}$, mapping to $x \in G^{\mathrm{reg}}$. Let $\mathbf{T} = \mathbf{Z}_{\mathbf{G}}^{\circ}(x)$. We say that $\tilde{x}$ is in the ***regular core*** of G if $\tilde{x} \in C(\tilde{T})$, the central core of $\tilde{T}$. Define $\tilde{G}^{\mathrm{creg}}$ to be the regular core of $\tilde{G}$.

Write G^{creg} for the image of $\tilde{G}^{\mathrm{creg}}$ in G. Since $C(\tilde{T}) \subset Z(\tilde{T})$, we find that

$$C(\tilde{G}) \cap \tilde{G}^{\mathrm{reg}} \subset \tilde{G}^{\mathrm{creg}} \subset \tilde{G}^{\mathrm{greg}}.$$

The following definition generalizes Definition 6.8 from tori to reductive groups.

Definition 6.18. — Suppose that $[\pi_1], [\pi_2] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$. We say that $[\pi_1]$ and $[\pi_2]$ belong to the same ***pouch*** if Θ_{π_1} and Θ_{π_2} (functions on $\tilde{G}^{\mathrm{reg}}$ by Theorem 6.12) coincide on $\tilde{G}^{\mathrm{creg}}$.

Lemma 6.19. — *Assume that F has characteristic zero. Suppose that $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$. Then there exists $\tilde{g} \in \tilde{G}^{\mathrm{creg}}$ such that $\Theta_\pi(\tilde{g}) \neq 0$.*

Proof. — From [**47**, Théorème 4.3.2], in characteristic zero, there exists an open neighborhood $0 \in U \subset \mathfrak{g}_F$ in the Lie algebra of $\mathbf{G}$ on which the local character expansion of Harish-Chandra and Howe is valid:

$$\Theta_\pi(\exp(X)) = \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi) \cdot \hat{\mu}_{\mathcal{O}}(X), \text{ for all } X \in U \cap \mathfrak{g}_F^{\mathrm{reg}}.$$

Here $\mathfrak{g}^{\mathrm{reg}}$ denotes the regular semisimple locus. We use the fact that the cover $\tilde{G} \to G$ splits over an open subgroup, and any two such splittings coincide on a (possibly smaller) open subgroup. This allows us to interpret $\exp(X)$ as an element of $\tilde{G}$, when X is close to $0 \in \mathfrak{g}_F$.

On the other hand, for any maximal torus $\mathbf{T} \subset \mathbf{G}$, the subgroup $C(\tilde{T})$ is open of finite index in $\tilde{T}$. There are finitely many $\mathbf{G}(F)$-conjugacy classes of F-rational maximal tori in $\mathbf{G}$. It follows that for a sufficiently small open neighborhood $0 \in U \subset \mathfrak{g}_F$, $\exp(U \cap \mathfrak{g}^{\mathrm{reg}}) \subset \tilde{G}^{\mathrm{creg}}$.

Hence, if $\Theta_\pi(\tilde{g}) = 0$ for all $\tilde{G}^{\mathrm{creg}}$, then we find that $c_{\mathcal{O}}(\pi) = 0$ for all nilpotent orbits $\mathcal{O}$. But this cannot occur; the character Θ_π of an irreducible admissible representation cannot vanish in a neighborhood of the identity. □

Proposition 6.20. — *Suppose that F has characteristic zero. If $[\pi_1]$ and $[\pi_2]$ belong to the same pouch in $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$, then the central core character of $[\pi_1]$ equals the central core character of $[\pi_2]$.*

Proof. — Fix an element $\tilde{g} \in \tilde{G}^{\mathrm{creg}}$ for which $\Theta_{\pi_1}(\tilde{g}) \neq 0$. If $\tilde{c} \in C(\tilde{G})$, we claim that $\tilde{c} \cdot \tilde{g} \in \tilde{G}^{\mathrm{creg}}$ as well. Indeed, since $\tilde{c} \in C(\tilde{G})$ and $C(\tilde{G}) \subset Z(\tilde{G})$, we find that $\mathbf{T} = \mathbf{Z}_{\mathbf{G}}^{\circ}(\tilde{g}) = \mathbf{Z}_{\mathbf{G}}^{\circ}(\tilde{c} \cdot \tilde{g})$. In particular, $\tilde{c} \cdot \tilde{g} \in \tilde{G}^{\mathrm{reg}}$. Since $\tilde{g} \in C(\tilde{T})$ and $\tilde{c} \in C(\tilde{G}) \subset C(\tilde{T})$, we find that $\tilde{c} \cdot \tilde{g} \in C(\tilde{T})$. Thus $\tilde{c} \cdot \tilde{g} \in \tilde{G}^{\mathrm{creg}}$ as claimed.

We now find that

$$\omega_{\pi_1}(\tilde{c}) \cdot \Theta_{\pi_1}(\tilde{g}) = \Theta_{\pi_1}(\tilde{c} \cdot \tilde{g}) = \Theta_{\pi_2}(\tilde{c} \cdot \tilde{g}) = \omega_{\pi_2}(\tilde{c}) \cdot \Theta_{\pi_2}(\tilde{g}) = \omega_{\pi_2}(\tilde{c}) \cdot \Theta_{\pi_1}(\tilde{g}).$$

Hence $\omega_{\pi_1}(\tilde{c}) = \omega_{\pi_2}(\tilde{c})$. □

6.5. Twisting. — Let $\xi \colon G \to \mathbb{C}^\times$ be a smooth homomorphism. Then ξ pulls back to a smooth character of $\tilde{G}$ as well. When $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$, the twist $[\xi \cdot \pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$ as well. This gives an action

$$\mathrm{Tw} \colon \mathrm{Hom}(G, \mathbb{C}^\times) \times \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}) \to \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}).$$

It is possible that $[\pi] = [\xi \cdot \pi]$, even if ξ is nontrivial.

Definition 6.21. — Let $\xi \colon G \to \mathbb{C}^\times$ be a smooth homomorphism. We say that ξ is ***genuinely trivial*** if $\xi(g) = 1$ for all $g \in G^{\mathrm{greg}}$. We say that ξ is core-trivial if $\xi(g) = 1$ for all $g \in G^{\mathrm{creg}}$.

Proposition 6.22. — *Assume that F has characteristic zero. Suppose that $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$ and $\xi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ is genuinely trivial. Then $[\pi] = [\xi \cdot \pi]$.*

Proof. — If $\xi(g) = 1$ for all $g \in G^{\mathrm{greg}}$, then

$$\Theta_\pi(\tilde{g}) = \Theta_\pi(\tilde{g}) \cdot \xi(g) = \Theta_{\xi \cdot \pi}(\tilde{g})$$

for all $\tilde{g} \in \tilde{G}^{\mathrm{greg}}$ mapping to $g \in G^{\mathrm{greg}}$. But Θ_π and $\Theta_{\xi \cdot \pi}$ are supported on $\tilde{G}^{\mathrm{greg}}$, and so we find that $\Theta_\pi = \Theta_{\xi \cdot \pi}$. Hence $[\pi] = [\xi \cdot \pi]$ by Theorem 6.13. □

Proposition 6.23. — *Suppose that $[\pi] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}})$ and $\xi \in \mathrm{Hom}(G, \mathbb{C}^\times)$ is core-trivial. Then $[\pi]$ and $[\xi \cdot \pi]$ belong to the same pouch.*

Proof. — This follows almost from the definition; if $\xi(g) = 1$ for all $g \in G^{\mathrm{creg}}$ then

$$\Theta_\pi(\tilde{g}) = \Theta_\pi(\tilde{g}) \cdot \xi(g) = \Theta_{\xi \cdot \pi}(\tilde{g})$$

for all $\tilde{g} \in \tilde{G}^{\mathrm{creg}}$ mapping to $g \in G^{\mathrm{creg}}$. Hence $[\pi]$ and $[\xi \cdot \pi]$ belong to the same pouch. □

On the dual group side, let $Z^\vee = \mathrm{Hom}(Y/Y^{\mathrm{sc}}, \mathbb{C}^\times)$ denote the center of the dual group $G^\vee$. If $\eta \in H^1(\mathcal{W}_F, Z^\vee)$, then one may associate a character $\xi \in \mathrm{Hom}(G, \mathbb{C}^\times)$. Let $\tilde{Z}^\vee = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times)$ denote the center of the dual group $\tilde{G}^\vee$. Then restriction gives a Galois-equivariant homomorphism $Z^\vee \to \tilde{Z}^\vee$, and thus a homomorphism

$$H^1(\mathcal{W}_F, Z^\vee) \to H^1(\mathcal{W}_F, \tilde{Z}^\vee).$$

Twisting a ${}^{\mathsf{L}}\tilde{G}$-valued Weil parameter by a $\tilde{Z}^\vee$-valued cocycle, we find a map

$${}^{\mathsf{L}}\mathrm{Tw} \colon H^1(\mathcal{W}_F, Z^\vee) \times \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}).$$

We might expect $\eta \in H^1(\mathcal{W}_F, Z^\vee)$ to correspond to a core-trivial character $\xi \in \mathrm{Hom}(G, \mathbb{C}^\times)$, if its image in $H^1(\mathcal{W}_F, \tilde{Z}^\vee)$ is trivial.

6.6. Langlands classification. — Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$, defined over F. Let $\mathbf{A}$ be a maximal F-split torus in $\mathbf{G}$, whose centralizer $\mathbf{T}$ is a maximal torus in $\mathbf{B}$. Let $\Phi_F = \Phi_F(\mathbf{G}, \mathbf{A})$ be the resulting set of relative roots, and Φ_F^+ the positive roots determined by $\mathbf{B}$, and Δ_F the simple positive roots therein. Our treatment of the Langlands classification closely follows Ban and Jantzen [7], who extend the Langlands classification to coverings of p-adic groups.

A ***standard parabolic*** will mean an F-parabolic subgroup of $\mathbf{G}$ containing $\mathbf{B}$. If $\mathbf{P} = \mathbf{MN}$ is a standard parabolic subgroup of $\mathbf{G}$, write $\Delta_{F,M} \subset \Delta_F$ for the corresponding set of simple roots. Let $\mathbf{A}_M$ be the maximal split torus contained in the center of $\mathbf{M}$, a subgroup of $\mathbf{A}$. Define $\mathfrak{a}_M^* = \mathrm{Hom}_F(\mathbf{M}, \mathbf{G}_m) \otimes_{\mathbb{Z}} \mathbb{R}$. Restriction of a character from $\mathbf{M}$ to $\mathbf{A}_M$ extends to an identification of real vector spaces,

$$\mathfrak{a}_M^* = \mathrm{Hom}_F(\mathbf{M}, \mathbf{G}_m) \otimes_{\mathbb{Z}} \mathbb{R} \equiv \mathrm{Hom}(\mathbf{A}_M, \mathbf{G}_m) \otimes_{\mathbb{Z}} \mathbb{R}.$$

An inclusion of standard Levi subgroups $\mathbf{L} \subset \mathbf{M}$ gives an inclusion $\mathbf{A}_M \subset \mathbf{A}_L$. The identifications above yield $\mathbb{R}$-linear maps,

$$r_M^L \colon \mathfrak{a}_L^* \twoheadrightarrow \mathfrak{a}_M^*, \quad i_M^L \colon \mathfrak{a}_M^* \hookrightarrow \mathfrak{a}_L^*.$$

Write $\mathfrak{a}^* = \mathrm{Hom}(\mathbf{A}, \mathbf{G}_m) \otimes \mathbb{R}$. We find an injective linear map $i_M \colon \mathfrak{a}_M^* \hookrightarrow \mathfrak{a}^*$ and a surjective linear map $r_M \colon \mathfrak{a}^* \twoheadrightarrow \mathfrak{a}_M^*$. The relative Weyl group $W_F = W_F(\mathbf{G}, \mathbf{A})$ acts on $\mathfrak{a}^*$, and there is a unique-up-to-scaling W_F-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}^*$.

If $\nu = \xi \otimes r \in \mathfrak{a}_M^*$ is a basic tensor, and $\tilde{m} \in \tilde{M}$ lies over $m \in M$, define

$$\exp(\nu)\tilde{m} = |\xi(m)|^r.$$

This extends to give a homomorphism

$$\exp \colon \mathfrak{a}_M^* \to \mathrm{Hom}(\tilde{M}, \mathbb{R}_{>0}^\times).$$

Elements $\exp(\nu) \in \mathrm{Hom}(\tilde{M}, \mathbb{R}_{>0}^\times)$ are ***unramified characters*** of $\tilde{M}$.

Within the real vector space $\mathfrak{a}_M^*$, define an open cone,

$$(\mathfrak{a}_M)_+^* = \{\nu \in \mathfrak{a}_M^* : \langle \nu, r_M(\alpha) \rangle > 0 \text{ for all } \alpha \in \Delta_F - \Delta_{F,M}\}.$$

Now, fix a standard parabolic subgroup $\mathbf{P} = \mathbf{MN}$. The central extension $\mathbf{K}_2 \hookrightarrow \mathbf{G}' \twoheadrightarrow \mathbf{G}$ splits uniquely over $\mathbf{N}$, and thus we have a canonical splitting $N \hookrightarrow \tilde{N}$. Pulling back $\tilde{\mathbf{G}}$ to covers $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{M}}$, we find a semidirect product structure

$$\tilde{P} = \tilde{M} \ltimes N.$$

The adjoint action of $\tilde{M}$ on N has the same modular character δ_P as the adjoint action of M on N.

Suppose that $(\sigma, \hat{H})$ is an ε-genuine irreducible unitary representation of $\tilde{M}$. Choose $\nu \in \mathfrak{a}_M^*$, so $\exp(\nu) \colon \tilde{M} \to \mathbb{R}_{>0}^\times$ is an unramified character. Let H be the subset of $\hat{H}$

consisting of *smooth vectors.* Define

$$I^\infty(\tilde{\mathbf{P}},\sigma,\nu) = \{f\colon \tilde{G} \to H : f \text{ is smooth, and}$$
$$f(n\tilde{m}\tilde{g}) = \delta_P(\tilde{m})^{1/2}\sigma(\tilde{m})\exp(\nu)(\tilde{m}) \text{ for all } n \in N, \tilde{m} \in \tilde{M}, \tilde{g} \in \tilde{G}\}.$$

Let $I(\tilde{\mathbf{P}},\sigma,\nu)$ denote the space of K-finite vectors therein. We find that $I(\tilde{\mathbf{P}},\sigma,\nu)$ is an ε-genuine admissible representation of $\tilde{G}$. This is a result of Ban and Jantzen [**7**, §3] in the nonarchimedean case, and follows from Borel and Wallach [**14**, §III.3.2] in the archimedean case.

Theorem 6.24 **(Langlands classification).** — *If σ is an ε-genuine irreducible* tempered *representation of $\tilde{M}$ and $\nu \in (\mathfrak{a}_M)^*_+$, then the admissible representation $I(\tilde{\mathbf{P}},\sigma,\nu)$ has a unique irreducible quotient, which we call $J(\tilde{\mathbf{P}},\sigma,\nu)$. This $J(\tilde{\mathbf{P}},\sigma,\nu)$ is an ε-genuine irreducible admissible representation of $\tilde{G}$.*

If π is an ε-genuine irreducible admissible representation of $\tilde{G}$, then there exists a unique triple $(\mathbf{P},[\sigma],\nu)$, where $\mathbf{P} = \mathbf{MN}$ is a standard parabolic subgroup of $\mathbf{G}$, $[\sigma]$ is an equivalence class of ε-genuine irreducible tempered representations of $\tilde{M}$, and $\nu \in (\mathfrak{a}_M)^*_+$ such that π is equivalent to $J(\tilde{\mathbf{P}},\sigma,\nu)$.

Proof. — For covering groups in the nonarchimedean case, this is the main result of Ban and Jantzen [**7**, Theorem 1.1]. Their restrictions on the characteristic of F are unnecessary here as we work with covers $\tilde{\mathbf{G}}$ coming from central extensions of $\mathbf{G}$ by $\mathbf{K}_2$ rather than general topological extensions. Ban and Jantzen rely on the uniqueness of a splitting of $\tilde{N} \to N$, whereas we can exploit the uniqueness of a splitting of $\mathbf{N}' \to \mathbf{N}$ in the algebraic category.

Indeed, for the canonical splitting $\mathrm{can}\colon N \hookrightarrow \tilde{N}$, and every $m \in M$, we have

$$\mathrm{Int}(m)^{-1} \circ \mathrm{can} \circ \mathrm{Int}(m) = \mathrm{can},$$

since both arise from *algebraic* splittings of $\mathbf{K}_2 \hookrightarrow \mathbf{N}' \twoheadrightarrow \mathbf{N}$. This suffices to demonstrate that [**7**, Lemma 2.7, Proposition 2.11] hold in arbitrary characteristic, for our class of covers.

Similarly, if $\mathbf{U} \subset \mathbf{G}$ is *any* unipotent subgroup, and $a \in G$ centralizes $\mathbf{U}$, then the canonical splitting $\mathrm{can}\colon U \hookrightarrow \tilde{U}$ satisfies

$$\mathrm{Int}(a) \circ \mathrm{can} = \mathrm{can}\,.$$

This suffices to demonstrate that [**7**, Lemma 2.13] holds in arbitrary characteristic, for our class of covers. Thus the main results of [**7**] hold in arbitrary characteristic, for the nonarchimedean covering groups considered in this paper.

In the archimedean case, we refer to Borel and Wallach [**14**, IV, Corollary 4.6, Theorem 4.11] (based on earlier work of Harish-Chandra, Casselman, Miličić and Langlands). $\square$

On the L-group side, a Weil parameter $\phi \in \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}})$ is called tempered if its image is bounded, i.e., if the closure of its image in ${}^{\mathsf{L}}\tilde{G}$ is compact. Recall from Section 5.7

that a parabolic subgroup $\mathbf{P} = \mathbf{MN}$ yields an embedding of L-groups,

$$\begin{tikzcd} \tilde{M}^\vee \arrow[r, hook] \arrow[d, hook] & {}^{\mathsf{L}}\tilde{M} \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_F \arrow[d, "="] \\ \tilde{G}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{G} \arrow[r, two heads] & \mathrm{Gal}_F . \end{tikzcd}$$

This arises by pushing out an embedding of L-groups,

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, hook] & {}^{\mathsf{L}}\tilde{Z} \arrow[r, two heads] \arrow[d, hook] & \mathrm{Gal}_F \arrow[d, "="] \\ \tilde{Z}_M^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{Z}_M \arrow[r, two heads] & \mathrm{Gal}_F . \end{tikzcd}$$

Within $\tilde{Z}_M^\vee$, let $\tilde{Z}_{M,\mathrm{hyp}}^\vee$ be the subgroup of ***hyperbolic*** elements—those whose eigenvalues are positive real numbers. The following result gives a Langlands classification for parameters.

Proposition 6.25. — *For all $\phi \in \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}})$, there exists a unique triple $(\mathbf{P}, \phi_M, \eta)$, where $\mathbf{P} = \mathbf{MN}$ is a standard parabolic subgroup of $\mathbf{G}$, $\phi_M \in \mathbf{\Phi}_\varepsilon^{\mathrm{temp}}(\tilde{\mathbf{M}})$ is a tempered Weil parameter, and $\eta \in H^1(\mathscr{W}_F, \tilde{Z}_{M,\mathrm{hyp}}^\vee)$, such that ϕ is equivalent to $\eta * \phi_M$ (the parameter obtained by twisting ϕ_M by η, then including ${}^{\mathsf{L}}\tilde{M} \hookrightarrow {}^{\mathsf{L}}\tilde{G}$).*

Proof. — The proof in the uncovered setting, sketched in various sources, and treated thoroughly by Silberger and Zink in [**66**] carries over without significant changes. □

7. Spherical representations

Now suppose that $\mathcal{O}$ the ring of integers in a nonarchimedean local field F, with residue field $\mathbb{F}_q$. Let $\mathbf{G}$ be a quasisplit reductive group over $\mathcal{O}$. Let $\mathbf{A}$ be a maximal $\mathcal{O}$-split torus in $\mathbf{G}$, and let $\mathbf{T}$ be the centralizer of $\mathbf{A}$; then $\mathbf{T}$ is a maximal $\mathcal{O}$-torus in $\mathbf{G}$. Let $\mathbf{B} = \mathbf{TU}$ be an $\mathcal{O}$-Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. For convenience, let E/F be an unramified extension over which $\mathbf{T}$ splits, and let Fr be a generator of $\mathrm{Gal}(E/F)$ which corresponds to the geometric Frobenius automorphism over $\mathbb{F}_q$. Let Y be the cocharacter lattice of $\mathbf{T}$, viewed as a $\mathbb{Z}[\mathrm{Fr}]$-module. Let X be the character lattice of $\mathbf{T}$. Let W be the (absolute) Weyl group of $\mathbf{G}$ with respect to $\mathbf{T}$. Note that Y^{Fr} is the group of cocharacters of the split torus $\mathbf{A}$.

Let $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ be a degree n cover of $\mathbf{G}$ over $\mathcal{O}$; such covers are studied in [**79**]. The requirement that $\tilde{\mathbf{G}}$ is defined over $\mathcal{O}$ implies that n divides $q-1$. This "tameness" gives an exact sequence,

$$\mathbf{K}_2(\mathcal{O}) \hookrightarrow \mathbf{K}_2(F) \xrightarrow{\mathrm{Hilb}_n} \mu_n.$$

Define $G^\circ = \mathbf{G}(\mathcal{O})$ and $G = \mathbf{G}(F)$. Pushing out $\mathbf{G}'(F)$ via Hilb_n, the central extension $\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G$ is endowed with a splitting over the hyperspecial maximal compact subgroup $G^\circ \subset G$. We view G° as a subgroup of $\tilde{G}$ throughout. Similarly, we write

$T^\circ = \mathbf{T}(\mathscr{O})$ and $T = \mathbf{T}(F)$, and we find a central extension $\mu_n \hookrightarrow \tilde{T} \twoheadrightarrow T$ endowed with a splitting over T°. Define $W^\circ = N_{G^\circ}(T^\circ)/T^\circ$. The inclusion $N_{G^\circ}(T^\circ) \hookrightarrow \mathbf{N_G}(\mathbf{T})(\mathscr{O})$ yields an identification of W° with W^{Fr}.

Fix an injective character $\varepsilon\colon \mu_n \to \mathbb{C}^\times$ as usual. Suppose that (π, V) is an irreducible admissible ε-genuine representation of $\tilde{G}$. We say that (π, V) is ***spherical*** if the space of G°-fixed vectors V^{G° is nonzero. Define $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathscr{O})$ to be the set of equivalence classes of spherical irreducible admissible ε-genuine representations of $\tilde{G}$.

7.1. Hecke algebras. — The ε-genuine ***spherical Hecke algebra***, denoted $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ is the space of compactly supported functions $f\colon \tilde{G} \to \mathbb{C}$ which satisfy

$$f(\zeta k_1 \tilde{g} k_2) = \varepsilon(\zeta) \cdot f(\tilde{g}) \text{ for all } \zeta \in \mu_n, k_1, k_2 \in G^\circ, \tilde{g} \in \tilde{G}.$$

Here, we fix the Haar measure on $\tilde{G}$ for which G° has measure 1. Then $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ is a subalgebra of $\mathscr{H}_\varepsilon(\tilde{G})$,

$$\mathscr{H}_\varepsilon(\tilde{G}, G^\circ) = \mathrm{char}(G^\circ) * \mathscr{H}_\varepsilon(\tilde{G}) * \mathrm{char}(G^\circ).$$

When (π, V) is a smooth ε-genuine spherical representation of $\tilde{G}$, the set V^{G° of G°-fixed vectors forms a module over the Hecke algebra $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$. In this way, $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathscr{O})$ is identified with the set of equivalence classes of irreducible $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$-modules.

The ***support*** of $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ is

$$\{\tilde{g} \in \tilde{G} : f(\tilde{g}) \neq 0 \text{ for some } f \in \mathscr{H}_\varepsilon(\tilde{G}, G^\circ)\}.$$

Similarly, we may define the spherical Hecke algebra $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)$, its support, and the set of its irreducible modules $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/\mathscr{O})$.

The support of $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ is a set of the form $\mu_n G^\circ \Lambda G^\circ$, for some subset $\Lambda \subset \tilde{G}$. The Cartan decomposition gives

$$G = G^\circ Y^{\mathrm{Fr}} G^\circ;$$

in the sense that for any choice of uniformizing parameter ϖ, every element of G can be expressed as $k_1 y(\varpi) k_2$ for some $y \in Y^{\mathrm{Fr}}$. In this way, the support of $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ corresponds to a W°-stable subset of Y^{Fr}. We call this subset of Y^{Fr} the ***combinatorial support*** of $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$. Similarly, the support of $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)$ is determined by its combinatorial support, a subset of Y^{Fr}.

The supports of $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ and $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)$ are given by Mcnamara [**51**, Theorem 10.1], and by Wen Wei Li in [**48**, §3.2].

Proposition 7.1. — *The combinatorial supports of $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ and $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)$ coincide. They equal $Y_{Q,n}^{\mathrm{Fr}}$.*

From [**48**, Lemma 3.2.3], it is known that the Hecke algebra $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)$ is commutative. It is (noncanonically) isomorphic to a polynomial ring over $\mathbb{C}$ in r variables, where r is the rank of $\mathbf{A}$. The support of $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)$ is contained in the centralizer of T° in $\tilde{T}$ (see [**80**, §3.2]). It follows that $W^\circ = N_{G^\circ}(T^\circ)/T^\circ$ acts on $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)$ by the rule

$$ {}^w f(\tilde{t}) := f(\dot{w}^{-1}\tilde{t}\dot{w}) \text{ for all } w \in W^\circ \text{ represented by } \dot{w} \in N_{G^\circ}(T^\circ). $$

The Satake isomorphism is a ring homomorphism from the spherical Hecke algebra of an unramified p-adic reductive group to the corresponding Hecke algebra of a maximal torus. For covering groups, this has been considered by McNamara [**51**] and Wen Wei Li [**48**].

Write $\delta : \tilde{T} \to \mathbb{R}_{>0}^\times$ for the modular character for the adjoint action of $\tilde{T}$ on U. Suppose that $f \in \mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$. Define the ***Satake transform*** of f, a function $\mathscr{S} f \colon \tilde{T} \to \mathbb{C}$, by

$$ [\mathscr{S} f](\tilde{t}) = \delta(\tilde{t})^{-1/2} \int_U f(u\tilde{t}) du. $$

From [**48**, Proposition 3.2.5], we have the following result.

Theorem 7.2. — *The Satake transform is a ring isomorphism from $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ to $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)^{W^\circ}$.*

Corollary 7.3. — *The Hecke algebra $\mathscr{H}_\varepsilon(\tilde{G}, G^\circ)$ is a finitely-generated commutative $\mathbb{C}$-algebra. If (π, V) is an irreducible ε-genuine spherical representation of $\tilde{G}$, then V^{G° is one-dimensional.*

Corollary 7.4. — *The Satake isomorphism defines a bijection,*

$$ \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O}) \equiv \mathrm{Hom}(\mathscr{H}_\varepsilon(\tilde{G}, G^\circ), \mathbb{C}) \xrightarrow{\mathscr{S}^*} \mathrm{Hom}(\mathscr{H}_\varepsilon(\tilde{T}, T^\circ)^{W^\circ}, \mathbb{C}) \equiv \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/\mathcal{O})/W^{\mathrm{Fr}}. $$

In other words, the Satake isomorphism gives a bijection from the set of equivalence classes of ε-genuine spherical irreps of $\tilde{G}$ to the W^{Fr}-orbits on the set of equivalence classes of ε-genuine spherical irreps of $\tilde{T}$. In applying the Satake isomorphism, the W^{Fr} action arises from the W° action on the Hecke algebra. This coincides with another action described below.

For $w \in W^{\mathrm{Fr}}$, conjugation by w defines a homomorphism $\mathrm{Int}(w) \colon \mathbf{G} \to \mathbf{G}$, which restricts to a homomorphism $\mathrm{Int}(w) \colon \mathbf{T} \to \mathbf{T}$. For any representative $\dot{w} \in \mathbf{N}_{\mathbf{G}}(\mathbf{T})(\mathcal{O})$, we find a well-aligned homomorphism of covers from $\tilde{\mathbf{T}}$ to itself, as in Section 2.6,

$$ \begin{tikzcd} \mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathbf{T}' \arrow[r, two heads] \arrow[d, "\mathrm{Int}(\dot{w})"] & \mathbf{T} \arrow[d, "\mathrm{Int}(w)"] \\ \mathbf{K}_2 \arrow[r, hook] & \mathbf{T}' \arrow[r, two heads] & \mathbf{T}. \end{tikzcd} $$

Pulling back via $\mathrm{Int}(\dot{w})$, we find a map, $\mathrm{Int}(\dot{w})^* \colon \mathbf{\Pi}_\varepsilon(\tilde{T}/\mathcal{O}) \to \mathbf{\Pi}_\varepsilon(\tilde{T}/\mathcal{O})$. Explicitly, if (π, V) is an ε-genuine spherical irrep of $\tilde{T}$, then $\mathrm{Int}(\dot{w})^*(\pi, V) = (\pi \circ \dot{w}^{-1}, V)$, where

$$ [\pi \circ \dot{w}^{-1}](\tilde{t}) := \pi(\dot{w}^{-1}\tilde{t}\dot{w}). $$

If $\ddot{w} = \dot{w}\tau$ for some $\tau \in T^\circ$, then

$$ \pi \circ \ddot{w}^{-1}(\tilde{t}) = \pi(\ddot{w}^{-1}\tilde{t}\ddot{w}) = \pi(\tau^{-1}\dot{w}^{-1}\tilde{t}\dot{w}\tau) = \pi(\tau)^{-1}[\pi \circ \ddot{w}^{-1}](\tilde{t})\pi(\tau). $$

Hence $\pi(\tau)$ gives an equivalence of representations from $(\pi \circ \dot{w}^{-1}, V)$ to $(\pi \circ \ddot{w}^{-1}, V)$. Therefore, we find a well-defined map (independent of representative),

$$\mathrm{Int}(w)^* \colon \mathbf{\Pi}_\varepsilon(\tilde{T}/\mathcal{O}) \to \mathbf{\Pi}_\varepsilon(\tilde{T}/\mathcal{O}).$$

This gives an action of W^{Fr} on $\mathbf{\Pi}_\varepsilon(\tilde{T}/\mathcal{O})$ which coincides with the action obtained from the W° action on the Hecke algebra $\mathcal{H}_\varepsilon(\tilde{T}, T^\circ)$.

8. Automorphic representations

Now let F be a global field, $\mathcal{V}$ the set of places of F, and $\mathbf{G}$ a quasisplit reductive group over F. Write $G_F = \mathbf{G}(F)$, $G_v = \mathbf{G}(F_v)$ for all $v \in \mathcal{V}$, and $G_{\mathbb{A}} = \mathbf{G}(\mathbb{A})$, where $\mathbb{A}$ denotes the ring of adeles of F.

Let $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ be a degree n cover of $\mathbf{G}$ over F. We may assume (see [**18**, §10.4]) that $\mathbf{G}$ and $\mathbf{G}'$ are defined over the ring $\mathcal{O}_{\mathcal{S}}$ of $\mathcal{S}$-integers in F, for some finite subset $\mathcal{S} \subset \mathcal{V}$.

In this way, for every place $v \in \mathcal{V}$, we find a central extension of locally compact groups, $\mu_n \hookrightarrow \tilde{G}_v \twoheadrightarrow G_v$, endowed with splittings $G_v^\circ = \mathbf{G}_v(\mathcal{O}_v) \hookrightarrow \tilde{G}_v$ for all $v \in \mathcal{V} - \mathcal{S}$. Define $\tilde{\tilde{G}}_{\mathbb{A}}$ to be the restricted direct product of the extensions $\tilde{G}_v$, with respect to the compact open subgroups G_v° (defined at almost all places):

$$\bigoplus_{v \in \mathcal{V}} \mu_n \hookrightarrow \tilde{\tilde{G}}_{\mathbb{A}} \twoheadrightarrow G_{\mathbb{A}}.$$

Pushing out via the product map $\bigoplus_{v \in \mathcal{V}} \mu_n \to \mu_n$ yields a central extension $\tilde{G}_{\mathbb{A}}$ as in the diagram below:

$$\begin{tikzcd} \bigoplus_{v \in \mathcal{V}} \mu_n \arrow[r, hook] \arrow[d, two heads, "\Pi"] & \tilde{\tilde{G}}_{\mathbb{A}} \arrow[r, two heads] \arrow[d, two heads] & G_{\mathbb{A}} \arrow[d, "="] \\ \mu_n \arrow[r, hook] & \tilde{G}_{\mathbb{A}} \arrow[r, two heads] & G_{\mathbb{A}}. \end{tikzcd}$$

As explained in [**18**, §10.4], $\mu_n \hookrightarrow \tilde{G}_{\mathbb{A}} \twoheadrightarrow G_{\mathbb{A}}$ is a central extension of locally compact groups, split canonically over G_F. Write $\tilde{Z}_{\mathbb{A}}$ for the center of $\tilde{G}_{\mathbb{A}}$. Then G_F is discrete in $\tilde{G}_{\mathbb{A}}$ and $\tilde{Z}_{\mathbb{A}} G_F \backslash \tilde{G}_{\mathbb{A}}$ has finite volume with respect to a $\tilde{G}_{\mathbb{A}}$-invariant measure.

8.1. Admissible and unitary representations. — Our approach to the admissible and automorphic representations of $\tilde{G}_{\mathbb{A}}$ coincides with the toral case in [**80**, §4], based on Flath [**27**] and Borel-Jacquet [**12**]. Let $\mathbb{A}_{\mathrm{fin}}$ be the finite adeles. Write $\tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}}$ for the restricted direct product of $\tilde{G}_v$ with respect to the compact open subgroups G_v°, where the product is indexed by only the nonarchimedean places $v \in \mathcal{V}_{\mathrm{fin}}$. Then $\tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}}$ is a totally disconnected locally compact group, a central extension as below:

$$\bigoplus_{v \in \mathcal{V}_{\mathrm{fin}}} \mu_n \hookrightarrow \tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}} \twoheadrightarrow G_{\mathbb{A}_{\mathrm{fin}}}.$$

Let $\tilde{\tilde{K}}_{\mathrm{fin}}$ be an open compact subgroup of $\tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}}$.

Define the ε-genuine Hecke algebra,

$$\mathscr{H}_{\mathrm{fin},\varepsilon} = \{f \in C_c^\infty(\tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}}) : f((\zeta_v)\cdot \tilde{\tilde{g}}) = \prod_{v\in\mathcal{V}_{\mathrm{fin}}} \varepsilon(\zeta_v)\cdot f(\tilde{\tilde{g}})$$
$$\text{for all } (\zeta_v) \in \bigoplus_{v\in\mathcal{V}_{\mathrm{fin}}} \mu_n, \quad \tilde{\tilde{g}} \in \tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}}\}.$$

As in [**27**, Example 2], the Hecke algebra $\mathscr{H}_{\mathrm{fin},\varepsilon}$ is isomorphic to the tensor product of the ε-genuine Hecke algebras $\mathscr{H}_{v,\varepsilon} = \mathscr{H}_\varepsilon(\tilde{G}_v)$ with respect to the system of idempotents $\mathrm{char}(G_v^\circ)$ at almost all places. An ***admissible*** ε-genuine representation of $\tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}}$ is a representation (π, V) of $\tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}}$ such that

$$\pi((\zeta_v)\tilde{\tilde{g}}) = \prod_{v\in\mathcal{V}_{\mathrm{fin}}} \varepsilon(\zeta_v)\cdot\pi(\tilde{\tilde{g}}), \text{ for all } (\zeta_v) \in \bigoplus_{v\in\mathcal{V}_{\mathrm{fin}}} \mu_n, \tilde{\tilde{g}} \in \tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}},$$

and which decomposes as a direct sum of irreducible representations of $\tilde{\tilde{K}}_{\mathrm{fin}}$, each occurring with finite multiplicity.

The commutativity of the spherical Hecke algebras $\mathscr{H}(\tilde{G}_v, G_v^\circ)$ at almost all places implies the factorization of irreducible admissible representations. The following is adapted from [**80**, Proposition 4.1], and is a direct result of [**27**, Theorem 2].

Proposition 8.1. — *For every irreducible admissible ε-genuine representation π_{fin} of $\tilde{\tilde{G}}_{\mathbb{A}_{\mathrm{fin}}}$, there exists a unique family of equivalence classes $([\pi_v])_{v\in\mathcal{V}_{\mathrm{fin}}}$ of irreducible admissible ε-genuine representations of each $\tilde{G}_v$, such that π_v is spherical for almost all $v \in \mathcal{V}_{\mathrm{fin}}$ and π_{fin} is isomorphic to the restricted tensor product of representations $\bigotimes_{v\in\mathcal{V}_{\mathrm{fin}}} \pi_v$ with respect to some choice of nonzero spherical vectors at almost all places.*

At the archimedean places $v \in \mathcal{V}_\infty$, we have Lie groups $\mu_n \hookrightarrow \tilde{G}_v \twoheadrightarrow G_v$; for each such place, choose a maximal compact subgroup K_v with pullback $\tilde{K}_v \subset \tilde{G}_v$. Let $\mathfrak{g}_v$ denote the complexified Lie algebra of G_v. The admissible ε-genuine representations of $\tilde{G}_v$ (i.e., admissible ε-genuine $(\mathfrak{g}_v, \tilde{K}_v)$-modules) may be identified with the admissible modules over $\mathscr{H}_{v,\varepsilon} = \mathscr{H}_\varepsilon(\tilde{G}_v)$—the algebra of ε-genuine, left and right $\tilde{K}_v$-finite distributions on $\tilde{G}_v$ with support in $\tilde{K}_v$. Define $\mathscr{H}_{\infty,\varepsilon} = \bigotimes_{v\in\mathcal{V}_\infty} \mathscr{H}_{v,\varepsilon}$.

An ε-genuine ***admissible representation*** of $\tilde{G}_{\mathbb{A}}$ is an admissible module over the Hecke algebra $\mathscr{H}_{\mathbb{A},\varepsilon} := \mathscr{H}_{\mathrm{fin},\varepsilon} \otimes \mathscr{H}_{\infty,\varepsilon}$.

A ***unitary representation*** of $\tilde{G}_{\mathbb{A}}$ will mean a continuous representation of $\tilde{G}_{\mathbb{A}}$ by unitary operators on a Hilbert space. As in [**80**, §4], following [**27**, Theorem 4] and Moore [**54**, Lemma 6.3], the unitary representations of $\tilde{G}_{\mathbb{A}}$ can be factored as Hilbert space tensor products of unitary representations at all places (spherical almost everywhere). We summarize the results in the following theorem.

Theorem 8.2. — *The restricted tensor product gives a bijection between two sets:*

- *the set of families $([\pi_v])_{v\in\mathcal{V}}$, where $[\pi_v] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F_v)$ for all $v \in \mathcal{V}$ and $[\pi_v] \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O}_v)$ for almost all $v \in \mathcal{V}_{\mathrm{fin}}$;*

– *the set of irreducible admissible ε-genuine representations of $\tilde{G}_{\mathbb{A}}$.*

Similarly, in the unitary setting, we find a bijection between two sets:

– *the set of families $([\pi_v])_{v \in \mathcal{V}}$, where $[\pi_v] \in \mathbf{\Pi}_\varepsilon^{\mathrm{unit}}(\tilde{\mathbf{G}}/F_v)$ for all $v \in \mathcal{V}$ and $[\pi_v] \in \mathbf{\Pi}_\varepsilon^{\mathrm{unit}}(\tilde{\mathbf{G}}/\mathcal{O}_v)$ for almost all $v \in \mathcal{V}_{\mathrm{fin}}$;*
– *the set of irreducible admissible ε-genuine unitary representations of $\tilde{G}_{\mathbb{A}}$.*

8.2. Automorphic representations

Definition 8.3. — An ***ε-genuine automorphic central character*** (ε-genuine ***ACC***, for short) for $\tilde{G}_{\mathbb{A}}$ is a continuous homomorphism $\chi \colon \tilde{Z}_{\mathbb{A}} \to \mathbb{C}^\times$ such that χ is trivial on $G_F \cap \tilde{Z}_{\mathbb{A}}$ and the restriction of χ to μ_n coincides with ε.

Proposition 8.4. — *Every ε-genuine ACC $\chi \colon \tilde{Z}_{\mathbb{A}} \to \mathbb{C}^\times$ extends uniquely to a continuous character $G_F \tilde{Z}_{\mathbb{A}} \to \mathbb{C}^\times$ which is trivial on G_F.*

Proof. — Multiplication gives a surjective continuous homomorphism,

$$G_F \times \tilde{Z}_{\mathbb{A}} \twoheadrightarrow G_F \tilde{Z}_{\mathbb{A}}.$$

Since $G_F \times \tilde{Z}_{\mathbb{A}}$ is a σ-compact locally compact topological group, the open mapping theorem implies that the multiplication map above is an open map. Hence $\tilde{Z}_{\mathbb{A}}$ (the image of the open subgroup $\{1\} \times \tilde{Z}_{\mathbb{A}}$ under multiplication) is an open subgroup of $G_F \tilde{Z}_{\mathbb{A}}$. Thus the inclusion $Z_{\mathbb{A}} \hookrightarrow G_F Z_{\mathbb{A}}$ induces a topological isomorphism of locally compact abelian groups,

$$\frac{\tilde{Z}_{\mathbb{A}}}{G_F \cap \tilde{Z}_{\mathbb{A}}} \xrightarrow{\sim} \frac{G_F \cdot \tilde{Z}_{\mathbb{A}}}{G_F}.$$

The result follows immediately. □

The following definition is lifted with slight adaptation from [**12**, §4.2] (and generalizes [**80**, Definition 4.9]

Definition 8.5. — A function $f \colon \tilde{G}_{\mathbb{A}} \to \mathbb{C}$ is an ε-genuine ***automorphic form*** if it satisfies the following conditions.

1. $f(\zeta \gamma \tilde{g}) = \varepsilon(\zeta) \cdot f(\tilde{g})$ for all $\gamma \in G_F$, $\zeta \in \mu_n$, and $\tilde{g} \in \tilde{G}_{\mathbb{A}}$.
2. Pulling back, f is locally constant on $\tilde{G}_{\mathbb{A}_{\mathrm{fin}}}$ and smooth on $\tilde{G}_v$ for all $v \in \mathcal{V}_\infty$.
3. There is a simple element $\xi = \xi_{\mathrm{fin}} \otimes \xi_\infty \in \mathcal{H}_{\mathrm{fin},\varepsilon} \otimes \mathcal{H}_{\infty,\varepsilon}$ such that $f * \xi = f$.
4. For every $y \in \tilde{G}_{\mathbb{A}}$ and every place $v \in \mathcal{V}_\infty$, the function $\tilde{G}_v \to \mathbb{C}$, $x \mapsto f(x \cdot y)$ has moderate growth.
5. The span of $\{f * \xi : \xi \in \mathcal{H}_{\mathrm{fin}} \otimes \mathcal{H}_\infty\}$ is an admissible representation of $\tilde{G}_{\mathbb{A}}$.

The space of ε-genuine automorphic forms will be written $\mathcal{AF}_\varepsilon(\tilde{G}_{\mathbb{A}})$.

Definition 8.6. — An ε-genuine ***automorphic representation*** of $\tilde{G}_{\mathbb{A}}$ is an irreducible admissible subquotient of $\mathcal{AF}_\varepsilon(\tilde{G}_{\mathbb{A}})$. Define $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/F)$ to be the set of equivalence classes (as admissible representations of $\tilde{G}_{\mathbb{A}}$) of ε-genuine automorphic representations of $\tilde{G}_{\mathbb{A}}$.

By Schur's Lemma, every ε-genuine automorphic representation of $\tilde{G}_{\mathbb{A}}$ has an ε-genuine automorphic central character. If χ is an ε-genuine ACC, then we also write χ for its extension to $G_F \cdot \tilde{Z}_{\mathbb{A}}$. If χ is *unitary*, define $L^2_\chi(G_F \backslash \tilde{G}_{\mathbb{A}})$ to be the unitary representation of $\tilde{G}_{\mathbb{A}}$, induced (in the sense of Blattner [**10**]) from the unitary character χ of $G_F \cdot \tilde{Z}_{\mathbb{A}}$. Let $L^{2,\mathrm{cusp}}_\chi(G_F \backslash \tilde{G}_{\mathbb{A}})$ be the cuspidal subspace, discussed for example in the work of Moeglin and Waldspurger [**53**].

An L^2 ***automorphic representation*** of $\tilde{\mathbf{G}}$ with central character χ is an irreducible unitary subrepresentation of $L^2_\chi(G_F \backslash \tilde{G}_{\mathbb{A}})$. An automorphic representation is called ***cuspidal*** if it lies within $L^{2,\mathrm{cusp}}_\chi(G_F \backslash \tilde{G}_{\mathbb{A}})$. Define $\mathbf{\Pi}^{\mathrm{unit}}_\chi(\tilde{\mathbf{G}}/F)$ to be the set of equivalence classes (as unitary representations of $\tilde{G}_{\mathbb{A}}$) of L^2 automorphic representations of $\tilde{\mathbf{G}}$ with central character χ. Let $\mathbf{\Pi}^{\mathrm{cusp}}_\chi(\tilde{\mathbf{G}}/F)$ be the subset consisting of cuspidal automorphic representations.

From the work of Harish-Chandra, Gelfand, and Piatetski-Shapiro, see also [**53**, §I.2.18], the cuspidal automorphic spectrum decomposes as nicely as possible.

Proposition 8.7. — *Let χ be a unitary ε-genuine ACC. Then $L^{2,\mathrm{cusp}}_\chi(G_F \backslash \tilde{G}_{\mathbb{A}})$ decomposes discretely into Hilbert space direct sum of ε-genuine unitary irreps, each with finite multiplicity.*

$$L^{2,\mathrm{cusp}}_\chi(G_F \backslash \tilde{G}_{\mathbb{A}}) \cong \widehat{\bigoplus}_{\pi \in \mathbf{\Pi}^{\mathrm{cusp}}_\varepsilon(\tilde{\mathbf{G}}/F)} m_\pi \cdot [\pi].$$

PART III
PARAMETERS FOR SPLIT TORI

9. A tale of two functors

Here we consider the simplest case in which we can give a one-to-one parameterization of irreducible genuine representations by Weil parameters: the case of *sharp* covers of split tori over local fields. This case underlies more difficult cases to follow.

Here F will denote a local field, and $\mathbf{T}$ a split torus over F. Fix a separable closure $\bar{F}/F$. Let Y be the cocharacter lattice of $\mathbf{T}$, and X the character lattice. We identify $T = \mathbf{T}(F)$ with $Y \otimes F^\times$; for $y \in Y$ and $u \in F^\times$, we write u^y for $y \otimes u \in T$. Similarly, we write $T^\vee = X \otimes \mathbb{C}^\times$ for the complex dual torus; for $x \in X$ and $z \in \mathbb{C}^\times$, we write z^x instead of $x \otimes z$.

Let n be a positive integer for which $\mu_n = \{\zeta \in F^\times : \zeta^n = 1\}$ has n elements. Fix an injective character $\varepsilon \colon \mu_n \hookrightarrow \mathbb{C}^\times$.

Let $\mathsf{Cov}_n(\mathbf{T})$ be the category of covers $\tilde{\mathbf{T}} = (\mathbf{T}', n)$ of $\mathbf{T}$ of degree n over F, as in Definition 1.2. Each such cover yields a quadratic form $Q \colon Y \to \mathbb{Z}$ (its first Brylinski-Deligne invariant) and hence a lattice $Y_{Q,n} \subset Y$. Let $\mathsf{Cov}^\sharp_n(\mathbf{T})$ be the full subcategory whose objects are those extensions $(\mathbf{T}', n)$ for which $Y = Y_{Q,n}$, and for which Q is even-valued if n is odd. These are the ***sharp covers*** of degree n, and they form a Picard groupoid.

If $\alpha\colon \mathbf{S} \xrightarrow{\sim} \mathbf{T}$ is an isomorphism of split tori, then pulling back gives an equivalence of Picard groupoids $\alpha^*\colon \mathsf{Cov}_n^\sharp(\mathbf{T}) \to \mathsf{Cov}_n^\sharp(\mathbf{S})$.

9.1. The functor of genuine characters. — Suppose that $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$ is a sharp degree n cover of $\mathbf{T}$ over F. Let $Q\colon Y \to \mathbb{Z}$ be its first Brylinski-Deligne invariant. The sharp cover $\tilde{\mathbf{T}}$ defines a short exact sequences of locally compact *abelian* groups (the construction of [**18**, §10.3], abelian by [**77**, Proposition 4.1]),

$$\mu_n \hookrightarrow \tilde{T} \twoheadrightarrow T.$$

Define $\mathbf{\Pi}(\mathbf{T}) = \mathrm{Hom}(T, \mathbb{C}^\times)$ (continuous homomorphisms). Recall that

$$\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}) = \mathrm{Hom}_\varepsilon(\tilde{T}, \mathbb{C}^\times),$$

the set of continuous ε-genuine homomorphisms (those that restrict to ε on μ_n). Then $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$ is a $\mathbf{\Pi}(\mathbf{T})$-torsor by twisting.

A morphism $f\colon \tilde{\mathbf{T}}_1 \to \tilde{\mathbf{T}}_2$ in $\mathsf{Cov}_n^\sharp(\mathbf{T})$ yields an isomorphism $f\colon \tilde{T}_1 \to \tilde{T}_2$ (an isomorphism of extensions of T by μ_n). The map $\chi \mapsto \chi \circ f$ gives a morphism of $\mathbf{\Pi}(\mathbf{T})$-torsors from $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_2)$ to $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_1)$.

Proposition 9.1. — *The construction above defines a contravariant additive functor of Picard groupoids,*

$$\mathbf{\Pi}_\varepsilon\colon \mathsf{Cov}_n^\sharp(\mathbf{T}) \to \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})).$$

Proof. — The only thing to check is compatibility with Baer sums. If $\tilde{\mathbf{T}} = \tilde{\mathbf{T}}_1 \dot{+} \tilde{\mathbf{T}}_2$ is the Baer sum of two objects of $\mathsf{Cov}_n^\sharp(\mathbf{T})$, then the resulting extension $\mu_n \hookrightarrow \tilde{T} \twoheadrightarrow T$ is the Baer sum of $\tilde{T}_1$ and $\tilde{T}_2$. If $t \in T$, then the elements of $\tilde{T}$ lying over t are given by pairs $(\tilde{t}_1, \tilde{t}_2) \in \tilde{T}_1 \times \tilde{T}_2$ lying over t, modulo the relation $(\tilde{t}_1\zeta, \tilde{t}_2) = (\tilde{t}_1, \zeta\tilde{t}_2)$ for all $\zeta \in \mu_n$.

If $\chi_1 \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_1)$ and $\chi_2 \in \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_2)$, then define

$$\chi(\tilde{t}_1, \tilde{t}_2) = \chi_1(\tilde{t}_1) \cdot \chi_2(\tilde{t}_2).$$

Since both χ_1 and χ_2 are ε-genuine, $\chi(\tilde{t}_1\zeta, \tilde{t}_2) = \chi(\tilde{t}_1, \zeta\tilde{t}_2)$, and so χ descends to an ε-genuine character of $\tilde{T}$. The map $(\chi_1, \chi_2) \mapsto \chi$ defines a function $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_1) \times \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_2) \to \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$. Checking compatibility with twisting by $\mathbf{\Pi}(\mathbf{T})$, this defines an isomorphism of $\mathbf{\Pi}(\mathbf{T})$-torsors,

$$\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_1) \dot{+} \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_2) \xrightarrow{\sim} \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_1 \dot{+} \tilde{\mathbf{T}}_2).$$

This isomorphism defines an additive structure on the functor $\mathbf{\Pi}_\varepsilon$. $\square$

If $\alpha\colon \mathbf{S} \xrightarrow{\sim} \mathbf{T}$ is an isomorphism of split tori, and $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$, and $\tilde{\mathbf{S}} = \alpha^* \tilde{\mathbf{T}}$, then we find a commutative diagram of locally compact abelian groups with exact rows:

$$\begin{tikzcd} \mu_n \arrow[r, hook] \arrow[d, "="] & \tilde{S} \arrow[r, two heads] \arrow[d, "\tilde{\alpha}"] & S \arrow[d, "\alpha"] \\ \mu_n \arrow[r, hook] & \tilde{T} \arrow[r, two heads] & T. \end{tikzcd}$$

Pulling back characters gives a group isomorphism $\alpha^*\colon \mathbf{\Pi}(\mathbf{T}) \to \mathbf{\Pi}(\mathbf{S})$. Pushing out torsors via α^* gives an equivalence of Picard groupoids, which we also write $\alpha^*\colon \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})) \to \mathsf{Tors}(\mathbf{\Pi}(\mathbf{S}))$. Pulling back genuine characters gives a bijection $\tilde{\alpha}^*\colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{S}})$. Allowing covers to vary, these bijections define a natural isomorphism $\tilde{\alpha}^*\colon \alpha^* \circ \mathbf{\Pi}_\varepsilon \Rightarrow \mathbf{\Pi}_\varepsilon \circ \alpha^*$ making the following diagram 2-commute:

$$\begin{tikzcd} \mathsf{Cov}_n^\sharp(\mathbf{T}) \arrow[r, "\mathbf{\Pi}_\varepsilon"] \arrow[d, "\alpha^*"] & \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})) \arrow[d, "\alpha^*"] \\ \mathsf{Cov}_n^\sharp(\mathbf{S}) \arrow[r, "\mathbf{\Pi}_\varepsilon"] & \mathsf{Tors}(\mathbf{\Pi}(\mathbf{S})). \end{tikzcd} \tag{9.1}$$

9.2. The functor of Weil parameters. — Since $\mathbf{T}$ is a split torus over F, and $Y = Y_{Q,n}$, the L-group of $\mathbf{T}$ (see Section 5.4) is a central extension of locally compact groups,

$$T^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{T} \twoheadrightarrow \mathrm{Gal}_F,$$

with $T^\vee = X \otimes \mathbb{C}^\times$. Since $\mathbf{T}$ is split, the L-group is well-defined up to unique isomorphism, without specifying a base point in the associated gerbe.

Define $\mathbf{\Phi}(\mathbf{T}) = \mathrm{Hom}(\mathscr{W}_F, T^\vee)$, the *abelian group* of Weil parameters for $\mathbf{T}$. Define $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}})$ to be the set of Weil parameters from $\mathscr{W}_F$ to ${}^{\mathsf{L}}\tilde{T}$. Then $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}})$ is naturally a torsor for $\mathbf{\Phi}(\mathbf{T})$.

A morphism $f\colon \tilde{\mathbf{T}}_1 \to \tilde{\mathbf{T}}_2$ in $\mathsf{Cov}_n^\sharp(\mathbf{T})$ is automatically a well-aligned isomorphism. It induces an equivalence of L-groups, by Section 5.5:

$$\begin{tikzcd} T^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{L}}\tilde{T}_2 \arrow[r, two heads] \arrow[d, "{}^{\mathsf{L}}f"] & \mathrm{Gal}_F \arrow[d, "="] \\ T^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{T}_1 \arrow[r, two heads] & \mathrm{Gal}_F\,. \end{tikzcd}$$

Composition with ${}^{\mathsf{L}}f$ defines an isomorphism of $\mathbf{\Phi}(\mathbf{T})$-torsors, from ${}^{\mathsf{L}}\tilde{T}_2$-valued parameters to ${}^{\mathsf{L}}\tilde{T}_1$-valued parameters.

Proposition 9.2. — *This defines a contravariant additive functor of Picard groupoids,*

$$\mathbf{\Phi}_\varepsilon\colon \mathsf{Cov}_n^\sharp(\mathbf{T}) \to \mathsf{Tors}(\mathbf{\Phi}(\mathbf{T})).$$

Proof. — We have described such a functor $\mathbf{\Phi}_\varepsilon$, at the level of objects and morphisms. To check that $\mathbf{\Phi}_\varepsilon$ respects the additive structure, we must see how the construction of the L-group behaves for the Baer sum of two covers. So consider two objects $\tilde{\mathbf{T}}_1$ and $\tilde{\mathbf{T}}_2$ of $\mathsf{Cov}_n^\sharp(\mathbf{T})$, and write $\tilde{\mathbf{T}}$ for their Baer sum. Write $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}$ for the second

Brylinski-Deligne invariants of $\tilde{\mathbf{T}}_1$, $\tilde{\mathbf{T}}_2$ and $\tilde{\mathbf{T}}$, respectively. The construction of the second Brylinski-Deligne invariant respects Baer sums, and so we find extensions of sheaves on $F_{\text{ét}}$,

$$\mathscr{G}_m \hookrightarrow \mathscr{D}_i \twoheadrightarrow Y, \text{ for } i = 1, 2,$$

and $\mathscr{D} = \mathscr{D}_1 \dotplus \mathscr{D}_2$. The $\mathscr{H}\!om(Y, \mathscr{G}_m)$-torsor of splittings $\mathscr{S}\!pl(\mathscr{D})$ is naturally identified with the contraction of the torsors $\mathscr{S}\!pl(\mathscr{D}_1)$ and $\mathscr{S}\!pl(\mathscr{D}_2)$. This gives an equivalence from the gerbe of nth roots of $\mathscr{S}\!pl(\mathscr{D})$ to the contraction of gerbes,

$$\sqrt[n]{\mathscr{S}\!pl(\mathscr{D})} \xrightarrow{\sim} \sqrt[n]{\mathscr{S}\!pl(\mathscr{D}_1)} \dotplus \sqrt[n]{\mathscr{S}\!pl(\mathscr{D}_2)}.$$

If z_1, z_2 are base points (over some finite Galois F'/F), objects of $\sqrt[n]{\mathscr{S}\!pl(\mathscr{D}_1)}$ and $\sqrt[n]{\mathscr{S}\!pl(\mathscr{D}_2)}$ respectively, then we find a base point $z = z_1 \wedge z_2$ of $\sqrt[n]{\mathscr{S}\!pl(\mathscr{D})}$.

This defines a map, for all $\gamma \in \mathrm{Gal}_F$

$$\mathrm{Hom}(z_1, {}^\gamma z_1) \times \mathrm{Hom}(z_2, {}^\gamma z_2) \to \mathrm{Hom}(z, {}^\gamma z).$$

Assembling these, we find an isomorphism of extensions of Gal_F by $T^\vee$,

$$\pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{z}) \xrightarrow{\sim} \pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}_1), \bar{z}_1) \dotplus \pi_1(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}_2), \bar{z}_2). \tag{9.2}$$

Let Q, Q_1, Q_2 be the first Brylinski-Deligne invariants of $\tilde{\mathbf{T}}$, $\tilde{\mathbf{T}}_1$, $\tilde{\mathbf{T}}_2$, respectively. Then $Q = Q_1 + Q_2$. This gives an isomorphism of extensions of Gal_F by $T^\vee$,

$$(\tau_Q)_* \widetilde{\mathrm{Gal}}_F \xrightarrow{\sim} (\tau_{Q_1})_* \widetilde{\mathrm{Gal}}_F \dotplus (\tau_{Q_2})_* \widetilde{\mathrm{Gal}}_F. \tag{9.3}$$

Taking the Baer sums of (9.2) and (9.3), we find an L-equivalence of L-groups,

$${}^{\mathsf{L}}\tilde{T} \xrightarrow{\sim} {}^{\mathsf{L}}\tilde{T}_1 \dotplus {}^{\mathsf{L}}\tilde{T}_2.$$

This L-equivalence defines an isomorphism of $\mathbf{\Phi}(\mathbf{T})$-torsors, from the ${}^{\mathsf{L}}\tilde{T}$-valued Weil parameters $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}})$ to the contraction of $\mathbf{\Phi}(\mathbf{T})$-torsors $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}_1) \dotplus \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}_2)$. This defines the additive structure on the functor $\mathbf{\Phi}_\varepsilon$ as required. □

If $\alpha\colon \mathbf{S} \xrightarrow{\sim} \mathbf{T}$ is an isomorphism of split tori, and $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$, and $\tilde{\mathbf{S}} = \alpha^* \tilde{\mathbf{T}}$, then we find a well-aligned isomorphism,

$$\begin{array}{ccccc} \mathbf{K}_2 & \hookrightarrow & \mathbf{S}' & \twoheadrightarrow & \mathbf{S} \\ \downarrow{=} & & \downarrow{\alpha'} & & \downarrow{\alpha} \\ \mathbf{K}_2 & \hookrightarrow & \mathbf{T}' & \twoheadrightarrow & \mathbf{T}. \end{array}$$

This defines an equivalence of L-groups,

$$\begin{array}{ccccc} T^\vee & \hookrightarrow & {}^{\mathsf{L}}\tilde{T} & \twoheadrightarrow & \mathrm{Gal}_F \\ \downarrow{\alpha^\vee} & & \downarrow{{}^{\mathsf{L}}\alpha} & & \downarrow{=} \\ S^\vee & \hookrightarrow & {}^{\mathsf{L}}\tilde{S} & \twoheadrightarrow & \mathrm{Gal}_F. \end{array}$$

The isomorphism $\alpha^\vee\colon T^\vee \to S^\vee$ gives a group isomorphism $\alpha^\vee\colon \mathbf{\Phi}(\mathbf{T}) \to \mathbf{\Phi}(\mathbf{S})$. Composition with ${}^{\mathsf{L}}\alpha$ gives a bijection ${}^{\mathsf{L}}\alpha\colon \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{S}})$. Allowing covers to vary, these bijections define a natural isomorphism ${}^{\mathsf{L}}\alpha \colon \alpha^\vee \circ \mathbf{\Phi}_\varepsilon \Rightarrow \mathbf{\Phi}_\varepsilon \circ \alpha^\vee$ making the following diagram 2-commute:

$$\begin{tikzcd} \mathsf{Cov}_n^\sharp(\mathbf{T}) \arrow[r, "\mathbf{\Phi}_\varepsilon"] \arrow[d, "\alpha^*"] & \mathsf{Tors}(\mathbf{\Phi}(\mathbf{T})) \arrow[d, "\alpha^\vee"] \\ \mathsf{Cov}_n^\sharp(\mathbf{S}) \arrow[r, "\mathbf{\Phi}_\varepsilon"] & \mathsf{Tors}(\mathbf{\Phi}(\mathbf{S})). \end{tikzcd} \tag{9.4}$$

9.3. The goal. — Class field theory gives a group isomorphism (the simplest case of [**45**]),

$$\mathscr{L} \colon \mathbf{\Pi}(\mathbf{T}) \xrightarrow{\sim} \mathbf{\Phi}(\mathbf{T}).$$

Indeed, if $\mathrm{rec} \colon \mathscr{W}_F^{\mathrm{ab}} \xrightarrow{\sim} F^\times$ is the isomorphism of class field theory (normalized so that a geometric Frobenius corresponds to a uniformizing parameter, in the nonarchimedean case), then $\mathscr{L}$ is the isomorphism given by composing the sequence of isomorphisms,

$$\begin{aligned} \mathscr{L} \colon \mathbf{\Pi}(\mathbf{T}) &= \mathrm{Hom}(T, \mathbb{C}^\times) = \mathrm{Hom}(Y \otimes F^\times, \mathbb{C}^\times) \\ &\xrightarrow{\mathrm{can}_Y} \mathrm{Hom}(F^\times, X \otimes \mathbb{C}^\times) \\ &\xrightarrow{\mathrm{rec}^*} \mathrm{Hom}(\mathscr{W}_F, X \otimes \mathbb{C}^\times) = \mathrm{Hom}(\mathscr{W}_F, T^\vee) = \mathbf{\Phi}(\mathbf{T}). \end{aligned}$$

Here and later, $\mathrm{can}_Y \colon \mathrm{Hom}(Y \otimes F^\times, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}(F^\times, X \otimes \mathbb{C}^\times)$ is the natural isomorphism arising from the duality of X and Y.

The goal for the next few sections is the construction of a natural isomorphism $\mathscr{L}_\varepsilon \colon \mathscr{L} \circ \mathbf{\Pi}_\varepsilon \Rightarrow \mathbf{\Phi}_\varepsilon$ of additive functors, which makes the following diagram of Picard groupoids and additive functors 2-commute:

$$\begin{tikzcd} & & \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})) \arrow[dd, "\mathscr{L}"] \\ \mathsf{Cov}_n^\sharp(\mathbf{T}) \arrow[urr, "\mathbf{\Pi}_\varepsilon"] \arrow[drr, "\mathbf{\Phi}_\varepsilon"'] \arrow[r, phantom, "\Downarrow \mathscr{L}_\varepsilon ?"] & & \\ & & \mathsf{Tors}(\mathbf{\Phi}(\mathbf{T})). \end{tikzcd} \tag{9.5}$$

Here we abuse notation slightly, and write $\mathscr{L}$ not only for the isomorphism $\mathbf{\Pi}(\mathbf{T}) \xrightarrow{\sim} \mathbf{\Phi}(\mathbf{T})$, but also for the equivalence of Picard groupoids given by pushing out via $\mathscr{L}$, $\mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})) \xrightarrow{\sim} \mathsf{Tors}(\mathbf{\Pi}(\mathbf{\Phi}(\mathbf{T})))$.

This pushout isomorphism $\mathscr{L}$ is naturally isomorphic to the functor which sends an $\mathbf{\Pi}(\mathbf{T})$-torsor V to the $\mathbf{\Phi}(\mathbf{T})$-torsor with underlying set V and action given by

$$\phi * v = \mathscr{L}^{-1}(\phi) * v \text{ for all } v \in V, \phi \in \mathbf{\Phi}(\mathbf{T}).$$

In this way, viewing $\mathscr{L}$ as the identity map on underlying sets, the natural isomorphism $\mathscr{L}_\varepsilon$ will give a bijection from $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$ to $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}})$, for any cover $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$.

We will also prove that $\mathscr{L}_\varepsilon$ is compatible with pullbacks, in the following sense. If $\alpha\colon \mathbf{S} \to \mathbf{T}$ is an isomorphism of split tori, and $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$, and $\tilde{\mathbf{S}} = \alpha^* \tilde{\mathbf{T}}$, then $\mathscr{L}$ is compatible with pullbacks in the sense that the following diagram of groups and isomorphisms commutes:

$$\begin{tikzcd} \mathbf{\Pi}(\mathbf{T}) \arrow[r, "\mathscr{L}"] \arrow[d, "\alpha^*"] & \mathbf{\Phi}(\mathbf{T}) \arrow[d, "\alpha^*"] \\ \mathbf{\Pi}(\mathbf{S}) \arrow[r, "\mathscr{L}"] & \mathbf{\Phi}(\mathbf{S}). \end{tikzcd}$$

We will find a commutative diagram of additive functors and natural isomorphisms:

$$\begin{tikzcd} \alpha^* \circ \mathscr{L} \circ \mathbf{\Pi}_\varepsilon \arrow[r, Rightarrow, "\mathscr{L}_\varepsilon"] \arrow[d, Rightarrow] & \alpha^* \circ \mathbf{\Phi}_\varepsilon \arrow[dd, Rightarrow, "{}^{\mathsf{L}}\alpha"] \\ \mathscr{L} \circ \alpha^* \circ \mathbf{\Pi}_\varepsilon \arrow[d, Rightarrow, "\tilde{\alpha}^*"] & \\ \mathscr{L} \circ \mathbf{\Pi}_\varepsilon \circ \alpha^* \arrow[r, Rightarrow, "\mathscr{L}_\varepsilon"] & \mathbf{\Phi}_\varepsilon \circ \alpha^*. \end{tikzcd} \tag{9.6}$$

In other words, given an ε-genuine character of a cover $\tilde{T}$, we may proceed in two directions: first, we may parameterize it by a ${}^{\mathsf{L}}\tilde{T}$-valued Weil parameter, and then push that parameter to find a ${}^{\mathsf{L}}\tilde{S}$-valued Weil parameter. Second, we may pull back the character to find an ε-genuine character of the cover $\tilde{S}$, and then parameterize by an ${}^{\mathsf{L}}\tilde{S}$-valued Weil parameter. The resulting ${}^{\mathsf{L}}\tilde{S}$-valued Weil parameters will be the same, according to the diagram above.

Thus $\mathscr{L}_\varepsilon \colon \mathscr{L} \circ \mathbf{\Pi}_\varepsilon \overset{\sim}{\Rightarrow} \mathbf{\Phi}_\varepsilon$ will parameterize ε-genuine characters by Weil parameters, for sharp covers of split tori. Formulating $\mathscr{L}_\varepsilon$ as a natural isomorphism of additive functors means that our parameterization of ε-genuine representations will be compatible with morphisms of covers, Baer sums, twisting, and pullbacks (via isomorphisms of tori). The proof requires some intermediate steps, and we will proceed through two more functors and three natural isomorphisms, according to the roadmap below:

$$\mathscr{L} \circ \mathbf{\Pi}_\varepsilon \xRightarrow{\mathscr{D}} \mathrm{rec}^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \overset{\mathscr{P}}{\Longleftrightarrow} \mathrm{rec}^* \circ \mathbf{\Phi}_\varepsilon^{\mathsf{A}} \xLeftarrow{\mathscr{A}} \mathbf{\Phi}_\varepsilon. \tag{9.7}$$

10. Deligne's construction

Here we utilize an idea of Deligne (personal communication) which relates the genuine characters of one extension to splittings of a "dual" extension. This provides a natural isomorphism from the functor $\mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon$ to a functor $\mathbf{\Pi}_\varepsilon^{\mathsf{D}}$ which will be easier to work with later. We fully credit Deligne with the elegant idea of this construction, and we take responsibility for any sloppiness in exposition.

10.1. Dual extensions. — We give a very general construction here; suppose that J and C are locally compact abelian groups (with composition written multiplicatively), and X and Y are finite-rank free abelian groups in duality as before. There is a group isomorphism,

$$\mathrm{Ext}(Y \otimes J, C) \cong \mathrm{Ext}(J, X \otimes C),$$

where Ext denotes the group of isomorphism classes of extensions (of locally compact abelian groups). This follows from the fact that Ext distributes over finite direct sums. We can beef this up to an equivalence of Picard groupoids as follows: Let $\mathsf{Ext}(Y \otimes J, C)$ be the Picard groupoid of extensions of locally compact abelian groups, and E an object therein,

$$C \hookrightarrow E \twoheadrightarrow Y \otimes J.$$

Tensoring with X yields an extension of locally compact abelian groups,

$$X \otimes C \hookrightarrow X \otimes E \twoheadrightarrow X \otimes Y \otimes J.$$

Pulling back via the canonical inclusion $\iota \colon \mathbb{Z} \hookrightarrow X \otimes Y$ yields an extension

$$X \otimes C \hookrightarrow \iota^*(X \otimes E) \twoheadrightarrow J.$$

Proposition 10.1. — *This defines an equivalence of Picard groupoids,*

$$\mathsf{D}_Y = \iota^*(X \otimes \bullet) \colon \mathsf{Ext}(Y \otimes J, C) \to \mathsf{Ext}(J, X \otimes C).$$

Proof. — We begin by checking that the construction D_Y gives a bijection on isomorphism classes. Choosing a $\mathbb{Z}$-basis $\{y_1, \ldots, y_r\}$ of Y, and dual basis $\{x_1, \ldots, x_r\}$ of X yields group isomorphisms $\mathrm{Ext}(Y \otimes J, C) \cong \bigoplus_{i=1}^r \mathrm{Ext}(J, C)$ and $\mathrm{Ext}(J, X \otimes C) \cong \bigoplus_{i=1}^r \mathrm{Ext}(J, C)$.

The canonical map $\iota \colon \mathbb{Z} \hookrightarrow X \otimes Y$ satisfies $\iota(1) = \sum_{i=1}^r x_i \otimes y_i$. It follows that the construction gives a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}(Y \otimes J, C) & \xrightarrow{\mathsf{D}_Y} & \mathrm{Ext}(J, X \otimes C) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \bigoplus_{i=1}^r \mathrm{Ext}(J, C) & \xrightarrow{\bigoplus \mathsf{D}_{\mathbb{Z}}} & \bigoplus_{i=1}^r \mathrm{Ext}(J, C). \end{array}$$

But the functor $\mathsf{D}_{\mathbb{Z}}$ is (naturally isomorphic to) the identity functor on the category $\mathsf{Ext}(J, C)$. Thus the bottom row of the commutative diagram is an isomorphism, and so the top row is an isomorphism.

To demonstrate that D_Y is an equivalence of categories, we trace through an automorphism of an extension $C \hookrightarrow E \twoheadrightarrow Y \otimes J$. Such an automorphism is given by an element $a \in \mathrm{Hom}(Y \otimes J, C)$ (a continuous homomorphism). With respect to the chosen basis, $a = \prod \xi_i^{x_i}$ for some family of homomorphisms $\xi_i \colon J \to C$. In other words, $a(j^y) = \prod_i \xi_i(j)^{\langle x_i, y \rangle}$.

If $e \in E$ lies over $j^y \in Y \otimes J$, then the automorphism $\eta_a \in \mathrm{Aut}(E)$ is given by

$$\eta_a(e) = e \cdot \prod_i \xi_i(j)^{\langle x_i, y \rangle}.$$

The resulting automorphism of $X \otimes C \hookrightarrow X \otimes E \twoheadrightarrow X \otimes Y \otimes J$ is characterized by

$$\eta_{a,X}(x \otimes e) = (x \otimes e) \cdot \prod_i \xi_i(j)^{\langle x_i, y \rangle \cdot x},$$

if $x \in X$, and $e \in E$ lies over $j^y \in Y \otimes J$. Thus in the pullback via $\iota \colon \mathbb{Z} \hookrightarrow X \otimes Y$, we find that the automorphism of $\mathsf{D}_Y E := \iota^*(X \otimes E)$ is given by

$$\mathsf{D}_Y \eta_a(\tilde{j}) = \tilde{j} \cdot \prod_i \xi_i(j)^{\langle x_i, y_i \rangle \cdot x_i} = \tilde{j} \cdot \prod_i \xi_i(j)^{x_i},$$

for all $\tilde{j} \in \mathsf{D}_Y E$ lying over $j \in J$.

The automorphism group of E is canonically identified with $\mathrm{Hom}(Y \otimes J, C)$. The automorphism group of $\mathsf{D}_Y E$ is canonically identified with $\mathrm{Hom}(J, X \otimes C)$. The computations above demonstrate that the functor D_Y coincides on automorphism groups with the canonical isomorphism,

$$\mathsf{D}_Y = \mathrm{can}_Y \colon \mathrm{Hom}(Y \otimes J, C) \xrightarrow{\sim} \mathrm{Hom}(J, X \otimes C).$$

Therefore D_Y is an equivalence of Picard groupoids. Compatibility with Baer sums is left to the reader. $\square$

The equivalence D_Y depends functorially on C and J as well. For example, if $j \colon J_1 \to J_2$ is a homomorphism of locally compact abelian groups, then a natural isomorphism makes the square below 2-commute:

$$\text{(10.1)} \qquad \begin{array}{ccc} \mathsf{Ext}(Y \otimes J_2, C) & \xrightarrow{\mathsf{D}_Y} & \mathsf{Ext}(J_2, X \otimes C) \\ \big\downarrow{\scriptstyle j^*} & & \big\downarrow{\scriptstyle j^*} \\ \mathsf{Ext}(Y \otimes J_1, C) & \xrightarrow{\mathsf{D}_Y} & \mathsf{Ext}(J_1, X \otimes C). \end{array}$$

Here the vertical arrows are additive functors given by pullback via j.

The equivalence D_Y also depends functorially on Y, at least for isomorphisms. Let $\alpha \colon Y_1 \to Y_2$ be an isomorphism of finite-rank free abelian groups, and write $\alpha^\vee \colon X_2 \to X_1$ for the dual isomorphism. Then a natural isomorphism makes the square below 2-commute:

$$\text{(10.2)} \qquad \begin{array}{ccc} \mathsf{Ext}(Y_2 \otimes J, C) & \xrightarrow{\mathsf{D}_Y} & \mathsf{Ext}(J, X_2 \otimes C) \\ \big\downarrow{\scriptstyle \alpha^*} & & \big\downarrow{\scriptstyle \alpha^\vee_*} \\ \mathsf{Ext}(Y_1 \otimes J, C) & \xrightarrow{\mathsf{D}_Y} & \mathsf{Ext}(J, X_1 \otimes C). \end{array}$$

10.2. Genuine characters and splittings. — Now consider an extension of locally compact abelian groups,

$$\mu_n \hookrightarrow E \twoheadrightarrow Y \otimes J,$$

Pushing out via $\varepsilon \colon \mu_n \hookrightarrow \mathbb{C}^\times$ yields a commutative diagram

$$\begin{tikzcd}
\mu_n \arrow[r, hook] \arrow[d, hook, "\varepsilon"] & E \arrow[r, two heads] \arrow[d, hook] & Y \otimes J \arrow[d, "="] \\
\mathbb{C}^\times \arrow[r, hook] & \varepsilon_* E \arrow[r, two heads] & Y \otimes J.
\end{tikzcd}$$

Giving an ε-genuine character of E is the same, by the universal property of pushouts, as giving a splitting of the bottom row of this diagram. In this way, we find an isomorphism of $\mathrm{Hom}(Y \otimes J, \mathbb{C}^\times)$-torsors,

$$\mathrm{Hom}_\varepsilon(E, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Spl}(\varepsilon_* E). \tag{10.3}$$

Applying the functor $\mathsf{D}_Y = \iota^*(X \otimes \bullet)$ to $\varepsilon_* E$ yields an extension

$$X \otimes \mathbb{C}^\times \hookrightarrow \mathsf{D}_Y \varepsilon_* E \twoheadrightarrow J.$$

A splitting of $\varepsilon_* E$ is the same as an isomorphism from $\varepsilon_* E$ to the trivial extension (i.e., zero object) in the Picard groupoid $\mathsf{Ext}(Y \otimes J, \mathbb{C}^\times)$. Since D_Y is an equivalence of Picard groupoids, it defines a bijection,

$$\mathsf{D}_Y \colon \mathrm{Spl}(\varepsilon_* E) \xrightarrow{\sim} \mathrm{Spl}(\mathsf{D}_Y \varepsilon_* E). \tag{10.4}$$

Assembling (10.3) and (10.4), we have a bijection,

$$\mathrm{Hom}_\varepsilon(E, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Spl}(\mathsf{D}_Y \varepsilon_* E), \tag{10.5}$$

which intertwines the $\mathrm{Hom}(Y \otimes J, \mathbb{C}^\times)$-action on $\mathrm{Hom}_\varepsilon(E, \mathbb{C}^\times)$ with the $\mathrm{Hom}(J, X \otimes \mathbb{C}^\times)$ action on $\mathrm{Spl}(\mathsf{D}_Y \varepsilon_* E)$, the intertwining passing through the isomorphism can_Y.

10.3. A natural isomorphism. — If $\mu_n \hookrightarrow \tilde{T} \twoheadrightarrow T$ arises from a sharp cover $\tilde{\mathbf{T}}$ as before, then define ${}^{\mathsf{D}}\tilde{T} = \mathsf{D}_Y \varepsilon_* \tilde{T}$. This is an extension of locally compact abelian groups,

$$X \otimes \mathbb{C}^\times \hookrightarrow {}^{\mathsf{D}}\tilde{T} \twoheadrightarrow F^\times.$$

Define $\mathbf{\Pi}_\varepsilon^{\mathsf{D}}(\tilde{\mathbf{T}}) = \mathrm{Spl}({}^{\mathsf{D}}\tilde{T})$. We have described a bijection,

$$\mathscr{D} \colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}) = \mathrm{Hom}_\varepsilon(\tilde{T}, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Spl}({}^{\mathsf{D}}\tilde{T}) = \mathbf{\Pi}_\varepsilon^{\mathsf{D}}(\tilde{\mathbf{T}}).$$

Define $\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T}) = \mathrm{Hom}(F^\times, X \otimes \mathbb{C}^\times)$, so $\mathrm{can}_Y \colon \mathbf{\Pi}(\mathbf{T}) \to \mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})$ is an isomorphism. We find that $\mathbf{\Pi}_\varepsilon^{\mathsf{D}}(\tilde{\mathbf{T}})$ is naturally a $\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})$-torsor. Moreover, $\mathscr{D}$ intertwines the $\mathbf{\Pi}(\mathbf{T})$-torsor structure on $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$ with the $\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})$-torsor structure on $\mathbf{\Pi}_\varepsilon^{\mathsf{D}}(\tilde{\mathbf{T}})$, via can_Y.

Putting everything together, we find the following diagram of Picard groupoids and additive functors, made 2-commutative by $\mathscr{D}\colon \operatorname{can}_Y \circ \mathbf{\Pi}_\varepsilon \Rightarrow \mathbf{\Pi}_\varepsilon^{\mathsf{D}}$,

$$\begin{tikzcd}
\mathsf{Cov}_n^\sharp(\mathbf{T}) \arrow[r] \arrow[rr, bend left, "\mathbf{\Pi}_\varepsilon"] \arrow[drr, bend right, "\mathbf{\Pi}_\varepsilon^{\mathsf{D}}"'] & \mathsf{Ext}(Y \otimes F^\times, \mu_n) \arrow[r, "\mathrm{Hom}_\varepsilon(\bullet, \mathbb{C}^\times)"] \arrow[d, "\mathsf{D}_Y \varepsilon_*"] & \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})) \arrow[d, "\operatorname{can}_Y"] \\
& \mathsf{Ext}(F^\times, X \otimes \mathbb{C}^\times) \arrow[r, "\mathrm{Spl}"] & \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})).
\end{tikzcd}$$

For reference, we trace an object $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$ through this diagram:

$$\begin{tikzcd}
\tilde{\mathbf{T}} \arrow[r, mapsto] \arrow[rr, bend left, mapsto, "\mathbf{\Pi}_\varepsilon"] \arrow[drr, bend right, mapsto, "\mathbf{\Pi}_\varepsilon^{\mathsf{D}}"'] & \tilde{T} \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{Hom}_\varepsilon(\tilde{T}, \mathbb{C}^\times) \\
& {}^{\mathsf{D}}\tilde{T} \arrow[r, mapsto] & \mathrm{Spl}\left({}^{\mathsf{D}}\tilde{T}\right).
\end{tikzcd}$$

Compressing the diagram down a bit, we find that $\mathscr{D}\colon \operatorname{can}_Y \circ \mathbf{\Pi}_\varepsilon \Rightarrow \mathbf{\Pi}_\varepsilon^{\mathsf{D}}$ is a natural isomorphism of additive functors, making the following diagram of Picard groupoids and additive functors 2-commute:

$$\begin{tikzcd}
& \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})) \arrow[dd, "\operatorname{can}_Y"] \\
\mathsf{Cov}_n^\sharp(\mathbf{T}) \arrow[ur, "\mathbf{\Pi}_\varepsilon"] \arrow[dr, "\mathbf{\Pi}_\varepsilon^{\mathsf{D}}"'] \arrow[r, phantom, "\Downarrow \mathscr{D}"] & {} \\
& \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})).
\end{tikzcd} \tag{10.6}$$

Composing with rec^*, and recalling that $\mathscr{L} = \operatorname{rec}^* \circ \operatorname{can}_Y$, we have constructed a natural isomorphism

$$\mathscr{D}\colon \mathscr{L} \circ \mathbf{\Pi}_\varepsilon \Longrightarrow \operatorname{rec}^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D}}. \tag{10.7}$$

The natural isomorphism $\mathscr{D}$ is compatible with pullbacks in the following way. Suppose that $\alpha\colon \mathbf{S} \to \mathbf{T}$ is an isomorphism of split tori, $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$ and $\tilde{\mathbf{S}}$ is the pullback of $\tilde{\mathbf{T}}$. The following diagram of additive functors and natural isomorphisms commutes:

$$
\begin{tikzcd}
\alpha^* \circ \mathscr{L} \circ \mathbf{\Pi}_\varepsilon \arrow[r, Rightarrow, "\mathscr{D}"] \arrow[d, Rightarrow] & \alpha^* \circ \mathrm{rec}^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \arrow[dd, Rightarrow, "{}^{\mathsf{D}}\tilde{\alpha}"] \\
\mathscr{L} \circ \alpha^* \circ \mathbf{\Pi}_\varepsilon \arrow[d, Rightarrow, "\tilde{\alpha}^*"] & \\
\mathscr{L} \circ \mathbf{\Pi}_\varepsilon \circ \alpha^* \arrow[r, Rightarrow, "\mathscr{D}"] & \mathrm{rec}^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \circ \alpha^*.
\end{tikzcd}
\tag{10.8}
$$

Indeed, rec^* and $\mathscr{L}$ are compatible with pullback. From (10.2), the construction of the "dual extension" is compatible with pullback. In particular, α induces an isomorphism ${}^{\mathsf{D}}\tilde{\alpha}\colon {}^{\mathsf{D}}\tilde{T} \to {}^{\mathsf{D}}\tilde{S}$ lying over $\alpha^\vee \colon T^\vee \to S^\vee$.

Given an ε-genuine character of $\tilde{T}$, we may proceed in two directions: first, one can construct a splitting of ${}^{\mathsf{D}}\tilde{T}$, and then push via ${}^{\mathsf{D}}\tilde{\alpha}$ to find a splitting of ${}^{\mathsf{D}}\tilde{S}$. Second, we can pull back to find an ε-genuine character of $\tilde{S}$, and use that to construct a splitting of ${}^{\mathsf{D}}\tilde{S}$. The above diagram states that the resulting splitting of ${}^{\mathsf{D}}\tilde{S}$ will be the same.

11. Weil parameters as splittings

The natural isomorphism $\mathscr{D}\colon \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \Rightarrow \mathbf{\Pi}_\varepsilon^{\mathsf{D}}$ relates genuine characters to splittings of a central extension. Here we define a natural isomorphism which relates Weil parameters to splittings of another central extension.

Begin with the L-group $T^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{T} \twoheadrightarrow \mathrm{Gal}_F$. Since $H^2(\mathrm{Gal}_F, \mathbb{C}^\times) = 0$, this extension splits. While there is no canonical splitting of ${}^{\mathsf{L}}\tilde{T}$, the splitting over the commutator subgroup $[\mathrm{Gal}_F, \mathrm{Gal}_F]$ is canonical; indeed, two splittings of ${}^{\mathsf{L}}\tilde{T}$ differ by an element of $\mathrm{Hom}(\mathrm{Gal}_F, T^\vee)$, and such a homomorphism is trivial on $[\mathrm{Gal}_F, \mathrm{Gal}_F]$ since $T^\vee$ is abelian. From this observation, we get an extension of *abelian* groups,

$$T^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{T}/[\mathrm{Gal}_F, \mathrm{Gal}_F] \twoheadrightarrow \mathrm{Gal}_F^{\mathrm{ab}}. \tag{11.1}$$

Let ${}^{\mathsf{A}}\tilde{T}$ be the pullback of (11.1) via $F^\times \xrightarrow{\mathrm{rec}^{-1}} \mathscr{W}_F^{\mathrm{ab}} \hookrightarrow \mathrm{Gal}_F^{\mathrm{ab}}$,

$$T^\vee \hookrightarrow {}^{\mathsf{A}}\tilde{T} \twoheadrightarrow F^\times.$$

(A stands for abelian.) Define $\mathbf{\Phi}_\varepsilon^{\mathsf{A}}(\tilde{\mathbf{T}}) = \mathrm{Spl}({}^{\mathsf{A}}\tilde{T})$, viewed as a $\mathrm{Hom}(F^\times, T^\vee)$-torsor.

Proposition 11.1. — *Every Weil parameter $\phi\colon \mathscr{W}_F \to {}^{\mathsf{L}}\tilde{T}$ descends uniquely to a homomorphism $\mathscr{W}_F^{\mathrm{ab}} \to {}^{\mathsf{A}}\tilde{T}$. This defines a bijection,*

$$\mathscr{A}\colon \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{\Phi}_\varepsilon^{\mathsf{A}}(\tilde{\mathbf{T}}),$$

which intertwines the $\mathbf{\Phi}(\mathbf{T}) = \mathrm{Hom}(\mathscr{W}_F, T^\vee)$-torsor structure on the left with the $\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T}) = \mathrm{Hom}(F^\times, T^\vee)$-torsor structure on the right, via the reciprocity isomorphism $\mathrm{rec}^*\colon \mathbf{\Pi}^{\mathsf{D}}(\mathbf{T}) \xrightarrow{\sim} \mathbf{\Phi}(\mathbf{T})$.

Proof. — Let $\sigma\colon \operatorname{Gal}_F \to {}^{\mathsf{L}}\tilde{T}$ be any splitting and write $\sigma'\colon \mathscr{W}_F \to {}^{\mathsf{L}}\tilde{T}$ for its pullback via the canonical map $\mathscr{W}_F \to \operatorname{Gal}_F$. If $\phi \in \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}})$ is a Weil parameter, then both σ' and ϕ are homomorphisms from $\mathscr{W}_F$ to ${}^{\mathsf{L}}\tilde{T}$, and there exists a unique function $\tau\colon \mathscr{W}_F \to T^\vee$ such that $\phi(w) = \sigma(w)\cdot\tau(w)$, for all $w \in \mathscr{W}_F$.

Since $T^\vee$ is contained in the center of ${}^{\mathsf{L}}\tilde{T}$, we find that $\tau\colon \mathscr{W}_F \to T^\vee$ is a homomorphism. Hence $\phi = \sigma\cdot\tau$ sends the commutator subgroup of $\mathscr{W}_F$ to the image of $[\operatorname{Gal}_F, \operatorname{Gal}_F]$ in ${}^{\mathsf{L}}\tilde{T}$. Thus ϕ descends uniquely to a homomorphism $\mathscr{W}_F^{\mathrm{ab}} \to {}^{\mathsf{A}}\tilde{T}$. Pulling back via the reciprocity isomorphism $\operatorname{rec}^{-1}\colon F^\times \xrightarrow{\sim} \mathscr{W}_F^{\mathrm{ab}}$ yields a homomorphism $\mathscr{A}(\phi)\colon F^\times \to {}^{\mathsf{A}}\tilde{T}$ as claimed. This homomorphism $\mathscr{A}(\phi)$ is a splitting of ${}^{\mathsf{A}}\tilde{T}$, and the construction of $\mathscr{A}(\phi)$ from ϕ is compatible with twisting by $\mathbf{\Phi}(\mathbf{T})$ throughout. $\square$

The construction of the L-group gives an additive functor of Picard categories, $\mathsf{Cov}_n^\sharp(\mathbf{T}) \to \mathsf{CExt}(\operatorname{Gal}_F, T^\vee)$ (the latter category is a Picard groupoid with one isomorphism class, since $H^2(\operatorname{Gal}_F, T^\vee)$ is trivial). The construction of ${}^{\mathsf{A}}\tilde{T}$ from the (split) extension ${}^{\mathsf{L}}\tilde{T}$ comes from another additive functor of Picard categories, $\mathsf{CExt}(\operatorname{Gal}_F, T^\vee) \to \mathsf{Ext}(F^\times, T^\vee)$. Here $\mathsf{Ext}(F^\times, T^\vee)$ is the Picard groupoid of extensions of locally compact abelian groups. It can be proven that every such extension splits. Finally we apply the additive functor of splittings Spl,

$$\operatorname{Spl}\colon \mathsf{Ext}(F^\times, T^\vee) \to \mathsf{Tors}(\operatorname{Hom}(F^\times, T^\vee)).$$

Recall that $\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})$ denotes $\operatorname{Hom}(F^\times, T^\vee)$. As the composition of three additive functors, $\mathbf{\Phi}_\varepsilon^{\mathsf{A}}$ is an additive functor,

$$\mathbf{\Phi}_\varepsilon^{\mathsf{A}}\colon \mathsf{Cov}_n^\sharp(\mathbf{T}) \to \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})).$$

The bijection $\phi \mapsto \mathscr{A}(\phi)$, from the set of Weil parameters to $\operatorname{Spl}({}^{\mathsf{A}}\tilde{T})$, gives a natural isomorphism of additive functors, $\mathscr{A}\colon \mathbf{\Phi}_\varepsilon \overset{\sim}{\Rightarrow} \operatorname{rec}^* \circ \mathbf{\Phi}_\varepsilon^{\mathsf{A}}$:

$$\text{(11.2)} \qquad \begin{tikzcd} & \mathsf{Tors}(\mathbf{\Phi}(\mathbf{T})) \\ \mathsf{Cov}_n^\sharp(\mathbf{T}) \arrow[ur, "\mathbf{\Phi}_\varepsilon"] \arrow[dr, "\mathbf{\Phi}_\varepsilon^{\mathsf{A}}"'] \arrow[r, phantom, "\Downarrow \mathscr{A}"] & {} \\ & \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})). \arrow[uu, "\operatorname{rec}^*"'] \end{tikzcd}$$

The natural isomorphism $\mathscr{A}$ is compatible with pullbacks in the following way. Suppose that $\alpha\colon \mathbf{S} \to \mathbf{T}$ is an isomorphism of split tori, $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$ and $\tilde{\mathbf{S}}$ is the pullback of $\tilde{\mathbf{T}}$. Then we find the L-morphism ${}^{\mathsf{L}}\tilde{\alpha}\colon {}^{\mathsf{L}}\tilde{T} \to {}^{\mathsf{L}}\tilde{S}$. This descends to a morphism of extensions ${}^{\mathsf{A}}\tilde{\alpha}$ as below:

$$\begin{tikzcd} T^\vee \arrow[r, hook] \arrow[d, "\alpha^\vee"] & {}^{\mathsf{A}}\tilde{T} \arrow[r, two heads] \arrow[d, "{}^{\mathsf{A}}\tilde{\alpha}"] & F^\times \arrow[d, "="] \\ S^\vee \arrow[r, hook] & {}^{\mathsf{A}}\tilde{S} \arrow[r, two heads] & F^\times. \end{tikzcd}$$

In this way, α yields a bijection ${}^{\mathsf{A}}\alpha\colon \mathrm{Spl}({}^{\mathsf{A}}\tilde{T}) \to \mathrm{Spl}({}^{\mathsf{A}}\tilde{S})$.

The following diagram of additive functors and natural isomorphisms commutes:

(11.3)
$$\begin{tikzcd} \alpha^* \circ \mathbf{\Phi}_\varepsilon \arrow[r, Rightarrow, "\mathscr{A}"] \arrow[d, Rightarrow, "{}^{\mathsf{L}}\tilde{\alpha}"] & \alpha^* \circ \mathrm{rec}^* \circ \mathbf{\Phi}_\varepsilon^{\mathsf{A}} \arrow[d, Rightarrow, "{}^{\mathsf{A}}\tilde{\alpha}"] \\ \mathbf{\Phi}_\varepsilon \circ \alpha^* \arrow[r, Rightarrow, "\mathscr{A}"] & \mathrm{rec}^* \circ \mathbf{\Phi}_\varepsilon^{\mathsf{A}} \circ \alpha^*. \end{tikzcd}$$

Given a Weil parameter valued in ${}^{\mathsf{L}}\tilde{T}$, we may proceed in two directions: first, one can push via ${}^{\mathsf{L}}\tilde{\alpha}$ to find a Weil parameter valued in ${}^{\mathsf{L}}\tilde{S}$, which descends to a splitting of ${}^{\mathsf{A}}\tilde{S}$. Second, one can descend the original parameter to find a splitting of ${}^{\mathsf{A}}\tilde{T}$, and then push via ${}^{\mathsf{A}}\tilde{\alpha}$ to find a splitting of ${}^{\mathsf{A}}\tilde{S}$. The above diagram states that the resulting splitting of ${}^{\mathsf{A}}\tilde{S}$ will be the same.

12. Parameterization

With (10.7) and (11.2), we have now constructed additive functors and natural isomorphisms as below:

$$\mathscr{L} \circ \mathbf{\Pi}_\varepsilon \xRightarrow{\mathscr{D}} \mathrm{rec}^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \qquad \mathrm{rec}^* \circ \mathbf{\Phi}_\varepsilon^{\mathsf{A}} \xLeftarrow{\mathscr{A}} \mathbf{\Phi}_\varepsilon.$$

The real work comes here, as we define a natural isomorphism of functors $\mathbf{\Pi}_\varepsilon^{\mathsf{D}} \Leftrightarrow \mathbf{\Phi}_\varepsilon^{\mathsf{A}}$ linking the middle terms. For this, we will define isomorphisms $\mathscr{P}(\tilde{\mathbf{T}})\colon {}^{\mathsf{D}}\tilde{T} \xrightarrow{\sim} {}^{\mathsf{A}}\tilde{T}$, in the category of extensions of $F^\times$ by $T^\vee$.

A morphism $j\colon \tilde{\mathbf{T}}_1 \to \tilde{\mathbf{T}}_2$ in $\mathsf{Cov}_n^\sharp(\mathbf{T})$ yields two morphisms in $\mathsf{Ext}(F^\times, T^\vee)$.

$${}^{\mathsf{D}}j\colon {}^{\mathsf{D}}\tilde{T}_2 \to {}^{\mathsf{D}}\tilde{T}_1, \quad {}^{\mathsf{A}}j\colon {}^{\mathsf{A}}\tilde{T}_2 \to {}^{\mathsf{A}}\tilde{T}_1.$$

We will demonstrate that our isomorphisms $\mathscr{P}$ are compatible with morphisms in $\mathsf{Cov}_n^\sharp(\mathbf{T})$, i.e., the following diagram commutes for all $j\colon \tilde{\mathbf{T}}_1 \to \tilde{\mathbf{T}}_2$:

$$\begin{tikzcd} {}^{\mathsf{D}}\tilde{T}_2 \arrow[r, "{}^{\mathsf{D}}j"] \arrow[d, "\mathscr{P}(\tilde{\mathbf{T}}_2)"] & {}^{\mathsf{D}}\tilde{T}_1 \arrow[d, "\mathscr{P}(\tilde{\mathbf{T}}_1)"] \\ {}^{\mathsf{A}}\tilde{T}_2 \arrow[r, "{}^{\mathsf{A}}j"] & {}^{\mathsf{A}}\tilde{T}_1. \end{tikzcd}$$

This will provide a natural isomorphism of torsors

$$\mathbf{\Pi}_\varepsilon^{\mathsf{D}}(\tilde{\mathbf{T}}) = \mathrm{Spl}({}^{\mathsf{D}}\tilde{T}) \xrightarrow{\mathscr{P}(\tilde{\mathbf{T}})} \mathrm{Spl}({}^{\mathsf{A}}\tilde{T}) = \mathbf{\Phi}_\varepsilon^{\mathsf{A}}(\tilde{\mathbf{T}}).$$

In the work that follows, we frequently refer to extensions which are "incarnated" by bimultiplicative cocycles, so we fix some terminology here. Suppose that J and C are locally compact abelian groups. A bimultiplicative cocycle

$$\theta\colon J \times J \to C$$

is a continuous function which factors through $J \otimes_{\mathbb{Z}} J \to C$. (The usual 2-cocycle identity follows from this). Such a cocycle allows one to define a central extension

$$C \hookrightarrow J \times_\theta C \twoheadrightarrow J,$$

where $J \times_\theta C = J \times C$ as sets, and multiplication is given by

$$(j_1, c_1) \cdot (j_2, c_2) = (j_1 j_2, c_1 c_2 \cdot \theta(j_1, j_2)).$$

We often write c instead of $(1, c)$, viewing C as a subgroup of $J \times_\theta C$. But we typically write $s(j) = (j, 1)$, using the letter s for the section (not often a homomorphism) from J to $J \times_\theta C$.

For a bimultiplicative cocycle θ, induction gives a formula for powers,

$$(j,1)^g = (j^g, 1) \cdot \theta(j,j)^{\frac{g(g-1)}{2}} \text{ for all } j \in J, g \in \mathbb{Z}_{>0}. \tag{12.1}$$

12.1. Incarnated covers. — Suppose that $\tilde{\mathbf{T}} = (\mathbf{T}', n)$ is a sharp cover ***incarnated*** by $C \in X \otimes X$ as in [**18**, §3]. We often utilize an ordered basis $\mathscr{B} = (y_1, \ldots, y_r)$ of Y and dual basis $(x_1, \ldots, x_r)$ of X in what follows. Write $C = \sum_{i,j} c_{\mathscr{B}}^{ij} x_i \otimes x_j$ with respect to this basis. Thus $\mathbf{T}' = \mathbf{T} \times_{\theta_C} \mathbf{K}_2$ (as sheaves of sets on F_{Zar}), where $\theta_C \colon \mathbf{T} \times \mathbf{T} \to \mathbf{K}_2$ is the bimultiplicative cocycle (a map of sheaves of sets),

$$\theta_C(t_1, t_2) = \prod_{i,j} \{x_i(t_1), x_j(t_2)\}^{c_{\mathscr{B}}^{ij}}.$$

Note that the right side depends on C, but not on the choice of basis. Hence $\mathbf{T}'$ depends only on C and not on the choice of basis.

The isomorphism class of $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$ is determined by the associated quadratic form $Q(y) = C(y,y)$. If $C_0 \in X \otimes X$, and $C_0(y,y) = C(y,y)$ for all $y \in Y$, define $A = C - C_0$, the alternating bilinear form represented by the matrix $(a_{\mathscr{B}}^{ij})$, $a_{\mathscr{B}}^{ij} = c_{\mathscr{B}}^{ij} - c_{\mathscr{B},0}^{ij}$. Write $\tilde{\mathbf{T}}_0$ for the degree n cover incarnated by C_0. Associated to A is an isomorphism $\iota_{\mathscr{B},A} \colon \tilde{\mathbf{T}}_0 \to \tilde{\mathbf{T}}$ given by

$$\iota_{\mathscr{B},A}(t, \kappa) = \left(t, \kappa \cdot \prod_{i<j} \{x_i(t), x_j(t)\}^{a_{\mathscr{B}}^{ij}} \right).$$

For any element $\hat{u} = \prod_i u_i^{x_i} \in X \otimes F^\times$, define an automorphism $j_{\hat{u}} \in \mathrm{Aut}(\tilde{\mathbf{T}})$ by

$$j_{\hat{u}}(t, \kappa) = \left(t, \kappa \cdot \prod_i \{x_i(t), u_i\} \right).$$

From [**18**, §3], it suffices to study incarnated covers and morphisms as above.

Proposition 12.1. — *The category $\mathsf{Cov}_n^\sharp(\mathbf{T})$ is equivalent to its full subcategory whose objects are the covers incarnated by elements of $X \otimes X$. Every morphism in this full subcategory can be expressed as the composition of a morphism $\iota_{\mathscr{B},A}$ (for some alternating form A) and an automorphism of type $j_{\hat{u}}$ (for some $\hat{u} \in X \otimes F^\times$).*

Proof. — The isomorphism classes in $\mathsf{Cov}_n^\sharp(\mathbf{T})$ are in bijection with quadratic forms $Q\colon Y \to \mathbb{Z}$ such that $Y = Y_{Q,n}$ and such that Q is even-valued if n is odd. Since every quadratic form $Q\colon Y \to \mathbb{Z}$ can be written as $Q(y) = C(y,y)$ for some $C \in X \otimes X$, we find that every isomorphism class contains an incarnated cover.

The proposition now follows, since the category $\mathsf{Cov}_n^\sharp(\mathbf{T})$ is a groupoid, the isomorphisms of type $\iota_{\mathscr{B},A}$ link all objects within the same isomorphism class, and the automorphism group of each object is identified with $X \otimes F^\times$ by [**18**, §3]. □

12.2. The extension ${}^{\mathsf{D}}\tilde{T}$. — For $\tilde{\mathbf{T}}$ a sharp cover incarnated by $C \in X \otimes X$, every ordered basis $\mathscr{B}$ of Y yields a section of ${}^{\mathsf{D}}\tilde{T}$. We describe this here, and track the dependence on basis and the effect of morphisms of covers.

First, $\tilde{\mathbf{T}}$ yields an extension $\mathbb{C}^\times \hookrightarrow \varepsilon_*\tilde{T} \twoheadrightarrow T$, with $\varepsilon_*\tilde{T} = T \times_{\theta_{C,\varepsilon}} \mathbb{C}^\times$ and incarnating cocycle

$$\theta_{C,\varepsilon}(t_1,t_2) = \prod_{i,j} \mathrm{Hilb}_n^\varepsilon(x_i(t_1), x_j(t_2))^{c_{\mathscr{B}}^{ij}}.$$

Hereafter we write $\mathrm{Hilb}_n^\varepsilon = \varepsilon \circ \mathrm{Hilb}_n\colon F^\times \times F^\times \to \boldsymbol{\mu}_n(\mathbb{C})$.

An element of $T = Y \otimes F^\times$ can be written uniquely as $\prod_i u_i^{y_i}$ with $u_i \in F^\times$ for all $1 \le i \le r$. Similarly, an element of $T^\vee = X \otimes \mathbb{C}^\times$ can be written uniquely as $\prod_i z_i^{x_i}$ with $z_i \in \mathbb{C}^\times$ for all $1 \le i \le r$.

To construct ${}^{\mathsf{D}}\tilde{T}$, we first tensor with X to obtain

$$X \otimes \mathbb{C}^\times \hookrightarrow X \otimes \varepsilon_*\tilde{T} \twoheadrightarrow X \otimes Y \otimes F^\times.$$

The basis $\mathscr{B}$ determines a section $s_{\mathscr{B}}^\otimes\colon X \otimes Y \otimes F^\times \to X \otimes \varepsilon_*\tilde{T} = X \otimes (T \times_{\theta_{C,\varepsilon}} \mathbb{C}^\times)$,

$$s_{\mathscr{B}}^\otimes\left(\prod_{i=1}^r \prod_{j=1}^r u_{ij}^{x_i \otimes y_j}\right) = \prod_{i=1}^r \underbrace{\left(\prod_{j=1}^r u_{ij}^{y_j}, 1\right)}_{\text{an element of } \varepsilon_*\tilde{T}}{}^{x_i} \in X \otimes \varepsilon_*\tilde{T}.$$

Define a bimultiplicative cocycle $\theta_{\mathscr{B}}^\otimes\colon (X \otimes Y \otimes F^\times) \times (X \otimes Y \otimes F^\times) \to X \otimes \mathbb{C}^\times$,

$$\theta_{\mathscr{B}}^\otimes\left(u^{x_i \otimes y_j}, v^{x_k \otimes y_\ell}\right) = \begin{cases} \mathrm{Hilb}_n^\varepsilon(u,v)^{c_{\mathscr{B}}^{j\ell} x_i}, & \text{if } i = k; \\ 1 & \text{otherwise.} \end{cases}$$

The section $s_{\mathscr{B}}^\otimes$ defines a group isomorphism,

$$\sigma_{\mathscr{B}}^\otimes\colon (X \otimes Y \otimes F^\times) \times_{\theta_{\mathscr{B}}^\otimes} (X \otimes \mathbb{C}^\times) \to X \otimes \varepsilon_*\tilde{T},$$

characterized by $\sigma_{\mathscr{B}}^\otimes(u^{x_i \otimes y_j}, t^\vee) = t^\vee \cdot s_{\mathscr{B}}^\otimes(u^{x_i \otimes y_j})$ for all $1 \le i,j \le r$, $u \in F^\times$.

Recall that ${}^{\mathsf{D}}\tilde{T} = \iota^*(X \otimes \varepsilon_* \tilde{T})$, where $\iota \colon \mathbb{Z} \hookrightarrow X \otimes Y$ sends 1 to $\sum_i x_i \otimes y_i$. Define $\theta_{\mathscr{B}} \colon F^\times \times F^\times \to X \otimes \mathbb{C}^\times$ to be the pullback of $\theta_{\mathscr{B}}^{\otimes}$ via ι. Thus

$$\begin{aligned}
\theta_{\mathscr{B}}(u,v) &= \theta_{\mathscr{B}}^{\otimes}\left(\prod_i u^{x_i \otimes y_i}, \prod_j v^{x_j \otimes y_j}\right) \\
&= \prod_{i,j} \theta_{\mathscr{B}}^{\otimes}\left(u^{x_i \otimes y_i}, v^{x_j \otimes y_j}\right) \\
&= \prod_{i} \theta_{\mathscr{B}}^{\otimes}\left(u^{x_i \otimes y_i}, v^{x_i \otimes y_i}\right) \\
&= \prod_i \mathrm{Hilb}_n^{\varepsilon}(u,v)^{c_{\mathscr{B}}^{ii} x_i} = \prod_i \mathrm{Hilb}_n^{\varepsilon}(u,v)^{Q(y_i) x_i}.
\end{aligned} \tag{12.2}$$

Pulling back $\sigma_{\mathscr{B}}^{\otimes}$ via ι gives a group isomorphism,

$$\sigma_{\mathscr{B}} \colon F^\times \times_{\theta_{\mathscr{B}}} T^\vee \xrightarrow{\sim} {}^{\mathsf{D}}\tilde{T},$$

characterized by $\sigma_{\mathscr{B}}(u, t^\vee) = t^\vee \cdot s_{\mathscr{B}}^{\otimes}(\iota(u))$ for all $u \in F^\times$.

Lemma 12.2. — *For all $u, v \in F^\times$, we have*

$$\theta_{\mathscr{B}}(u,v) = \prod_{i=1}^{r} \mathrm{Hilb}_n^{\varepsilon}(u,v)^{Q(y_i)x_i} = \tau_Q(\mathrm{Hilb}_2(u,v)).$$

In particular, $\theta_{\mathscr{B}}$ is independent of basis.

Remark 12.3. — If $\mathrm{char}(F) = 2$, then n is odd, so $\tau_Q(\pm 1) = 1$, and one should interpret the lemma as $\theta_{\mathscr{B}}(u,v) \equiv 1$ since the quadratic Hilbert symbol is not defined.

Proof. — Recall that $\tau_Q \colon \mu_2 \to T^\vee = X \otimes \mathbb{C}^\times$ is dual to the homomorphism $Y \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ given by $y \mapsto n^{-1}Q(y)$. Thus τ_Q can be expressed using the basis $\mathscr{B}$,

$$\tau_Q(-1) = \prod_i e^{2\pi i n^{-1} Q(y_i) \cdot x_i}.$$

Since $\tilde{\mathbf{T}}$ is a sharp cover, $Y = Y_{Q,n}$, and so $2Q(y) \in n\mathbb{Z}$ for all $y \in Y$. If n is odd (in particular if $\mathrm{char}(F) = 2$), this implies that $Q(y_i) \in n\mathbb{Z}$ for all $1 \leq i \leq r$, and so $\theta_{\mathscr{B}}(u,v) = 1 = \tau_Q(\mathrm{Hilb}_2(u,v))$ as claimed.

If n is even, then $\mathrm{Hilb}_2(u,v) = \mathrm{Hilb}_n(u,v)^{n/2}$ and so

$$\mathrm{Hilb}_n^{\varepsilon}(u,v)^{Q(y_i)} = \mathrm{Hilb}_2(u,v)^{2Q(y_i)/n} = \begin{cases} \mathrm{Hilb}_2(u,v), & \text{if } n^{-1}Q(y_i) \notin \mathbb{Z}; \\ 1, & \text{if } n^{-1}Q(y_i) \in \mathbb{Z}. \end{cases}$$

The result follows from our previous computation of $\theta_{\mathscr{B}}(u,v)$ in (12.2). □

Hereafter, we write simply θ rather than $\theta_{\mathscr{B}}$. Thus $\sigma_{\mathscr{B}}$ gives an isomorphism in $\mathsf{Ext}(F^\times, T^\vee)$,

$$\sigma_{\mathscr{B}} \colon F^\times \times_\theta T^\vee \to {}^{\mathsf{D}}\tilde{T}.$$

The cocycle $\theta(u,v) = \tau_Q(\mathrm{Hilb}_2(u,v))$ is independent of basis, but the isomorphism $\sigma_{\mathscr{B}}$ depends on the basis.

We trace through morphisms of extensions here. First, consider an element $\hat{u} = \prod_\ell u_\ell^{x_\ell} \in X \otimes F^\times = \mathrm{Aut}(\tilde{\mathbf{T}})$. This defines an automorphism of $\varepsilon_* \tilde{T}$, given by

$$j_{\hat{u}}(t,z) = \left(t, z \cdot \prod_\ell \mathrm{Hilb}_n^\varepsilon(x_\ell(t), u_\ell) \right).$$

On the other hand, for any extension $E \in \mathsf{Ext}(F^\times, T^\vee)$, $\hat{u}$ defines an automorphism $j_{\hat{u}}^\vee \in \mathrm{Aut}(E)$ by

$$j_{\hat{u}}^\vee(e) = e \cdot \prod_\ell \mathrm{Hilb}_n^\varepsilon(v, u_\ell)^{x_\ell}, \text{ for all } e \in E \text{ lying over } v \in F^\times. \tag{12.3}$$

In particular, the automorphism $j_{\hat{u}}^\vee$ coincides with ${}^{\mathsf{D}}j_{\hat{u}}$ on the extension ${}^{\mathsf{D}}\tilde{T}$. If $E_1, E_2 \in \mathsf{Ext}(F^\times, T^\vee)$, and $\eta \colon E_1 \to E_2$ is a morphism of extensions, then

$$j_{\hat{u}}^\vee \circ \eta = \eta \circ j_{\hat{u}}^\vee.$$

From this, we directly obtain compatibility of $\sigma_{\mathscr{B}}$ with automorphisms.

Proposition 12.4. — *The isomorphisms $\sigma_{\mathscr{B}}$ are compatible with automorphisms $j_{\hat{u}}$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} F^\times \times_\theta T^\vee & \xrightarrow{\sigma_{\mathscr{B}}} & {}^{\mathsf{D}}\tilde{T} \\ \downarrow{\scriptstyle j_{\hat{u}}^\vee} & & \downarrow{\scriptstyle j_{\hat{u}}^\vee = {}^{\mathsf{D}}j_{\hat{u}}} \\ F^\times \times_\theta T^\vee & \xrightarrow{\sigma_{\mathscr{B}}} & {}^{\mathsf{D}}\tilde{T}. \end{array}$$

Next we consider a morphism $\iota_{\mathscr{B},A} \colon \tilde{\mathbf{T}}_0 \to \tilde{\mathbf{T}}$, given by two elements $C_0, C \in X \otimes X$ with $C(y,y) = C_0(y,y)$ for all $y \in Y$. The matrix $(a_{\mathscr{B}}^{ij})$ is defined by $a_{\mathscr{B}}^{ij} = c_{\mathscr{B}}^{ij} - c_{\mathscr{B},0}^{ij}$, and the morphism of covers $\iota_{\mathscr{B},A}$ induces an isomorphism $\iota_{\mathscr{B},A} \colon \varepsilon_* \tilde{T}_0 \to \varepsilon_* \tilde{T}$,

$$\iota_{\mathscr{B},A}(t,z) = \left(t, z \cdot \prod_{i<j} \mathrm{Hilb}_n^\varepsilon(x_i(t), x_j(t))^{a_{\mathscr{B}}^{ij}} \right).$$

Write $\sigma_{0,\mathscr{B}} \colon F^\times \times_\theta T^\vee \to {}^{\mathsf{D}}\tilde{T}_0$ for the isomorphism defined analogously to $\sigma_{\mathscr{B}}$.

Proposition 12.5. — *The isomorphisms $\sigma_{\mathscr{B}}$ and $\sigma_{0,\mathscr{B}}$ fit into a commutative diagram:*

$$\begin{array}{ccc} F^\times \times_\theta T^\vee & \xrightarrow{\sigma_{\mathscr{B}}} & {}^{\mathsf{D}}\tilde{T} \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle {}^{\mathsf{D}}\iota_{\mathscr{B},A}} \\ F^\times \times_\theta T^\vee & \xrightarrow{\sigma_{0,\mathscr{B}}} & {}^{\mathsf{D}}\tilde{T}_0. \end{array}$$

Proof. — It suffices to check that the diagram commutes for elements of the form $(u,1) \in F^\times \times_\theta T^\vee$. Observe that the vertical map ${}^{\mathsf{D}}\iota_{\mathscr{B},A}$ is the restriction of a homomorphism $[X \otimes \iota_{\mathscr{B},A}] \colon X \otimes \varepsilon_* \tilde{T}_0 \to X \otimes \varepsilon_* \tilde{T}$,

$$[X \otimes \iota_{\mathscr{B},A}] \left(\prod_\ell (t_\ell, 1)^{x_\ell} \right) = \prod_\ell \left(t_\ell, \prod_{i<j} \mathrm{Hilb}_n^\varepsilon (x_i(t_\ell), x_j(t_\ell))^{a_{\mathscr{B}}^{ij}} \right)^{x_\ell}.$$

If $(u,1) \in F^\times \times_\theta T^\vee$, then

$$\sigma_{\mathscr{B}}(u,1) = s_{\mathscr{B}}^{\otimes} \left(\prod_\ell u^{x_\ell \otimes y_\ell} \right) = \prod_\ell (u^{y_\ell}, 1)^{x_\ell} \in X \otimes \varepsilon_* \tilde{T}.$$

Similarly,

$$\sigma_{0,\mathscr{B}}(u,1) = \prod_\ell (u^{y_\ell}, 1)^{x_\ell} \in X \otimes \varepsilon_* \tilde{T}_0.$$

Thus we find

$$\begin{aligned} {}^{\mathsf{D}}\iota_{\mathscr{B},A}(\sigma_{0,\mathscr{B}}(u,1)) &= [X \otimes \iota_{\mathscr{B},A}] \left(\prod_\ell (u^{y_\ell}, 1)^{x_\ell} \right) \\ &= \prod_\ell \left(u^{y_\ell}, \prod_{i<j} \mathrm{Hilb}_n^\varepsilon (x_i(u^{y_\ell}), x_j(u^{y_\ell}))^{a_{\mathscr{B}}^{ij}} \right)^{x_\ell} \\ &= \prod_\ell (u^{y_\ell}, 1)^{x_\ell} = \sigma_{\mathscr{B}}(u,1). \end{aligned}$$

The simplification of the inner product is based on the following observation: $x_i(u^{y_\ell}) = 1$ unless $i = \ell$ and $x_j(u^{y_\ell}) = 1$ unless $j = \ell$, but the product is indexed by only those i,j with $i < j$. □

12.3. Change of basis. — If $\mathscr{B}' = (y_1', \ldots, y_r')$ is another ordered basis of Y, then we find another isomorphism, $\sigma_{\mathscr{B}'} \colon F^\times \times_\theta T^\vee \to {}^{\mathsf{D}}\tilde{T}$. Consider the change of basis matrix $(g_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$, so that $x_i' = \sum_j g_{ij} x_j$ and $\sum_i g_{ij} y_i' = y_j$. For what follows, define

$$\begin{aligned} \Delta_{ij} &= c_{\mathscr{B}'}^{ii} \frac{g_{ij}(g_{ij}-1)}{2}, \quad \Gamma_j^{k\ell} = c_{\mathscr{B}'}^{k\ell} g_{kj} g_{\ell j}, \\ \chi(u) &= \mathrm{Hilb}_n^\varepsilon(u,u) \text{ for all } u \in F^\times. \end{aligned} \tag{12.4}$$

Proposition 12.6. — *Define the automorphism $f_{\mathscr{B},\mathscr{B}'} \in \mathrm{Aut}(F^\times \times_\theta T^\vee)$, in the category $\mathsf{Ext}(F^\times, T^\vee)$,*

$$f_{\mathscr{B},\mathscr{B}'}(u, t^\vee) = \left(u, t^\vee \cdot \prod_j \chi(u)^{\left(\sum_i \Delta_{ij} + \sum_{k<\ell} \Gamma_j^{k\ell}\right) x_j} \right).$$

This fits into a commutative diagram in $\mathsf{Ext}(F^\times, T^\vee)$,

$$\begin{tikzcd}
F^\times \times_\theta T^\vee \arrow[dd, "f_{\mathscr{B},\mathscr{B}'}"] \arrow[dr, "\sigma_{\mathscr{B}}"] & \\
& {}^{\mathsf{D}}\tilde{T} \\
F^\times \times_\theta T^\vee . \arrow[ur, "\sigma_{\mathscr{B}'}"'] &
\end{tikzcd}$$

Proof. — There exists a unique isomorphism $f_{\mathscr{B},\mathscr{B}'}$ making the diagram above commute, given by $f_{\mathscr{B},\mathscr{B}'}(u, t^\vee) = (u, \delta(u)t^\vee)$, where $\delta(u) = s_{\mathscr{B}}^{\otimes}(\iota(u)) \cdot s_{\mathscr{B}'}^{\otimes}(\iota(u))^{-1}$. Now we compute

$$\begin{aligned}
s_{\mathscr{B}'}^{\otimes}(\iota(u)) &= s_{\mathscr{B}'}^{\otimes}\left(\prod_i u^{x_i'\otimes y_i'}\right), \\
&= \prod_{i=1}^r (u^{y_i'}, 1)^{x_i'}, \quad \text{(noting that } (u^{y_i'}, 1) \in \varepsilon_* \tilde{T} = T \times_{\theta_{C,\varepsilon}} \mathbb{C}^\times) \\
&= \prod_{i=1}^r (u^{y_i'}, 1)^{\sum_{j=1}^r g_{ij} x_j}, \\
&= \prod_{i,j=1}^r (u^{y_i'}, 1)^{g_{ij} x_j}, \\
&= \prod_j \prod_i \left(u^{g_{ij} y_i'}, 1\right)^{x_j} \cdot \prod_{i,j} \chi(u)^{\Delta_{ij} \cdot x_j}, \quad \text{by (12.1)} \\
&= \prod_j \left(\prod_i u^{g_{ij} y_i'}, 1\right)^{x_j} \cdot \prod_{i,j} \chi(u)^{\Delta_{ij} \cdot x_j} \cdot \prod_j \prod_{k<\ell} \chi(u)^{\Gamma_j^{k\ell} x_j}, \\
&= \prod_j (u^{y_j}, 1)^{x_j} \cdot \prod_j \chi(u)^{(\sum_i \Delta_{ij} + \sum_{k<\ell} \Gamma_j^{k\ell}) x_j}, \\
&= s_{\mathscr{B}}^{\otimes}\left(\prod_j u^{x_j \otimes y_j}\right) \cdot \prod_j \chi(u)^{(\sum_i \Delta_{ij} + \sum_{k<\ell} \Gamma_j^{k\ell}) x_j}, \\
&= s_{\mathscr{B}}^{\otimes}(\iota(u)) \cdot \prod_j \chi(u)^{(\sum_i \Delta_{ij} + \sum_{k<\ell} \Gamma_j^{k\ell}) x_j}.
\end{aligned}$$

The proposition follows immediately. □

12.4. Parameters. — We begin with the same data, a degree n sharp cover $\tilde{\mathbf{T}}$ of a split torus $\mathbf{T}$ over F, incarnated by $C \in X \otimes X$, and an ordered basis $\mathscr{B} = (y_1, \ldots, y_r)$ of the cocharacter lattice Y. The second Brylinski-Deligne invariant of $\tilde{\mathbf{T}}$ is an extension $\mathscr{G}_m \hookrightarrow \mathscr{D} \twoheadrightarrow Y$ of sheaves on $F_{\text{ét}}$, and we can describe $\mathscr{D}$ in terms of $C \in X \otimes X$.

Tracing through the construction of the second Brylinski-Deligne invariant [**18**, §3.12], consider $\mathbf{T}'(F(\!(v)\!))$ for a formal parameter v. This fits into a short exact sequence

$$\mathbf{K}_2(F(\!(v)\!)) \hookrightarrow \mathbf{T}'(F(\!(v)\!)) \twoheadrightarrow \mathbf{T}(F(\!(v)\!)).$$

This extension is incarnated by the bimultiplicative cocycle θ_v satisfying

$$\theta_{\mathscr{B},v}(f^{y_i}, g^{y_j}) = \{f,g\}^{c^{ij}_{\mathscr{B}}}, \text{ for all } f,g \in F(\!(v)\!), 1 \leq i,j \leq r.$$

Pushing out via the tame symbol $\partial \colon \mathbf{K}_2(F(\!(v)\!)) \to \mathbf{K}_1(F) = F^\times$, and pulling back via $Y \to \mathbf{T}(F(\!(v)\!))$, $y \mapsto v^y$, we find the extension $F^\times \hookrightarrow \mathscr{D}[F] \twoheadrightarrow Y$. This extension is incarnated by the bimultiplicative cocycle $\theta_{\mathscr{B},D}$ satisfying

$$\theta_{\mathscr{B},D}(y_i,y_j) = \partial\{v,v\}^{c^{ij}_{\mathscr{B}}} = (-1)^{c^{ij}_{\mathscr{B}}}.$$

This construction applies equally well over a finite Galois extension of F. Thus the basis $\mathscr{B}$ determines an isomorphism

$$\sigma_{\mathscr{B},D} \colon Y \times_{\theta_{\mathscr{B},D}} \mathscr{G}_m \to \mathscr{D},$$

of extensions of sheaves of groups on $F_{\text{ét}}$. Sharpness of $\tilde{\mathbf{T}}$ implies that $\mathscr{D}$ is a sheaf of *abelian* groups on $F_{\text{ét}}$.

The choice of basis $\mathscr{B}$ gives a splitting of $\mathscr{D}$: since Y is a constant sheaf of free abelian groups, there exists a unique splitting $d_{\mathscr{B}} \colon Y \to \mathscr{D}$ which satisfies $d_{\mathscr{B}}(y_i) = \sigma_{\mathscr{B},D}(y_i, 1)$, for all $1 \leq i \leq r$. Thus the choice of basis trivializes the $\mathscr{H}\!om(Y, \mathscr{G}_m) = \hat{\mathscr{T}}$-torsor $\mathscr{S}\!pl(\mathscr{D})$. This trivialization, in turn, neutralizes the gerbe of nth roots $\sqrt[n]{\mathscr{S}\!pl(\mathscr{D})}$ over F. Explicitly, define $z_{\mathscr{B}} \in \sqrt[n]{\mathscr{S}\!pl(\mathscr{D})}[F]$ by $z_{\mathscr{B}} = (\hat{\mathscr{T}}, h_{\mathscr{B}})$ with $h_{\mathscr{B}} \colon \hat{\mathscr{T}} \to \mathscr{S}\!pl(\mathscr{D})$ given by

$$h_{\mathscr{B}}(\hat{a}) = \hat{a}^n * d_{\mathscr{B}}. \tag{12.5}$$

This neutralizes the pushout $\mathsf{E}_\varepsilon(\tilde{\mathbf{T}})$, and we find an isomorphism in $\mathsf{CExt}(\mathrm{Gal}_F, T^\vee)$,

$$\lambda_{\mathscr{B}} \colon \mathrm{Gal}_F \times T^\vee \to \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), z_{\mathscr{B}}). \tag{12.6}$$

The Baer sum with $(\tau_Q)_* \widetilde{\mathrm{Gal}}_F$ yields an isomorphism in $\mathsf{CExt}(\mathrm{Gal}_F, T^\vee)$,

$$\lambda_{\mathscr{B}} \colon (\tau_Q)_* \widetilde{\mathrm{Gal}}_F \to {}^{\mathsf{L}}\tilde{T}_{z_{\mathscr{B}}}. \tag{12.7}$$

Here ${}^{\mathsf{L}}\tilde{T}_{z_{\mathscr{B}}}$ denotes the L-group with respect to the base point $z_{\mathscr{B}}$. Note that $(\tau_Q)_* \widetilde{\mathrm{Gal}}_F = \mathrm{Gal}_F \times T^\vee$ as sets, with multiplication given by the cocycle,

$$(\gamma_1, \gamma_2) \mapsto \tau_Q\left(\mathrm{Hilb}_2(\mathrm{rec}\,\gamma_1, \mathrm{rec}\,\gamma_2)\right).$$

From (12.7), taking the quotient by $[\mathrm{Gal}_F, \mathrm{Gal}_F]$, and pulling back via $F^\times \to \mathscr{W}_F^{\mathrm{ab}} \to \mathrm{Gal}_F^{\mathrm{ab}}$, we find an isomorphism in $\mathsf{Ext}(F^\times, T^\vee)$,

$$\lambda_{\mathscr{B}} \colon F^\times \times_\theta T^\vee \to {}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}}, \tag{12.8}$$

where we recall that $\theta(u,v) = \tau_Q(\mathrm{Hilb}_2(u,v))$. For what follows, it helps to describe $\lambda_{\mathscr{B}} \colon F^\times \times_\theta T^\vee \to {}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}}$ in more detail. For an element $(u,1) \in F^\times \times_\theta T^\vee$, take the following steps:

- Choose $\gamma \in \mathscr{W}_F \subset \mathrm{Gal}_F$ such that $\mathrm{rec}(\gamma) = u$.
- Since $z_{\mathscr{B}}$ is defined over F, $\mathrm{Hom}(z_{\mathscr{B}}, {}^\gamma z_{\mathscr{B}}) = \mathrm{Hom}(z_{\mathscr{B}}, z_{\mathscr{B}})$. Thus we have an element $\mathrm{Id}_{\mathscr{B},\gamma} \in \mathrm{Hom}(z_{\mathscr{B}}, {}^\gamma z_{\mathscr{B}}) \subset \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), z_{\mathscr{B}})$.
- We also have an element $(\gamma, 1) \in (\tau_Q)_* \widetilde{\mathrm{Gal}}_F$ lying over $\gamma \in \mathrm{Gal}_F$. Hence
$$((\gamma,1), \mathrm{Id}_{\mathscr{B},\gamma}) \in {}^{\mathsf{L}}\tilde{T}_{z_{\mathscr{B}}} = (\tau_Q)_* \widetilde{\mathrm{Gal}}_F \dotplus \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{T}}), z_{\mathscr{B}}).$$
- $\lambda_{\mathscr{B}}(u,1)$ is the image of $((\gamma,1), \mathrm{Id}_{\mathscr{B},\gamma})$ in the quotient ${}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}}$.

As in Proposition 12.4, the isomorphisms $\lambda_{\mathscr{B}}$ are compatible with automorphisms of covers. If $\hat{u} \in X \otimes F^\times$, then we have the resulting automorphism $j_{\hat{u}} \in \mathrm{Aut}(\tilde{\mathbf{T}})$. As before, let $j_{\hat{u}}^\vee$ denote the automorphism of any extension in $\mathsf{Ext}(F^\times, T^\vee)$ associated to $\hat{u}$. Since $j_{\hat{u}}$ is well-aligned, we also get an automorphism of the L-group ${}^{\mathsf{L}}\tilde{T}_{z_{\mathscr{B}}}$, which descends to an automorphism ${}^{\mathsf{A}}j_{\hat{u}}$ of ${}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}}$.

Proposition 12.7. — *For all $\hat{u} \in X \otimes F^\times$, $j_{\hat{u}}^\vee = {}^{\mathsf{A}}j_{\hat{u}}$, making the following diagram commute:*

$$\begin{array}{ccc} F^\times \times_\theta T^\vee & \xrightarrow{\lambda_{\mathscr{B}}} & {}^{\mathsf{A}}\tilde{T}_{\mathscr{B}} \\ \downarrow{\scriptstyle j_{\hat{u}}^\vee} & & \downarrow{\scriptstyle j_{\hat{u}}^\vee = {}^{\mathsf{A}}j_{\hat{u}}} \\ F^\times \times_\theta T^\vee & \xrightarrow{\lambda_{\mathscr{B}}} & {}^{\mathsf{A}}\tilde{T}_{\mathscr{B}}. \end{array}$$

Proof. — We trace the automorphism $j_{\hat{u}}$ through the construction of ${}^{\mathsf{L}}\tilde{T}_{z_{\mathscr{B}}}$. It has no effect on $(\tau_Q)_* \widetilde{\mathrm{Gal}}_F$. On the other hand, the automorphism $j_{\hat{u}}$ corresponds to the automorphism of $\mathscr{G}_m \hookrightarrow \mathscr{D} \twoheadrightarrow Y$,
$$d \mapsto d \cdot \hat{u}(y), \text{ for all } d \in \mathscr{D} \text{ lying over } y \in Y.$$
Here and later, $\hat{u} \in X \otimes F^\times = \mathrm{Hom}(Y, F^\times)$, so $\hat{u}(y) \in F^\times$.

If $s \in \mathscr{S}\!p\ell(\mathscr{D})$, then $j_{\hat{u}}^*(s) = \hat{u}^{-1} * s$ is the pullback of s via $j_{\hat{u}}$. (See Section 3.4 for conventions on well-aligned functoriality). From this, $j_{\hat{u}}$ gives an equivalence of gerbes $\sqrt[n]{j_{\hat{u}}^*}\colon \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})} \to \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})}$, which sends the base point $z_{\mathscr{B}} = (\hat{\mathscr{T}}, h_{\mathscr{B}})$ to $z'_{\mathscr{B}} = (\hat{\mathscr{T}}, \hat{u} \circ h_{\mathscr{B}})$. Here $\hat{u} \circ h_{\mathscr{B}}\colon \hat{\mathscr{T}} \to \mathscr{S}\!p\ell(\mathscr{D})$ satisfies
$$[\hat{u} \circ h_{\mathscr{B}}](\hat{a}) = \hat{a}^n \hat{u}^{-1} * d_{\mathscr{B}}. \tag{12.9}$$
Choose any isomorphism $\rho\colon z_{\mathscr{B}} \to z'_{\mathscr{B}}$ in $\sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})}[\bar{F}]$. Concretely, $\rho\colon \hat{\mathscr{T}} \to \hat{\mathscr{T}}$ must be a map of $\hat{\mathscr{T}}$-torsors, so there exists $\hat{\tau} \in \hat{\mathscr{T}}$ such that $\rho(\hat{a}) = \hat{\tau} * \hat{a}$, for all $\hat{a} \in \hat{\mathscr{T}}$. Moreover, since ρ intertwines $h_{\mathscr{B}}$ and $\hat{u} \circ h_{\mathscr{B}}$, (12.5) and (12.9) imply that $\hat{\tau}^n = \hat{u}$.

Now, for any $f \in \pi_1^{\text{ét}}(\sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})}, z_{\mathscr{B}}) = \mathrm{Hom}(z_{\mathscr{B}}, z_{\mathscr{B}})$, define an isomorphism $f' \in \pi_1^{\text{ét}}(\sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})}, z'_{\mathscr{B}})$ by the rule $f'(\hat{a}) = f(\hat{a})$ for all $\hat{a} \in \hat{\mathscr{T}}$. (Here we note that $z_{\mathscr{B}} = (\hat{\mathscr{T}}, h_{\mathscr{B}})$ and $z'_{\mathscr{B}} = (\hat{\mathscr{T}}, \hat{u} \circ h_{\mathscr{B}})$, so f and f' are both given by functions from $\hat{\mathscr{T}}$ to itself). Then f' is the image of f, under the equivalence of gerbes $\sqrt[n]{j_{\hat{u}}}$.

Define
$$^{\rho}f \in \pi_1^{\text{ét}}(\sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})}, z_{\mathscr{B}}) = {}^\gamma\rho^{-1} \circ f' \circ \rho \in \mathrm{Hom}(z_{\mathscr{B}}, {}^\gamma z_{\mathscr{B}}).$$

Then we compute

$$ {}^{\rho}f(\hat{a}) = {}^{\gamma}\rho^{-1}(f'(\rho(\hat{a}))) = {}^{\gamma}\rho^{-1}(f'(\hat{\tau} * \hat{a})) = \frac{\hat{\tau}}{\gamma(\hat{\tau})} * f(\hat{a}). $$

Pushing out via ε and taking the Baer sum with $(\tau_Q)_* \widetilde{\mathrm{Gal}}_F$, we find that $f \mapsto {}^{\rho}f$ induces an automorphism ${}^{\mathsf{L}}j_{\hat{u}}$ of the extension

$$ T^{\vee} \hookrightarrow {}^{\mathsf{L}}\tilde{T}_{z_{\mathscr{B}}} \twoheadrightarrow \mathrm{Gal}_F, $$

given by

$$ {}^{\mathsf{L}}j_{\hat{u}}(f) = \varepsilon\left(\frac{\sqrt[n]{\hat{u}}}{\gamma(\sqrt[n]{\hat{u}})}\right) \cdot f, $$

for all f lying over $\gamma \in \mathrm{Gal}_F$. For $u_\ell \in F^{\times}$ and $\gamma \in \mathscr{W}_F \subset \mathrm{Gal}_F$, the relationship between the Artin symbol and Hilbert symbol gives

$$ \frac{\gamma \sqrt[n]{u_\ell}}{\sqrt[n]{u_\ell}} = \mathrm{Hilb}_n(u_\ell, \mathrm{rec}\,\gamma) = \mathrm{Hilb}_n(\mathrm{rec}\,\gamma, u_\ell)^{-1}. $$

Hence, ${}^{\mathsf{L}}j_{\hat{u}}$ descends to an automorphism ${}^{\mathsf{A}}j_{\hat{u}}$ of the extension

$$ T^{\vee} \hookrightarrow {}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}} \twoheadrightarrow F^{\times}, $$

given by

$$ {}^{\mathsf{A}}j_{\hat{u}}(e) = e \cdot \prod_{\ell} \mathrm{Hilb}_n^{\varepsilon}(v, u_\ell)^{x_\ell}, $$

for all $e \in {}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}}$ lying over $v \in F^{\times}$. Hence ${}^{\mathsf{A}}j_{\hat{u}} = j_{\hat{u}}^{\vee}$ as defined in (12.3). □

Next, as in Proposition 12.5, consider two elements $C_0, C \in X \otimes X$ such that $C_0(y,y) = C(y,y)$ for all $y \in Y$. Let $A = C - C_0$, giving an isomorphism $\iota_{\mathscr{B},A} \colon \tilde{\mathbf{T}}_0 \to \tilde{\mathbf{T}}$ of the incarnated covers. Write $\mathscr{D}_0$, $d_{0,\mathscr{B}}$, $z_{0,\mathscr{B}}$, ${}^{\mathsf{A}}\tilde{T}_{0,z_{0,\mathscr{B}}}$, etc., for the analogs of $\mathscr{D}$, $d_{\mathscr{B}}$, $z_{\mathscr{B}}$, ${}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}}$, using C_0 instead of C.

Proposition 12.8. — *The homomorphism* ${}^{\mathsf{A}}\iota_{\mathscr{B},A} \colon {}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}} \to {}^{\mathsf{A}}\tilde{T}_{0,z_{0,\mathscr{B}}}$*, induced by the well-aligned isomorphism* $\iota_{\mathscr{B},A}$*, fits into a commutative square*

$$ \begin{array}{ccc} F^{\times} \times_{\theta} T^{\vee} & \xrightarrow{\lambda_{\mathscr{B}}} & {}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}} \\ \big\downarrow = & & \big\downarrow {}^{\mathsf{A}}\iota_{\mathscr{B},A} \\ F^{\times} \times_{\theta} T^{\vee} & \xrightarrow{\lambda_{0,\mathscr{B}}} & {}^{\mathsf{A}}\tilde{T}_{0,z_{0,\mathscr{B}}}. \end{array} $$

Proof. — The isomorphism $\iota_{\mathscr{B},A} \colon \tilde{\mathbf{T}}_0 \to \tilde{\mathbf{T}}$ gives an isomorphism of extensions below:

$$ \begin{array}{ccccc} \mathscr{G}_m & \hookrightarrow & \mathscr{D}_0 & \twoheadrightarrow & Y \\ \big\downarrow = & & \big\downarrow \iota_{\mathscr{B},A} & & \big\downarrow = \\ \mathscr{G}_m & \hookrightarrow & \mathscr{D} & \twoheadrightarrow & Y. \end{array} $$

Using the presentation of $\mathscr{D}_0$ and $\mathscr{D}$ by cocycles from C_0 and C, respectively, we find that

$$\iota_{\mathscr{B},A}(y,1) = \left(y, \prod_{i<j} (-1)^{\langle x_i, y\rangle \cdot \langle x_j, y\rangle \cdot a_{\mathscr{B}}^{ij}} \right),$$

for all $y \in Y$. Recall that the basis $\mathscr{B}$ gave splittings of $\mathscr{D}_0$ and $\mathscr{D}$ respectively, satisfying $d_0(y_i) = (y_i, 1)$ and $d(y_i) = (y_i, 1)$. Since $\iota_{\mathscr{B},A}(y_i,1) = (y_i,1)$, we find that $\iota_{\mathscr{B},A} \circ d_0 = d$.

We may use $\iota_{\mathscr{B},A}$ to pull back splittings of $\mathscr{D}$ to splittings of $\mathscr{D}_0$. We find that this map of $\hat{\mathscr{T}}$-torsors

$$\iota_{\mathscr{B},A}^* \colon \mathscr{S}\!p\ell(\mathscr{D}) \to \mathscr{S}\!p\ell(\mathscr{D}_0)$$

sends the trivialization d to the trivialization d_0. It follows that $\iota_{\mathscr{B},A}^*$ gives a functor of gerbes,

$$\iota_{\mathscr{B},A}^* \colon \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})} \to \sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D}_0)}$$

which sends the base point $z_{\mathscr{B}}$ to the base point $z_{0,\mathscr{B}}$. As both are base points defined over F, we find that $\iota_{\mathscr{B},A}^*$ sends the identity morphism $\mathrm{Id}_{\mathscr{B},\gamma} \in \mathrm{Hom}(z_{\mathscr{B}}, {}^\gamma z_{\mathscr{B}}) = \mathrm{Hom}(z_{\mathscr{B}}, z_{\mathscr{B}})$ to the corresponding identity morphism $\mathrm{Id}_{0,\mathscr{B},\gamma} \in \mathrm{Hom}(z_{0,\mathscr{B}}, {}^\gamma z_{0,\mathscr{B}})$.

It follows that ${}^{\mathsf{A}}\iota_{\mathscr{B},A}$ sends $\lambda_{\mathscr{B}}(u,1)$ to $\lambda_{0,\mathscr{B}}(u,1)$ for all $u \in F^\times$. The proposition follows. $\square$

12.5. Change of basis. — If $\mathscr{B}' = (y_1', \ldots, y_r')$ is another basis, then we find from it another splitting $d_{\mathscr{B}'} \colon Y \to \mathscr{D}$ characterized by $d_{\mathscr{B}}'(y_i') = (y_i', 1)$. This provides another trivialization of the torsor $\mathscr{S}\!p\ell(\mathscr{D})$, and thus another object $z_{\mathscr{B}'} = (\hat{\mathscr{T}}, h_{\mathscr{B}'})$ of $\sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})}[F]$.

An isomorphism $\iota \colon z_{\mathscr{B}} \to z_{\mathscr{B}'}$ in $\sqrt[n]{\mathscr{S}\!p\ell(\mathscr{D})}[\bar{F}]$ produces an isomorphism ${}^{\mathsf{L}}\iota \colon {}^{\mathsf{L}}T_{z_{\mathscr{B}}} \to {}^{\mathsf{L}}T_{z_{\mathscr{B}'}}$ in $\mathsf{CExt}(\mathrm{Gal}_F, T^\vee)$, and any two such isomorphisms ι_1, ι_2 produce the *same* isomorphism ${}^{\mathsf{L}}\iota_1 = {}^{\mathsf{L}}\iota_2$. It follows that there is a unique automorphism $r_{\mathscr{B},\mathscr{B}'}$ of $F^\times \times_\theta T^\vee$ in the category $\mathsf{Ext}(F^\times, T^\vee)$, which makes the following diagram commute:

$$\begin{tikzcd}
F^\times \times_\theta T^\vee \arrow[r, "\lambda_{\mathscr{B}}"] \arrow[d, "r_{\mathscr{B},\mathscr{B}'}"] & {}^{\mathsf{A}}T_{z_{\mathscr{B}}} \arrow[d, "{}^{\mathsf{A}}\iota"] \\
F^\times \times_\theta T^\vee \arrow[r, "\lambda_{\mathscr{B}'}"] & {}^{\mathsf{A}}T_{z_{\mathscr{B}'}}.
\end{tikzcd}$$

Consider the change of basis matrix $(g_{ij}) \in \mathrm{GL}_r(\mathbb{Z})$, so that $x_i' = \sum_j g_{ij} x_j$ and $\sum_i g_{ij} y_i' = y_j$. Define Δ_{ij}, $\Gamma_j^{k\ell}$, and χ as in (12.4).

Proposition 12.9. — *The automorphism $r_{\mathscr{B},\mathscr{B}'} \in \mathrm{Aut}(F^\times \times_\theta T^\vee)$ satisfies*

$$r_{\mathscr{B},\mathscr{B}'}(u, t^\vee) = \left(u, t^\vee \cdot \prod_j \chi(u)^{\left(\sum_i \Delta_{ij} + \sum_{k<\ell} \Gamma_j^{k\ell}\right) \cdot x_j} \right).$$

Proof. — Giving an isomorphism $\iota\colon z_{\mathscr{B}} \to z_{\mathscr{B}'}$ is the same as giving a morphism $f\colon \hat{\mathscr{T}} \to \hat{\mathscr{T}}$ of $\hat{\mathscr{T}}$-torsors, $f(\hat{a}) = \hat{e} \cdot \hat{a}$, satisfying $d_{\mathscr{B}'} = \hat{e}^n * d_{\mathscr{B}}$. This condition on $\hat{e}$ is equivalent to the following, for all $1 \leq j \leq r$.

$$\begin{aligned}
\hat{e}(y_j)^n &= d_{\mathscr{B}}(y_j)^{-1} \cdot d_{\mathscr{B}'}(y_j), \\
&= (y_j, 1)^{-1} \cdot d_{\mathscr{B}'} \left(\sum_i g_{ij} y_i' \right), \\
&= (y_j, 1)^{-1} \cdot \prod_i (y_i', 1)^{g_{ij}}, \\
&= (y_j, 1)^{-1} \cdot \prod_i (g_{ij} y_i', 1) \cdot (-1)^{\Delta_{ij}}, \\
&= (y_j, 1)^{-1} \cdot \left(\sum_i g_{ij} y_i', 1 \right) \cdot \prod_i (-1)^{\Delta_{ij}} \cdot \prod_{k<\ell} (-1)^{\Gamma_j^{k\ell}}, \\
&= \prod_i (-1)^{\Delta_{ij}} \cdot \prod_{k<\ell} (-1)^{\Gamma_j^{k\ell}}.
\end{aligned}$$

In other words, giving a morphism $\iota\colon z_{\mathscr{B}} \to z_{\mathscr{B}'}$ (defined over $\bar{F}$) is the same as giving an element $\hat{e} \in \hat{\mathscr{T}}[\bar{F}] = X \otimes \bar{F}^\times$ satisfying

$$\hat{e}^n = \prod_j (-1)^{\left(\sum_i \Delta_{ij} + \sum_{k<\ell} \Gamma_j^{k\ell}\right) \cdot x_j}. \tag{12.10}$$

Now consider an element $(u, 1) \in F^\times \times_\theta T^\vee$. Let $\gamma \in \mathscr{W}_F$ be an element such that $\operatorname{rec}(\gamma) = u$, and $\mathrm{Id}_{\mathscr{B},\gamma} \in \operatorname{Hom}(z_{\mathscr{B}}, {}^\gamma z_{\mathscr{B}})$ the resulting element of $\pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), z_{\mathscr{B}})$. Similarly, let $\mathrm{Id}_{\mathscr{B}',\gamma} \in \operatorname{Hom}(z_{\mathscr{B}'}, {}^\gamma z_{\mathscr{B}'})$ be the resulting element of $\pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), z_{\mathscr{B}'})$

The following diagram of objects and morphisms in $\sqrt[n]{\mathscr{S}pl(\mathscr{D})}$ commutes

$$\begin{array}{ccc}
z_{\mathscr{B}} & \xrightarrow{\mathrm{Id}_{\mathscr{B},\gamma}} & {}^\gamma z_{\mathscr{B}} = z_{\mathscr{B}} \\
{\scriptstyle \hat{e}}\downarrow & & \downarrow{\scriptstyle {}^\gamma \hat{e}} \\
z_{\mathscr{B}'} & \xrightarrow{({}^\gamma \hat{e}/\hat{e}) \cdot \mathrm{Id}_{\mathscr{B}',\gamma}} & {}^\gamma z_{\mathscr{B}'} = z_{\mathscr{B}'}.
\end{array}$$

Here we label morphisms among $z_{\mathscr{B}}$ and $z_{\mathscr{B}}'$ by elements of $\hat{\mathscr{T}}$, since all morphisms are morphisms of $\hat{\mathscr{T}}$-torsors from $\hat{\mathscr{T}}$ to itself. Projecting from ${}^{\mathsf{L}}\tilde{T}_{z_{\mathscr{B}}}$ to ${}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}}$, we find that

$${}^{\mathsf{A}}\iota\left(\lambda_{\mathscr{B}}^{\mathsf{A}}(u, 1)\right) = \frac{{}^\gamma \hat{e}}{\hat{e}} \cdot \lambda_{\mathscr{B}'}^{\mathsf{A}}(u, 1).$$

The relationship between the Artin symbol and Hilbert symbol gives

$$\varepsilon \left(\frac{\gamma(\sqrt[n]{-1})}{\sqrt[n]{-1}} \right) = \mathrm{Hilb}_n^\varepsilon(-1, u) = \mathrm{Hilb}_n^\varepsilon(u, u) = \chi(u),$$

since $\mathrm{rec}(\gamma) = u$. Hence Equation (12.10) yields

$$^{\mathsf{A}}\iota(\lambda^{\mathsf{A}}_{\mathscr{B}}(u,1)) = \lambda^{\mathsf{A}}_{\mathscr{B}'}(u,1) \cdot \prod_j \chi(u)^{(\sum_i \Delta_{ij} + \sum_{k<\ell} \Gamma_j^{k\ell}) \cdot x_j}.$$

The proposition follows. □

12.6. Isomorphism of functors. — For each basis $\mathscr{B}$ of Y, and each cover $\tilde{\mathbf{T}}$ of $\mathbf{T}$ incarnated by C, we have constructed isomorphisms $\sigma_{\mathscr{B}} \colon F^\times \times_\theta T^\vee \to {}^{\mathsf{D}}\tilde{T}$ and $\lambda_{\mathscr{B}} \colon F^\times \times_\theta T^\vee \to {}^{\mathsf{A}}\tilde{T}$. Here we write ${}^{\mathsf{A}}\tilde{T}$ rather than ${}^{\mathsf{A}}\tilde{T}_{z_{\mathscr{B}}}$, since ${}^{\mathsf{A}}\tilde{T}$ is defined up to unique isomorphism by $\tilde{\mathbf{T}}$ without reference to base point.

There exists a unique isomorphism $\mathscr{P}_{\mathscr{B}}$ in $\mathsf{Ext}(F^\times, T^\vee)$ which makes the following diagram commute

$$\begin{tikzcd} & {}^{\mathsf{D}}\tilde{T} \arrow[dd, "\mathscr{P}_{\mathscr{B}}"] \\ F^\times \times_\theta T^\vee \arrow[ur, "\sigma_{\mathscr{B}}"] \arrow[dr, "\lambda_{\mathscr{B}}"'] & \\ & {}^{\mathsf{A}}\tilde{T}. \end{tikzcd}$$

Theorem 12.10. — *If $\mathscr{B}$ and $\mathscr{B}'$ are two bases of Y, then $\mathscr{P}_{\mathscr{B}} = \mathscr{P}_{\mathscr{B}'}$.*

Proof. — Propositions 12.6 and 12.9 imply that the change of basis functions $f_{\mathscr{B},\mathscr{B}'}$ and $r_{\mathscr{B},\mathscr{B}'}$ coincide, yielding

$$\begin{aligned} \mathscr{P}_{\mathscr{B}'} &= \lambda_{\mathscr{B}'} \circ \sigma_{\mathscr{B}'}^{-1} \\ &= \lambda_{\mathscr{B}} \circ r_{\mathscr{B},\mathscr{B}'} \circ (\sigma_{\mathscr{B}} \circ f_{\mathscr{B},\mathscr{B}'}^{-1})^{-1} \\ &= \lambda_{\mathscr{B}} \circ r_{\mathscr{B},\mathscr{B}'} f_{\mathscr{B},\mathscr{B}'} \circ \sigma_{\mathscr{B}}^{-1} \\ &= \lambda_{\mathscr{B}} \circ \sigma_{\mathscr{B}}^{-1} = \mathscr{P}_{\mathscr{B}}. \end{aligned}$$

Here we use the fact that $r_{\mathscr{B},\mathscr{B}'}$ and $f_{\mathscr{B},\mathscr{B}'}$ not only coincide, but they have order 1 or 2 (the character χ has values in ± 1). Thus $r_{\mathscr{B},\mathscr{B}'} f_{\mathscr{B},\mathscr{B}'} = \mathrm{Id}$ in $\mathrm{Aut}(F^\times \times_\theta T^\vee)$. □

From this theorem, we write simply $\mathscr{P} \colon {}^{\mathsf{D}}\tilde{T} \xrightarrow{\sim} {}^{\mathsf{A}}\tilde{T}$ without reference to a basis. Recall that $\mathbf{\Pi}^{\mathsf{D}}_{\varepsilon}(\tilde{\mathbf{T}}) = \mathrm{Spl}({}^{\mathsf{D}}\tilde{T})$, and $\mathbf{\Phi}^{\mathsf{A}}_{\varepsilon}(\tilde{\mathbf{T}}) = \mathrm{Spl}({}^{\mathsf{A}}\tilde{T})$. Thus $\mathscr{P}$ defines an isomorphism of $\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})$-torsors,

$$\mathscr{P}(\tilde{\mathbf{T}}) \colon \mathbf{\Pi}^{\mathsf{D}}_{\varepsilon}(\tilde{\mathbf{T}}) \xrightarrow{\sim} \mathbf{\Phi}^{\mathsf{A}}_{\varepsilon}(\tilde{\mathbf{T}}).$$

Theorem 12.11. — *Allowing the cover to vary, the system of isomorphisms $\mathscr{P}$ defines a natural isomorphism of additive functors,*

$$\mathscr{P} \colon \mathsf{Cov}^{\sharp}_n(\mathbf{T}) \overset{\sim}{\Rightarrow} \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})).$$

Proof. — It suffices to work with the full subcategory of $\mathsf{Cov}^{\sharp}_n$ given by the incarnated covers—those that arise from elements $C \in X \otimes X$. We have constructed isomorphisms $\mathscr{P} \colon \mathbf{\Pi}^{\mathsf{D}}_{\varepsilon}(\tilde{\mathbf{T}}) \xrightarrow{\sim} \mathbf{\Phi}^{\mathsf{A}}_{\varepsilon}(\tilde{\mathbf{T}})$ for each such incarnated cover $\tilde{\mathbf{T}}$. For naturality, recall that all morphisms among incarnated covers are composites of automorphisms $j_{\hat{u}}$ for $\hat{u} \in X \otimes F^\times$ and isomorphisms $\iota_{\mathscr{B},A}$ for alternating forms $A \in X \otimes X$.

Consider first an automorphism $j_{\hat{u}}$ and a fixed cover $\tilde{\mathbf{T}}$. In the diagram below, Proposition 12.4 gives the commutativity of the top face and Proposition 12.7 the bottom face. Front and back faces commute by the definition of $\mathscr{P}$. Hence the right face commutes:

$$\begin{tikzcd}
 & & & {}^{\mathsf{D}}\tilde{T} \arrow[dd, "\mathscr{P}"] \\
 & F^\times \times_\theta T^\vee \arrow[urr, "\sigma_{\mathscr{B}}"] \arrow[drr, dashed, "\lambda_{\mathscr{B}}"'] & {}^{\mathsf{D}}\tilde{T} \arrow[ur, "{}^{\mathsf{D}}j_{\hat{u}}"'] \arrow[dd, "\mathscr{P}"] & \\
F^\times \times_\theta T^\vee \arrow[ur, "j_{\hat{u}}^\vee"] \arrow[urr, "\sigma_{\mathscr{B}}"'] \arrow[drr, "\lambda_{\mathscr{B}}"] & & & {}^{\mathsf{A}}\tilde{T} \\
 & & {}^{\mathsf{A}}\tilde{T}. \arrow[ur, "{}^{\mathsf{A}}j_{\hat{u}}"'] &
\end{tikzcd}$$

Next consider $C_0, C \in X \otimes X$ such that $C_0(y,y) = C(y,y)$, and let $A = C - C_0$. This gives an isomorphism $\iota_{\mathscr{B},A} \colon \tilde{\mathbf{T}}_0 \to \tilde{\mathbf{T}}$. In the diagram below, Proposition 12.5 gives commutativity of the top triangle and Proposition 12.8 the bottom. Left and right triangles commute by the definition of $\mathscr{P}$. Hence the outer square commutes:

$$\begin{tikzcd}
{}^{\mathsf{D}}\tilde{T} \arrow[rr, "{}^{\mathsf{D}}\iota_{\mathscr{B},A}"] \arrow[dd, "\mathscr{P}(\mathbf{T})"'] & & {}^{\mathsf{D}}\tilde{T}_0 \arrow[dd, "\mathscr{P}(\mathbf{T}_0)"] \\
 & F^\times \times_\theta T^\vee \arrow[ul, "\sigma_{\mathscr{B}}"] \arrow[ur, "\sigma_{0,\mathscr{B}}"'] \arrow[dl, "\lambda_{\mathscr{B}}"'] \arrow[dr, "\lambda_{0,\mathscr{B}}"] & \\
{}^{\mathsf{A}}\tilde{T} \arrow[rr, "{}^{\mathsf{A}}\iota_{\mathscr{B},A}"'] & & {}^{\mathsf{A}}\tilde{T}_0.
\end{tikzcd}$$

By Proposition 12.1, commutativity of the previous two diagrams implies that $\mathscr{P}$ is a natural isomorphism of functors.

To check that $\mathscr{P}$ is a natural isomorphism of *additive* functors, we may use the monoidal structure on incarnated covers arising from addition of elements of $X \otimes X$. In other words, if $\tilde{\mathbf{T}}_1$ is incarnated by $C_1 \in X \otimes X$, and $\tilde{\mathbf{T}}_2$ is incarnated by $C_2 \in X \otimes X$, then $\tilde{\mathbf{T}} = \tilde{\mathbf{T}}_1 \dotplus \tilde{\mathbf{T}}_2$ may be identified with the cover incarnated by $C_1 + C_2$. Such identifications allow us to identify the following, for incarnated covers:

$${}^{\mathsf{D}}\tilde{T}_1 \dotplus {}^{\mathsf{D}}\tilde{T}_2 \equiv {}^{\mathsf{D}}\tilde{T}, \quad {}^{\mathsf{A}}\tilde{T}_1 \dotplus {}^{\mathsf{A}}\tilde{T}_2 \equiv {}^{\mathsf{A}}\tilde{T}.$$

As $\mathscr{P}$ is defined by identifying extensions given by the same cocycle θ, we find that $\mathscr{P}$ is compatible with the additive structure of the functors $\mathbf{\Pi}_\varepsilon^{\mathsf{D}}$ and $\mathbf{\Phi}_\varepsilon^{\mathsf{A}}$. $\square$

The natural isomorphism $\mathscr{P}$ is compatible with pullbacks in the following way. If $\alpha \colon \mathbf{S} \to \mathbf{T}$ is an isomorphism of split tori, and $\tilde{\mathbf{T}} \in \mathsf{Cov}_n^\sharp(\mathbf{T})$ is incarnated by $C \in X \otimes X$, and $\tilde{\mathbf{S}}$ is the pullback of $\tilde{\mathbf{T}}$, then $\tilde{\mathbf{S}}$ is incarnated by the pullback of C. If $\mathscr{B}$ is a basis of Y, then we may pull back $\mathscr{B}$ to form a basis of the cocharacter lattice of $\mathbf{S}$. We find a commutative diagram of groups and isomorphisms:

$$\begin{tikzcd}
 & & & {}^{\mathsf{D}}\tilde{S} \arrow[dd, "\mathscr{P}_S"] \\
 & F^\times \times_\theta S^\vee \arrow[urr, "\sigma_{\mathscr{B},S}"] \arrow[drr, dashed, "\lambda_{\mathscr{B},S}"] & {}^{\mathsf{D}}\tilde{T} \arrow[ur, "{}^{\mathsf{D}}\tilde{\alpha}"] \arrow[dd, "\mathscr{P}_T"] & \\
F^\times \times_\theta T^\vee \arrow[ur, "\alpha^\vee"] \arrow[urr, "\sigma_{\mathscr{B},T}"] \arrow[drr, "\lambda_{\mathscr{B},T}"] & & & {}^{\mathsf{A}}\tilde{S} \\
 & & {}^{\mathsf{A}}\tilde{T}. \arrow[ur, "{}^{\mathsf{A}}\tilde{\alpha}"] &
\end{tikzcd}$$

Indeed, the definition of $\sigma_{\mathscr{B}}$ and $\lambda_{\mathscr{B}}$ (for S and T) yields the commutativity of the top and bottom squares. The front and back faces commute by definitions of $\mathscr{P}_T$ and $\mathscr{P}_S$, respectively. This makes the right face commute as well.

We find a commutative diagram of additive functors and natural isomorphisms:

$$\begin{tikzcd}
\alpha^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \arrow[r, Rightarrow, "\mathscr{P}"] \arrow[d, Rightarrow, "{}^{\mathsf{D}}\tilde{\alpha}"] & \alpha^* \circ \mathbf{\Phi}_\varepsilon^{\mathsf{A}} \arrow[d, Rightarrow, "{}^{\mathsf{A}}\tilde{\alpha}"] \\
\mathbf{\Pi}_\varepsilon^{\mathsf{D}} \circ \alpha^* \arrow[r, Rightarrow, "\mathscr{P}"] & \mathbf{\Phi}_\varepsilon^{\mathsf{A}} \circ \alpha^*.
\end{tikzcd} \tag{12.11}$$

This completes the chain of additive functors and natural isomorphisms, compatible with pullbacks throughout by (10.8), (12.11), (11.3):

$$\mathrm{rec}_F^* \circ \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \xRightarrow{\mathscr{D}} \mathrm{rec}_F^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \xRightarrow{\mathscr{P}} \mathrm{rec}_F^* \circ \mathbf{\Phi}_\varepsilon^{A} \xLeftarrow{\mathscr{A}} \mathbf{\Phi}_\varepsilon.$$

Assembling commutative diagrams, we have proven the following result.

Theorem 12.12. — *Define $\mathscr{L} = \mathrm{rec}_F^* \circ \mathrm{can}_Y$ and $\mathscr{L}_\varepsilon = \mathscr{A}^{-1} \circ \mathscr{P} \circ \mathscr{D}$. Then $\mathscr{L}_\varepsilon \colon \mathscr{L} \circ \mathbf{\Pi}_\varepsilon \Rightarrow \mathbf{\Phi}_\varepsilon$ is a natural isomorphism of additive functors:*

$$\begin{tikzcd}
 & \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})) \arrow[dd, "\mathscr{L}"] \\
\mathsf{Cov}_n^\sharp(\mathbf{T}) \arrow[ur, "\mathbf{\Pi}_\varepsilon"] \arrow[dr, "\mathbf{\Phi}_\varepsilon"'] \arrow[r, phantom, "\Downarrow \mathscr{L}_\varepsilon"] & \\
 & \mathsf{Tors}(\mathbf{\Phi}(\mathbf{T})).
\end{tikzcd}$$

When $\alpha\colon \mathbf{S} \to \mathbf{T}$ *is an isomorphism of split tori, the natural isomorphisms* $\mathscr{L}_\varepsilon$ *for* $\mathbf{S}$ *and* $\mathbf{T}$ *commute with pullback as in the diagram* (9.6)*.*

This theorem gives a parameterization of genuine characters by Weil parameters, functorial in the (sharp) cover, and compatibly with Langlands' parameterization of characters of algebraic tori over local fields.

13. The integral case

Now consider $\mathbf{T}$, a split torus over $\mathscr{O}$, the ring of integers in a nonarchimedean local field F. We may consider the category $\mathsf{Cov}_n^\sharp(\mathbf{T}/\mathscr{O})$ of sharp covers defined over $\mathscr{O}$. By [79], the objects of $\mathsf{Cov}_n(\mathbf{T}/\mathscr{O})$ are classified by pairs $(Q, \mathscr{D})$ just as in the case of tori over fields, and each object is isomorphic to one incarnated by an element $C \in X \otimes X$. The automorphism group of a cover $\tilde{\mathbf{T}} = (\mathbf{T}', n)$ is naturally isomorphic to $\mathrm{Hom}(Y, \mathscr{O}^\times) = X \otimes \mathscr{O}^\times$.

Given a sharp cover $\tilde{\mathbf{T}} = (\mathbf{T}', n)$ over $\mathscr{O}$, write $T^\circ = \mathbf{T}(\mathscr{O})$, $T = \mathbf{T}(F)$, and $\mu_n \hookrightarrow \tilde{T} \twoheadrightarrow T$ for the resulting extension of locally compact abelian groups. As $\tilde{\mathbf{T}}$ is defined over $\mathscr{O}$, this comes with a splitting $\sigma^\circ\colon T^\circ \hookrightarrow \tilde{T}$. Thus we may consider the set $\mathrm{Hom}_\varepsilon(\tilde{T}/\sigma^\circ(T^\circ), \mathbb{C}^\times) \subset \mathrm{Hom}_\varepsilon(\tilde{T}, \mathbb{C}^\times)$ of T°-spherical ε-genuine characters.

13.1. Parameterization by splittings. — We abbreviate

$$\mathbf{\Pi}^\circ(\mathbf{T}) = \mathbf{\Pi}(\mathbf{T}/\mathscr{O}) = \mathrm{Hom}(T/T^\circ, \mathbb{C}^\times) \equiv T^\vee,$$
$$\mathbf{\Pi}_\varepsilon^\circ(\tilde{\mathbf{T}}) = \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/\mathscr{O}) = \mathrm{Hom}_\varepsilon(\tilde{T}/\sigma^\circ(T^\circ), \mathbb{C}^\times).$$

Then $\mathbf{\Pi}_\varepsilon^\circ(\tilde{\mathbf{T}})$ is a $\mathbf{\Pi}^\circ(\mathbf{T})$-torsor. Write

$$\iota^\circ\colon \mathbf{\Pi}^\circ(\mathbf{T}) \hookrightarrow \mathbf{\Pi}(\mathbf{T}) = \mathbf{\Pi}(\mathbf{T}/F),$$

for the inclusion of the T°-spherical characters of T into the group of all characters of T. There is a natural isomorphism of $\mathbf{\Pi}(\mathbf{T})$-torsors,

$$(\iota^\circ)_* \mathbf{\Pi}_\varepsilon^\circ(\tilde{\mathbf{T}}) \xrightarrow{\sim} \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}).$$

We identified $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$ with the set of splittings of an extension ${}^{\mathsf{D}}\tilde{T}$. This works as well for T°-spherical characters as follows. The cover $\tilde{\mathbf{T}}$ over $\mathscr{O}$ yields a commutative diagram with exact rows, with top row split by σ°:

$$\begin{tikzcd}
\mathbb{C}^\times \arrow[r, hook] \arrow[d, "="] & \varepsilon_* \tilde{T}^\circ \arrow[r, two heads] \arrow[d, hook] & T^\circ \arrow[l, bend right, "\sigma^\circ"'] \arrow[d, hook] \\
\mathbb{C}^\times \arrow[r, hook] & \varepsilon_* \tilde{T} \arrow[r, two heads] & T.
\end{tikzcd}$$

Recalling that $\iota\colon \mathbb{Z} \hookrightarrow X \otimes Y$ was the canonical inclusion, and $\mathsf{D}_Y = \iota^*(X \otimes \bullet)$, we may tensor the above diagram by X and pull back via ι to form a commutative diagram with exact rows, again with top row split by a homomorphism we call ${}^{\mathsf{D}}\sigma^\circ$,

$$\begin{tikzcd} T^\vee \arrow[r, hook] \arrow[d, "="] & \mathsf{D}_Y \varepsilon^* \tilde{T}^\circ \arrow[r, two heads] \arrow[d, hook] & \mathcal{O}^\times \arrow[l, bend right, "{}^{\mathsf{D}}\sigma^\circ"'] \arrow[d, hook] \\ T^\vee \arrow[r, hook] & {}^{\mathsf{D}}\tilde{T} \arrow[r, two heads] & F^\times . \end{tikzcd}$$

Just as there is a natural bijection $\mathscr{D}\colon \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}) \to \mathbf{\Pi}_\varepsilon^{\mathsf{D}}(\tilde{\mathbf{T}}) = \mathrm{Spl}({}^{\mathsf{D}}\tilde{T})$, the T°-spherical characters correspond to those splittings from $\mathrm{Spl}({}^{\mathsf{D}}\tilde{T})$ which pull back to the splitting ${}^{\mathsf{D}}\sigma^\circ$ of the top row above. Thus define $\mathbf{\Pi}_\varepsilon^{\mathsf{D},\circ}(\tilde{\mathbf{T}})$ to be the subset of splittings pulling back to ${}^{\mathsf{D}}\sigma^\circ$. The commutative diagrams above extend to a diagram of categories and functors below.

Proposition 13.1. — *There is a natural isomorphism of additive functors,*

$$\mathscr{D}^\circ\colon \mathrm{can}_Y^\circ \circ \mathbf{\Pi}_\varepsilon^\circ \Rightarrow \mathbf{\Pi}_\varepsilon^{\mathsf{D},\circ}.$$

Proof. — Consider the following diagram of Picard categories and additive functors.

$$\begin{tikzcd} & & \mathsf{Tors}(T^\vee) \arrow[dd, "\iota_*^\circ"] \\ \mathsf{Cov}_n^\sharp(\mathbf{T}/\mathcal{O}) \arrow[r, "\mathbf{\Pi}_\varepsilon^\circ"'] \arrow[urr, "\mathbf{\Pi}_\varepsilon^{\mathsf{D},\circ}"] \arrow[dd, "\iota_*^\circ"] & \mathsf{Tors}(\mathbf{\Pi}^\circ(\mathbf{T})) \arrow[ur, "\mathrm{can}_Y^\circ"'] \arrow[dd, "\iota_*^\circ"] & \\ & & \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}(\mathbf{T})) \\ \mathsf{Cov}_n^\sharp(\mathbf{T}/F) \arrow[r, "\mathbf{\Pi}_\varepsilon"'] \arrow[urr, "\mathbf{\Pi}_\varepsilon^{\mathsf{D}}"] & \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})). \arrow[ur, "\mathrm{can}_Y"'] & \end{tikzcd}$$

The bottom face is the diagram (10.6), in which we found a natural isomorphism

$$\mathscr{D}\colon \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \Rightarrow \mathbf{\Pi}_\varepsilon^{\mathsf{D}}.$$

The lateral faces of the prism 2-commute by the commutative diagrams discussed just above. There are natural isomorphisms

$$\alpha\colon \iota_*^\circ \circ \mathrm{can}_Y^\circ \circ \mathbf{\Pi}_\varepsilon^\circ \Rightarrow \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \circ \iota_*^\circ, \quad \beta\colon \iota_*^\circ \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D},\circ} \Rightarrow \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \circ \iota_*^\circ.$$

There is a unique natural isomorphism $\mathscr{D}^\circ\colon \mathrm{can}_Y^\circ \circ \mathbf{\Pi}_\varepsilon^\circ \Rightarrow \mathbf{\Pi}_\varepsilon^{\mathsf{D},\circ}$ making the following diagram of functors and natural isomorphisms commute.

$$\begin{tikzcd} \iota_*^\circ \circ \mathrm{can}_Y^\circ \circ \mathbf{\Pi}_\varepsilon^\circ \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\mathscr{D}^\circ"] & \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \circ \iota_*^\circ \arrow[d, Rightarrow, "\mathscr{D}"] \\ \iota_*^\circ \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D},\circ} \arrow[r, Rightarrow, "\beta"] & \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \circ \iota_*^\circ . \end{tikzcd}$$

□

13.2. Unramified Weil parameters. — We continue to write $\iota^\circ\colon \mathcal{O} \hookrightarrow F$ for the inclusion, and for all maps and functors resulting from this inclusion. Given a sharp cover $\tilde{\mathbf{T}}$ over $\mathcal{O}$, and fixing $\varepsilon\colon \mu_n \hookrightarrow \mathbb{C}^\times$ as always, we have constructed the L-group

$$T^\vee \hookrightarrow {}^{\mathsf{L}}T^\circ \twoheadrightarrow \mathrm{Gal}_{\mathcal{O}} = \langle \mathrm{Fr} \rangle_{\mathrm{prof}}.$$

This is an extension of *abelian* groups, defined uniquely up to unique isomorphism from $\tilde{\mathbf{T}}$ and ε.

The construction of the L-group is compatible with base extension from $\mathcal{O}$ to F (see, e.g., Section 3.6), and we find a commutative diagram,

$$\begin{tikzcd} T^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{L}}\tilde{T} \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Gal}_F \arrow[d, two heads] \arrow[l, bend right, dashed, "\tau^\circ"'] \\ T^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{T}^\circ \arrow[r, two heads] & \mathrm{Gal}_{\mathcal{O}}. \end{tikzcd} \tag{13.1}$$

This identifies the complex L-group ${}^{\mathsf{L}}\tilde{T}$ with the pullback of ${}^{\mathsf{L}}\tilde{T}^\circ$. As such, the top row is endowed with a splitting $\tau^\circ\colon \mathcal{J}_F \to {}^{\mathsf{L}}\tilde{T}$ over the inertial subgroup of Gal_F.

The set of Weil parameters $\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}) = \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}/F)$ consists of continuous homomorphisms $\rho\colon \mathcal{W}_F \to {}^{\mathsf{L}}\tilde{T}$, lying over the canonical map $\mathcal{W}_F \to \mathrm{Gal}_F$. A subset is given by the unramified Weil parameters $\mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{T}}) = \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}/\mathcal{O})$, consisting of those $\rho\colon \mathcal{W}_F \to {}^{\mathsf{L}}\tilde{T}$ whose restriction to inertia coincides with τ°. The unramified parameters $\mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{T}})$ are identified, in turn, with the homomorphisms $\mathcal{W}_{\mathcal{O}} = \langle \mathrm{Fr} \rangle \to {}^{\mathsf{L}}\tilde{T}^\circ$ lying over the canonical map $\mathcal{W}_{\mathcal{O}} \to \mathrm{Gal}_{\mathcal{O}}$. The set $\mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{T}})$ is naturally a torsor for $\mathbf{\Phi}^\circ(\mathbf{T}) = \mathrm{Hom}(\mathcal{W}_{\mathcal{O}}, T^\vee)$, the set of unramified parameters for $\mathbf{T}$.

These sets of Weil parameters, $\mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{T}}) \subset \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}})$ can be identified with splittings sequences of exact sequences as follows: the canonical splitting of ${}^{\mathsf{L}}\tilde{T}$ over $[\mathrm{Gal}_F, \mathrm{Gal}_F]$ coincides with the splitting τ° on $[\mathrm{Gal}_F, \mathrm{Gal}_F] \cap \mathcal{J}$. In this way, the previous commutative diagram (14.1) gives a commutative diagram,

$$\begin{tikzcd} T^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{L}}\tilde{T}/[\mathrm{Gal}_F, \mathrm{Gal}_F] \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Gal}_F^{\mathrm{ab}} \arrow[d, two heads] \\ T^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{T}^\circ \arrow[r, two heads] & \mathrm{Gal}_{\mathcal{O}}. \end{tikzcd}$$

Pulling back via $\mathcal{W}_F^{\mathrm{ab}} \hookrightarrow \mathrm{Gal}_F^{\mathrm{ab}}$, and $\mathcal{W}_{\mathcal{O}} \hookrightarrow \mathrm{Gal}_{\mathcal{O}}$, and via the reciprocity isomorphisms $\mathrm{rec}_F\colon \mathcal{W}_F^{\mathrm{ab}} \xrightarrow{\sim} F^\times$, $\mathrm{rec}_{\mathcal{O}}\colon \mathcal{W}_{\mathcal{O}} \xrightarrow{\sim} \mathbb{Z}$, we obtain a commutative diagram of locally compact abelian groups,

$$\begin{tikzcd} T^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{A}}\tilde{T} \arrow[r, two heads] \arrow[d, two heads] & F^\times \arrow[d, two heads, "\mathrm{val}"] \\ T^\vee \arrow[r, hook] & {}^{\mathsf{A}}\tilde{T}^\circ \arrow[r, two heads] & \mathbb{Z}. \end{tikzcd}$$

This identifies ${}^{\mathsf{A}}\tilde{T}$ wth the pullback of ${}^{\mathsf{A}}\tilde{T}^\circ$ via $\mathrm{val}\colon F^\times \to \mathbb{Z}$. In particular, ${}^{\mathsf{A}}\tilde{T}$ is endowed with a canonical splitting ${}^{\mathsf{A}}\lambda^\circ\colon \mathcal{O}^\times \to {}^{\mathsf{A}}\tilde{T}$. Recall that $\mathbf{\Phi}_\varepsilon^{\mathsf{A}}(\tilde{\mathbf{T}})$ denotes the set of splittings $\mathrm{Spl}({}^{\mathsf{A}}\tilde{T})$, and define $\mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}(\tilde{\mathbf{T}})$ to be the set of splittings of ${}^{\mathsf{A}}\tilde{T}^{\circ}$. Equivalently, $\mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}(\tilde{\mathbf{T}})$ is the set of splittings in $\mathrm{Spl}({}^{\mathsf{A}}\tilde{T})$ which restrict to ${}^{\mathsf{A}}\lambda^{\circ}$ on $\mathscr{O}^{\times}$. This gives an inclusion $\mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}(\tilde{\mathbf{T}}) \hookrightarrow \mathbf{\Phi}_{\varepsilon}^{\mathsf{A}}(\tilde{\mathbf{T}})$.

Proposition 13.2. — *There is a natural isomorphism of additive functors,*

$$\mathscr{A}^{\circ} \colon \mathbf{\Phi}_{\varepsilon}^{\circ} \overset{\sim}{\Rightarrow} \mathrm{rec}_{\mathscr{O}}^{*} \circ \mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}.$$

Proof. — Consider the following diagram of Picard categories and additive functors.

$$\begin{tikzcd}
 & & \mathsf{Tors}(T^{\vee}) \arrow[dl, "\mathrm{rec}_{\mathscr{O}}^{*}"'] \arrow[dd, "\iota_{*}^{\circ}"] \\
\mathsf{Cov}_{n}^{\sharp}(\mathbf{T}/\mathscr{O}) \arrow[urr, "\mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}"] \arrow[r, "\mathbf{\Phi}_{\varepsilon}^{\circ}"'] \arrow[dd, "\iota_{*}^{\circ}"] & \mathsf{Tors}(\mathbf{\Phi}^{\circ}(\mathbf{T})) \arrow[dd, "\iota_{*}^{\circ}"] & \\
 & & \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}(\tilde{\mathbf{T}})) \arrow[dl, "\mathrm{rec}_{F}^{*}"'] \\
\mathsf{Cov}_{n}^{\sharp}(\mathbf{T}/F) \arrow[urr, "\mathbf{\Phi}_{\varepsilon}^{\mathsf{A}}"] \arrow[r, "\mathbf{\Phi}_{\varepsilon}"'] & \mathsf{Tors}(\mathbf{\Phi}(\mathbf{T})). &
\end{tikzcd}$$

The proof mirrors Proposition 13.1. The bottom face 2-commutes via the natural isomorphism $\mathscr{A}$ from (11.2). The lateral faces 2-commute by examining the commutative diagrams just above. This family of natural isomorphisms determines a natural isomorphism $\mathscr{A}^{\circ}$ making the top face 2-commute as well. ☐

13.3. Spherical/Unramified comparison. — The connection between genuine characters and Weil parameters was made through a sequence of three natural isomorphisms,

$$\mathrm{rec}_{F}^{*} \circ \mathrm{can}_{Y} \circ \mathbf{\Pi}_{\varepsilon} \xRightarrow{\mathscr{D}} \mathrm{rec}_{F}^{*} \circ \mathbf{\Pi}_{\varepsilon}^{\mathsf{D}} \xRightarrow{\mathscr{P}} \mathrm{rec}_{F}^{*} \circ \mathbf{\Phi}_{\varepsilon}^{A} \xLeftarrow{\mathscr{A}} \mathbf{\Phi}_{\varepsilon}.$$

We have found a spherical analog of $\mathscr{D}$ and an unramified analog of $\mathscr{A}$, giving the following sequence of functors and natural isomorphisms,

$$\mathrm{rec}_{\mathscr{O}}^{*} \circ \mathrm{can}_{Y} \circ \mathbf{\Pi}_{\varepsilon}^{\circ} \xRightarrow{\mathscr{D}^{\circ}} \mathrm{rec}_{\mathscr{O}}^{*} \circ \mathbf{\Pi}_{\varepsilon}^{\mathsf{D},\circ} \qquad \mathrm{rec}_{\mathscr{O}}^{*} \circ \mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ} \xLeftarrow{\mathscr{A}^{\circ}} \mathbf{\Phi}_{\varepsilon}^{\circ}.$$

The natural isomorphism $\mathscr{P} \colon \mathbf{\Pi}_{\varepsilon}^{\mathsf{D}} \Rightarrow \mathbf{\Phi}_{\varepsilon}^{\mathsf{A}}$ arose from the full subcategory of incarnated covers, where any basis of Y yields a pair of isomorphisms $\sigma_{\mathscr{B}}$ and $\lambda_{\mathscr{B}}$,

$$\begin{tikzcd}
 & {}^{\mathsf{D}}\tilde{T} \arrow[dd, "\mathscr{P}"] \\
F^{\times} \times_{\theta} T^{\vee} \arrow[ur, "\sigma_{\mathscr{B}}"] \arrow[dr, "\lambda_{\mathscr{B}}"'] & \\
 & {}^{\mathsf{A}}\tilde{T}.
\end{tikzcd}$$

A unique isomorphism $\mathscr{P}$ (independent of the choice of basis) makes this diagram commute; it yields a natural isomorphism from $\mathrm{Spl}({}^{\mathsf{D}}\tilde{T})$ to $\mathrm{Spl}({}^{\mathsf{A}}\tilde{T})$.

For sharp covers $\tilde{\mathbf{T}}$ over $\mathcal{O}$ incarnated by $C \in X \otimes X$, the same steps yield isomorphisms $\sigma_{\mathscr{B}}^{\circ}$ and $\lambda_{\mathscr{B}}^{\circ}$,

$$\begin{tikzcd} & {}^{\mathsf{D}}\tilde{T} \arrow[dd, "\mathscr{P}"] \\ \mathcal{O}^{\times} \times_{\theta} T^{\vee} \arrow[ur, "\sigma_{\mathscr{B}}^{\circ}"] \arrow[dr, "\lambda_{\mathscr{B}}^{\circ}"'] & \\ & {}^{\mathsf{A}}\tilde{T}. \end{tikzcd}$$

As the Hilbert symbol of order n is trivial on $\mathcal{O}^{\times} \times \mathcal{O}^{\times}$ (as $\tilde{\mathbf{T}}$ is defined over $\mathcal{O}$, n is coprime to the residue characteristic), we find that $\mathcal{O}^{\times} \times_{\theta} T^{\vee} = \mathcal{O}^{\times} \times T^{\vee}$. In this way, $\sigma_{\mathscr{B}}^{\circ}$ is a splitting of ${}^{\mathsf{D}}\tilde{T}^{\circ}$ over $\mathcal{O}^{\times}$, and $\lambda_{\mathscr{B}}^{\circ}$ is a splitting of ${}^{\mathsf{A}}\tilde{T}^{\circ}$ over $\mathcal{O}^{\times}$. These coincide with the canonical splittings ${}^{\mathsf{D}}\sigma^{\circ}$ and ${}^{\mathsf{A}}\lambda^{\circ}$ described in the previous two sections.

In this way, the isomorphism $\mathscr{P}$ sets up a bijection between those splittings of ${}^{\mathsf{D}}\tilde{T}$ which restrict to ${}^{\mathsf{D}}\sigma^{\circ}$, and those splittings of ${}^{\mathsf{A}}\tilde{T}$ which restrict to ${}^{\mathsf{A}}\lambda^{\circ}$. In other words, $\mathscr{P}$ restricts to a bijection

$$\mathscr{P}^{\circ} \colon \mathbf{\Pi}_{\varepsilon}^{\mathsf{D},\circ}(\tilde{\mathbf{T}}) \xrightarrow{\sim} \mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}(\tilde{\mathbf{T}}).$$

Proposition 13.3. — *The following diagram of Picard categories and additive functors 2-commutes:*

$$\begin{tikzcd} \mathsf{Cov}_n^{\sharp}(\mathbf{T}/\mathcal{O}) \arrow[r, bend left, "\mathbf{\Pi}_{\varepsilon}^{\mathsf{D},\circ}"] \arrow[r, bend right, "\mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}"'] \arrow[d, "\iota_*^{\circ}"] & \mathsf{Tors}(\mathrm{Hom}(F^{\times}/\mathcal{O}^{\times}, T^{\vee})) \arrow[d, "\iota_*^{\circ}"] \\ \mathsf{Cov}_n^{\sharp}(\mathbf{T}/F) \arrow[r, bend left, "\mathbf{\Pi}_{\varepsilon}^{\mathsf{D}}"] \arrow[r, bend right, "\mathbf{\Phi}_{\varepsilon}^{\mathsf{A}}"'] & \mathsf{Tors}(\mathrm{Hom}(F^{\times}, T^{\vee})). \end{tikzcd}$$

Proof. — The natural isomorphism $\mathscr{P}$ makes the bottom face of the cylinder 2-commute. As $\mathscr{P}$ pulls back to a bijection $\mathscr{P}^{\circ}$ from $\mathbf{\Pi}_{\varepsilon}^{\mathsf{D},\circ}(\tilde{\mathbf{T}})$ to $\mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}(\tilde{\mathbf{T}})$, for any sharp cover $\tilde{\mathbf{T}}$, we find a unique natural isomorphism $\mathscr{P}^{\circ} \colon \mathbf{\Pi}_{\varepsilon}^{\mathsf{D},\circ} \Rightarrow \mathbf{\Phi}_{\varepsilon}^{\mathsf{A},\circ}$ making the diagram 2-commute. □

Assembling these propositions, we find that our parameterization makes spherical characters correspond to unramified parameters. We have two sequences of functors and natural isomorphisms,

$$\mathrm{rec}_F^* \circ \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \xRightarrow{\mathscr{D}} \mathrm{rec}_F^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D}} \xRightarrow{\mathscr{P}} \mathrm{rec}_F^* \circ \mathbf{\Phi}_\varepsilon^{A} \xRightarrow{\mathscr{A}^{-1}} \mathbf{\Phi}_\varepsilon,$$

$$\mathrm{rec}_{\mathcal{O}}^* \circ \mathrm{can}_Y^\circ \circ \mathbf{\Pi}_\varepsilon^\circ \xRightarrow{\mathscr{D}^\circ} \mathrm{rec}_{\mathcal{O}}^* \circ \mathbf{\Pi}_\varepsilon^{\mathsf{D},\circ} \xRightarrow{\mathscr{P}^\circ} \mathrm{rec}_{\mathcal{O}}^* \circ \mathbf{\Phi}_\varepsilon^{\mathsf{A},\circ} \xRightarrow{(\mathscr{A}^\circ)^{-1}} \mathbf{\Phi}_\varepsilon^\circ.$$

Recall that $\mathscr{L} = \mathrm{rec}_F^* \circ \mathrm{can}_Y \colon \mathrm{Hom}(T, \mathbb{C}^\times) \to \mathrm{Hom}(\mathscr{W}_F, T^\vee)$. Define analogously $\mathscr{L}^\circ = \mathrm{rec}_{\mathcal{O}}^* \circ \mathrm{can}_Y^\circ \colon \mathrm{Hom}(T/T^\circ, \mathbb{C}^\times) \to \mathrm{Hom}(\mathscr{W}_{\mathcal{O}}, T^\vee)$. Recall that $\mathscr{L}_\varepsilon = \mathscr{A}^{-1} \circ \mathscr{P} \circ \mathscr{D}$, and define analogously $\mathscr{L}_\varepsilon^\circ = (\mathscr{A}^\circ)^{-1} \circ \mathscr{P}^\circ \circ \mathscr{D}^\circ$.

Theorem 13.4. — *The following diagram of Picard categories and additive functors 2-commutes, via the natural isomorphisms $\mathscr{L}_\varepsilon^\circ$ (on the top face) and $\mathscr{L}_\varepsilon$ (on the bottom face):*

$$\begin{tikzcd}
 & & \mathsf{Tors}(\mathbf{\Phi}^\circ(\mathbf{T})) \arrow[dd, "\iota_*^\circ"] \\
\mathsf{Cov}_n^\sharp(\mathbf{T}/\mathcal{O}) \arrow[urr, "\mathbf{\Pi}_\varepsilon^\circ"] \arrow[r, "\mathbf{\Phi}_\varepsilon^\circ"'] \arrow[dd, "\iota_*^\circ"] & \mathsf{Tors}(\mathbf{\Pi}^\circ(\mathbf{T})) \arrow[ur, "\mathscr{L}^\circ"'] \arrow[dd, "\iota_*^\circ"] & \\
 & & \mathsf{Tors}(\mathbf{\Phi}(\mathbf{T})) \\
\mathsf{Cov}_n^\sharp(\mathbf{T}/F) \arrow[urr, "\mathbf{\Pi}_\varepsilon"] \arrow[r, "\mathbf{\Phi}_\varepsilon"'] & \mathsf{Tors}(\mathbf{\Pi}(\mathbf{T})). \arrow[ur, "\mathscr{L}"'] &
\end{tikzcd}$$

This theorem gives a parameterization of spherical genuine characters of $\tilde{T}$ by unramified Weil parameters, functorial in the choice of sharp cover, and compatible with the parameterization of all genuine characters by all Weil parameters. As in the case of local fields, one may verify that the natural isomorphisms $\mathscr{L}_\varepsilon^\circ$ are compatible with pullbacks, for isomorphisms of split tori $\mathbf{S} \to \mathbf{T}$ over $\mathcal{O}$.

14. Global case

Now we consider sharp covers $\tilde{\mathbf{T}}$ of a split torus $\mathbf{T}$ over a *global* field F. The methods are much the same as the previous section, with the inclusion $\mathcal{O} \hookrightarrow F$ replaced by the inclusion $F \hookrightarrow \mathbb{A}$. Thus we leave a few details to the reader, to adapt proofs from the previous section as needed.

Given a sharp cover $\tilde{\mathbf{T}} = (\mathbf{T}', n)$ over F, write $T_F = \mathbf{T}(F)$ and $T_{\mathbb{A}} = \mathbf{T}(\mathbb{A})$, and $\mu_n \hookrightarrow \tilde{T}_{\mathbb{A}} \twoheadrightarrow T_{\mathbb{A}}$ for the resulting extension of locally compact abelian groups. As $\tilde{\mathbf{T}}$ is defined over F, this comes with a splitting $\sigma_F \colon T_F \hookrightarrow \tilde{T}_{\mathbb{A}}$.

14.1. Parameterization by splittings. — Define $\mathbf{\Pi}_{\mathbb{A}}(\mathbf{T}) = \mathrm{Hom}(T_{\mathbb{A}}, \mathbb{C}^\times)$, the group of continuous characters, and $\mathbf{\Pi}_{\mathbb{A},\varepsilon}(\tilde{\mathbf{T}}) = \mathrm{Hom}_\varepsilon(\tilde{T}_{\mathbb{A}}, \mathbb{C}^\times)$ for the $\mathbf{\Pi}_{\mathbb{A}}(\mathbf{T})$-torsor of ε-genuine continuous characters. To give such a character, it is equivalent to give genuine characters of $\tilde{T}_v$ for all places v of F, almost all of which are T_v°-spherical; see [**77**] and [**80**, §4] for details.

We abbreviate,

$$\mathbf{\Pi}_F(\mathbf{T}) = \mathbf{\Pi}(\mathbf{T}/F) = \mathrm{Hom}(T_{\mathbb{A}}/T_F, \mathbb{C}^\times),$$
$$\mathbf{\Pi}_{F,\varepsilon}(\tilde{\mathbf{T}}) = \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/F) = \mathrm{Hom}_\varepsilon(\tilde{T}/\sigma_F(T), \mathbb{C}^\times).$$

Then $\mathbf{\Pi}_{F,\varepsilon}(\tilde{\mathbf{T}})$ is a $\mathbf{\Pi}_F(\mathbf{T})$-torsor. Write $\iota_F \colon \mathbf{\Pi}_F(\mathbf{T}) \hookrightarrow \mathbf{\Pi}_{\mathbb{A}}(\mathbf{T})$ for the inclusion of the automorphic characters of $T_{\mathbb{A}}$ into the group of all continuous characters of $T_{\mathbb{A}}$. There is a natural isomorphism of $\mathbf{\Pi}(\mathbf{T})$-torsors,

$$(\iota_F)_* \left(\mathbf{\Pi}_{F,\varepsilon}(\tilde{\mathbf{T}})\right) \xrightarrow{\sim} \mathbf{\Pi}_{\mathbb{A},\varepsilon}(\tilde{\mathbf{T}}).$$

Define

$$\mathbf{\Pi}_{\mathbb{A}}^{\mathsf{D}}(\mathbf{T}) = \mathrm{Hom}(\mathbb{A}^\times, T^\vee), \quad \mathbf{\Pi}_F^{\mathsf{D}}(\mathbf{T}) = \mathrm{Hom}(\mathbb{A}^\times/F^\times, T^\vee),$$

so there are natural isomorphisms

$$\mathrm{can}_{\mathbb{A},Y} \colon \mathbf{\Pi}_{\mathbb{A}}(\mathbf{T}) \xrightarrow{\sim} \mathbf{\Pi}_{\mathbb{A}}^{\mathsf{D}}(\mathbf{T}), \quad \mathrm{can}_{F,Y} \colon \mathbf{\Pi}_F(\mathbf{T}) \xrightarrow{\sim} \mathbf{\Pi}_F^{\mathsf{D}}(\mathbf{T}).$$

We may identify $\mathbf{\Pi}_{\mathbb{A},\varepsilon}(\tilde{\mathbf{T}})$ with the set of splittings of an extension ${}^{\mathsf{D}}\tilde{T}_{\mathbb{A}}$, just as in the local case. Indeed, begin with the extension $\mu_n \hookrightarrow \tilde{T}_{\mathbb{A}} \twoheadrightarrow T_{\mathbb{A}}$. Push out via ε to get an extension $\mathbb{C}^\times \hookrightarrow \varepsilon_* \tilde{T}_{\mathbb{A}} \twoheadrightarrow T_{\mathbb{A}}$. As $T_{\mathbb{A}} = Y \otimes \mathbb{A}^\times$, we may tensor with X and pull back via $\mathbb{Z} \hookrightarrow X \otimes Y$ to obtain an extension,

$$T^\vee \hookrightarrow {}^{\mathsf{D}}\tilde{T}_{\mathbb{A}} \twoheadrightarrow \mathbb{A}^\times.$$

Define $\mathbf{\Pi}_{\mathbb{A},\varepsilon}^{\mathsf{D}}(\tilde{\mathbf{T}}) = \mathrm{Spl}({}^{\mathsf{D}}\tilde{T}_{\mathbb{A}})$, a $\mathbf{\Pi}_{\mathbb{A}}^{\mathsf{D}}(\mathbf{T})$-torsor. As in the local case, there is a natural bijection $\mathscr{D}_{\mathbb{A}} \colon \mathbf{\Pi}_{\mathbb{A},\varepsilon}(\tilde{\mathbf{T}}) \to \mathbf{\Pi}_{\mathbb{A},\varepsilon}^{\mathsf{D}}(\tilde{\mathbf{T}})$, and this is compatible with pullbacks to local fields. The splitting σ_F defines a splitting ${}^{\mathsf{D}}\sigma_F$ in the following commutative diagram:

$$\begin{tikzcd}
T^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{D}}\tilde{T}_F \arrow[r, two heads] \arrow[d, hook] & F^\times \arrow[l, bend right, "{}^{\mathsf{D}}\sigma_F"'] \arrow[d, hook] \\
T^\vee \arrow[r, hook] & {}^{\mathsf{D}}\tilde{T}_{\mathbb{A}} \arrow[r, two heads] & \mathbb{A}^\times .
\end{tikzcd}$$

The automorphic genuine characters $\mathbf{\Pi}_{F,\varepsilon}(\tilde{\mathbf{T}}) \subset \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}})$ correspond to the subset $\mathbf{\Pi}_{F,\varepsilon}^{\mathsf{D}}(\tilde{\mathbf{T}}) \subset \mathbf{\Pi}_{\mathbb{A},\varepsilon}^{\mathsf{D}}(\tilde{\mathbf{T}})$ of splittings which restrict to ${}^{\mathsf{D}}\sigma_F$ on $F^\times$. Then $\mathbf{\Pi}_{F,\varepsilon}^{\mathsf{D}}(\tilde{\mathbf{T}})$ is a $\mathbf{\Pi}_F^{\mathsf{D}}(\mathbf{T})$-torsor. The commutative diagrams above extend to a diagram of categories and functors below.

Proposition 14.1. — *There is a natural isomorphism of additive functors,*

$$\mathscr{D}_F \colon \mathrm{can}_{F,Y} \circ \mathbf{\Pi}_{F,\varepsilon} \overset{\sim}{\Rightarrow} \mathbf{\Pi}_{F,\varepsilon}^{\mathsf{D}}.$$

Proof. — Consider the following diagram of Picard categories and additive functors:

$$\begin{tikzcd}
 & & & \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}_F(\mathbf{T})) \arrow[dd, "\iota_{F,*}"] \\
\mathsf{Cov}^\sharp_n(\mathbf{T}) \arrow[r, "\mathbf{\Pi}_{F,\varepsilon}"'] \arrow[urrr, "\mathbf{\Pi}^{\mathsf{D}}_{F,\varepsilon}"] \arrow[dd, "="] & \mathsf{Tors}(\mathbf{\Pi}_F(\mathbf{T})) \arrow[urr, "\mathrm{can}_{F,Y}"'] \arrow[dd, "\iota_{F,*}"] & & \\
 & & & \mathsf{Tors}(\mathbf{\Pi}^{\mathsf{D}}_{\mathbb{A}}(\mathbf{T})) \\
\mathsf{Cov}^\sharp_n(\mathbf{T}) \arrow[r, "\mathbf{\Pi}_{\mathbb{A},\varepsilon}"'] \arrow[urrr, "\mathbf{\Pi}^{\mathsf{D}}_{\mathbb{A},\varepsilon}"] & \mathsf{Tors}(\mathbf{\Pi}_{\mathbb{A}}(\mathbf{T})). \arrow[urr, "\mathrm{can}_{\mathbb{A},Y}"'] & &
\end{tikzcd}$$

The natural isomorphisms $\mathscr{D}$ and $\mathscr{D}^\circ$ define a natural isomorphism $\mathscr{D}_{\mathbb{A}}$ making the bottom face 2-commute. Indeed, $\mathbf{\Pi}_{\mathbb{A},\varepsilon}(\tilde{\mathbf{T}})$ gives the set of continuous genuine characters of $\tilde{T}_{\mathbb{A}}$—these are described in turn by families of genuine characters of $\tilde{T}_v$ for all places, almost all of which are T_v°-spherical. But these are described, by splittings of ${}^{\mathsf{D}}\tilde{T}_v$ for all places, coinciding with the splitting σ_v° at all T_v°-spherical places.

As in the local integral case, the lateral faces 2-commute, from which we find a unique natural isomorphism $\mathscr{D}_F$ making the top face 2-commute by pulling back $\mathscr{D}_{\mathbb{A}}$. □

14.2. Global Weil parameters. — Given a sharp cover $\tilde{\mathbf{T}}$ over F, we have constructed the L-group,

$$T^\vee \hookrightarrow {}^{\mathsf{L}}T \twoheadrightarrow \mathrm{Gal}_F\,.$$

The construction of the L-group is compatible with base extension from F to F_v, and we find a commutative diagram for every place v of F,

$$\begin{tikzcd}
T^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{L}}\tilde{T}_v \arrow[r, hook] \arrow[d, two heads] & \mathrm{Gal}_{F_v} \arrow[d, hook] \\
T^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{T} \arrow[r, two heads] & \mathrm{Gal}_F\,.
\end{tikzcd} \tag{14.1}$$

This identifies the L-group ${}^{\mathsf{L}}\tilde{T}_v$ with the pullback of ${}^{\mathsf{L}}\tilde{T}$.

The set of Weil parameters $\mathbf{\Phi}_{F,\varepsilon}(\tilde{\mathbf{T}}) = \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}/F)$ consists of continuous homomorphisms $\rho\colon \mathscr{W}_F \to {}^{\mathsf{L}}\tilde{T}$, lying over the canonical map $\mathscr{W}_F \to \mathrm{Gal}_F$. As in the local case, the cohomology $H^2(\mathrm{Gal}_F, T^\vee)$ is trivial, yielding the (noncanonical) splitting of ${}^{\mathsf{L}}\tilde{T}$. From this it follows that the L-groups split canonically over $[\mathrm{Gal}_F, \mathrm{Gal}_F]$. Taking the quotient ${}^{\mathsf{L}}\tilde{T}/[\mathrm{Gal}_F, \mathrm{Gal}_F]$ and pulling back via $\mathbb{A}^\times/F^\times \xrightarrow{\mathrm{rec}_F^{-1}} \mathscr{W}_F^{\mathrm{ab}} \to \mathrm{Gal}_F^{\mathrm{ab}}$ yields an extension

$$T^\vee \hookrightarrow {}^{\mathsf{A}}\tilde{T}_F \twoheadrightarrow \mathbb{A}^\times/F^\times.$$

Pulling back via $F_v^\times \hookrightarrow \mathbb{A}^\times$ gives a commutative diagram,

$$\begin{tikzcd} T^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{A}}\tilde{T}_v \arrow[r, two heads] \arrow[d] & F_v^\times \arrow[d] \\ T^\vee \arrow[r, hook] & {}^{\mathsf{A}}\tilde{T}_F \arrow[r, two heads] & \mathbb{A}^\times / F^\times. \end{tikzcd}$$

The local extensions $T^\vee \hookrightarrow {}^{\mathsf{A}}\tilde{T}_v \twoheadrightarrow F_v^\times$ are endowed with splittings σ_v° over $\mathcal{O}_v^\times$ for almost all v. Define the restricted product,

$${}^{\mathsf{A}}\tilde{T}_{\mathbb{A}} = \frac{\{(\tilde{t}_v)_{v \in \mathcal{V}} \in \prod_v {}^{\mathsf{A}}\tilde{T}_v : \tilde{t}_v \in \sigma_v^\circ(\mathcal{O}_v^\times) \text{ for almost all } v\}}{\operatorname{Ker}\left(\bigoplus_v T^\vee \xrightarrow{\Pi} T^\vee\right)}.$$

This fits into a commutative diagram with exact rows,

$$\begin{tikzcd} T^\vee \arrow[r, hook] \arrow[d, "="] & {}^{\mathsf{A}}\tilde{T}_{\mathbb{A}} \arrow[r, two heads] \arrow[d, two heads] & \mathbb{A}^\times \arrow[d, two heads] \\ T^\vee \arrow[r, hook] & {}^{\mathsf{A}}\tilde{T}_F \arrow[r, two heads] & \mathbb{A}^\times / F^\times. \end{tikzcd}$$

The construction of ${}^{\mathsf{A}}\tilde{T}_{\mathbb{A}}$ gives a splitting ${}^{\mathsf{A}}\lambda_F \colon F^\times \to {}^{\mathsf{A}}\tilde{T}_{\mathbb{A}}$ of the top row over $F^\times$.

Define $\mathbf{\Phi}_{F,\varepsilon}^{\mathsf{A}}(\tilde{\mathbf{T}}) = \operatorname{Spl}({}^{\mathsf{A}}\tilde{T}_F)$; this is in natural bijection with the set $\mathbf{\Phi}_{F,\varepsilon}(\tilde{\mathbf{T}})$ of Weil parameters for ${}^L\tilde{T}_F$. Analogously, define $\mathbf{\Phi}_{\mathbb{A},\varepsilon}^{\mathsf{A}}(\tilde{\mathbf{T}}) = \operatorname{Spl}({}^{\mathsf{A}}\tilde{T}_{\mathbb{A}})$. Pullback yields an inclusion $\operatorname{Spl}({}^{\mathsf{A}}\tilde{T}_F) \hookrightarrow \operatorname{Spl}({}^{\mathsf{A}}\tilde{T}_{\mathbb{A}})$, whose image consists of splittings of ${}^{\mathsf{A}}\tilde{T}_{\mathbb{A}}$ which restrict to the splitting ${}^{\mathsf{A}}\lambda_F$ on $F^\times$.

Tracing through the definitions, $\operatorname{Spl}({}^{\mathsf{A}}\tilde{T}_{\mathbb{A}})$ is in natural bijection with the set of families $(\phi_v)_{v \in \mathcal{V}}$ of Weil parameters in $\mathbf{\Phi}_{v,\varepsilon}(\tilde{\mathbf{T}})$ at every place, for which ϕ_v is unramified almost everywhere.

Define $\mathbf{\Phi}_F(\mathbf{T})$ to be the set of Weil parameters for $\mathbf{T}$, i.e., $\mathbf{\Phi}_F(\mathbf{T}) = \operatorname{Hom}(\mathcal{W}_F, T^\vee)$. Define $\mathbf{\Phi}_{\mathbb{A}}(\mathbf{T})$ to be the set of families $(\phi_v)_{v \in \mathcal{V}}$ of Weil parameters in $\mathbf{\Phi}_v(\mathbf{T}) = \operatorname{Hom}(\mathcal{W}_{F_v}, T^\vee)$, which are unramified almost everywhere.

Proposition 14.2. — *There is a natural isomorphism of additive functors,*

$$\mathscr{A}_F \colon \mathbf{\Phi}_{F,\varepsilon} \overset{\sim}{\Rightarrow} \operatorname{rec}_F^* \circ \mathbf{\Phi}_{F,\varepsilon}^{\mathsf{A}}.$$

Proof. — Consider the following diagram of Picard categories and additive functors:

$$\begin{tikzcd}
 & & \mathsf{Tors}(\mathbf{\Pi}_F^{\mathsf{D}}(\mathbf{T})) \arrow[dl, "\mathrm{rec}_F^*"'] \arrow[dd, "\iota_{F,*}"] \\
\mathsf{Cov}_n^{\sharp}(\mathbf{T}) \arrow[r, "\mathbf{\Phi}_{F,\varepsilon}"'] \arrow[urr, "\mathbf{\Phi}^{\mathsf{A}}_{F,\varepsilon}"] \arrow[dd, "="] & \mathsf{Tors}(\mathbf{\Phi}_F(\mathbf{T})) \arrow[dd, "\iota_{F,*}"] & \\
 & & \mathsf{Tors}(\mathbf{\Pi}_{\mathbb{A}}^{\mathsf{D}}(\mathbf{T})) \arrow[dl, "\prod_v \mathrm{rec}_v^*"] \\
\mathsf{Cov}_n^{\sharp}(\mathbf{T}) \arrow[r, "\mathbf{\Phi}_{\mathbb{A},\varepsilon}"'] \arrow[urr, "\mathbf{\Phi}^{\mathsf{A}}_{\mathbb{A},\varepsilon}"] & \mathsf{Tors}(\mathbf{\Phi}_{\mathbb{A}}(\mathbf{T})). &
\end{tikzcd}$$

The natural isomorphisms $\mathscr{A}$ and $\mathscr{A}^\circ$ in the local and unramified cases yield a natural isomorphism $\mathscr{A}_{\mathbb{A}}$ which makes the bottom face 2-commute. The lateral faces 2-commute by the local-global compatibility of our constructions. A natural isomorphism $\mathscr{A}_F$ making the top face 2-commute follows. $\square$

14.3. Global comparison. — Over local fields, we have found a connection between genuine characters and Weil parameters through a sequence of three natural isomorphisms,

$$\mathrm{rec}^*_{F_v} \circ \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \xRightarrow{\mathscr{D}_v} \mathrm{rec}^*_{F_v} \circ \mathbf{\Pi}^{\mathsf{D}}_\varepsilon \xRightarrow{\mathscr{P}_v} \mathrm{rec}^*_{F_v} \circ \mathbf{\Phi}^{\mathsf{A}}_{v,\varepsilon} \xRightarrow{\mathscr{A}_v^{-1}} \mathbf{\Phi}_{v,\varepsilon}.$$

Each term has an unramified counterpart

$$\mathrm{rec}^*_{\mathscr{O}_v} \circ \mathrm{can}_Y \circ \mathbf{\Pi}^\circ_\varepsilon \xRightarrow{\mathscr{D}_v^\circ} \mathrm{rec}^*_{\mathscr{O}_v} \circ \mathbf{\Pi}^{\mathsf{D},\circ}_{v,\varepsilon} \xRightarrow{\mathscr{P}_v^\circ} \mathrm{rec}^*_{\mathscr{O}_v} \circ \mathbf{\Phi}^{\mathsf{A},\circ}_{v,\varepsilon} \xRightarrow{(\mathscr{A}_v^\circ)^{-1}} \mathbf{\Phi}^\circ_{v,\varepsilon}.$$

Local-global compatibility gives an adelic version

$$\mathrm{rec}^*_{\mathbb{A}} \circ \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \xRightarrow{\mathscr{D}_{\mathbb{A}}} \mathrm{rec}^*_{\mathbb{A}} \circ \mathbf{\Pi}^{\mathsf{D}}_{\mathbb{A},\varepsilon} \xRightarrow{\mathscr{P}_{\mathbb{A}}} \mathrm{rec}^*_{\mathbb{A}} \circ \mathbf{\Phi}^{\mathsf{A}}_{\mathbb{A},\varepsilon} \xRightarrow{\mathscr{A}_{\mathbb{A}}^{-1}} \mathbf{\Phi}_{\mathbb{A},\varepsilon}.$$

We have found an automorphic version, as below:

$$\mathrm{rec}^*_F \circ \mathrm{can}_Y \circ \mathbf{\Pi}_\varepsilon \xRightarrow{\mathscr{D}_F} \mathrm{rec}^*_F \circ \mathbf{\Pi}^{\mathsf{D}}_{F,\varepsilon} \qquad \mathrm{rec}^*_{\mathbb{A}} \circ \mathbf{\Phi}^{\mathsf{A}}_{F,\varepsilon} \xRightarrow{\mathscr{A}_F^{-1}} \mathbf{\Phi}_{F,\varepsilon}.$$

To link the middle terms, it suffices as before to consider the incarnated covers, on which any basis of Y yields a pair of isomorphisms $\sigma_{\mathbb{A},\mathscr{B}}$ and $\lambda_{\mathbb{A},\mathscr{B}}$,

$$\begin{tikzcd}
 & {}^{\mathsf{D}}\tilde{T}_{\mathbb{A}} \arrow[dd, "\mathscr{P}_{\mathbb{A}}"] \\
\mathbb{A}^\times \times_\theta T^\vee \arrow[ur, "\sigma_{\mathbb{A},\mathscr{B}}"] \arrow[dr, "\lambda_{\mathbb{A},\mathscr{B}}"'] & \\
 & {}^{\mathsf{A}}\tilde{T}_{\mathbb{A}}.
\end{tikzcd}$$

A unique isomorphism $\mathscr{P}_{\mathbb{A}}$ makes this diagram commute, and is independent of basis; it yields an isomorphism from $\mathrm{Spl}({}^{\mathsf{D}}\tilde{T}_{\mathbb{A}})$ to $\mathrm{Spl}({}^{\mathsf{A}}\tilde{T}_{\mathbb{A}})$, from which we find the natural isomorphism $\mathscr{P}_{\mathbb{A}} \colon \mathbf{\Pi}^{\mathsf{D}}_{\mathbb{A},\varepsilon} \Rightarrow \mathbf{\Phi}^{\mathsf{A}}_{\mathbb{A},\varepsilon}$.

As the global Hilbert symbol of order n is trivial on $F^\times \times F^\times$ (Hilbert reciprocity), we find that $F^\times \times_\theta T^\vee = F^\times \times T^\vee$. In this way, $\sigma_{\mathbb{A},\mathscr{B}}$ gives a splitting $\sigma_{F,\mathscr{B}}$ of ${}^{\mathsf{D}}\tilde{T}_{\mathbb{A}}$ over $F^\times$, and $\lambda_{\mathbb{A},\mathscr{B}}$ gives a splitting $\lambda_{F,\mathscr{B}}$ of ${}^{\mathsf{A}}\tilde{T}_{\mathbb{A}}$ over $F^\times$. These coincide with the canonical splittings ${}^{\mathsf{D}}\sigma_F$ and ${}^{\mathsf{A}}\lambda_F$ described before.

In this way, the isomorphism $\mathscr{P}_{\mathbb{A}}$ sets up a bijection $\mathscr{P}_F$ between those splittings of ${}^{\mathsf{D}}\tilde{T}_{\mathbb{A}}$ which restrict to ${}^{\mathsf{D}}\sigma_F$ on $F^\times$, and those splittings of ${}^{\mathsf{A}}\tilde{T}_{\mathbb{A}}$ which restrict to ${}^{\mathsf{A}}\lambda_F$ on $F^\times$. In other words, $\mathscr{P}_F$ gives a bijection,

$$\mathscr{P}_F \colon \mathbf{\Pi}^{\mathsf{D}}_{F,\varepsilon}(\tilde{\mathbf{T}}) \xrightarrow{\sim} \mathbf{\Phi}^{\mathsf{A}}_{F,\varepsilon}(\tilde{\mathbf{T}}).$$

Naturality of this bijection is the following.

Proposition 14.3. — *There is a natural isomorphism of additive functors,* $\mathscr{P}_F \colon \mathbf{\Pi}^{\mathsf{D}}_{F,\varepsilon} \Rightarrow \mathbf{\Phi}^{\mathsf{A}}_{F,\varepsilon}$.

Proof. — Consider the following diagram of Picard categories and additive functors

$$\begin{tikzcd}
\mathsf{Cov}^\sharp_n(\mathbf{T}) \arrow[r, bend left, "\mathbf{\Pi}^{\mathsf{D}}_{F,\varepsilon}"] \arrow[r, bend right, "\mathbf{\Phi}^{\mathsf{A}}_{F,\varepsilon}"'] \arrow[d, "="] & \mathsf{Tors}(\mathrm{Hom}(\mathbb{A}^\times/F^\times, T^\vee)) \arrow[d, "\iota_{F,*}"] \\
\mathsf{Cov}^\sharp_n(\mathbf{T}) \arrow[r, bend left, "\mathbf{\Pi}^{\mathsf{D}}_{\mathbb{A},\varepsilon}"] \arrow[r, bend right, "\mathbf{\Phi}^{\mathsf{A}}_{\mathbb{A},\varepsilon}"'] & \mathsf{Tors}(\mathrm{Hom}(\mathbb{A}^\times, T^\vee)).
\end{tikzcd}$$

The natural isomorphism $\mathscr{P}_{\mathbb{A}}$ makes the bottom face of the cylinder 2-commute. As $\mathscr{P}_{\mathbb{A}}$ pulls back to a bijection $\mathscr{P}_F$ from $\mathbf{\Pi}^{\mathsf{D}}_{F,\varepsilon}(\tilde{\mathbf{T}})$ to $\mathbf{\Phi}^{\mathsf{A}}_{F,\varepsilon}(\tilde{\mathbf{T}})$, we find a unique natural isomorphism $\mathscr{P}_F \colon \mathbf{\Pi}^{\mathsf{D}}_{F,\varepsilon} \Rightarrow \mathbf{\Phi}^{\mathsf{A}}_{F,\varepsilon}$ making the diagram 2-commute. □

Theorem 14.4. — *Define* $\mathscr{L}_F = \mathrm{rec}_F \circ \mathrm{can}_{F,Y}$ *and* $\mathscr{L}_\varepsilon = \mathscr{A}_F^{-1} \circ \mathscr{P}_F \circ \mathscr{D}_F$. *This gives a 2-commutative diagram of Picard groupoids and additive functors*

$$\begin{tikzcd}
 & \mathsf{Tors}(\mathbf{\Pi}_F(\mathbf{T})) \arrow[d, "\mathscr{L}_F"] \\
\mathsf{Cov}^\sharp_n(\mathbf{T}) \arrow[ur, "\mathbf{\Pi}_{F,\varepsilon}"] \arrow[dr, "\mathbf{\Phi}_{F,\varepsilon}"'] \arrow[r, phantom, "\Downarrow \mathscr{L}_\varepsilon"] & {} \\
 & \mathsf{Tors}(\mathbf{\Phi}_F(\mathbf{T})).
\end{tikzcd}$$

This theorem gives a parameterization of genuine automorphic characters of $\tilde{T}_{\mathbb{A}}$ by global Weil parameters with values with ${}^{\mathsf{L}}\tilde{T}$, functorial in the choice of sharp cover, and compatible with the previous local parameterization. As in the case of local fields, one may verify that the natural isomorphisms $\mathscr{L}_\varepsilon$ are compatible with pullbacks, for isomorphisms of split tori $\mathbf{S} \to \mathbf{T}$ over F.

15. Split tori

Let S be the spectrum of a local field, a global field, or the ring of integers in a nonarchimedean local field. Let $\mathbf{T}$ be a split torus over S, with character lattice X and cocharacter lattice Y (constant sheaves on $S_{\text{ét}}$). Let $\tilde{\mathbf{T}}$ be a degree n cover of $\mathbf{T}$ over S. Let $Q\colon Y \to \mathbb{Z}$ be the first Brylinski-Deligne invariant. Following Assumption 3.1, we assume that Q is even-valued if n is odd. But we do *not* assume $\tilde{\mathbf{T}}$ is a sharp cover here, i.e., we do not assume $Y = Y_{Q,n}$.

We can parameterize the following sets of irreducible genuine representations.

Local case: The cover $\tilde{\mathbf{T}}$ over a local field F yields a central extension $\mu_n \hookrightarrow \tilde{T} \twoheadrightarrow T$. Recall that $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/F)$ is the set of equivalence classes of irreducible ε-genuine admissible representations of $\tilde{T}$.

Local integral case: The cover $\tilde{\mathbf{T}}$ over $\mathscr{O}$ (the ring of integers in a nonarchimedean local field) yields a central extension $\mu_n \hookrightarrow \tilde{T} \twoheadrightarrow T$ and a splitting $T^\circ = \mathbf{T}(\mathscr{O}) \hookrightarrow \tilde{T}$. Recall that $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/\mathscr{O})$ is the set of equivalence classes of irreducible T°-spherical ε-genuine representations of $\tilde{T}$.

Global case: The cover $\tilde{\mathbf{T}}$ over a global field F yields a central extension $\mu_n \hookrightarrow \tilde{T}_{\mathbb{A}} \twoheadrightarrow T_{\mathbb{A}}$ and a splitting $T_F = \mathbf{T}(F) \hookrightarrow \tilde{T}_{\mathbb{A}}$. Recall that $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/F)$ is the set of equivalence classes of automorphic ε-genuine representations of $\tilde{T}_{\mathbb{A}}$.

Let $\mathbf{T}_{Q,n}$ denote the split torus over S with cocharacter lattice $Y_{Q,n}$, and define

$$\iota_{Q,n}\colon \mathbf{T}_{Q,n} \to \mathbf{T}$$

to be the isogeny (of tori over S) corresponding to the inclusion $Y_{Q,n} \hookrightarrow Y$. Pulling back the cover $\tilde{\mathbf{T}}$ via $\iota_{Q,n}$ yields a degree n cover $\tilde{\mathbf{T}}_{Q,n}$ of $\mathbf{T}_{Q,n}$. Note that $\tilde{\mathbf{T}}_{Q,n}$ is a sharp cover of $\mathbf{T}_{Q,n}$. Viewing $\iota_{Q,n}$ as a well-aligned homomorphism of covers $\tilde{\mathbf{T}}_{Q,n} \to \tilde{\mathbf{T}}$, we find a canonical identification of L-groups, ${}^{\mathsf{L}}\tilde{T} \equiv {}^{\mathsf{L}}\tilde{T}_{Q,n}$.

The results of this section and of **[80]** imply the following.

Theorem 15.1. — *In all three cases above, pulling back the central character via the isogeny $\iota_{Q,n}$ gives a one-to-one function (bijective in the local integral case)*

$$\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/S) \to \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}_{Q,n}/S).$$

Composing with the parameterization $\mathscr{L}_\varepsilon$, this gives a one-to-one parameterization (bijective in the local integral case)

$$\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/S) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}/S).$$

In the local integral case, the parameterization is bijective. In the case of local fields or global fields, we would like to characterize the images of $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}/S) \to \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}/S)$—to find the "relevant" parameters for covers of tori. But we leave such a characterization for a future paper.

PART IV
OTHER PARAMETERIZATIONS

16. Spherical/Unramified parameterization

Let $\mathbf{G}$ be a quasisplit reductive group over $\mathcal{O}$, the ring of integers in a nonarchimedean local field F. Let $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ be a degree n cover of $\mathbf{G}$ defined over $\mathcal{O}$. Write $G = \mathbf{G}(F)$ and $G^\circ = \mathbf{G}(\mathcal{O})$. Then we have a central extension of locally compact groups,

$$\mu_n \hookrightarrow \tilde{G} \twoheadrightarrow G,$$

and a splitting $G^\circ \hookrightarrow \tilde{G}$.

We have constructed an L-group,

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}_{\mathcal{O}},$$

where $\tilde{G}^\vee$ is a complex reductive group and $\mathrm{Gal}_{\mathcal{O}} = \langle \mathrm{Fr} \rangle_{\mathrm{prof}}$. This L-group is well-defined up to L-equivalence, and the L-equivalence is uniquely determined up to unique natural isomorphism.

16.1. Parameterization. — Let $\mathbf{\Pi}_\varepsilon^\circ(\tilde{\mathbf{G}}) = \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O})$ denote the set of equivalence classes of ε-genuine irreducible G°-spherical representations of $\tilde{G}$. Write $\mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{G}}) = \mathbf{\Phi}_\varepsilon(\tilde{\mathbf{G}}/\mathcal{O})$ for the set of equivalence classes (i.e., $\tilde{G}^\vee$-orbits) of unramified Weil parameters $\phi \colon \mathcal{W}_{\mathcal{O}} = \langle \mathrm{Fr} \rangle \to {}^{\mathsf{L}}\tilde{G}$. In five steps below, we define a bijection

$$\mathcal{L}_\varepsilon(\tilde{\mathbf{G}}) \colon \mathbf{\Pi}_\varepsilon^\circ(\tilde{\mathbf{G}}) \xrightarrow{\sim} \mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{G}}).$$

Thus the ε-genuine irreducible spherical representations of $\tilde{G}$ are parameterized (bijectively) by Weil parameters.

For this parameterization, let $\mathbf{A}$ be a maximal $\mathcal{O}$-split torus in $\mathbf{G}$, and let $\mathbf{T}$ be the centralizer of $\mathbf{A}$ in $\mathbf{G}$. Then $\mathbf{T}$ is a maximally split maximal torus in $\mathbf{G}$. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$. Let $\mathcal{W}$ be the Weyl group of the pair $(\mathbf{G}, \mathbf{T})$, viewed as a sheaf of finite groups on $\mathcal{O}_{\text{ét}}$. Write $W = \mathcal{W}(\bar{F})$ and $W^\circ = W^{\mathrm{Fr}}$.

Let $\mathbf{A}_{Q,n}$ be the $\mathcal{O}$-split torus with cocharacter lattice $Y_{Q,n}^{\mathrm{Fr}}$. Then the inclusion $Y_{Q,n}^{\mathrm{Fr}} \hookrightarrow Y$ defines a W°-equivariant homomorphism of $\mathcal{O}$-tori $\mathbf{A}_{Q,n} \to \mathbf{T}$. Let $\tilde{\mathbf{A}}_{Q,n}$ be the sharp cover of $\mathbf{A}_{Q,n}$ obtained by pulling back the cover $\tilde{\mathbf{T}}$.

16.1.1. *Satake step.* — From Corollary 7.4, the Satake isomorphism $\mathscr{S}\colon \mathscr{H}_\varepsilon(\tilde{G}, G^\circ) \xrightarrow{\sim} \mathscr{H}_\varepsilon(\tilde{T}, T^\circ)^{W^\circ}$ gives a bijection

$$\mathscr{S}^*\colon \mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{G}}) \xrightarrow{\sim} \frac{\mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{T}})}{W^\circ}.$$

16.1.2. *Support step.* — From Proposition 7.1, restriction gives a W°-equivariant isomorphism of Hecke algebras, $\mathscr{H}_\varepsilon(\tilde{T}, T^\circ) \xrightarrow{\sim} \mathscr{H}_\varepsilon(\tilde{A}_{Q,n}, A^\circ_{Q,n})$. See also [**80**, §3.3] for a description of this Hecke algebra. This gives a bijection,

$$\frac{\mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{T}})}{W^\circ} \xrightarrow{\sim} \frac{\mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n})}{W^\circ}.$$

16.1.3. *Parameterization for sharp covers of split tori.* — Our parameterization of Theorem 13.4 gives a bijection,

$$\mathscr{L}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n})\colon \mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n}) \to \mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n}).$$

Each $w \in W^\circ$ can be represented by an element $\dot{w} \in \mathbf{G}(\mathscr{O})$. The action of W° on the sets above can be untangled through the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathbf{A}'_{Q,n} \arrow[r, two heads] \arrow[d, "\mathrm{Int}(\dot{w})"] & \mathbf{A}_{Q,n} \arrow[d, "="] \\
\mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathrm{Int}(w)^*\mathbf{A}'_{Q,n} \arrow[r, two heads, "\mathrm{Int}(w)^{-1}\circ p"] \arrow[d, "="] & \mathbf{A}_{Q,n} \arrow[d, "\mathrm{Int}(w)"] \\
\mathbf{K}_2 \arrow[r, hook] & \mathbf{A}'_{Q,n} \arrow[r, two heads, "p"] & \mathbf{A}_{Q,n}.
\end{tikzcd} \tag{16.1}$$

The top two rows of (16.1) give a morphism in the category of covers $\mathsf{Cov}^\sharp_n(\mathbf{A}_{Q,n})$,

$$\mathrm{Int}(\dot{w})\colon \tilde{\mathbf{A}}_{Q,n} \to \mathrm{Int}(w)^*\tilde{\mathbf{A}}_{Q,n}.$$

The functoriality of $\mathscr{L}_\varepsilon$ for such morphisms gives a commutative diagram

$$\begin{tikzcd}
\mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n}) \arrow[r, "\mathscr{L}_\varepsilon(\tilde{\mathbf{A}}_{Q,n})"] \arrow[d, "\mathscr{L}_\varepsilon(\mathrm{Int}(\dot{w}))"] & \mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n}) \arrow[d, "\mathscr{L}_\varepsilon(\mathrm{Int}(\dot{w}))"] \\
\mathbf{\Pi}^\circ_\varepsilon(\mathrm{Int}(w)^*\tilde{\mathbf{A}}_{Q,n}) \arrow[r, "\mathscr{L}_\varepsilon(\mathrm{Int}(w)^*\tilde{\mathbf{A}}_{Q,n})"] & \mathbf{\Phi}^\circ_\varepsilon(\mathrm{Int}(w)^*\tilde{\mathbf{A}}_{Q,n}).
\end{tikzcd}$$

Compatibility of $\mathscr{L}_\varepsilon$ with pullbacks, combined with the bottom two rows of (16.1), gives a commutative diagram,

$$\begin{tikzcd}
\mathbf{\Pi}^\circ_\varepsilon(\mathrm{Int}(w)^*\tilde{\mathbf{A}}_{Q,n}) \arrow[r, "\mathscr{L}_\varepsilon(\mathrm{Int}(w)^*\tilde{\mathbf{A}}_{Q,n})"] \arrow[d] & \mathbf{\Phi}^\circ_\varepsilon(\mathrm{Int}(w)^*\tilde{\mathbf{A}}_{Q,n}) \arrow[d] \\
\mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n}) \arrow[r, "\mathscr{L}_\varepsilon(\tilde{\mathbf{A}}_{Q,n})"] & \mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n}).
\end{tikzcd}$$

Combining these, we find that the parameterization $\mathscr{L}_\varepsilon \colon \mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n}) \to \mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n})$ is equivariant for the action of W°. This gives a bijective parameterization

$$\mathscr{L}_\varepsilon \colon \frac{\mathbf{\Pi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n})}{W^\circ} \xrightarrow{\sim} \frac{\mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n})}{W^\circ}.$$

16.1.4. *Split and unramified parameters.* — The inclusion $\tilde{\mathbf{A}}_{Q,n} \to \tilde{\mathbf{T}}$ is a well-aligned morphism of covers, giving an L-morphism

$$\begin{tikzcd}
\mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times) = \tilde{T}^\vee \arrow[r, hook] \arrow[d, two heads, "\nu"] & {}^{\mathsf{L}}\tilde{T} \arrow[r, two heads] \arrow[d, two heads, "\nu"] & \mathrm{Gal}_{\mathcal{O}} \arrow[d, "="] \\
\mathrm{Hom}(Y_{Q,n}^{\mathrm{Fr}}, \mathbb{C}^\times) = \tilde{A}_{Q,n}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{A}_{Q,n} \arrow[r, two heads] & \mathrm{Gal}_{\mathcal{O}}.
\end{tikzcd}$$

Since the inclusion $\tilde{\mathbf{A}}_{Q,n} \hookrightarrow \tilde{\mathbf{T}}$ is $W^\circ = W^{\mathrm{Fr}}$-equivariant, we find that the L-morphism $\nu \colon {}^{\mathsf{L}}\tilde{T} \to {}^{\mathsf{L}}\tilde{A}_{Q,n}$ satisfies

$$\nu \circ {}^{\mathsf{L}}\mathrm{Int}(\dot{w}) \text{ is naturally isomorphic to } {}^{\mathsf{L}}\mathrm{Int}(\dot{w}) \circ \nu,$$

for all $w \in W^{\mathrm{Fr}}$ and all representatives $\dot{w} \in G^\circ$ for w.

The method below follows [**11**, §6] very closely; minimal changes are required. Giving an equivalence class of Weil parameters $\mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n})$ is the same as giving an $\tilde{A}^\vee_{Q,n}$-conjugacy class of elements ${}^{\mathsf{L}}\tilde{A}_{Q,n}$ lying over $\mathrm{Fr} \in \mathrm{Gal}_{\mathcal{O}}$. Similarly, giving an equivalence class of Weil parameters in $\mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{T}})$ is the same as giving a $\tilde{T}^\vee$-conjugacy class of elements in ${}^{\mathsf{L}}\tilde{T}$ lying over $\mathrm{Fr} \in \mathrm{Gal}_{\mathcal{O}}$.

As $\nu \colon {}^{\mathsf{L}}\tilde{T} \to {}^{\mathsf{L}}\tilde{A}_{Q,n}$ is surjective and W^{Fr}-equivariant, we find that ν gives a surjective map,

$$\bar{\nu} \colon \frac{\mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{T}})}{W^{\mathrm{Fr}}} \twoheadrightarrow \frac{\mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n})}{W^{\mathrm{Fr}}}.$$

If $[\phi], [\phi'] \in \mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{T}})$ and $\nu([\phi])$ and $\nu([\phi'])$ are in the same W^{Fr}-oribt, then there exist $t, t' \in {}^{\mathsf{L}}\tilde{T}$ and $w \in W^{\mathrm{Fr}}$ represented by $\dot{w} \in G^\circ$, such that $\phi(\mathrm{Fr}) = t$ and $\phi'(\mathrm{Fr}) = t'$, and $\nu(t) = {}^{\mathsf{L}}\mathrm{Int}(\dot{w})(\nu(t'))$. Thus $\nu(t) = \nu(\mathrm{Int}(\dot{w})t')$, and so

$$t^{-1} \cdot \mathrm{Int}(\dot{w})t' \in \mathrm{Ker}(\tilde{T}^\vee \twoheadrightarrow \tilde{A}^\vee_{Q,n}).$$

The following sequence is exact, by the same arguments as in [**11**, §6.3]:

$$\mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times) \xrightarrow{\mathrm{Fr}-1} \mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times) \xrightarrow{\mathrm{Res}} \mathrm{Hom}(Y^{\mathrm{Fr}}_{Q,n}, \mathbb{C}^\times).$$

It follows that $t^{-1} \cdot \mathrm{Int}(\dot{w})t' = \tau / \mathrm{Fr}(\tau)$ for some $\tau \in \tilde{T}^\vee$. Hence

$$\mathrm{Int}(\dot{w})t' = \tau t \tau^{-1}.$$

Hence if $\nu([\phi])$ and $\nu([\phi'])$ are in the same W^{Fr}-orbit, then $[\phi]$ and $[\phi']$ are in the same W^{Fr}-orbit. Therefore, $\bar{\nu}$ gives a bijective map,

$$\bar{\nu} \colon \frac{\mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{T}})}{W^{\mathrm{Fr}}} \xrightarrow{\sim} \frac{\mathbf{\Phi}^\circ_\varepsilon(\tilde{\mathbf{A}}_{Q,n})}{W^{\mathrm{Fr}}}.$$

16.1.5. *Semisimple twisted conjugacy classes.* — The Weyl group W is canonically isomorphic to the Weyl group of $\tilde{G}^\vee$ with respect to $\tilde{T}^\vee$, as finite groups with Gal_S-action. In this way, any element $w \in W^{\mathrm{Fr}}$ corresponds to an element $w^\vee \in (W^\vee)^{\mathrm{Fr}}$, which may be represented by a Gal_S-invariant element $n^\vee \in \tilde{N}^\vee \subset \tilde{G}^\vee$, where $\tilde{N}^\vee$ is the normalizer of $\tilde{T}^\vee$ in $\tilde{G}^\vee$ (see [**11**, Lemma 6.2]). From Theorem 5.13, the L-morphism ${}^{\mathsf{L}}\mathrm{Int}(\dot{w})$ of ${}^{\mathsf{L}}\tilde{T}$ is naturally isomorphic to the L-morphism $\mathrm{Int}(n^\vee)$. This gives a bijection

$$\frac{\mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{T}})}{W^{\mathrm{Fr}}} \leftrightarrow \frac{\mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{T}})}{(\tilde{N}^\vee)^{\mathrm{Fr}}}.$$

Finally, from [**11**, Lemma 6.5], inclusion provides a bijection,

$$\frac{\mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{T}})}{(\tilde{N}^\vee)^{\mathrm{Fr}}} \xrightarrow{\sim} \mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{G}}).$$

Theorem 16.1. — *Assembling the bijections in the five steps above, we have constructed a bijection*

$$\mathscr{L}_\varepsilon(\tilde{\mathbf{G}})\colon \mathbf{\Pi}_\varepsilon^\circ(\tilde{\mathbf{G}}) \xrightarrow{\sim} \mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{G}}).$$

16.2. Automorphic L-functions. — The spherical/unramified parameterization provides a definition of automorphic L-functions, almost everywhere. Let F be a global field. Let $\mathscr{V}$ be the set of places of F, and let $\mathscr{S}$ be a finite subset of $\mathscr{V}$ containing all archimedean places. Suppose that $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ is a degree n cover of a quasisplit reductive group $\mathbf{G}$ over the ring of $\mathscr{S}$-integers $\mathscr{O}_{\mathscr{S}}$ (see [**79**, §3.2] for a classification). This defines a central extension $\mu_n \hookrightarrow \tilde{G}_{\mathbb{A}} \twoheadrightarrow G_{\mathbb{A}}$, endowed with splittings over G_F and over $G_v^\circ = \mathbf{G}(\mathscr{O}_v)$ for all $v \notin \mathscr{S}$.

Let π be an ε-genuine automorphic representation of $\tilde{G}_{\mathbb{A}}$. Factorization yields ε-genuine spherical irreducible representations $[\pi_v]$, for almost all nonarchimedean places $v \in \mathscr{V} - \mathscr{S}$. Each equivalence class $[\pi_v]$ yields an equivalence class of unramified parameters $[\sigma_v] \in \mathbf{\Phi}_\varepsilon^\circ(\tilde{\mathbf{G}})$ by our parameterization $\mathscr{L}_\varepsilon(\tilde{\mathbf{G}}_v)$ of Theorem 16.1. Each unramified parameter yields a well-defined semisimple $\tilde{G}^\vee$-conjugacy class $g_v = \sigma_v(\mathrm{Fr}) \in {}^{\mathsf{L}}\tilde{G}_v$. Local-global compatibility in the construction of the L-group identify g_v with semisimple conjugacy classes in the globally-defined L-group ${}^{\mathsf{L}}G$.

Let $\rho\colon {}^{\mathsf{L}}\tilde{G} \to \mathrm{GL}(V)$ be a representation of the L-group on a finite-dimensional complex vector space V. Then, for almost all nonarchimedean places v (with residue fields of cardinality q_v), we have a local L-function

$$L_v(\pi, \rho, s) = \det(\mathrm{Id}_V - q_v^{-s}\rho(g_v)|V)^{-1}.$$

We define the ***automorphic L-function*** away from $\mathscr{S}$,

$$L_{\mathscr{S}}(\pi, \rho, s) = \prod_{v \notin S} L_v(\pi, \rho, s).$$

From the remarks of [**18**, §10.5], another choice $\mathscr{S}'$ of places and an $\mathscr{O}_{\mathscr{S}'}$-model of $\tilde{\mathbf{G}}$ will yield an automorphic L-function $L_{\mathscr{S}'}(\pi, \rho, s)$ which agrees with $L_{\mathscr{S}}(\pi, \rho, s)$ at almost all places. In this way the automorphic L-function is well-defined from π

and ρ, up to factors at finitely many places. For split groups, the convergence of automorphic L-functions, in a suitable right half-plane, appears in this volume in the work of Gan and Gao [**29**, §13.4].

Remark 16.2. — Langlands' argument (see [**11**, §13] or [**65**, §2.5] for treatments), together with the spectral decomposition of automorphic forms by Moeglin and Waldspurger [**53**, §III.2.6], should imply the absolute convergence of $L_{\mathcal{S}}(\pi, \rho, s)$ in some right half-plane. To make this more precise, one needs the Macdonald formula for the spherical function. Progress in this direction has been made in the work of McNamara [**51**] and the thesis of Fan Gao [**30**, §4.3].

17. Sharp covers of anisotropic real tori

Let $\mathbf{T}$ be an *anisotropic* torus over $\mathbb{R}$, and let $\tilde{\mathbf{T}}$ be a degree 2 *sharp* cover of $\mathbf{T}$ over $\mathbb{R}$. Write $Y = \mathscr{Y}[\mathbb{C}]$ for the cocharacter lattice, and σ for complex conjugation. Thus $\sigma(y) = -y$ for all $y \in Y$. The cover $\tilde{\mathbf{T}}$ yields an extension,

$$\mu_2 \hookrightarrow \tilde{T} \twoheadrightarrow T.$$

There is a natural identification of Lie groups $T = Y \otimes U(1)$, from which we identify the topological fundamental group $\pi_1(T) \equiv Y$. Connectedness of T implies that $\tilde{T}$ is abelian, and that this extension is rigid; there are no nontrivial automorphisms of the extension $\tilde{T} \in \mathsf{Ext}(T, \mu_2)$. The extension $\tilde{T}$ is determined, up to unique isomorphism, by an associated homomorphism $\kappa \colon \pi_1(T) = Y \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

The homomorphism κ may be constructed via Pontryagin duality as follows. The Pontryagin dual of T, $\mathbf{\Pi}(\mathbf{T}) = \mathrm{Hom}(T, U(1))$, is naturally identified with the character lattice $X = \mathscr{X}[\mathbb{C}]$. Write X_κ for the Pontryagin dual of $\tilde{T}$, fitting into a short exact sequence

$$X \hookrightarrow X_\kappa \twoheadrightarrow \mathbb{Z}_{/2}.$$

The genuine characters of $\tilde{T}$ are then identified with the elements of $X_\kappa^- = X_\kappa - X$,

$$\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}) \equiv X_\kappa^- = X_\kappa - X.$$

For each such genuine character $x \in X_\kappa^-$, $2x \in X$; thus there exists a unique $\xi \in \frac{1}{2}X$ such that $2x = 2\xi$. The map $x \mapsto \xi$ provides an embedding, equivariant for translation by X (i.e., for twisting by $\mathbf{\Pi}(\mathbf{T})$), $X_\kappa^- \hookrightarrow \frac{1}{2}X$. As X_κ^- is an X-torsor, there exists a unique $\kappa \in \frac{1}{2}X/X$ such that

$$X_\kappa^- \equiv \{\xi \in \tfrac{1}{2}X : \xi = \kappa \bmod X\}.$$

The element $\kappa \in \frac{1}{2}X/X$ can also be viewed as a homomorphism

$$\kappa \colon \pi_1(T) = Y \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

This κ is the homomorphism which determines the double cover $\tilde{T}$ uniquely up to unique isomorphism.

The Brylinski-Deligne invariants of $\tilde{\mathbf{T}}$ consist of a quadratic form $Q\colon Y \to \mathbb{Z}$ and a $\mathrm{Gal}_{\mathbb{R}}$-equivariant extension

$$\mathbb{C}^\times \hookrightarrow D \twoheadrightarrow Y.$$

Recall that $\tilde{\mathbf{T}}$ is a sharp cover; thus $Y = Y_{Q,2}$ and D is commutative. For each $y \in Y$, write $\iota_y\colon \mathbb{Z} \hookrightarrow Y$ for the homomorphism $n \mapsto n \cdot y$. We get a $\mathrm{Gal}_{\mathbb{R}}$-equivariant extension

$$\mathbb{C}^\times \hookrightarrow \iota_y^* D \twoheadrightarrow \mathbb{Z},$$

where $\sigma \in \mathrm{Gal}_{\mathbb{R}}$ acts by $n \mapsto -n$ on $\mathbb{Z}$, and by complex conjugation on $\mathbb{C}^\times$. Following [**18**, §12.6], such $\mathrm{Gal}_{\mathbb{R}}$-equivariant extensions of $\mathbb{Z}$ by $\mathbb{C}^\times$ are classified up to isomorphism by elements of $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. (Brylinski and Deligne use $\mathbb{Z}_{/2}$ instead, but our normalization has advantages to follow). Indeed, take $d \in \iota_y^* D$ lying over $1 \in \mathbb{Z}$. Then $d \cdot \sigma(d)$ lies over $1 + (-1) = 0 \in \mathbb{Z}$, and so $d\sigma(d) \in \mathbb{C}^\times$. Since $d\sigma(d) = \sigma(d\sigma(d))$, we have $d\sigma(d) \in \mathbb{R}^\times$. Define $\eta(y) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ so that

$$\mathrm{sgn}(d \cdot \sigma(d)) = e^{2\pi i \eta(y)}.$$

The function η is independent of choices along the way, and defines a homomorphism

$$\eta\colon Y \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

The next proposition is a direct consequence of [**18**, §12.6, Proposition 12.7]. It gives a practical recipe for determining the topological cover $\tilde{T}$ from the Brylinski-Deligne invariants of $\tilde{\mathbf{T}}$.

Proposition 17.1. — *The homomorphism $\kappa\colon Y \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is given by*

$$\kappa(y) = \eta(y) + \tfrac{1}{2}Q(y) \pmod{\mathbb{Z}}.$$

To see that the Brylinski-Deligne framework is sufficiently strong for work on Lie groups, the following may be of interest.

Corollary 17.2. — *Every topological double cover $\mu_2 \hookrightarrow \tilde{T} \twoheadrightarrow T$ arises from a sharp cover $\tilde{\mathbf{T}} \in \mathsf{Cov}_2^\sharp(\mathbf{T})$ over $\mathbb{R}$.*

Proof. — A topological double cover $\tilde{T}$ arises from a homomorphism $\kappa\colon Y \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Choose a basis $y_1, \ldots, y_r \in Y$, and define a quadratic form $Q\colon Y \to \mathbb{Z}$ by

$$Q(a_1 y_1 + \cdots + a_r y_r) = 2\kappa(y_1)a_1^2 + \cdots + 2\kappa(y_r)a_r^2.$$

Then $\frac{1}{2}Q(y) = \kappa(y)$, mod $\mathbb{Z}$. Moreover, $Y = Y_{Q,2}$ for this quadratic form. By the classification of [**18**], there exists a central extension $\mathbf{K}_2 \hookrightarrow \mathbf{T}' \twoheadrightarrow \mathbf{T}$ over $\mathbb{R}$ with first invariant Q and trivial second invariant. The previous proposition gives a unique isomorphism from $\tilde{T}$ to the topological double cover associated to $\mathbf{T}'$. □

Now, we work on the side of the L-group. The dual group $T^\vee = \tilde{T}^\vee = X \otimes \mathbb{C}^\times$ is endowed with the $\mathrm{Gal}_{\mathbb{R}}$-action $\sigma(t^\vee) = (t^\vee)^{-1}$ for all $t^\vee \in T^\vee$. The L-group is a short exact sequence of complex groups,

$$T^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{T} \twoheadrightarrow \mathrm{Gal}_{\mathbb{R}}.$$

As $\kappa \in \frac{1}{2}X/X$, we may view $e^{2\pi i\kappa}$ as an element of $T^\vee_{[2]} = X \otimes \mu_2$. Explicitly,

$$y(e^{2\pi i\kappa}) = e^{2\pi i\kappa(y)}, \text{ for all } y \in Y.$$

Lemma 17.3. — *The lifts $\gamma \in {}^{\mathsf{L}}\tilde{T}$ of $\sigma \in \mathrm{Gal}_{\mathbb{R}}$ form a single orbit under $T^\vee$-conjugation, and for any such lift γ,*

$$\gamma^2 = e^{2\pi i\kappa}.$$

Proof. — If γ_1, γ_2 are two lifts of σ, then $\gamma_1 = t^\vee\gamma_2$ for some $t^\vee \in T^\vee$. Since squaring is surjective on $T^\vee$, there exists $\tau^\vee \in T^\vee$ such that $\tau^\vee/\sigma(\tau^\vee) = (\tau^\vee)^2 = t^\vee$. Hence

$$\tau^\vee \cdot \gamma_1 \cdot (\tau^\vee)^{-1} = \frac{\tau^\vee}{\sigma(\tau^\vee)} \cdot \gamma_1 = t^\vee \cdot \gamma_1 = \gamma_2.$$

It follows that $\gamma_2^2 = \gamma_1^2$ and the lifts of σ form a single $T^\vee$ orbit. Therefore, to prove the theorem, it suffices to prove that $\gamma^2 = e^{2\pi i\kappa}$ for a single lift γ of σ.

Recall that ${}^{\mathsf{L}}\tilde{T}$ is the Baer sum of two extensions of $\mathrm{Gal}_{\mathbb{R}}$ by $T^\vee$,

$${}^{\mathsf{L}}\tilde{T} = (\tau_Q)_*\widetilde{\mathrm{Gal}}_{\mathbb{R}} \dotplus \pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{z}).$$

Let γ_Q be a lift of σ in the extension $(\tau_Q)_*\widetilde{\mathrm{Gal}}_{\mathbb{R}}$. In the metaGalois group $\mu_2 \hookrightarrow \widetilde{\mathrm{Gal}}_{\mathbb{R}} \twoheadrightarrow \mathrm{Gal}_{\mathbb{R}}$, the square of any lift of σ equals -1 (since $\mathrm{Hilb}_2(-1,-1) = -1$). Hence we find $\gamma_Q^2 = \tau_Q(-1)$. In other words,

$$y(\gamma_Q^2) = (-1)^{Q(y)} = e^{\pi i Q(y)} \text{ for all } y \in Y.$$

Next we construct an object $\bar{z}$ of $\mathbf{E}_\varepsilon(\tilde{\mathbf{T}})[\mathbb{C}]$ and a lift of σ in $\pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{z})$. Let $\hat{T} = \mathrm{Hom}(Y, \mathbb{C}^\times)$, with $\mathrm{Gal}_{\mathbb{R}}$-action arising from $y \mapsto -y$ on Y, and complex conjugation on $\mathbb{C}^\times$. Then $\mathrm{Spl}(D)$ is the $\hat{T}$-torsor of splittings of D. Choose any $s \in \mathrm{Spl}(D)$. The $\mathrm{Gal}_{\mathbb{R}}$-action on D gives an action on splittings; we have

$${}^\sigma s(y) = \sigma(s(\sigma(y))) = \sigma(s(-y)) = \sigma(s(y))^{-1} \text{ for all } y \in Y.$$

Define a map $h\colon \hat{T} \to \mathrm{Spl}(D)$ by

$$h(\hat{t}) = \hat{t}^2 * s.$$

Then $\bar{z} = (\hat{T}, h) \in \sqrt{\mathrm{Spl}(D)}$ is a square root of the $\hat{T}$-torsor $\mathrm{Spl}(D)$, i.e., $\bar{z}$ is an object of $\mathbf{E}_\varepsilon(\tilde{\mathbf{T}})[\mathbb{C}]$. Its complex conjugate ${}^\sigma\bar{z}$ is given by ${}^\sigma\bar{z} = (\hat{T}, {}^\sigma h)$, where

$${}^\sigma h(\hat{t}) = \hat{t}^2 * {}^\sigma s.$$

Choose $f \in \mathrm{Hom}(\bar{z}, {}^\sigma\bar{z}) \subset \pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{z})$. Then, as a morphism of $\hat{T}$ torsors, there exists $\hat{\tau} \in \hat{T}$ such that $f\colon \hat{T} \to \hat{T}$ satisfies $f(\hat{a}) = \hat{\tau} * \hat{a}$ for all $\hat{a}$. The condition ${}^\sigma h \circ f = h$ implies that

$$\hat{a}^2(y)s(y) = \hat{a}^2(y)\hat{\tau}^2(y) \cdot \sigma(s(y))^{-1}$$

for all $y \in Y$. Hence, if $y \in Y$ and $d = s(y)$, then

$$\hat{\tau}(y) = \sqrt{d\sigma(d)} \in \mathbb{C}^\times \quad \text{(for some square root).}$$

The square of $f \in \mathrm{Hom}(\bar{z}, {}^\sigma\bar{z}) \subset \pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{T}}), \bar{z})$ is the element of $\mathrm{Hom}(\bar{z}, \bar{z})$ given by composing morphisms

$$\bar{z} \xrightarrow{f} {}^\sigma\bar{z} \xrightarrow{{}^\sigma f} {}^\sigma({}^\sigma\bar{z}) = \bar{z}.$$

As a function from $\hat{T}$ to $\hat{T}$, we find that $f^2(\hat{a}) = {}^\sigma\hat{\tau} \cdot \hat{\tau} \cdot \hat{a}$, for all $\hat{a} \in \hat{T}$. In other words, $f^2 = {}^\sigma\hat{\tau} \cdot \hat{\tau}$ as an element of $\hat{T}_{[2]} \subset T^\vee$. We compute

$$y(f^2) = [{}^\sigma\hat{\tau} \cdot \hat{\tau}](y) = \sigma(\hat{\tau}(-y)) \cdot \tau(y) = \frac{\sqrt{d\sigma(d)}}{\sigma\sqrt{d\sigma(d)}} = \mathrm{sgn}(d\sigma(d)) = e^{2\pi i \eta(y)}.$$

Finally, define $\gamma = \gamma_Q \dotplus f \in {}^{\mathsf{L}}\tilde{T}$. From Proposition 17.1, we find

$$y(\gamma^2) = y(\gamma_Q^2) \cdot y(f^2) = e^{\pi i Q(y) + 2\pi i \eta(y)} = e^{2\pi i \kappa(y)}. \qquad \square$$

The Weil group of $\mathbb{R}$ is $\mathscr{W}_\mathbb{R} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$; the map from $\mathscr{W}_\mathbb{R}$ to $\mathrm{Gal}_\mathbb{R}$ sends $\mathbb{C}^\times$ to Id and j to σ. Recall that $jzj^{-1} = \bar{z}$ for all $z \in \mathbb{C}^\times$, and $j^2 = -1$. A continuous homomorphism from $\mathbb{C}^\times$ to $T^\vee = \mathrm{Hom}(Y, \mathbb{C}^\times)$ has the form $z \mapsto z^{x_1 - x_2}(z\bar{z})^{x_2}$ for some $x_1, x_2 \in X \otimes \mathbb{C}$ satisfying $x_1 - x_2 \in X$. We abuse notation slightly, and write $z^{x_1}\bar{z}^{x_2}$ for such a homomorphism, keeping in mind that $x_1 - x_2 \in X$ is necessary for this expression to make sense.

Recall that $\kappa \in \frac{1}{2}X/X = \mathrm{Hom}(Y, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$, and

$$X_\kappa^- = \{\xi \in \tfrac{1}{2}X : \xi = \kappa \bmod X\}.$$

Theorem 17.4. — *For every Weil parameter $\phi\colon \mathscr{W}_\mathbb{R} \to {}^{\mathsf{L}}\tilde{T}$, there exists a unique element $\xi \in X_\kappa^-$ such that*

$$\phi(z) = (z/\bar{z})^\xi := z^{2\xi}(z\bar{z})^{-\xi}.$$

The map $\phi \mapsto \xi$ defines a bijection,

$$\mathbf{\Phi}_\varepsilon(\tilde{\mathbf{T}}) = \frac{\mathrm{Par}(\mathscr{W}_\mathbb{R}, {}^{\mathsf{L}}\tilde{T})}{T^\vee - \textit{conjugation}} \xrightarrow{\sim} X_\kappa^- \equiv \mathbf{\Pi}_\varepsilon(\tilde{\mathbf{T}}).$$

Proof. — Giving a Weil parameter $\phi\colon \mathscr{W}_\mathbb{R} \to {}^{\mathsf{L}}\tilde{T}$ is the same as giving a homomorphism $\phi(z) = z^{x_1}\bar{z}^{x_2}$ for some $x_1, x_2 \in X \otimes \mathbb{C}$, $x_1 - x_2 \in X$, and an element $\gamma = \phi(j) \in {}^{\mathsf{L}}\tilde{T}$ lying over $\sigma \in \mathrm{Gal}_\mathbb{R}$, which satisfy the following conditions:

- $\gamma^2 = \phi(-1) = (-1)^{x_1 - x_2}$;
- $\gamma\phi(z)\gamma^{-1} = \phi(\bar{z})$ for all $z \in \mathbb{C}^\times$.

From the previous lemma, $T^\vee$ acts transitively on the lifts of σ in ${}^{\mathsf{L}}\tilde{T}$, and all lifts γ satisfy $\gamma^2 = e^{2\pi i\kappa}$. We find that giving a $T^\vee$-orbit on $\mathrm{Par}(\mathscr{W}_\mathbb{R}, {}^{\mathsf{L}}\tilde{T})$ is the same as giving a homomorphism $\phi(z) = z^{x_1}\bar{z}^{x_2}$ for some $x_1, x_2 \in X \otimes \mathbb{C}$, $x_1 - x_2 \in X$, which satisfies the following conditions:

- $\phi(-1) = (-1)^{x_1 - x_2} = e^{2\pi i\kappa}$;
- $\phi(\bar{z}) = \bar{z}^{x_1}z^{x_2} = \sigma(\phi(z)) = z^{-x_1}\bar{z}^{-x_2}$.

The second condition is equivalent to the condition $x_1 = -x_2$; if $\xi = x_1 = -x_2$, note that $2\xi = x_1 - x_2 \in X$ and so $\xi \in \frac{1}{2}X$. In this case, the first condition is equivalent to $(-1)^{2\xi} = e^{2\pi i \xi} = e^{2\pi i \kappa}$. We find that giving a $T^\vee$-orbit on $\mathrm{Par}(\mathcal{W}_{\mathbb{R}}, {}^{\mathsf{L}}\tilde{T})$ is the same as giving an element $\xi \in \frac{1}{2}X$ such that $\xi = \kappa \bmod X$. This defines the bijection. □

18. Discrete series for covers of real semisimple groups

18.1. Harish-Chandra classification. — Suppose here that $\mathbf{G}$ is a *semisimple* quasisplit group over $\mathbb{R}$, with Borel subgroup $\mathbf{B}$ containing a maximally split maximal torus $\mathbf{T}$. Let $\tilde{\mathbf{G}} = (\mathbf{G}', 2)$ be a double cover of $\mathbf{G}$ defined over $\mathbb{R}$. Then we find a topological double cover over Lie groups,

$$\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow G.$$

Remark 18.1. — If, in addition, $\mathbf{G}$ is simply-connected and absolutely simple, then there is a unique up to unique isomorphism double cover $\tilde{\mathbf{G}}$ whose first Brylinski-Deligne invariant takes the value 1 on all short coroots. The resulting double cover $\mu_2 \hookrightarrow \tilde{G} \twoheadrightarrow G$ coincides with Deligne's canonical central extension, after [**18**, Construction 9.3, 10.3]. In this case, Prasad and Rapinchuk prove that $\tilde{G}$ is the unique nontrivial 2-fold cover of G [**56**, Theorem 8.4 and §8.5]. Thus the framework of Brylinski-Deligne is sufficient to work with the most interesting nonlinear double covers of semisimple Lie groups.

Let G° be the identity component of the Lie group G, and let $\tilde{G}^\circ$ be the identity component of the Lie group $\tilde{G}$. Then G° and $\tilde{G}^\circ$ are connected real semisimple Lie groups with finite center. The Harish-Chandra classification and the description of discrete series by Atiyah and Schmid (see [**6**]) apply to G° and $\tilde{G}^\circ$.

Let $\mathbf{\Pi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}})$ be the set of equivalence classes of irreducible ε-genuine *discrete series* representations of $\tilde{G}$. Here $\varepsilon\colon \boldsymbol{\mu}_2(\mathbb{R}) \to \boldsymbol{\mu}_2(\mathbb{C})$ is the identity map, so we often refer to "genuine representations" without mention of ε. Any irreducible discrete series representation of $\tilde{G}$ restricts to a finite direct sum of irreducible discrete series representations of $\tilde{G}^\circ$, all with the same infinitesimal character. By Frobenius reciprocity, all irreducible discrete series representations of $\tilde{G}^\circ$ occur in such a restriction.

The set $\mathbf{\Pi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}})$ is nonempty if and only if there exists a compact Cartan subgroup of $\tilde{G}^\circ$; such a compact Cartan subgroup exists if and only if there exists a maximal torus $\mathbf{S} \subset \mathbf{G}$, defined and anisotropic over $\mathbb{R}$. Suppose that $\mathbf{S}$ is such a torus.

Let $S = \mathbf{S}(\mathbb{R})$ and $\tilde{S}$ its preimage in $\tilde{G}$. Let K be a maximal compact subgroup of G containing S, and K° its identity component. Write $\tilde{K}$ for the preimage of K in $\tilde{G}$, and $\tilde{K}^\circ$ for the identity component of $\tilde{K}$.

Consider a genuine irreducible discrete series representation (π, V) of $\tilde{G}$; the $\tilde{K}$-finite vectors therein form an irreducible admissible $(\mathfrak{g}, \tilde{K})$-module. As such, it has an infinitesimal character $\chi \colon \mathfrak{Z} \to \mathbb{C}$, where $\mathfrak{Z}$ denotes the center of the universal enveloping algebra of $\mathfrak{g}$. The Harish-Chandra isomorphism gives an isomorphism of $\mathbb{C}$-algebras,

$$\mathfrak{Z} \xrightarrow{\sim} \mathbb{C}[\mathfrak{s}]^W,$$

where $\mathfrak{s}$ is the complexified Lie algebra of $\mathbf{S}$, and W is the Weyl group of $\mathbf{G}$ with respect to $\mathbf{S}$. Let X be the cocharacter lattice of $\mathbf{S}$. Then $\operatorname{Hom}(\mathfrak{s}, \mathbb{C})$ is naturally isomorphic to $X \otimes \mathbb{C}$. Thus the infinitesimal character defines a map,

$$\inf \colon \mathbf{\Pi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}}) \to \frac{X \otimes \mathbb{C}}{W - \text{conjugation}}.$$

As $S = \mathbf{S}(\mathbb{R}) \equiv Y \otimes U(1)$, the Pontryagin dual of S is identified with X. The set of genuine characters $\mathbf{\Pi}_\varepsilon(\tilde{\mathbf{S}})$ of $\tilde{S}$ is identified with a X-coset $\kappa + X \subset \frac{1}{2}X$. Here, the element

$$\kappa \in \tfrac{1}{2}X/X = \operatorname{Hom}(Y, \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}) = \operatorname{Hom}(\pi_1(S), \tfrac{1}{2}\mathbb{Z}/\mathbb{Z})$$

determines the double cover $\tilde{S}$ up to unique isomorphism.

Theorem 18.2. — *Let ρ be the half-sum of the positive roots of $\mathbf{G}$ (with respect to some Borel subgroup containing $\mathbf{S}$). Let $X_{\mathbb{Q}}^{\mathrm{reg}}$ denote the regular (=nonsingular) locus in $X_{\mathbb{Q}} = X \otimes \mathbb{Q}$. The infinitesimal character provides a finite-to-one surjective map,*

$$\inf \colon \mathbf{\Pi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}}) \twoheadrightarrow \frac{(\kappa + \rho + X) \cap X_{\mathbb{Q}}^{\mathrm{reg}}}{W - \textit{conjugation}}.$$

Proof. — This follows directly from [**6**, Corollary 6.13], together with our remarks on the restriction from $\tilde{G}$ to $\tilde{G}^\circ$. □

In the uncovered case, the fibers of the infinitesimal character map are precisely the L-packets (under our semisimplicity assumption). In what follows, we parameterize the genuine discrete series representations by discrete series Weil parameters, valued in the L-group of the cover $\tilde{\mathbf{G}}$.

18.2. Discrete series parameters. — Our treatment here is largely based on the work of Langlands [**44**, §3]. We have reproduced the structure and arguments of Langlands, making adaptations and additions where necessary. As before, $\mathbf{G}$ is a *semisimple* quasisplit group over $\mathbb{R}$ and $\tilde{\mathbf{G}}$ is a double cover of $\mathbf{G}$ over $\mathbb{R}$. The L-group of $\tilde{\mathbf{G}}$ fits into a short exact sequence,

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}_{\mathbb{R}} = \{1, \sigma\}.$$

Recall that the dual group $\tilde{G}^\vee$ is a pinned complex reductive group, with $\tilde{G}^\vee \supset \tilde{B}^\vee \supset \tilde{T}^\vee$, associated to the root datum

$$\tilde{\Psi}^\vee = (Y_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, X_{Q,n}, \tilde{\Phi}, \tilde{\Delta}).$$

A *discrete series Weil parameter* is a Weil parameter $\phi \colon \mathscr{W}_{\mathbb{R}} \to {}^{\mathsf{L}}\tilde{G}$ whose image is not contained in any proper parabolic subgroup ${}^{\mathsf{L}}\tilde{P}$. (As in [**44**], a parabolic subgroup of ${}^{\mathsf{L}}\tilde{G}$ is a subgroup ${}^{\mathsf{L}}\tilde{P}$ whose intersection with $\tilde{G}^\vee$ is a parabolic subgroup and whose projection to $\mathrm{Gal}_{\mathbb{R}}$ is surjective). Let $\mathbf{\Phi}_{\varepsilon}^{\mathrm{disc}}(\tilde{\mathbf{G}})$ be the set of equivalence classes of discrete series Weil parameters.

Suppose that ϕ_0 is a discrete series Weil parameter. Recall that $\mathcal{W}_{\mathbb{R}} = \mathbb{C}^\times \sqcup \mathbb{C}^\times j$, where $j^2 = -1$ and $jzj^{-1} = \sigma(z)$ for all $z \in \mathbb{C}^\times$.

Lemma 18.3. — *ϕ_0 is equivalent to a parameter ϕ such that $\phi(\mathbb{C}^\times) \subset \tilde{T}^\vee$ and $\phi(j)$ normalizes $\tilde{T}^\vee$.*

Proof. — The reasoning of [**44**] based on [**67**, §II, Theorem 5.16] demonstrates that $\phi(\mathbb{C}^\times) \subset \tilde{G}^\vee$ is contained in a maximal torus which is normalized by $\phi(\mathcal{W}_{\mathbb{R}})$. Replacing ϕ by a $\tilde{G}^\vee$-conjugate parameter if necessary, we may assume that $\phi(\mathbb{C}^\times) \subset \tilde{T}^\vee$ and $\phi(\mathcal{W}_{\mathbb{R}})$ normalizes $\tilde{T}^\vee$. $\square$

Let ϕ be a discrete series Weil parameter, with $\phi(\mathbb{C}^\times) \subset \tilde{T}^\vee$ and $\gamma = \phi(j)$ normalizing $\tilde{T}^\vee$. Then conjugation by γ is a $\mathbb{C}$-algebraic automorphism of $\tilde{T}^\vee$, and so it arises from

$$\gamma \colon Y_{Q,n} \to Y_{Q,n}.$$

Lemma 18.4. — *For all $y \in Y_{Q,n}$, $\gamma(y) = -y$. Equivalently, for all $t^\vee \in \tilde{T}^\vee$,*

$$\gamma t^\vee \gamma^{-1} = (t^\vee)^{-1}.$$

Proof. — Since $j^2 = -1$, γ acts as an automorphism of order 2 of $Y_{Q,n}$. Thus to demonstrate that γ acts as -1 on $Y_{Q,n}$, it suffices to demonstrate that γ has no fixed points in $Y_{Q,n} \otimes \mathbb{Q}$.

If $\lambda \in Y_{Q,n} \otimes \mathbb{Q}$ and $\gamma\lambda = \lambda$, then define $P_\lambda^\vee$ to be the parabolic subgroup of $\tilde{G}^\vee$ containing $\tilde{T}^\vee$, whose roots are those $\tilde{\alpha}^\vee \in \tilde{\Phi}^\vee$ for which

$$\tilde{\alpha}(\lambda) \geq 0.$$

The conjugation action of $\gamma \to \mathrm{Aut}(\tilde{G}^\vee)$ stabilizes $\tilde{P}^\vee$; indeed, it stabilizes $\tilde{T}^\vee$ and the set of roots of $\tilde{P}^\vee$. Hence the group ${}^{\mathsf{L}}\tilde{P}$ generated by $\tilde{P}^\vee$ and γ is a parabolic subgroup of ${}^{\mathsf{L}}\tilde{G}$. This contradicts the assumption that ϕ is a discrete series Weil parameter. $\square$

Now we find that the discrete series Weil parameter ϕ, with $\phi(\mathbb{C}^\times) \subset \tilde{T}^\vee$, satisfies

$$\phi(z) = z^{x_1}\bar{z}^{x_2} \in \tilde{T}^\vee,$$

for some $x_1, x_2 \in X_{Q,n} \otimes \mathbb{C}$ satisfying $x_1 - x_2 \in X_{Q,n}$. Since $jzj^{-1} = \bar{z}$, the previous lemma implies

$$z^{x_2}\bar{z}^{x_1} = \phi(\bar{z}) = \gamma\phi(z)\gamma^{-1} = \phi(z)^{-1} = z^{-x_1}\bar{z}^{-x_2}.$$

Hence $x_2 = -x_1$. Since $x_1 - x_2 \in X_{Q,n}$, we find that $2x_1 \in X$. Defining $\xi = x_1 \in \frac{1}{2}X_{Q,n}$, we find that

$$\phi(z) = (z/\bar{z})^{\xi} = \arg(z)^{2\xi} \text{ for some } \xi \in \tfrac{1}{2}X_{Q,n}.$$

In particular, we find

$$\gamma^2 = \phi(j^2) = \phi(-1) = e^{2\pi i\xi}.$$

Lemma 18.5. — *The element $\xi \in \frac{1}{2}X_{Q,n}$ is regular. In particular, the centralizer of $\phi(\mathbb{C}^\times)$ is the torus $\tilde{T}^\vee$.*

Proof. — This proof follows [**11**, §10.5]. If ξ were singular, then the centralizer of $\phi(\mathbb{C}^\times)$ would contain a nontrivial semisimple subgroup $H^\vee \subset \tilde{G}^\vee$, normalized by γ. But then $\mathrm{Int}(\gamma)$, as an involution of $H^\vee$, would fix (pointwise) a nontrivial torus $S^\vee$ in $H^\vee$. Thus $\mathrm{Im}(\phi)$ would be contained in the centralizer of $S^\vee$, which is contained in a proper parabolic subgroup of $\tilde{G}^\vee$, a contradiction. □

Proposition 18.6. — *Every discrete series Weil parameter for $\tilde{\mathbf{G}}$ is equivalent to a discrete series Weil parameter ϕ satisfying $\phi(z) = (z/\bar{z})^\xi \in \tilde{T}^\vee$ and $\phi(j) = \gamma \in {}^{\mathsf{L}}\tilde{G}$, where ξ and γ satisfy the following conditions:*

(1) *$\xi \in \frac{1}{2}X_{Q,n}$ is regular.*
(2) *γ lies over σ and $\gamma^2 = e^{2\pi i \xi}$*
(3) *$\gamma t^\vee \gamma^{-1} = (t^\vee)^{-1}$ for all $t^\vee \in \tilde{T}^\vee$.*

This gives a bijection, from the set of equivalence classes of discrete series Weil parameters $\mathbf{\Phi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}})$ to the set of $\tilde{N}^\vee$-orbits on the set of pairs (ξ, γ) satisfying (1), (2), (3) *above.*

Proof. — The previous lemmata demonstrate that every discrete series Weil parameter for $\tilde{\mathbf{G}}$ is equivalent to a parameter $\phi(z) = (z/\bar{z})^\xi$, $\phi(j) = \gamma$, with ξ and γ satisfying the conditions above. What remains is to trace through equivalence of parameters.

If ϕ_0 is a discrete series Weil parameter for $\tilde{\mathbf{G}}$, then there exists $g^\vee \in \tilde{G}^\vee$ such that $\mathrm{Int}(g^\vee)\phi_0(z) \in \tilde{T}^\vee$ for all $z \in \mathbb{C}^\times$. Moreover, the regularity of ξ in the previous lemma implies that $g^\vee$ is uniquely determined up to $\tilde{N}^\vee$.

Hence, to understand the $\tilde{G}^\vee$-orbits on $\mathbf{\Phi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}})$, it suffices to verify that $\tilde{N}^\vee$ acts on the set of pairs (ξ, γ) satisfying (1), (2), and (3). If $\xi \in \frac{1}{2}X_{Q,n}$ is regular, and $n^\vee \in \tilde{N}^\vee$, then $\mathrm{Int}(n^\vee)\xi \in \frac{1}{2}X_{Q,n}$ is regular, so (1) is stable under the action of $\tilde{N}^\vee$. If $\gamma \in {}^{\mathsf{L}}\tilde{G}$ lies over $\sigma \in \mathrm{Gal}_\mathbb{R}$, and $\gamma^2 = e^{2\pi i \xi}$, then $\mathrm{Int}(n^\vee)\gamma$ lies over $\sigma \in \mathrm{Gal}_\mathbb{R}$ and

$$(\mathrm{Int}(n^\vee)\gamma)^2 = \mathrm{Int}(n^\vee)\gamma^2 = \mathrm{Int}(n^\vee)e^{2\pi i \xi} = e^{2\pi i \cdot \mathrm{Int}(n^\vee)\xi}.$$

Hence (1) and (2) are stable under the action of $\tilde{N}^\vee$. Finally, if $\gamma t^\vee \gamma^{-1} = (t^\vee)^{-1}$ for all $t^\vee \in \tilde{T}^\vee$, then

$$\mathrm{Int}(n^\vee)\gamma \cdot t^\vee \cdot \mathrm{Int}(n^\vee)\gamma^{-1} = \mathrm{Int}(n^\vee)\left(\gamma\left(\mathrm{Int}(n^\vee)^{-1}t^\vee\right)\gamma^{-1}\right) = (t^\vee)^{-1}.$$

Hence (1), (2), and (3) are stable under the action of $\tilde{N}^\vee$. □

Now we examine a pair (ξ, γ) satisfying the conditions (1), (2), (3) of the above proposition. Recall that the L-group ${}^{\mathsf{L}}\tilde{G}$ is constructed as

$${}^{\mathsf{L}}\tilde{G} = \frac{{}^{\mathsf{L}}\tilde{Z} \ltimes \tilde{G}^\vee}{\langle \zeta, \zeta^{-1} : \zeta \in \tilde{Z}^\vee \rangle}.$$

Here $\tilde{Z}^\vee$ is the center of $\tilde{G}^\vee$ and ${}^{\mathsf{L}}\tilde{Z}$ fits into a short exact sequence

$$\tilde{Z}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{Z} \twoheadrightarrow \mathrm{Gal}_\mathbb{R}.$$

In this way, ${}^{\mathsf{L}}\tilde{T}$ can be viewed as a subgroup of ${}^{\mathsf{L}}\tilde{G}$,

$$ {}^{\mathsf{L}}\tilde{T} = \frac{{}^{\mathsf{L}}\tilde{Z} \ltimes \tilde{T}^\vee}{\langle \zeta, \zeta^{-1} : \zeta \in \tilde{Z}^\vee \rangle}. $$

Hence $\gamma = \zeta n^\vee$ for some $n^\vee \in \tilde{G}^\vee$ and some $\zeta \in {}^{\mathsf{L}}\tilde{Z}$ lying over $\sigma \in \mathrm{Gal}_{\mathbb{R}}$.

Lemma 18.7. — *For any pair (ξ, γ) satisfying conditions (1), (2), (3), we have $\gamma = \zeta n^\vee$ for some $\zeta \in {}^{\mathsf{L}}\tilde{Z}$ and $n^\vee \in \tilde{G}^\vee$. The element $n^\vee$ is contained in $\tilde{N}^\vee$, the normalizer of $\tilde{T}^\vee$, and represents a σ-fixed involution $w^\vee \in \tilde{W}^\vee$. This involution acts on $Y_{Q,n}$ by $w^\vee(y) = -\sigma(y)$.*

Proof. — Conjugation by ζ and conjugation by $\gamma = \zeta n^\vee$ stabilize $\tilde{T}^\vee$. Hence conjugation by $\zeta^{-1} \cdot \gamma = n^\vee$ stabilizes $\tilde{T}^\vee$. Thus $n^\vee \in \tilde{N}^\vee$, the normalizer of $\tilde{T}^\vee$ in $\tilde{G}^\vee$. Write $w^\vee$ for the associated element of the Weyl group $\tilde{W}^\vee = \tilde{N}^\vee / \tilde{T}^\vee$.

For all $t^\vee \in T^\vee$, we compute

$$ \begin{aligned} (t^\vee)^{-1} = \gamma t^\vee \gamma^{-1} &= \zeta n^\vee \cdot t^\vee \cdot (n^\vee)^{-1} \zeta^{-1} \\ &= \zeta n^\vee \zeta^{-1} \cdot \zeta t^\vee \zeta^{-1} \cdot \zeta (n^\vee)^{-1} \zeta^{-1} \\ &= \sigma({}^{w^\vee} t^\vee). \end{aligned} $$

Hence $w^\vee = -\sigma$ as automorphisms of $Y_{Q,n}$. It follows that $\sigma \circ w^\vee = w^\vee \circ \sigma$, and $w^\vee \in (\tilde{W}^\vee)^\sigma$ is a Galois-fixed element of the Weyl group.

Note that $\gamma^2 = \phi(-1) \in \tilde{T}^\vee$. Hence

$$ \zeta n^\vee \zeta n^\vee = \zeta\zeta \cdot \zeta^{-1} n^\vee \zeta \cdot n^\vee = \zeta^2 \sigma(n^\vee) n^\vee = \phi(-1) \in \tilde{T}^\vee. $$

Since $\zeta^2 \in \tilde{T}^\vee$, we find that ${}^\sigma n^\vee n^\vee \in \tilde{T}^\vee$ too. Since ${}^\sigma w^\vee = w^\vee$, we find that $w^\vee$ is an involution in $\tilde{W}^\vee$. □

The function $(\xi, \gamma) \mapsto \xi$ descends to a function

$$ \inf \colon \mathbf{\Phi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}}) \to \frac{\frac{1}{2} X_{Q,n}}{W - \text{conjugation}}. $$

Proposition 18.8. — *The fibers of the function* inf *have cardinality zero or one.*

Proof. — We demonstrate that, if (ξ, γ_1) and (ξ, γ_2) are two pairs (with the same ξ) satisfying the conditions (1), (2), (3), then there exists $t^\vee \in \tilde{T}^\vee$ such that $\gamma_2 = t^\vee \gamma_1 (t^\vee)^{-1}$. Indeed, there exist $\zeta_1, n_1^\vee, \zeta_2, n_2^\vee$ such that

$$ \gamma_1 = \zeta_1 n_1^\vee, \quad \gamma_2 = \zeta_2 n_2^\vee. $$

The elements $n_1^\vee$ and $n_2^\vee$ represent the same element $w^\vee \in \tilde{W}^\vee$ by the previous Lemma. Thus there exists $s^\vee \in \tilde{T}^\vee$ such that $n_2^\vee = n_1^\vee s^\vee$. The elements $\zeta_1, \zeta_2 \in {}^{\mathsf{L}}\tilde{Z}$ lie over $\sigma \in \mathrm{Gal}_{\mathbb{R}}$, and so there exists $z^\vee \in \tilde{Z}^\vee$ satisfying $\zeta_2 = \zeta_1 z^\vee$. We have

$$ \gamma_1 s^\vee z^\vee = \zeta_1 n_1^\vee s^\vee z^\vee = \zeta_1 z^\vee \cdot n_1^\vee s^\vee = \zeta_2 n_2^\vee = \gamma_2. $$

Since squaring is surjective on $\tilde{T}^\vee$, let $t^\vee \in \tilde{T}^\vee$ be such that $(t^\vee)^{-2} = s^\vee z^\vee$. The previous Lemma describes the action of $w^\vee$ on $\tilde{T}^\vee$, yielding

$$t^\vee \gamma_1 (t^\vee)^{-1} = t^\vee \zeta_1 n_1^\vee (t^\vee)^{-1} = \zeta_1 n_1^\vee \cdot {}^{\sigma w^\vee} t^\vee (t^\vee)^{-1} = \gamma_1 (t^\vee)^{-2} = \gamma_1 s^\vee z^\vee = \gamma_2. \quad \square$$

To understand the image of inf, we must understand which pairs (ξ, γ) may occur.

Lemma 18.9. — *Suppose that there exists $w^\vee \in \tilde{W}^\vee$ such that $w^\vee$ acts on Y via $-\sigma$. Suppose that ξ is a regular element of $\frac{1}{2} X_{Q,n}$. Let ρ be the half-sum of the positive roots of $\mathbf{G}$ with respect to $\mathbf{T}$. Then ξ occurs in the image of* inf *if and only if there exists $\zeta \in {}^{\mathsf{L}}\tilde{Z}$ lying over $\sigma \in \mathrm{Gal}_{\mathbb{R}}$ such that*

$$e^{2\pi i \xi} = e^{2\pi i \rho} \cdot \zeta^2. \tag{18.1}$$

Proof. — For one direction, suppose that (ξ, γ) satisfies conditions (1), (2), and (3). Condition (2) states that $\gamma^2 = e^{2\pi i \xi}$. We also know that $\gamma = \zeta n^\vee$ for some $\zeta \in {}^{\mathsf{L}}\tilde{Z}$ and $n^\vee \in \tilde{N}^\vee$ representing $w^\vee$. Hence

$$e^{2\pi i \xi} = \gamma^2 = (\zeta n^\vee)^2 = \zeta^2 \cdot {}^\sigma n^\vee n^\vee. \tag{18.2}$$

The computation of ${}^\sigma n^\vee \cdot n^\vee$ follows from a combinatorial lemma of Langlands, [**44**, Lemma 3.2]:

$${}^\sigma n^\vee n^\vee = e^{2\pi i \rho}. \tag{18.3}$$

To compare to [**44**], our $n^\vee$ corresponds to Langlands' a, our ρ corresponds to Langlands' δ, and our assumption that $\mathbf{G}$ is semisimple implies that Langlands' $\mu^\wedge(a)$ equals 1. Langlands' Lemma (18.3) and (18.2) imply that (18.1) holds.

Conversely, suppose that $\xi \in \frac{1}{2} X_{Q,n}$ is regular and (18.1) holds for some $\zeta \in {}^{\mathsf{L}}\tilde{Z}$ lying over σ. Define $\gamma = \zeta n^\vee$, where $n^\vee \in \tilde{N}^\vee$ represents $w^\vee \in \tilde{W}^\vee$ (acting by $-\sigma$ on Y). Suppose ξ and ζ satisfy (18.1). Define a discrete series Weil parameter ϕ by setting

$$\phi(z) = (z/\bar{z})^\xi, \quad \phi(j) = \gamma.$$

To see that ϕ defines a discrete series Weil parameter, it suffices to show that $\gamma^2 = (-1)^{2\xi}$ and $\gamma \phi(z) \gamma^{-1} = \phi(\bar{z})$. These follow from (18.1) and the fact that $n^\vee$ represents the appropriate involution $w^\vee$ in $\tilde{W}^\vee$. $\square$

Remark 18.10. — If $s^\vee \in \tilde{Z}^\vee$, then we find

$$s^\vee = n^\vee s^\vee (n^\vee)^{-1} = {}^\sigma (s^\vee)^{-1},$$

since $w^\vee$ acts on $\tilde{T}^\vee$ by $t^\vee \mapsto {}^\sigma (t^\vee)^{-1}$. Hence

$$(\zeta s^\vee)^2 = \zeta^2 s^{\vee \sigma} s^\vee = \zeta^2.$$

It follows that if (18.1) holds for some $\zeta \in {}^{\mathsf{L}}\tilde{Z}$ lying over σ, then (18.1) holds for *all* $\zeta \in {}^{\mathsf{L}}\tilde{Z}$ lying over σ.

The previous lemma reduces the study of discrete series Weil parameters to the study of ζ^2 for $\zeta \in {}^{\mathsf{L}}\tilde{Z}$. This can be related, in turn, to the L-group of an anisotropic maximal torus in $\mathbf{G}$. The following lemma guarantees the existence of such a torus, and is essentially the same as [**44**, Lemma 3.1]

Lemma 18.11. — *If there exists a discrete series Weil parameter $\phi \in \mathbf{\Phi}_\varepsilon^{\mathrm{disc}}(\tilde{\mathbf{G}})$, then there exists a maximal $\mathbb{R}$-torus $\mathbf{S} \subset \mathbf{G}$ which is anisotropic, i.e., $S = \mathbf{S}(\mathbb{R})$ is compact.*

Proof. — Let $w \in W^\sigma$ be the $\mathrm{Gal}_\mathbb{R}$-fixed element of W corresponding to $w^\vee$ above. Then $w^\vee$ acts on Y as $-\sigma$, since Y contains $Y_{Q,n}$ as a finite-index subgroup. Since $w = {}^\sigma w$ and $w^2 = 1$, we find an element

$$[\eta] \in H^1(\mathrm{Gal}_\mathbb{R}, W), \quad \eta^\vee(\mathrm{Id}) = 1, \eta^\vee(\sigma) = w.$$

Since we assume $\mathbf{G}$ is quasisplit, a result of Raghunathan [**57**] (proven earlier by Gillé [**31**]) implies that η occurs as the "type" of a maximal $\mathbb{R}$-torus $\mathbf{S} \subset \mathbf{G}$.

The character lattice of $\mathbf{S}$ is isomorphic to Y, but with the Galois action twisted by the cocycle η

$$\sigma_S(y) = w(\sigma(y)).$$

But $w = -\sigma$ on Y and so $w(\sigma(y)) = -y$. Therefore σ_S has no fixed points in Y, and so $\mathbf{S}$ is anisotropic. □

Now fix such an anisotropic maximal $\mathbb{R}$-torus $\mathbf{S} \subset \mathbf{G}$. We elaborate on the connection between $\mathbf{S}$ and the σ-fixed involution in W here. Since all maximal tori are conjugate over $\mathbb{C}$, there exists $g \in G_\mathbb{C} = \mathbf{G}(\mathbb{C})$ such that $\mathrm{Int}(g)\mathbf{S} = \mathbf{T}$. Since $\mathbf{S}$ and $\mathbf{T}$ are defined over $\mathbb{R}$, we find that $\mathrm{Int}({}^\sigma g)\mathbf{S} = \mathbf{T}$ as well. Define $\dot{w} = {}^\sigma g \cdot g^{-1}$. Then we find that $\mathrm{Int}(\dot{w})\mathbf{T} = \mathbf{T}$, and so $\dot{w} \in \mathbf{N}_\mathbf{G}(\mathbf{T})(\mathbb{C})$.

Write Y_S for the cocharacter lattice of $\mathbf{S}$, so $\mathrm{Int}(g) \colon Y_S \to Y$ is an isomorphism of groups. If $\dot{w}$ represents $w \in W$, then the following diagram commutes:

$$\begin{array}{ccc} Y_S & \xrightarrow{\mathrm{Int}(g)} & Y \\ \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle w \circ \sigma} \\ Y_S & \xrightarrow{\mathrm{Int}(g)} & Y. \end{array}$$

Since $\mathbf{S}$ is anisotropic, $\sigma \colon Y_S \to Y_S$ is multiplication by -1. Hence $\mathrm{Int}(g)^{-1} \circ w \circ \sigma \circ \mathrm{Int}(g) = -\mathrm{Id}$, as automorphisms of Y, and so $w = -\sigma$ as an automorphism of Y. Therefore ${}^\sigma w = w$ and $w \in W^{\mathrm{Gal}_\mathbb{R}}$. Since $\mathbf{G}$ is quasisplit, the $\mathrm{Gal}_\mathbb{R}$-fixed element w can be represented by a $\mathrm{Gal}_\mathbb{R}$-fixed element of $\mathbf{N}_\mathbf{G}(\mathbf{T})(\mathbb{C})$. Hence we may assume, without loss of generality, that

$${}^\sigma \dot{w} = \dot{w} = {}^\sigma g \cdot g^{-1} \in \mathbf{G}(\mathbb{R}).$$

Since ${}^\sigma \dot{w} \cdot \dot{w} = 1$ and ${}^\sigma \dot{w} = \dot{w}$, we find that $\dot{w}^2 = 1$ and $w^2 = 1$. Note that $w \in W^{\mathrm{Gal}_\mathbb{R}}$ corresponds to $w^\vee \in \tilde{W}^\vee$ discussed earlier.

The inner automorphism $\mathrm{Int}(g) \colon \mathbf{G}_\mathbb{C} \to \mathbf{G}_\mathbb{C}$ lifts canonically to an automorphism $\mathrm{Int}(g) \colon \mathbf{G}'_\mathbb{C} \to \mathbf{G}'_\mathbb{C}$ (central extensions of $\mathbf{G}_\mathbb{C}$ by $\mathbf{K}_2$ on the big Zariski site over $\mathbb{C}$).

Writing $\mathbf{S}'$ and $\mathbf{T}'$ for the pullbacks of $\mathbf{G}'$, and extending scalars to $\mathbb{C}$, we find a commutative diagram of sheaves of groups on $\mathbb{C}_{\mathrm{Zar}}$ with exact rows:

$$\begin{tikzcd} \mathbf{K}_2 \arrow[r, hook] \arrow[d, "="] & \mathbf{S}'_{\mathbb{C}} \arrow[r, two heads] \arrow[d, "\mathrm{Int}(g)"] & \mathbf{S}_{\mathbb{C}} \arrow[d, "\mathrm{Int}(g)"] \\ \mathbf{K}_2 \arrow[r, hook] & \mathbf{T}'_{\mathbb{C}} \arrow[r, two heads] & \mathbf{T}_{\mathbb{C}}. \end{tikzcd} \tag{18.4}$$

From the diagram (18.4), the first Brylinski-Degline invariant of $\mathbf{S}'$ is the quadratic form

$$Q_S(y) = Q(\mathrm{Int}(g)y) \text{ for all } y \in Y_S.$$

We abbreviate $Y^\sharp = Y_{Q,n}$ and $Y_S^\sharp = Y_{S,Q_S,2}$ in what follows. Then $\mathrm{Int}(g)$ restricts to an isomorphism from $Y_S^\sharp$ to $Y^\sharp$.

Write $\hat{T} = \hat{\mathscr{T}}[\mathbb{C}] = \mathrm{Hom}(Y^\sharp, \mathbb{C}^\times)$, and similarly $\hat{S} = \mathrm{Hom}(Y_S^\sharp, \mathbb{C}^\times)$. The underlying groups of $\hat{T}$ and $\hat{S}$ are the same as those of $T^\vee$ and $S^\vee$, but the $\mathrm{Gal}_{\mathbb{R}}$-actions are different: in $\hat{T}$ and $\hat{S}$, we consider $\mathbb{C}^\times$ as a $\mathrm{Gal}_{\mathbb{R}}$-module by complex conjugation, while in $T^\vee$ and $S^\vee$, we consider $\mathbb{C}^\times$ as a trivial $\mathrm{Gal}_{\mathbb{R}}$-module.

The action of W on $Y^\sharp$ yields a homomorphism $\hat{w}\colon \hat{T} \to \hat{T}$. The isomorphism $\mathrm{Int}(g)\colon Y_S^\sharp \to Y^\sharp$ yields a pullback isomorphism $\hat{g}\colon \hat{T} \to \hat{S}$, fitting into a commutative diagram:

$$\begin{tikzcd} \hat{T} \arrow[r, "\hat{g}"] \arrow[d, "\sigma\circ\hat{w}"] & \hat{S} \arrow[d, "\sigma"] \\ \hat{T} \arrow[r, "\hat{g}"] & \hat{S}. \end{tikzcd}$$

Taking the $\mathbb{C}(\!(v)\!)$-points in (18.4), pushing out via $\mathbf{K}_2(\mathbb{C}(\!(v)\!)) \twoheadrightarrow \mathbb{C}^\times$, and pulling back via $Y_S^\sharp \to \mathbf{S}(\mathbb{C}(\!(v)\!))$ and $Y^\sharp \hookrightarrow \mathbf{T}(\mathbb{C}(\!(v)\!))$ yields a commutative diagram of abelian groups, relating the second Brylinski-Deligne invariants of $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{T}}$:

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, hook] \arrow[d, "="] & D_S^\sharp \arrow[r, two heads] \arrow[d, "\mathrm{Int}(g)"] & Y_S^\sharp \arrow[d, "\mathrm{Int}(g)"] \\ \mathbb{C}^\times \arrow[r, hook] & D^\sharp \arrow[r, two heads] & Y^\sharp. \end{tikzcd} \tag{18.5}$$

Recall that $\dot{w} \in G = \mathbf{G}(\mathbb{R})$ and $\dot{w} \cdot g = {}^\sigma g$. It follows that, while $\mathrm{Int}(g)\colon D_S^\sharp \to D^\sharp$ is typically not $\mathrm{Gal}_{\mathbb{R}}$-equivariant, the following diagram commutes:

$$\begin{tikzcd} D_S^\sharp \arrow[r, "\mathrm{Int}(g)"] \arrow[d, "\sigma"] & D^\sharp \arrow[d, "\mathrm{Int}(\dot{w})\circ\sigma"] \\ D_S^\sharp \arrow[r, "\mathrm{Int}(g)"] & D^\sharp. \end{tikzcd}$$

Let $\mathrm{Spl}(D^\sharp)$ be the $\hat{T}$-torsor of splittings of $D^\sharp$, and $\mathrm{Spl}(D_S^\sharp)$ the $\hat{S}$-torsor of splittings of $D_S^\sharp$. The $\mathrm{Gal}_{\mathbb{R}}$ actions on $D^\sharp$ and $D_S^\sharp$ yield $\mathrm{Gal}_{\mathbb{R}}$-actions on $\mathrm{Spl}(D^\sharp)$ and

$\mathrm{Spl}(D_S^\sharp)$, compatible with those on $\hat{T}$ and $\hat{S}$. Moreover, if $s \in \mathrm{Spl}(D^\sharp)$ then we find a splitting

$$g^* s = \mathrm{Int}(g)^{-1} \circ s \circ \mathrm{Int}(g) \in \mathrm{Spl}(D_S^\sharp).$$

Similarly, we may construct a splitting $\dot{w}^* s = \mathrm{Int}(\dot{w})^{-1} \circ s \circ \mathrm{Int}(\dot{w}) \in \mathrm{Spl}(D^\sharp)$. The following diagrams commute, exhibiting the interactions of Galois actions, torsor structure, $\mathrm{Int}(\dot{w})$, and $\mathrm{Int}(g)$:

$$\begin{array}{ccccccc} \mathrm{Spl}(D^\sharp) & \xrightarrow{g^*} & \mathrm{Spl}(D_S^\sharp) & & \hat{T} \times \mathrm{Spl}(D^\sharp) & \xrightarrow{*} & \mathrm{Spl}(D^\sharp) \\ \downarrow{\scriptstyle \dot{w}^* \circ \sigma} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \hat{g} \times g^*} & & \downarrow{\scriptstyle g^*} \\ \mathrm{Spl}(D^\sharp) & \xrightarrow{g^*} & \mathrm{Spl}(D_S^\sharp) & & \hat{S} \times \mathrm{Spl}(D_S^\sharp) & \xrightarrow{*} & \mathrm{Spl}(D_S^\sharp). \end{array}$$

Let $\hat{g}_* \mathrm{Spl}(D^\sharp)$ denote the pushout of the torsor $\mathrm{Spl}(D^\sharp)$ via $\hat{g}\colon \hat{T} \to \hat{S}$. Thus $\hat{g}_* \mathrm{Spl}(D^\sharp)$ has the same underlying set $\mathrm{Spl}(D^\sharp)$, but is now viewed as a $\hat{S}$-torsor. The commutativity of the diagram above (on the right) demonstrates that

$$g^* : \hat{g}_* \mathrm{Spl}(D^\sharp) \xrightarrow{\sim} \mathrm{Spl}(D_S^\sharp)$$

is an isomorphism of $\hat{S}$-torsors.

Write $\sqrt{\mathrm{Spl}(D^\sharp)}$ for the groupoid of square roots of the $\hat{T}$-torsor $\mathrm{Spl}(D^\sharp)$. Similarly, write $\sqrt{\mathrm{Spl}(D_S^\sharp)}$ for the groupoid of square roots of the $\hat{S}$-torsor $\mathrm{Spl}(S)$. If $\bar{x} = (H, h) \in \sqrt{\mathrm{Spl}(D^\sharp)}$, then pushing out yields a square root $(\hat{g}_* H, \hat{g}_* h) \in \sqrt{\hat{g}_* \mathrm{Spl}(D^\sharp)}$. Composing with the isomorphism of $\hat{S}$-torsors $g^*\colon \hat{g}_* \mathrm{Spl}(D^\sharp) \to \mathrm{Spl}(D_S^\sharp)$, we find an object $g_* \bar{x} = (\hat{g}_* H, g^* \circ \hat{g}_* h) \in \sqrt{\mathrm{Spl}(D_S^\sharp)}$.

The diagrams and discussions above define functors of groupoids,

$$\begin{aligned} \sigma_* &: \sqrt{\mathrm{Spl}(D^\sharp)} \to \sqrt{\mathrm{Spl}(D^\sharp)}, \\ \sigma_{S,*} &: \sqrt{\mathrm{Spl}(D_S^\sharp)} \to \sqrt{\mathrm{Spl}(D_S^\sharp)}, \\ \dot{w}_* &: \sqrt{\mathrm{Spl}(D^\sharp)} \to \sqrt{\mathrm{Spl}(D^\sharp)}, \\ g_* &: \sqrt{\mathrm{Spl}(D^\sharp)} \to \sqrt{\mathrm{Spl}(D_S^\sharp)}. \end{aligned}$$

Lemma 18.12. — *Let $\dot{w}$ be any element of $\mathbf{N_G}(\mathbf{T})(\mathbb{C})$ satisfying $\dot{w}^2 = 1$ (e.g, the element $\dot{w} = {}^\sigma g \cdot g^{-1}$ from before). Let $\bar{x} = (H, h)$ be any object of $\sqrt{\mathrm{Spl}(D^\sharp)}$. Then there exists an isomorphism $\dot{\rho}\colon \bar{x} \to \dot{w}_* \bar{x}$ in the groupoid $\sqrt{\mathrm{Spl}(D^\sharp)}$ such that the morphism*

$$\dot{\rho}^2 : \bar{x} \xrightarrow{\dot{\rho}} \dot{w}_* \bar{x} \xrightarrow{\dot{w}_* \dot{\rho}} \dot{w}_* \dot{w}_* \bar{x} = \bar{x}$$

equals the identity.

Proof. — Since w is an involution in W, there exist strongly orthogonal roots $\alpha_1, \ldots, \alpha_\ell$ with associated root reflections $w_1, \ldots, w_\ell \in W$, such that $w = \prod_{i=1}^\ell w_i$

(see, for example, [**52**, Proposition 1.1]). Let $q_i = Q(\alpha_i^\vee)$ and let $n_i = n_{\alpha_i}$. Thus $n_i = 1$ if q_i is even, and $n_i = 2$ if q_i is odd. Define

$$\beta = \sum_{i=1}^{\ell} q_i \tilde{\alpha}_i \in X_{Q,n}.$$

Orthogonality implies that $w(\beta) = -\beta$.

Also, orthogonality implies that the coroots $\tilde{\alpha}_i^\vee$ are $\mathbb{Q}$-linearly independently, and so there exists a splitting $s \in \mathrm{Spl}(D^\sharp)$ which is aligned in the sense that

$$s(\tilde{\alpha}_i^\vee) = [e_i]^{n_i}, \text{ for all } 1 \leq i \leq \ell. \tag{18.6}$$

As in Lemma 3.16, if $d \in D^\sharp$ lies over $y \in Y^\sharp$, we have

$$\mathrm{Int}(\dot{w})d = d \cdot \prod_{i=1}^{\ell} \left([e_i]^{-n_i \langle \tilde{\alpha}_i, y\rangle} \cdot (-1)^{q_i \varepsilon(-n_i \langle \tilde{\alpha}_i, y \rangle)} \right).$$

Here and below, $\varepsilon(N) = N(N+1)/2$ for any integer N. As in Theorem 3.17, recall that $\varepsilon(2N) = N$ modulo 2, and $\varepsilon(-N) - N = \varepsilon(N)$ modulo 2.

When s is aligned as in (18.6), we compute

$$\begin{aligned}
[\dot{w}^*(s)](y) &= \mathrm{Int}(\dot{w}) s \left(y - \sum_{j=1}^{\ell} \langle \tilde{\alpha}_j, y \rangle \tilde{\alpha}_j^\vee \right), \\
&= \mathrm{Int}(\dot{w}) s(y) \cdot \prod_j \mathrm{Int}(\dot{w}) s(\tilde{\alpha}_j^\vee)^{-\langle \tilde{\alpha}_j, y\rangle}, \\
&= s(y) \cdot \prod_i [e_i]^{-n_i \langle \tilde{\alpha}_i, y \rangle} \cdot (-1)^{q_i \varepsilon(-n_i \langle \tilde{\alpha}_i, y\rangle)} \\
&\quad \cdot \prod_j \left(s(\tilde{\alpha}_j^\vee)^{-\langle \tilde{\alpha}_j, y \rangle} \cdot \prod_i [e_i]^{n_i \langle \tilde{\alpha}_j, y \rangle \langle \tilde{\alpha}_i, \tilde{\alpha}_j^\vee \rangle} \cdot (-1)^{-q_i \langle \tilde{\alpha}_j, y \rangle \varepsilon(-n_i \langle \tilde{\alpha}_i, \tilde{\alpha}_j^\vee \rangle)} \right), \\
&= s(y) \cdot (-1)^{q_i \varepsilon(-n_i \langle \tilde{\alpha}_i, y \rangle) + q_i \langle \tilde{\alpha}_i, y \rangle \varepsilon(-2n_i)}, \quad \text{(by (18.6) and orthogonality)} \\
&= s(y) \cdot (-1)^{q_i \varepsilon(-n_i \langle \tilde{\alpha}_i, y \rangle) - q_i n_i \langle \tilde{\alpha}_i, y \rangle}, \\
&= s(y) \cdot \prod_{i=1}^{\ell} (-1)^{q_i \varepsilon(n_i \langle \tilde{\alpha}_i, y \rangle)}.
\end{aligned}$$

If $n_i = 1$, then q_i is even and the exponent $q_i \varepsilon(n_i \langle \tilde{\alpha}, y \rangle)$ is even. If $n_i = 2$ then $\varepsilon(n_i \langle \tilde{\alpha}, y \rangle)$ has the same parity as $\langle \tilde{\alpha}, y \rangle$. In both cases, we find

$$\prod_{i=1}^{\ell} (-1)^{q_i \varepsilon(n_i \langle \tilde{\alpha}_i, y \rangle)} = \prod_{i=1}^{\ell} (-1)^{q_i \langle \tilde{\alpha}_i, y \rangle}.$$

Hence

$$\mathrm{Int}(\dot{w}) s = (-1)^\beta * s = e^{i\pi\beta} * s.$$

Since $h\colon H \to \mathrm{Spl}(D^\sharp)$ is surjective, let $a \in H$ be such that $h(a) = s$, with s aligned as above. Let $\rho\colon \bar{x} \to \dot{w}_* \bar{x}$ be any isomorphism in the connected groupoid $\sqrt{\mathrm{Spl}(D^\sharp)}$. Then, as a function from H to H, we have $\rho(a) = \hat{r} * a$ for some $\hat{r} \in \hat{T}$. We have

$$s = h(a) = \mathrm{Int}(\dot{w})(h(\rho(a))) = \mathrm{Int}(\dot{w})(h(\hat{r} * a)) = \hat{w}(\hat{r}^2) * \mathrm{Int}(\dot{w})(s),$$

and so

$$\hat{w}(\hat{r})^2 = e^{i\pi\beta} \in \hat{T}.$$

Since $w(\beta) = -\beta$, $\hat{w}(e^{i\pi\beta}) = e^{-i\pi\beta} = e^{i\pi\beta}$, and so $\hat{r}^2 = e^{i\pi\beta}$ too. Define

$$\dot{\rho} = \hat{r}^{-1} e^{i\pi\beta/2} \circ \rho, \quad \dot{a} = e^{i\pi\beta/4} * a \in H.$$

Note that $\hat{r}^{-1} e^{i\pi\beta/2} \in \hat{T}_{[2]}$, and so $\dot{\rho}\colon \bar{x} \to \dot{w}_* \bar{x}$ is again a morphism in $\sqrt{\mathrm{Spl}(D^\sharp)}$.

As in Lemma 5.10, we compute

$$\begin{aligned}
\dot{\rho}(\dot{a}) &= \hat{r}^{-1} e^{i\pi\beta/2} * \rho(e^{i\pi\beta/4} * a), \\
&= \hat{r}^{-1} e^{i\pi\beta/2} e^{-i\pi\beta/4} * \rho(a), \quad (\text{since } \hat{w}(e^{i\pi\beta/4}) = e^{-i\pi\beta/4}) \\
&= \hat{r}^{-1} e^{i\pi\beta/2} e^{-i\pi\beta/4} \hat{r} * a, \\
&= \hat{r}^{-1} e^{i\pi\beta/2} e^{-i\pi\beta/4} \hat{r} e^{-i\pi\beta/4} * \dot{a} = \dot{a}.
\end{aligned}$$

Hence $\dot{\rho}(\dot{\rho}(\dot{a})) = \dot{a}$. But $\dot{\rho} \circ \dot{\rho}$ is a $\hat{T}$-torsor isomorphism from H to itself, and so $\dot{\rho}^2 = \mathrm{Id}$. □

Now fix an object $\bar{z} = (H, h, j)$ of the groupoid $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})[\mathbb{C}]$. Define $\bar{x} = (H, h)$, an object of $\sqrt{\mathrm{Spl}(D^\sharp)}$. Pushing out yields an object $g_* \bar{x} \in \sqrt{\mathrm{Spl}(D_S^\sharp)}$.

Theorem 18.13. — *There exists an L-morphism ${}^{\mathsf{L}}g$ making the following diagram commute:*

$$\begin{tikzcd}
\tilde{Z}^\vee \arrow[r, hook] \arrow[d, "g^\vee"] & {}^{\mathsf{L}}\tilde{Z} \arrow[r, two heads] \arrow[d, "{}^{\mathsf{L}}g"] & \mathrm{Gal}_\mathbb{R} \arrow[d, "="] \\
\tilde{S}^\vee \arrow[r, hook] & {}^{\mathsf{L}}\tilde{S} \arrow[r, two heads] & \mathrm{Gal}_\mathbb{R}.
\end{tikzcd}$$

Proof. — We begin by producing an L-morphism $I(g)$ making the following diagram commute:

$$\begin{tikzcd}
\tilde{Z}^\vee \arrow[r, hook] \arrow[d, "g^\vee"] & \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z}) \arrow[r, two heads] \arrow[d, "I(g)"] & \mathrm{Gal}_\mathbb{R} \arrow[d, "="] \\
\tilde{S}^\vee \arrow[r, hook] & \pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{S}}), g_* \bar{x}) \arrow[r, two heads] & \mathrm{Gal}_\mathbb{R}.
\end{tikzcd}$$

Consider a morphism $f\colon \bar{z} \to \sigma_* \bar{z}$ in the groupoid $\mathsf{E}_\varepsilon(\tilde{\mathbf{G}})[\mathbb{C}]$. Thus f is an element of $\pi_1^{\text{ét}}(\mathsf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z})$ lying over σ. f restricts to an isomorphism from $\bar{x} \to \sigma_* \bar{x}$ in the groupoid $\mathsf{E}_\varepsilon(\tilde{\mathbf{T}})[\mathbb{C}]$.

Choose an isomorphism $\rho\colon \bar{x} \to \dot{w}_* \bar{x}$ in the groupoid $\sqrt{\mathrm{Spl}(D^\sharp)}$ such that $\rho^2 = \mathrm{Id}$, as in the previous lemma. From the proof of Theorem 5.13, the following diagram in the groupoid $\mathbf{E}_\varepsilon(\tilde{\mathbf{T}})[\mathbb{C}]$ commutes:

$$\begin{tikzcd} \bar{x} \arrow[r, "f"] \arrow[d, "\rho"] & \sigma_* \bar{x} \arrow[d, "\rho"] \\ \dot{w}_* \bar{x} \arrow[r, "f"] & \dot{w}_* \sigma_* \bar{x} = \sigma_* \dot{w}_* \bar{x}. \end{tikzcd}$$

The equality $\dot{w}_* \sigma_* \bar{x} = \sigma_* \dot{w}_* \bar{x}$ follows from the fact that $\dot{w}$ is $\mathrm{Gal}_{\mathbb{R}}$-invariant. It follows that $f^2 = (\rho f)^2 \in \tilde{Z}^\vee$:

$$\begin{tikzcd} \bar{x} \arrow[r, "f"] \arrow[rrrr, bend right, "f \circ f"'] & \sigma_* \bar{x} \arrow[r, "\rho"] & \sigma_* \dot{w}_* \bar{x} = \dot{w}_* \sigma_* \bar{x} \arrow[r, "f"] & \dot{w}_* \sigma_* \sigma_* \bar{x} = \dot{w}_* \bar{x} \arrow[r, "\rho = \rho^{-1}"] & \bar{x}. \end{tikzcd}$$

To compute $g^\vee(f^2)$, we apply the functor g_* throughout:

$$\begin{tikzcd} g_* \bar{x} \arrow[r, "g_* f"] \arrow[rrrr, bend right, "g^\vee(f \circ f)"'] & g_* \sigma_* \bar{x} \arrow[r, "g_* \rho"] & g_* \sigma_* \dot{w}_* \bar{x} = g_* \dot{w}_* \sigma_* \bar{x} \arrow[r, "g_* f"] & g_* \dot{w}_* \sigma_* \sigma_* \bar{x} = g_* \dot{w}_* \bar{x} \arrow[r, "g_* \rho"] & g_* \bar{x}. \end{tikzcd}$$

But $g_* \dot{w}_* \sigma_* \bar{x} = \sigma_* g_* \bar{x}$, so the diagram above yields a commutative diagram in $\mathbf{E}_\varepsilon(\tilde{\mathbf{S}})[\mathbb{C}]$,

(18.7)
$$\begin{tikzcd} g_* \bar{x} \arrow[r, "g_*(\rho f)"] \arrow[rr, bend right, "g^\vee(f \circ f)"'] & \sigma_* g_* \bar{x} \arrow[r, "g_*(\rho f)"] & g_* \bar{x}. \end{tikzcd}$$

Now we define the L-morphism $I(g)$ as follows. Since f lies over $\sigma \in \mathrm{Gal}_{\mathbb{R}}$,

$$\pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z}) = \tilde{Z}^\vee \sqcup f \cdot \tilde{Z}^\vee.$$

Define $I(g)\zeta = g^\vee(\zeta)$ for all $\zeta \in \tilde{Z}^\vee$. Define $I(g) f = g_*(\rho f)$. To see that $I(g)$ extends uniquely to a homomorphism from $\pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{G}}), \bar{z})$ to $\pi_1^{\text{ét}}(\mathbf{E}_\varepsilon(\tilde{\mathbf{S}}), g_* \bar{x})$, it suffices to observe two facts:

- For all $\zeta \in \tilde{Z}^\vee$, we have $g^\vee \sigma(\zeta) = g^\vee(\sigma(w^\vee(\zeta))) = \sigma(g^\vee(\zeta))$, and so $g^\vee$ is $\mathrm{Gal}_{\mathbb{R}}$-equivariant.
- Diagram (18.7) implies that $I(g) f^2 = g^\vee(f^2) = (g_*(\rho f))^2 = (I(g) f)^2$.

To finish the proof, we recall that the first Brylinski-Deligne invariant of $\tilde{\mathbf{S}}$ satisfies $Q_S(y) = Q(\mathrm{Int}(g) y)$ for all $y \in Y_S$. It follows quickly that $\tau_{Q_S}\colon \mu_2 \to \tilde{S}^\vee$ satisfies $\tau_{Q_S}(\pm 1) = g^\vee(\tau_Q(\pm 1))$. Hence we find a L-morphism τ_g making the following diagram commute:

$$\begin{tikzcd} \tilde{Z}^\vee \arrow[r, hook] \arrow[d, "g^\vee"] & (\tau_Q)_* \widetilde{\mathrm{Gal}}_{\mathbb{R}} \arrow[r, two heads] \arrow[d, "\tau_g"] & \mathrm{Gal}_{\mathbb{R}} \arrow[d, "="] \\ \tilde{S}^\vee \arrow[r, hook] & (\tau_{Q_S})_* \widetilde{\mathrm{Gal}}_{\mathbb{R}} \arrow[r, two heads] & \mathrm{Gal}_{\mathbb{R}}\,. \end{tikzcd}$$

The Baer sum yields the desired L-morphism ${}^{\mathsf{L}} g = I(g) \dotplus \tau_g$. □

Corollary 18.14. — *Let $\rho \in \frac{1}{2}X$ be the half-sum of the positive roots for $\mathbf{G}$ with respect to $\mathbf{B} \supset \mathbf{T}$. Let $\kappa \in \frac{1}{2}X_{Q,n}/X_{Q,n}$ be the element defining the double cover $\tilde{S}_{Q,n} \to S_{Q,n}$. Then the map* inf *gives a bijection,*

$$\mathbf{\Phi}_{\varepsilon}^{\mathrm{disc}}(\tilde{\mathbf{G}}) \to \frac{(\kappa + \rho + X_{Q,n}) \cap X_{\mathbb{Q}}^{\mathrm{reg}}}{W - \textit{conjugation}}.$$

Proof. — From Lemma 18.9, we find that $\xi \in \frac{1}{2}X_{Q,n}$ is in the image of inf if and only if ξ is regular and $e^{2\pi i \xi} = e^{2\pi i \rho} \cdot \zeta^2$ for some $\zeta \in {}^{\mathsf{L}}\tilde{Z}$ lying over $\sigma \in \mathrm{Gal}_{\mathbb{R}}$. The previous Theorem, together with Lemma 17.3 implies that $g^{\vee}(\zeta^2) = (I(g)\zeta)^2 = e^{2\pi i \kappa} \in \tilde{S}^{\vee}$. The result follows immediately. $\square$

Recall that the infinitesimal character provided a finite-to-one surjective map,

$$\inf \colon \mathbf{\Pi}_{\varepsilon}^{\mathrm{disc}}(\tilde{\mathbf{G}}) \twoheadrightarrow \frac{(\kappa + \rho + X) \cap X_{\mathbb{Q}}^{\mathrm{reg}}}{W - \mathrm{conjugation}}.$$

The W-equivariant inclusion $X \hookrightarrow X_{Q,n}$ uniquely determines a parameterization of discrete series $\mathscr{L}_{\varepsilon}^{\mathrm{disc}}$.

Theorem 18.15. — *There is a unique finite-to-one function $\mathscr{L}_{\varepsilon}^{\mathrm{disc}} \colon \mathbf{\Pi}_{\varepsilon}^{\mathrm{disc}}(\tilde{\mathbf{G}}) \to \mathbf{\Phi}_{\varepsilon}^{\mathrm{disc}}(\tilde{\mathbf{G}})$, making the following diagram commute:*

$$\begin{tikzcd}
\mathbf{\Pi}_{\varepsilon}^{\mathrm{disc}}(\tilde{\mathbf{G}}) \arrow[r, two heads, "\inf"] \arrow[d, "\mathscr{L}_{\varepsilon}^{\mathrm{disc}}"] & \frac{(\kappa+\rho+X)\cap X_{\mathbb{Q}}^{\mathrm{reg}}}{W-\textit{conjugation}} \arrow[d, hook] \\
\mathbf{\Phi}_{\varepsilon}^{\mathrm{disc}}(\tilde{\mathbf{G}}) \arrow[r, "\inf"] & \frac{(\kappa+\rho+X_{Q,n})\cap X_{\mathbb{Q}}^{\mathrm{reg}}}{W-\textit{conjugation}}.
\end{tikzcd}$$

Proof. — This follows from the bijectivity of the bottom row, and the finite-to-one nature of the top row. $\square$

This theorem provides a finite-to-one parameterization of the genuine discrete series representations of $\tilde{G}$, by discrete series Weil parameters valued in ${}^{\mathsf{L}}\tilde{G}$.

Appendix

Torsors, gerbes, and fundamental groups

Let S be a connected scheme, and $S_{\text{ét}}$ the étale site. Our treatment of sheaves on $S_{\text{ét}}$ follows [**22**, §II]. Recall that a geometric point of S is a morphism of schemes $\bar{s} \colon \mathrm{Spec}(\bar{F}) \to S$, where $\bar{F}$ is a separably closed field.

An ***open étale neighborhood*** of $\bar{s}$ is an étale morphism $U \to S$ endowed with a lift $\bar{u} \colon \mathrm{Spec}(\bar{F}) \to U$ of the geometric point $\bar{s}$. If $\bar{s}$ is a geometric point, we write $\pi_1^{\text{ét}}(S, \bar{s})$ for the étale fundamental group. When $\bar{s}$ is fixed, we define $\mathrm{Gal}_S = \pi_1^{\text{ét}}(S, \bar{s})$.

A.1. Local systems on $S_{\text{ét}}$

Definition A.1. — A ***local system on*** $S_{\text{ét}}$ is a locally constant sheaf $\mathscr{J}$ of sets on $S_{\text{ét}}$.

When $\mathscr{J}$ is a local system on $S_{\text{ét}}$ and $U \to S$ is étale, we write $\mathscr{J}[U]$ for the set of sections over U and we write $\mathscr{J}_U$ for the local system on $U_{\text{ét}}$ obtained by restriction. If $\bar{s}$ is a geometric point of S, then the fiber $\mathscr{J}_{\bar{s}}$ is the inductive limit $\varinjlim_U \mathscr{J}[U]$, over open étale neighborhoods of $\bar{s}$. By local constancy, $\mathscr{J}_{\bar{s}} = \mathscr{J}[U]$ for some such open étale neighborhood. Often in this paper we work locally on $S_{\text{ét}}$ and abuse notation a bit by writing $j \in \mathscr{J}$ rather than $j \in \mathscr{J}[U]$ (for an étale $U \to S$).

More generally, if C is a category, then one may work with local systems on $S_{\text{ét}}$ of objects of C, or "C-valued local systems". If $\mathsf{C} \to \mathsf{D}$ is a functor, then one finds a corresponding functor from the category of C-valued local systems to the category of D-valued local systems. Fibers of such C-valued local systems over geometric points make sense in this generality, by local constancy.

Example A.2. — *Let $\mathscr{M}$ be a local system on $S_{\text{ét}}$ of finitely-generated abelian groups, and let R be a commutative ring. Then* $\operatorname{Spec}(R[\mathscr{M}])$ *will denote the local system on $S_{\text{ét}}$ of affine group schemes over R given by*

$$\operatorname{Spec}(R[\mathscr{M}])[U] = \operatorname{Spec}(R[\mathscr{M}[U]]).$$

We will work with local systems of groups, local systems of affine group schemes over $\mathbb{Z}$, local systems of root data, etc.

A.2. Torsors on $S_{\text{ét}}$

Definition A.3. — Let $\mathscr{G}$ be a sheaf of groups on $S_{\text{ét}}$. A ***$\mathscr{G}$-torsor*** is a locally nonempty sheaf of sets $\mathscr{V}$ on $S_{\text{ét}}$, endowed with an action $*\colon \mathscr{G} \times \mathscr{V} \to \mathscr{V}$, such that

$$\mathscr{G} \times \mathscr{V} \to \mathscr{V} \times \mathscr{V}, \quad (g, v) \mapsto (g * v, v)$$

is an isomorphism of sheaves of sets on $S_{\text{ét}}$. Morphisms of $\mathscr{G}$-torsors are morphisms of sheaves on $S_{\text{ét}}$ which intertwine the $\mathscr{G}$-action. The ***category of $\mathscr{G}$-torsors*** will be denoted $\mathsf{Tors}(\mathscr{G})$.

If $\mathscr{V}$ is a $\mathscr{G}$-torsor, we write $[\mathscr{V}]$ for its isomorphism class. The isomorphism classes of $\mathscr{G}$-torsors form a pointed set denoted $H^1_{\text{ét}}(S, \mathscr{G})$. The set is pointed by the isomorphism class of the ***neutral*** $\mathscr{G}$-torsor: $\mathscr{G}$ itself, as a $\mathscr{G}$-torsor by left-multiplication. If $\mathscr{V}$ is a $\mathscr{G}$-torsor, $U \to S$ is étale, and $v \in \mathscr{V}[U]$, then write $v_{U'}$ for the restriction of v to any further étale $U' \to U$. For any such U', and any $w \in \mathscr{V}[U']$, there exists a unique $g \in \mathscr{G}[U']$ satisfying $w = g * v_{U'}$. Allowing U' to vary, the map $w \mapsto g$ gives a $\mathscr{G}_U$-torsor isomorphism from $\mathscr{V}_U$ to $\mathscr{G}_U$. Thus we say that the point $v \in \mathscr{V}[U]$ ***neutralizes*** the torsor $\mathscr{V}$ over U.

The category of torsors has more structure in the abelian case. If $\mathscr{A}$ is a sheaf of *abelian* groups on $S_{\text{ét}}$, then the category $\mathsf{Tors}(\mathscr{A})$ inherits a monoidal structure.

Namely, if $\mathscr{V}_1$ and $\mathscr{V}_2$ are two $\mathscr{A}$-torsors, define

$$\mathscr{V}_1 \dotplus \mathscr{V}_2 = \frac{\mathscr{V}_1 \times \mathscr{V}_2}{(a * v_1, v_2) \sim (v_1, a * v_2)}.$$

With this monoidal structure, the trivial torsor $\mathscr{A}$ as zero object, and obvious isomorphisms for commutativity and associativity and unit, the category $\mathsf{Tors}(\mathscr{A})$ becomes a Picard groupoid (i.e., a strictly commutative Picard category, in the terminology of [**34**, Exposé XVIII §1.4]. The pointed set of isomorphism classes $H^1_{\text{ét}}(S, \mathscr{A})$ becomes an abelian group, with

$$[\mathscr{V}_1] + [\mathscr{V}_2] := [\mathscr{V}_1 \dotplus \mathscr{V}_2].$$

The group $H^1_{\text{ét}}(S, \mathscr{A})$ is identified with the étale cohomology with coefficients $\mathscr{A}$.

Suppose that $f\colon \mathscr{A} \to \mathscr{G}$ is a homomorphism of sheaves of groups on $S_{\text{ét}}$, with $\mathscr{A}$ abelian, and f central (i.e., f factors through the inclusion of the center $\mathscr{Z} \hookrightarrow \mathscr{G}$). If $\mathscr{V}$ is an $\mathscr{A}$-torsor, then we write $f_* \mathscr{V}$ for the pushout,

$$f_* \mathscr{V} := \frac{\mathscr{G} \times \mathscr{V}}{(g, a * v) \sim (f(a) \cdot g, v)}.$$

This operation of ***pushing out torsors*** defines a functor,

$$f_* \colon \mathsf{Tors}(\mathscr{A}) \to \mathsf{Tors}(\mathscr{G}).$$

If $g \in \mathscr{G}$ and $v \in \mathscr{V}$, we write $g \wedge v$ for its image in $f_* \mathscr{V}$. Then

$$g \wedge (a * v) = g f(a) \wedge v = f(a) g \wedge v.$$

The $\mathscr{G}$-torsor structure is given by

$$\gamma * (g \wedge v) = (\gamma g) \wedge v, \text{ for all } \gamma, g \in \mathscr{G}, v \in \mathscr{V}.$$

Suppose that $c\colon \mathscr{A}_1 \to \mathscr{A}_2$ is a homomorphism of sheaves of abelian groups on $S_{\text{ét}}$. Let $\mathscr{V}_1$ be an $\mathscr{A}_1$-torsor and $\mathscr{V}_2$ an $\mathscr{A}_2$-torsor. A map of torsors $\pi\colon \mathscr{V}_1 \to \mathscr{V}_2$ ***lying over*** c means a morphism of sheaves of sets on $S_{\text{ét}}$ satisfying

$$\pi(a_1 * v_1) = c(a_1) * \pi(v_1), \text{ for all } a_1 \in \mathscr{A}_1, v_1 \in \mathscr{V}_1.$$

Such a map factors uniquely through $c_* \mathscr{V}_1$.

A short exact sequence of sheaves of abelian groups on $S_{\text{ét}}$,

$$\mathscr{A} \xhookrightarrow{\alpha} \mathscr{B} \xrightarrow{\beta} \mathscr{C}, \tag{A.1}$$

yields two more constructions of torsors.

First, the sequence yields a boundary map in cohomology, $\partial\colon H^0_{\text{ét}}(S, \mathscr{C}) \to H^1_{\text{ét}}(S, \mathscr{A})$. There is a corresponding map from global sections of $\mathscr{C}$ to objects of the category of $\mathscr{A}$-torsors as follows.

Begin with $c \in \mathscr{C}[S]$ and write $[c]$ to consider it as an element of $H^0_{\text{ét}}(S, \mathscr{C})$. For any étale $U \to S$, write $c_U \in \mathscr{C}[U]$ for the restriction of c to U. Define ∂c to be the sheaf on $S_{\text{ét}}$ whose sections are given by

$$\partial c[U] = \{b \in \mathscr{B}[U] : \beta(b) = c_U\}.$$

The sheaf ∂c is naturally an $\mathscr{A}$-torsor; the equivalence class $[\partial c] \in H^1_{\text{ét}}(S, \mathscr{A})$ coincides with $\partial[c]$. The sheaf ∂c is called the ***torsor of liftings*** of c via β.

Next, write $\mathscr{H}om(\mathscr{C}, \mathscr{B})$ for the sheaf of homomorphisms ("sheaf-hom") from $\mathscr{C}$ to $\mathscr{B}$. This is a sheaf of abelian groups on $S_{\text{ét}}$, and there is a subsheaf of sets $\mathscr{S}pl(\mathscr{B})$ consisting of those homomorphisms which split the extension (A.1). This subsheaf $\mathscr{S}pl(\mathscr{B})$ is naturally a $\mathscr{H}om(\mathscr{C}, \mathscr{A})$-torsor, called the ***torsor of splittings***. This corresponds to a familiar map in the cohomology of sheaves of abelian groups,

$$\operatorname{Ext}(\mathscr{C}, \mathscr{A}) \to H^1_{\text{ét}}(S, \mathscr{H}om(\mathscr{C}, \mathscr{A})).$$

A.3. Gerbes on $S_{\text{ét}}$. — Here we introduce a class of gerbes on $S_{\text{ét}}$. In what follows, let $\mathscr{A}$ be a sheaf of *abelian* groups on $S_{\text{ét}}$.

Definition A.4. — A ***gerbe on $S_{\text{ét}}$ banded by $\mathscr{A}$*** is a (strict) stack E on $S_{\text{ét}}$ of groupoids such that E is locally nonempty, locally connected, and banded by $\mathscr{A}$.

We unravel this definition here, beginning with the data:

a (strict) stack E on $S_{\text{ét}}$ of groupoids: for each étale $U \to S$, we have a (possibly empty) groupoid $\mathsf{E}[U]$. For $\gamma\colon U' \to U$, a morphism of schemes étale over S, we have a pullback functor $\gamma^*\colon \mathsf{E}[U] \to \mathsf{E}[U']$;

banded by $\mathscr{A}$: for every object x of $\mathsf{E}[U]$, there is given an isomorphism $\mathscr{A}[U] \to \operatorname{Aut}(x)$ (written $\alpha \mapsto \alpha_x$).

This data satisfies additional axioms:

(strict) stack axioms: for each pair $\gamma\colon U' \to U$ and $\delta\colon U'' \to U'$, we require *equality* of functors $\delta^* \circ \gamma^* = (\gamma\delta)^*$. ("Strictness" refers to the requirement of equality rather than extra data of natural isomorphisms). Descent for objects and morphisms is effective;

locally nonempty: there exists a finite étale $U \to S$ such that $\mathsf{E}[U]$ is nonempty;

locally connected: for any étale $U \to S$ and pair of objects $x, y \in \mathsf{E}[U]$, there exists a finite étale $\gamma\colon U' \to U$ such that $\gamma^* x$ is isomorphic to $\gamma^* y$ in $\mathsf{E}[U']$;

banding: given a morphism $f\colon x \to y$ in $\mathsf{E}[U]$, and $\alpha \in \mathscr{A}[U]$, $\alpha_y \circ f = f \circ \alpha_x$. Also, given $\gamma\colon U' \to U$ étale, $\alpha_{\gamma^* x} = \gamma^* \alpha_x \in \operatorname{Aut}(\gamma^* x)$.

Remark A.5. — We will not require gerbes banded by nonabelian groups—commutativity greatly simplifies the theory. For a fuller treatment of gerbes, one can consult the original book of Giraud [**32**], work of Breen [**16**], [**15**], the book of Brylinski [**17**, Chapter V], the article of Deligne [**23**], the introduction of Garland and Patnaik [**55**], and the Stacks Project [**68**], among others. We don't keep track of universes along the way, while Giraud [**32**] is careful about set-theoretic subtleties. Our "strictness" assumption is typically referred to as an assumption that the fiberd category $\mathsf{E} \to S_{\text{ét}}$ is "split" (see [**73**]). The strictness assumption is not so restrictive, since every stack is equivalent to a strict stack (cf. [**73**, Theorem 3.45]). See also [**32**, §I.1].

If E is a gerbe on $S_{\text{ét}}$ banded by $\mathscr{A}$, and $U \to S$ is étale, then we write E_U for its restriction to $U_{\text{ét}}$; this is a gerbe on $U_{\text{ét}}$ banded by $\mathscr{A}_U$.

Our strictness assumption allows us to easily define the ***fiber*** of a gerbe E at a geometric point $\bar{s}$. This is the category $\mathsf{E}_{\bar{s}}$ whose object set is the direct limit $\varinjlim_U \mathsf{E}[U]$ of object sets, indexed by open étale neighborhoods of $\bar{s}$. Write $A = \mathscr{A}_{\bar{s}}$ for the fiber of $\mathscr{A}$ over $\bar{s}$. If $\bar{z}_1, \bar{z}_2$ are objects of $\mathsf{E}_{\bar{s}}$, then $\mathrm{Hom}(\bar{z}_1, \bar{z}_2)$ is naturally an A-torsor.

If $x, y \in \mathsf{E}[U]$, then for $\gamma\colon U' \to U$ define

$$\mathscr{H}om(x,y)[U'] = \mathrm{Hom}(\gamma^* x, \gamma^* y).$$

In this way, we construct a sheaf $\mathscr{H}om(x,y)$ of sets on $U_{\text{ét}}$. In particular, we find a sheaf of groups $\mathscr{A}ut(x)$ on $U_{\text{ét}}$. The banding provides an isomorphism of sheaves of groups, $\mathscr{A}_U \xrightarrow{\sim} \mathscr{A}ut(x)$, and $\mathscr{H}om(x,y)$ becomes an $\mathscr{A}_U$-torsor.

A.3.1. *Functors of gerbes.* — Suppose that $c\colon \mathscr{A}_1 \to \mathscr{A}_2$ is a homomorphism of sheaves of abelian groups on $S_{\text{ét}}$, and $\mathsf{E}_1, \mathsf{E}_2$ are gerbes on $S_{\text{ét}}$ banded by $\mathscr{A}_1, \mathscr{A}_2$, respectively. A ***functor of gerbes*** $\phi\colon \mathsf{E}_1 \to \mathsf{E}_2$, ***lying over*** c, is a (strict) functor of stacks lying over c. This entails the following:

- **(strict) functor of stacks:** for each étale $U \to S$, a functor of categories $\phi[U]\colon \mathsf{E}_1[U] \to \mathsf{E}_2[U]$. For every $\gamma\colon U' \to U$, with pullback functors γ_1^* in E_1 and γ_2^* in E_2, the "strictness" condition requires an *equality* of functors, $\gamma_2^* \circ \phi[U] = \phi[U'] \circ \gamma_1^*$;
- **lying over** c**:** the "lying over c" condition requires that, for each $\alpha_1 \in \mathscr{A}_1[U]$ with $\alpha_2 = c(\alpha_1)$, and object $x_1 \in \mathsf{E}_1[U]$ with $x_2 = \phi[U]x_1$, we have

$$(\alpha_2)_{x_2} = \phi[U]\left((\alpha_1)_{x_1}\right) \in \mathrm{Aut}(x_2).$$

Gerbes on $S_{\text{ét}}$ banded by a fixed sheaf of abelian groups $\mathscr{A}$ form a 2-category (in the sense of [**32**, §I.1.8]); if E_1 and E_2 are two such gerbes banded by the same $\mathscr{A}$, an ***equivalence of gerbes*** $\phi\colon \mathsf{E}_1 \to \mathsf{E}_2$ is a functor of gerbes lying over $\mathrm{Id}\colon \mathscr{A} \to \mathscr{A}$. Given two such equivalences of gerbes $\phi, \phi'\colon \mathsf{E}_1 \to \mathsf{E}_2$, a natural isomorphism $\phi \overset{\sim}{\Rightarrow} \phi'$ consists of natural isomorphisms of functors $\phi[U] \Rightarrow \phi'[U]$ for each U, compatible with pullback. This defines a 2-category of gerbes banded by $\mathscr{A}$, equivalences, and natural isomorphisms of equivalences.

Given two gerbes $\mathsf{E}_1, \mathsf{E}_2$ banded by $\mathscr{A}$, one may "contract" them to form another gerbe $\mathsf{E}_1 \dotplus \mathsf{E}_2$ banded by $\mathscr{A}$. The family of categories of torsors, $\mathbf{Tors}(\mathscr{A})$, given by $\mathbf{Tors}(\mathscr{A})[U] = \mathsf{Tors}(\mathscr{A}_U)$ (for each étale $U \to S$), with pullbacks given by restriction of sheaves, forms the ***neutral*** $\mathscr{A}$-gerbe on $S_{\text{ét}}$. $\mathsf{E} \dotplus \mathbf{Tors}(\mathscr{A})$ is equivalent to E, for any gerbe E banded by $\mathscr{A}$ (and the equivalence is determined up to unique natural isomorphism).

Suppose that E is a gerbe on $S_{\text{ét}}$ banded by $\mathscr{A}$, and x is an object of $\mathsf{E}[U]$ for some étale $U \to S$. Then, for $\gamma\colon U' \to U$ étale, and $y \in \mathsf{E}[U']$, we have a $\mathscr{A}_{U'}$-torsor $\mathscr{H}om(\gamma^* x, y)$. This map $y \mapsto \mathscr{H}om(\gamma^* x, y)$ extends to an equivalence of gerbes from E_U to $\mathbf{Tors}(\mathscr{A})_U$. In this way, we say that x ***neutralizes*** the gerbe E over U.

If $\mathbf{E}$ is a gerbe banded by $\mathscr{A}$, we write $[\mathbf{E}]$ for its equivalence class. The set of such equivalence classes is denoted $H^2_{\text{ét}}(S, \mathscr{A})$. This forms an abelian group, with zero corresponding to the neutral gerbe of $\mathscr{A}$-torsors, and addition arising from contraction. From [**32**], we identify $H^2_{\text{ét}}(S, \mathscr{A})$ with the étale cohomology of S with coeffiencts $\mathscr{A}$.

A.3.2. *Pushouts.* — Given a gerbe $\mathbf{E}_1$ banded by $\mathscr{A}_1$, and $c\colon \mathscr{A}_1 \to \mathscr{A}_2$ as above, one may construct a gerbe $c_* \mathbf{E}_1$ banded by $\mathscr{A}_2$ called the ***pushout*** of $\mathbf{E}_1$ by c. Any functor of gerbes $\phi\colon \mathbf{E}_1 \to \mathbf{E}_2$ lying over c factors through a functor $c_* \mathbf{E}_1 \to \mathbf{E}_2$ (the functor being determined uniquely up to unique natural isomorphism, see [**23**, §5.3]). The objects of $c_* \mathbf{E}_1$ are the same as those of $\mathbf{E}_1$. But, given two such objects $x, y \in \mathbf{E}_1[U]$, the morphism set $\mathrm{Hom}_{c_* \mathbf{E}_1}(x, y)$ is defined as the pushout of torsors,

$$\mathrm{Hom}_{c_* \mathbf{E}_1}(x, y) = c_* \mathrm{Hom}_{\mathbf{E}_1}(x, y).$$

If $x, y \in \mathbf{E}_1[U]$, $f \in \mathrm{Hom}_{\mathbf{E}_1}(x, y)$, and $\alpha_2 \in \mathscr{A}_2[U]$, we write $\alpha_2 \wedge f$ for the resulting morphism from x to y in $c_* \mathbf{E}_1$. If $\alpha_1 \in \mathscr{A}_1[U]$, then we have

$$\alpha_2 \wedge (\alpha_{1,y} \circ f) = (\alpha_2 \cdot c(\alpha_1)) \wedge f.$$

The pushout of gerbes corresponds to the map in cohomology,

$$H^2_{\text{ét}}(S, \mathscr{A}_1) \to H^2_{\text{ét}}(S, \mathscr{A}_2), \quad [\mathbf{E}_1] \mapsto c_*[\mathbf{E}_1].$$

See [**32**, Chapitre IV, §3.3, 3.4] for details.

A.3.3. *The gerbe of liftings.* — If $\mathscr{A} \overset{\alpha}{\hookrightarrow} \mathscr{B} \overset{\beta}{\twoheadrightarrow} \mathscr{C}$ is a short exact sequence of sheaves of abelian groups on $S_{\text{ét}}$, then the sequence of cohomology groups,

$$H^2_{\text{ét}}(S, \mathscr{A}) \xrightarrow{\alpha} H^2_{\text{ét}}(S, \mathscr{B}) \xrightarrow{\beta} H^2_{\text{ét}}(S, \mathscr{C})$$

is also exact. The analogous construction with gerbes is the following: suppose that $\mathbf{E}$ is a gerbe banded by $\mathscr{B}$, $\mathbf{F}$ is a gerbe banded by $\mathscr{C}$, and $\mathbf{p}\colon \mathbf{E} \to \mathbf{F}$ is a functor of gerbes lying over $\mathscr{B} \xrightarrow{\beta} \mathscr{C}$. If z is an S-object of $\mathbf{F}$ (neutralizing $\mathbf{F}$, so that $0 = [\mathbf{F}] \in H^2_{\text{ét}}(S, \mathscr{C})$), then cohomology suggests that $\mathbf{E}$ arises as the pushout of a gerbe banded by $\mathscr{A}$.

Indeed, we define the gerbe $\mathbf{p}^{-1}(z)$ as follows: the objects of $\mathbf{p}^{-1}(z)[U]$ are pairs (y, j) where y is an object of $\mathbf{E}$[U], and $j\colon \mathbf{p}(y) \to z$ is an isomorphism in $\mathbf{F}[U]$. The morphisms in $\mathbf{p}^{-1}(z)$ are those in $\mathbf{E}$ which are compatible with the isomorphisms to z. The gerbe $\mathbf{p}^{-1}(z)$ will be called the ***gerbe of liftings*** of z via p. It is a gerbe banded by $\mathscr{A}$, and there is a natural equivalence from $\alpha_* \mathbf{p}^{-1}(z)$ to $\mathbf{E}$, given by "forgetting j".

A.3.4. *The gerbe of nth roots.* — Suppose that $\mathscr{C}$ is a sheaf of abelian groups on $S_{\text{ét}}$, and the homomorphism $\mathscr{C} \xrightarrow{n} \mathscr{C}$ is surjective. An important example of a gerbe of liftings arises from the Kummer sequence $\mathscr{C}_{[n]} \hookrightarrow \mathscr{C} \overset{n}{\twoheadrightarrow} \mathscr{C}$.

Pushing out gives to a functor of gerbes, $n_*\colon \mathbf{Tors}(\mathscr{C}) \to \mathbf{Tors}(\mathscr{C})$, lying over $\mathscr{C} \overset{n}{\twoheadrightarrow} \mathscr{C}$. Given a $\mathscr{C}$-torsor $\mathscr{V}$, the gerbe of liftings of $\mathscr{V}$ via n_* will be called the ***gerbe of nth roots***, denoted $\sqrt[n]{\mathscr{V}}$. It is a gerbe on $S_{\text{ét}}$ banded by $\mathscr{C}_{[n]}$. The map which sends a $\mathscr{C}$-torsor to its gerbe of nth roots corresponds to the Kummer coboundary map $\varkappa\colon H^1_{\text{ét}}(S,\mathscr{C}) \to H^2_{\text{ét}}(S,\mathscr{C}_{[n]})$.

Explicitly, an object of $\sqrt[n]{\mathscr{V}}$ is a pair $(\mathscr{H}, h)$ where $\mathscr{H}$ is a $\mathscr{C}$-torsor, and $h\colon \mathscr{H} \to \mathscr{V}$ is a morphism of sheaves making the following diagram commute:

$$\begin{tikzcd} \mathscr{C} \times \mathscr{H} \arrow[r, "*"] \arrow[d, "n\times h"] & \mathscr{H} \arrow[d, "h"] \\ \mathscr{C} \times \mathscr{V} \arrow[r, "*"] & \mathscr{V}. \end{tikzcd}$$

The construction of the gerbe of nth roots is itself functorial. Consider a homomorphism of sheaves of abelian groups, $c\colon \mathscr{C}_1 \to \mathscr{C}_2$, and assume that $\mathscr{C}_1 \xrightarrow{n} \mathscr{C}_1$ and $\mathscr{C}_2 \xrightarrow{n} \mathscr{C}_2$ are surjective. Suppose that $\mathscr{V}_1$ is a $\mathscr{C}_1$-torsor, and $\mathscr{V}_2$ is a $\mathscr{C}_2$-torsor. Suppose that $f\colon \mathscr{V}_1 \to \mathscr{V}_2$ is a morphism of torsors lying over the homomorphism $c\colon \mathscr{C}_1 \to \mathscr{C}_2$. Then we find a functor of gerbes $\sqrt[n]{f}\colon \sqrt[n]{\mathscr{V}_1} \to \sqrt[n]{\mathscr{V}_2}$ lying over the homomorphism of bands $\mathscr{C}_{1,[n]} \to \mathscr{C}_{2,[n]}$.

A.4. Fundamental group. — Now let $\mathscr{A}$ be a *local system* of abelian groups on $S_{\text{ét}}$. Let $\mathbf{E}$ be a gerbe on $S_{\text{ét}}$ banded by $\mathscr{A}$. Let $\bar{F}$ be a separably closed field, let $\bar{s}\colon \operatorname{Spec}(\bar{F}) \to S$ be a geometric point, and recall that $\mathrm{Gal}_S = \pi_1^{\text{ét}}(S, \bar{s})$ denotes the étale fundamental group.

Suppose that $U \to S$ is a Galois cover, and $\bar{u}\colon \operatorname{Spec}(\bar{F}) \to U$ lifts the geometric point $\bar{s}$. Writing $\mathrm{Gal}_U = \pi_1^{\text{ét}}(U, \bar{u})$, we find a short exact sequence

$$\mathrm{Gal}_U \hookrightarrow \mathrm{Gal}_S \twoheadrightarrow \mathrm{Gal}(U/S).$$

If $\gamma \in \mathrm{Gal}_S$ we write γ_U for its image in $\mathrm{Gal}(U/S)$.

Suppose moreover that $\mathscr{A}_U$ is a constant sheaf and $\mathbf{E}[U]$ is a nonempty groupoid (i.e., $\mathbf{E}$ is neutral over U). Write $A = \mathscr{A}[U]$ for the resulting abelian group. Then A is endowed with an action of Gal_S that factors through the finite quotient $\mathrm{Gal}(U/S)$. An object $z \in \mathbf{E}[U]$ will be called a ***base point*** for the gerbe $\mathbf{E}$ (over U). The banding identifies A with the automorphism group of z.

Without loss of generality, pulling back to a larger Galois cover if necessary, we may assume that $\mathrm{Hom}(z, \gamma_U^* z)$ is nonempty for all $\gamma \in \mathrm{Gal}_S$. In this way, the base point $z \in \mathbf{E}[U]$ and $\gamma \in \mathrm{Gal}_S$ define an A-torsor,

$$\mathrm{Aut}_\gamma(z) := \mathrm{Hom}(z, \gamma_U^* z).$$

We write γ^* instead of γ_U^*, when there is little chance of confusion.

Define the ***étale fundamental group of the gerbe*** $\mathbf{E}$, at the base point z, by

$$\pi_1(\mathbf{E}, z) = \bigsqcup_{\gamma \in \mathrm{Gal}_S} \mathrm{Aut}_\gamma(z).$$

The group structure is given, for $\gamma_1, \gamma_2 \in \mathrm{Gal}_S$, by the following sequence:

$$\begin{aligned}\mathrm{Aut}_{\gamma_1}(z) \times \mathrm{Aut}_{\gamma_2}(z) &= \mathrm{Hom}(z, \gamma_1^* z) \times \mathrm{Hom}(z, \gamma_2^* z) \\ &\xrightarrow{\gamma_2^* \times \mathrm{Id}} \mathrm{Hom}(\gamma_2^* z, \gamma_2^* \gamma_1^* z) \times \mathrm{Hom}(z, \gamma_2^* z) \\ &\xrightarrow{\circ} \mathrm{Hom}(z, \gamma_2^* \gamma_1^* z) \\ &\xrightarrow{=} \mathrm{Hom}(z, (\gamma_1\gamma_2)^* z) = \mathrm{Aut}_{\gamma_1\gamma_2}(z).\end{aligned}$$

As $\mathrm{Aut}_{\mathrm{Id}}(z) = \mathrm{Aut}(z)$, the isomorphism $A \xrightarrow{\sim} \mathrm{Aut}(z)$, $\alpha \mapsto \alpha_z$, gives an extension of groups,

$$A \hookrightarrow \pi_1^{\text{ét}}(\mathbf{E}, z) \twoheadrightarrow \mathrm{Gal}_S \,. \tag{A.2}$$

If $\gamma \in \mathrm{Gal}_U \subset \mathrm{Gal}_S$, then $\gamma_U = \mathrm{Id}$ and so $\mathrm{Aut}_\gamma(z) = \mathrm{Aut}_{\mathrm{Id}}(z)$. In this way, we find a splitting $\mathrm{Gal}_U \hookrightarrow \pi_1(\mathbf{E}, z)$. In other words, $\pi_1^{\text{ét}}(\mathbf{E}, z)$ arises as the pullback of an extension of $\mathrm{Gal}(U/S)$ by A. The conjugation action of Gal_S on A, in the extension (A.2), coincides with the canonical action of $\mathrm{Gal}(U/S)$ on $A = \mathscr{A}[U]$.

The sequence (A.2) describes the fundamental group of a gerbe (with base point) as an extension of Gal_S by A. Here we analyze how this fundamental group depends on the choice of base point, and how it behaves under equivalence of gerbes.

Consider a further Galois cover $\delta \colon U' \to U$ and geometric base point $\bar{u}'$ lifting $\bar{u}$. By constancy of $\mathscr{A}_U$, we identify $A = \mathscr{A}[U] = \mathscr{A}[U']$. For all $\gamma \in \mathrm{Gal}_S$, we have $\gamma_U \circ \delta = \delta \circ \gamma_{U'}$. This defines an isomorphism of A-torsors, $\delta^* \colon \mathrm{Aut}_\gamma(z) \xrightarrow{\sim} \mathrm{Aut}_\gamma(\delta^* z)$, using the sequence below:

$$\begin{aligned}\mathrm{Aut}_\gamma(z) = \mathrm{Hom}(z, \gamma_U^* z) &\xrightarrow{\delta^*} \mathrm{Hom}(\delta^* z, \delta^* \gamma_U^* z) \\ &\xrightarrow{=} \mathrm{Hom}\left(\delta^* z, (\gamma_U \delta)^* z\right) \\ &\xrightarrow{=} \mathrm{Hom}\left(\delta^* z, (\delta \gamma_{U'})^* z\right) \\ &\xrightarrow{=} \mathrm{Hom}\left(\delta^* z, \gamma_{U'}^* \delta^* z\right) = \mathrm{Aut}_\gamma(\delta^* z).\end{aligned}$$

Putting these isomorphisms together, we find an isomorphism of extensions:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "="] & \pi_1^{\text{ét}}(\mathbf{E}, z) \arrow[r, two heads] \arrow[d, "\iota_\delta"] & \mathrm{Gal}_S \arrow[d, "="] \\ A \arrow[r, hook] & \pi_1^{\text{ét}}(\mathbf{E}, \delta^* z) \arrow[r, two heads] & \mathrm{Gal}_S \,. \end{tikzcd}$$

A further cover $\delta' \colon U'' \to U'$, with $\delta'' = \delta \circ \delta' \colon U'' \to U$, gives a commutative diagram in the category of extensions of Gal_S by A:

$$\begin{tikzcd} \pi_1^{\text{ét}}(\mathbf{E}, z) \arrow[r, "\iota_\delta"'] \arrow[rr, bend left, "\iota_{\delta''}"] & \pi_1^{\text{ét}}(\mathbf{E}, \delta^* z) \arrow[r, "\iota_{\delta'}"'] & \pi_1^{\text{ét}}(\mathbf{E}, (\delta')^* \delta^* z) = \pi_1^{\text{ét}}(\mathbf{E}, (\delta'')^* z). \end{tikzcd}$$

Define $\bar{z} \in \mathbf{E}_{\bar{s}}$ to be the image of the base point z in the direct limit. We call $\bar{z}$ a ***geometric base point*** for the gerbe $\mathbf{E}$. Define

$$\pi_1^{\text{ét}}(\mathbf{E}, \bar{z}) = \varinjlim_{U'} \pi_1^{\text{ét}}(\mathbf{E}, \delta^* z),$$

the direct limit over Galois covers $\delta \colon (U', \bar{u}') \to (U, \bar{u})$, via the isomorphisms ι_δ described above. This gives an extension of groups, depending (up to unique isomorphism) only on the *geometric* base point $\bar{z} \in \mathbf{E}_{\bar{s}}$:

$$A \hookrightarrow \pi_1^{\text{ét}}(\mathbf{E}, \bar{z}) \twoheadrightarrow \mathrm{Gal}_S .$$

This extension is also endowed with a family of splittings over finite-index subgroups $\mathrm{Gal}_U \subset \mathrm{Gal}_S$, arising from base points $z \in \mathbf{E}[U]$ mapping to $\bar{z}$. Having such splittings is useful for topological purposes, e.g., $\pi_1^{\text{ét}}(\mathbf{E}, \bar{z}) \to \mathrm{Gal}_S$ is naturally a continuous homomorphism of profinite groups when A is finite.

Consider a second geometric base point $\bar{z}_0 \in \mathbf{E}_{\bar{s}}$ (over the *same* $\bar{s} \to S$). There exists an isomorphism $\bar{f} \colon \bar{z}_0 \to \bar{z}$ in $\mathbf{E}_{\bar{s}}$. For a sufficiently large Galois cover $U \to S$, we may assume that $\bar{f} \colon \bar{z}_0 \to \bar{z}$ arises from a morphism $f \colon z_0 \to z$ in $\mathbf{E}[U]$.

Define $\iota_f \colon \mathrm{Aut}_\gamma(z_0) \to \mathrm{Aut}_\gamma(z)$ to be the bijection

$$\iota_f(\eta) = \gamma^* f \circ \eta \circ f^{-1}, \text{ for all } \eta \in \mathrm{Aut}_\gamma(z_0),$$

making the following diagram commute (in the groupoid $\mathbf{E}[U]$):

$$\begin{array}{ccc} z_0 & \xrightarrow{f} & z \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \iota_f(\eta)} \\ \gamma^* z_0 & \xrightarrow{\gamma^* f} & \gamma^* z. \end{array}$$

As γ varies over Gal_S, this provides an isomorphism of extensions, $\iota_f \colon \pi_1^{\text{ét}}(\mathbf{E}, z_0) \to \pi_1^{\text{ét}}(\mathbf{E}, z)$. Passing to the direct limit, we find an isomorphism of extensions depending only on $\bar{f} \colon \bar{z}_0 \to \bar{z}$,

$$\begin{array}{ccccc} A & \hookrightarrow & \pi_1^{\text{ét}}(\mathbf{E}, \bar{z}_0) & \twoheadrightarrow & \mathrm{Gal}_S \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \iota_{\bar{f}}} & & \downarrow{\scriptstyle =} \\ A & \hookrightarrow & \pi_1^{\text{ét}}(\mathbf{E}, \bar{z}) & \twoheadrightarrow & \mathrm{Gal}_S . \end{array}$$

Given *another* isomorphisms $\bar{g} \colon \bar{z}_0 \to \bar{z}$, there exists a unique element $\alpha \in A$ such that $\bar{g} = \alpha_{\bar{z}} \circ \bar{f}$. As for f, we may assume that $\bar{g}$ arises from $g \colon z_0 \to z$ in $\mathbf{E}[U]$. It follows that, for all $\eta \in \mathrm{Aut}_\gamma(z_0)$,

$$\begin{aligned} \iota_g(\eta) &= \gamma^* g \circ \eta \circ g^{-1} \\ &= \gamma^*(\alpha_z \circ f) \circ \eta \circ f^{-1} \circ \alpha_z^{-1} \\ &= \gamma^* \alpha_z \circ \iota_f(\eta) \circ \alpha_z^{-1} \\ &= \alpha_{\gamma^* z} \circ \iota_f(\eta) \circ \alpha_z^{-1}. \end{aligned}$$

In other words, we have $\iota_{\bar{g}} = \mathrm{Int}(\alpha) \circ \iota_{\bar{f}}$.

To summarize the relationship between gerbes banded by $\mathscr{A}$ and extensions of Gal_S by A, we have the following.

Theorem A.6. — *To each geometric base point $\bar{z} \in \mathsf{E}_{\bar{s}}$, we obtain an extension*

$$A \hookrightarrow \pi_1^{\text{ét}}(\mathsf{E}, \bar{z}) \twoheadrightarrow \mathrm{Gal}_S,$$

known as the fundamental group of the gerbe E at $\bar{z}$. For any two geometric base points $\bar{z}_0, \bar{z}$, we obtain a family *of isomorphisms of extensions*

$$\pi_1^{\text{ét}}(\mathsf{E}, \bar{z}_0) \to \pi_1^{\text{ét}}(\mathsf{E}, \bar{z}),$$

any two of which are related by $\mathrm{Int}(\alpha)$ *for a uniquely determined $\alpha \in A$.*

This theorem may seem more natural using 2-categorical language as follows: consider the 2-category $\mathsf{OpExt}(\mathrm{Gal}_S, \mathscr{A})$ whose objects are extensions of Gal_S by A in which the Gal_S action on $A = \mathscr{A}_{\bar{s}}$ coincides with that which arises from the local system $\mathscr{A}$. The morphisms in this category are isomorphisms of extensions (giving equality on Gal_S and A). Given two such morphisms ι, ι' sharing the same source and target, a natural transformation $\iota \Rightarrow \iota'$ is an element $\alpha \in A$ such that $\iota' = \mathrm{Int}(\alpha) \circ \iota$. A restatement of the above theorem is the following.

Theorem A.7. — *A gerbe E banded by $\mathscr{A}$, and a geometric point $\bar{s} \to S$, yield an object $\pi_1^{\text{ét}}(\mathsf{E}, \bar{s})$ of $\mathsf{OpExt}(\mathrm{Gal}_S, \mathscr{A})$, well-defined up to equivalence, the equivalence being uniquely determined up to unique natural isomorphism.*

Remark A.8. — If $\mathscr{A}$ is a constant sheaf, then the extension $A \hookrightarrow \pi_1^{\text{ét}}(\mathsf{E}, \bar{z}) \twoheadrightarrow \mathrm{Gal}_S$ is a central extension. It follows quickly from the theorem that we may define an extension

$$A \hookrightarrow \pi_1^{\text{ét}}(\mathsf{E}, \bar{s}) \twoheadrightarrow \mathrm{Gal}_S$$

up to unique isomorphism (without choice of geometric base point $\bar{z}$). In this case, the 2-category $\mathsf{OpExt}(\mathrm{Gal}_S, A)$ is an ordinary category: the only transformations are the identities.

Next, consider a functor of gerbes $\phi\colon \mathsf{E}_1 \to \mathsf{E}_2$ lying over a homomorphism $c\colon \mathscr{A}_1 \to \mathscr{A}_2$ of local systems of abelian groups. If $\bar{z}_1$ is a geometric base point for E_1 over $\bar{s}$, arising from $z_1 \in \mathsf{E}_1[U]$, then let $z_2 = \phi[U](z_1)$. Define $\bar{z}_2$ to be the resulting geometric base point of E_2. Our strictness assumption for functors of gerbes implies that the geometric base point $\bar{z}_2$ depends only on the geometric base point $\bar{z}_1$, and not on the choice of z_1.

For any $\gamma \in \mathrm{Gal}_S$, we obtain a map $\phi_\gamma\colon\ \mathrm{Aut}_\gamma(z_1) \to \mathrm{Aut}_\gamma(z_2)$, given by

$$\begin{aligned}\mathrm{Aut}_\gamma(z_1) = \mathrm{Hom}(z_1, \gamma^* z_1) &\xrightarrow{\phi[U]} \mathrm{Hom}(\phi[U](z_1), \phi[U](\gamma^* z_1)) \\ &= \mathrm{Hom}(z_2, \gamma^* \phi[U](z_1)) \\ &= \mathrm{Hom}(z_2, \gamma^* z_2) = \mathrm{Aut}_\gamma(z_2).\end{aligned}$$

Putting these together yields a homomorphism of extensions,

$$\begin{tikzcd} A_1 \arrow[r, hook] \arrow[d, "c"] & \pi_1^{\text{ét}}(\mathbf{E}_1, \bar{z}_1) \arrow[r, two heads] \arrow[d, "\phi"] & \mathrm{Gal}_S \arrow[d, "="] \\ A_2 \arrow[r, hook] & \pi_1^{\text{ét}}(\mathbf{E}_2, \bar{z}_2) \arrow[r, two heads] & \mathrm{Gal}_S\,. \end{tikzcd}$$

If $\phi, \phi' \colon \mathbf{E}_1 \to \mathbf{E}_2$ are two functors of gerbes lying over $c \colon \mathscr{A}_1 \to \mathscr{A}_2$, and $\phi(\bar{z}_1) = \phi'(\bar{z}_1) = \bar{z}_2$, then we find two such homomorphisms of extensions,

$$\phi, \phi' \colon \pi_1^{\text{ét}}(\mathbf{E}_1, \bar{z}_1) \to \pi_1^{\text{ét}}(\mathbf{E}_2, \bar{z}_2),$$

lying over $c \colon A_1 \to A_2$.

If $N \colon \phi \Rightarrow \phi'$ is a natural isomorphism of functors, then N determines an isomorphism $\phi(\bar{z}_1) \to \phi'(\bar{z}_1)$, whence an isomorphism $\bar{z}_2 \to \bar{z}_2$. Such an isomorphism is given by an element $\alpha_2 \in A_2 = \mathrm{Aut}(\bar{z}_2)$, and one may check that

$$\phi' = \mathrm{Int}(\alpha_2) \circ \phi \colon \pi_1^{\text{ét}}(\mathbf{E}_1, \bar{z}_1) \to \pi_1^{\text{ét}}(\mathbf{E}_2, \bar{z}_2).$$

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THE LANGLANDS-WEISSMAN PROGRAM FOR BRYLINSKI-DELIGNE EXTENSIONS

by

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Abstract. — We describe an evolving and conjectural extension of the Langlands program for a class of nonlinear covering groups of algebraic origin studied by Brylinski and Deligne. In particular, we describe the construction of an L-group extension of such a covering group (over a split reductive group) due to Weissman, study some of its properties and discuss a variant of it. Using this L-group extension, we describe a local Langlands correspondence for covering (split) tori and unramified genuine representations, using work of Savin, McNamara, Weissman and W.-W. Li. We then define the notion of automorphic (partial) L-functions attached to genuine automorphic representations of the covering groups of Brylinski and Deligne. Finally, we see how the L-group formalism explains certain anomalies in the representation theory of covering groups and examine some examples of Langlands functoriality such as base change.

Résumé **(Le programme de Langlands-Weissman pour les extensions de Brylinski-Deligne)**

Nous décrivons une extension conjecturale, en evolution, du programme de Langlands pour une classe de revêtements de groupes réductifs d'origine algébrique, étudiés par Brylinski et Deligne. Nous décrivons, en particulier, la construction, due à Weissman, d'une extension de L-groupe d'un tel revêtement de groupe (au-dessus d'un groupe réductif déployé). Nous étudions certaines de ses propriétés et discutons d'une variante de celui-ci. En utilisant cette extension de L-groupe, à l'aide du travail de Savin, McNamara, Weissman et W.-W. Li, nous décrivons une correspondance de Langlands locale pour des revêtements des tores (déployés) et pour les représentations non-ramifiées spécifiques. Nous définissons ensuite la notion de L-fonctions automorphes (partielles) attachées aux représentations automorphes spécifiques pour les groupes de Brylinski et Deligne. Enfin, nous verrons comment le formalisme de L-groupe explique certaines anomalies dans la théorie des représentations des revêtements de groupes réductifs et examinons quelques exemples de fonctorialité de Langlands tels que le changement de base.

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1. Introduction

One of the goals of the local Langlands program is to provide an arithmetic classification of the set of isomorphism classes of irreducible representations of a locally compact group $G = \mathbb{G}(F)$, where $\mathbb{G}$ is a connected reductive group over a local field F. Analogously, if k is a number field with ring of adeles $\mathbb{A}$, the global Langlands program postulates a classification of automorphic representations of $\mathbb{G}(\mathbb{A})$ in terms of Galois representations. In this proposed arithmetic classification, which has been realized in several important instances, a key role is played by the L-group ${}^L\mathbb{G}$ of $\mathbb{G}$. This key notion was introduced by Langlands in his re-interpretation of the Satake isomorphism in the theory of spherical functions and used by him to introduce the notion of *automorphic L-functions*. One of the main goals of this paper is to do the same for a class of nonlinear covering groups of "algebraic origin" studied by Brylinski-Deligne [**15**].

1.1. Covering groups. — The theory of the L-group is so far confined to the case when $\mathbb{G}$ is a connected reductive linear algebraic group. On the other hand, since Steinberg's beautiful paper [**59**], the structure theory of nonlinear covering groups of G (i.e., topological central extensions of G by finite groups) have been investigated by many mathematicians, notably Moore [**48**], Matsumoto [**42**], Deodhar [**21**], Deligne [**20**], Prasad-Raghunathan [**50, 51, 52**], and its relation to the reciprocity laws of abelian class field theory has been noted. In addition, nonlinear covering groups of G have repeatedly made their appearance in representation theory and the theory of automorphic forms. This goes way back to Jacobi's construction of his theta function, a holomorphic modular form of weight $1/2$, and a more recent instance is the work of Kubota [**30**] and the Shimura correspondence between integral and half integral weight modular forms. Both these examples concern automorphic forms and representations of the metaplectic group $\mathrm{Mp}_2(F)$, which is a nonlinear double cover of $\mathrm{SL}_2(F) = \mathrm{Sp}_2(F)$. As another example, the well-known Weil representation of $\mathrm{Mp}_{2n}(F)$ gives a representation theoretic incarnation of theta functions and has been a very useful tool in the construction of automorphic forms. Finally, much of Harish-Chandra's theory of local harmonic analysis and Langlands' theory of Eisenstein series continue to hold for such nonlinear covering groups (see [**47**] and [**36**]).

It is thus natural to wonder if the framework of the Langlands program can be extended to encompass the representation theory and the theory of automorphic forms of covering groups. There have been many attempts towards this end, such as Flicker [**23**], Kazhdan-Patterson [**28, 29**], Flicker-Kazhdan [**24**], Adams [**1, 2**], Savin [**57**] among others. However, these attempts have tended to focus on the treatment of specific families of examples rather than a general theory. This is understandable, for what is lacking is a structure theory which is sufficiently functorial. For example, the classification of nonlinear covering groups given in [**48, 21, 50, 51**] is given only when $\mathbb{G}$ is simply-connected and isotropic, in which case a universal cover exists.

1.2. Brylinski-Deligne theory. — A functorial structure theory was finally developed by Brylinski and Deligne [**15**]. More precisely, Brylinski-Deligne considered the category of multiplicative $\mathbb{K}_2$-torsors on a connected reductive group $\mathbb{G}$ over F; these are extensions of $\mathbb{G}$ by the sheaf $\mathbb{K}_2$ of Quillen's K_2 group in the category of sheaves of groups on the big Zariski site of $\mathrm{Spec}(F)$:

$$1 \longrightarrow \mathbb{K}_2 \longrightarrow \overline{\mathbb{G}} \longrightarrow \mathbb{G} \longrightarrow 1.$$

In other words, Brylinski and Deligne started with an extension problem in the world of algebraic geometry. Some highlights of [**15**] include:

- an elegant and functorial classification of this category in terms of enhanced root theoretic data, much like the classification of split connected reductive groups by their root data.
- the description of a functor from the category of multiplicative $\mathbb{K}_2$-torsors $\overline{\mathbb{G}}$ on $\mathbb{G}$ (together with an integer n such that $\#\mu_n(F) = n$, which determines the degree of the covering) to the category of topological central extensions $\overline{G}$ of G:

  $$1 \longrightarrow \mu_n \longrightarrow \overline{G} \longrightarrow G \longrightarrow 1.$$

  These topological central extensions may be considered of "algebraic origin" and can be constructed using cocycles which are essentially algebraic in nature.
- though this construction does not exhaust all topological central extensions, it captures a sufficiently large class of such extensions, and essentially all interesting examples which have been investigated so far; for example, it captures all such coverings of G when $\mathbb{G}$ is split and simply-connected (except perhaps in the case of type C_r over real numbers).

We shall give a more detailed discussion of the salient features of the Brylinski-Deligne theory in §2 and §3. Hence, the paper [**15**] provides a structure theory which is essentially algebraic and categorical, and may be perceived as a natural extension of Steinberg's original treatment [**59**] from the split simply connected case to general reductive groups.

1.3. Dual and L-groups. — One should expect that such a natural structure theory would elucidate the study of representations and automorphic forms of the Brylinski-Deligne covering groups $\overline{G}$, henceforth referred to as BD covering groups. Indeed, Brylinski and Deligne wrote in the introduction of [**15**]: "We hope that for k a global field, this will prove useful in the study of metaplectic automorphic forms, that is, the harmonic analysis of functions on $\tilde{G}(\mathbb{A})/G(k)$".

The first person to fully appreciate this is probably our colleague M. Weissman. In a series of papers [**66, 67, 27**], Weissman systematically exploited the Brylinski-Deligne theory to study the representation theory of covering tori, the unramified representations and the depth zero representations. This was followed by the work of several authors who discovered a "Langlands dual group" $\overline{G}^\vee$ for a BD covering group $\overline{G}$ (with $\mathbb{G}$ split) from different considerations. These include the work of Finkelberg-Lysenko [**22**] and Reich [**54**] in the framework of the geometric Langland program and the work of McNamara [44, 45] who established a Satake isomorphism and interpreted it in terms of the dual group $\overline{G}^\vee$. The dual group $\overline{G}^\vee$ was constructed by making a metaplectic modification of the root datum of $\mathbb{G}$.

In [70], Weissman built upon [44] and gave a construction of the "L-group" ${}^L\overline{G}$ of a *split* BD covering group $\overline{G}$. The construction in [70] is quite involved, and couched in the language of Hopf algebras. Moreover, with hindsight, it gives the correct notion only for a subclass of BD covering groups. In a foundational paper [73], Weissman gives a simpler and completely general revised construction of the L-group for an arbitrary quasi-split BD covering group (not necessarily split), using the framework of étale gerbes, thus laying the groundwork for an extension of the Langlands program to the setting of BD covering groups.

1.4. The L-group extension. — We shall describe in §4 Weissman's construction of the L-group of $\overline{G}$ for split $\mathbb{G}$ (given in the letter [69]), where one could be more down-to-earth and avoid the notion of gerbes. The fact that this more down-to-earth construction is equivalent to the more sophisticated one in [73] is shown in [72]. At this point, let us note that since $\mathbb{G}$ is split, one is inclined to simply take ${}^L\overline{G}$ as the direct product $\overline{G}^\vee \times W_F$, where W_F denotes the Weil group of F. At least, this is what one is conditioned to do by the theory of L-groups for linear reductive groups. However, Weissman realized that such an approach would be overly naive.

Indeed, the key insight of [70] is that the construction of the L-group of a BD covering group should be the functorial construction of an extension

$$1 \longrightarrow \overline{G}^\vee \longrightarrow {}^L\overline{G} \longrightarrow W_F \longrightarrow 1, \tag{1.1}$$

and an L-parameter for $\overline{G}$ should be a splitting of this short exact sequence. The point is that, *even if ${}^L\overline{G}$ is isomorphic to the direct product $\overline{G}^\vee \times W_F$*, it is not supposed to be equipped with a canonical isomorphism to $\overline{G}^\vee \times W_F$. This reflects the fact that there is no canonical irreducible genuine representation of $\overline{G}$, and hence there should not be any canonical L-parameter. Hence it would not be appropriate to say that the L-group of $\overline{G}$ "is" the direct product $\overline{G}^\vee$ with W_F.

As Weissman is the first person to make use of the full power of the Brylinski-Deligne structure theory for the purpose of representation theory and is the one who introduced the L-group extension, we shall call this evolving area the Langlands-Weissman program for BD extensions. As the adjective "evolving" is supposed to suggest, we caution the reader that the construction in [69, 73] may not be the final word on the L-group.

1.5. Results of this paper. — Against this backdrop, the purpose of this paper is to supplement the viewpoint of [70, 69, 73] concerning the L-group ${}^L\overline{G}$ in several ways. In particular, we shall study some properties of the L-group extension, suggest a variant of it and provide supporting evidence for its essential correctness. We summarize our results here:

(i) (The L-group extension) Firstly, we show that the L-group extension of a split BD covering group constructed in [**70, 69, 73**] is a split extension (see Proposition 6.9), but the L-group is *not* isomorphic to the direct product $\overline{G}^{\vee} \times W_F$ in general. This phenomenon can be seen already in the following simple family of BD covering groups

$$\overline{G}_\eta = (\mathrm{GL}_2(F) \times \mu_2)/i_\eta(F^\times) \cong \mathrm{GL}_2(F)/NE_\eta^\times$$

where $\eta \in F^\times/F^{\times 2}$, with corresponding quadratic étale algebra E_η, and

$$i_\eta(t) = (t, (\eta, t)_2),$$

with $(-,-)_2$ denoting the quadratic Hilbert symbol. Observe that the first projection defines a topological central extension

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G}_\eta \longrightarrow \mathrm{PGL}_2(F) \longrightarrow 1.$$

Then it turns out that $\overline{G}_\eta^{\vee} \cong \mathrm{SL}_2(\mathbb{C})$ and

$${}^L\overline{G}_\eta \cong \mathrm{SL}_2(\mathbb{C}) \rtimes_\eta W_F$$

where the action of W_F on $\mathrm{SL}_2(\mathbb{C})$ is by the conjugation action via the map $W_F \longrightarrow \mathrm{GL}_2(\mathbb{C})$ given by

$$w \mapsto \begin{pmatrix} \chi_\eta(w) & \\ & 1 \end{pmatrix},$$

with χ_η the quadratic character of W_F determined by η.

This family of BD covering groups is quite instructive, as it illustrates several interesting phenomena. For example, one can show that the covering splits over the hyperspecial maximal compact subgroup $\mathrm{PGL}_2(\mathcal{O}_F)$ if and only if $\eta \in \mathcal{O}_F^\times$ (see § 4.6). This seems to contradict [**44**, Thm. 4.2] (see [**45**] where a corrected version is given) where it was claimed that a BD covering for a split $\mathbb{G}$ is always split over a hyperspecial maximal compact subgroup K. In general, we treat this issue of splitting over K in §4; see especially Theorem 4.2.

Though the L-group extension may not be split, the inner class of $\mathrm{Aut}(\overline{G}^{\vee})$ induced by the conjugation action of W_F (via any choice of splitting) contains the trivial automorphism. Such a situation is familiar from the theory of endoscopy for linear algebraic groups, where in the definition of an endoscopic datum $(H, \mathcal{H}, s, \xi)$, the group $\mathcal{H}$ is a split extension of W_F by $H^\vee$ but not necessarily isomorphic to the L-group of H.

(ii) (Distinguished splittings) Secondly, we would like to argue that the L-group of a split BD covering group should be (the isomorphic class of) the extension (1.1), *together with a finite set of distinguished splittings* (which we will define and which give rise to isomorphisms ${}^L\overline{G} \cong \overline{G}^{\vee} \times W_F$ if they exist). If the degree of the covering is $n = 1$, so that $\overline{G} = G$, then the set of distinguished splittings is a singleton, so that ${}^L\overline{G}$ "is" $\overline{G}^{\vee} \times W_F$ in this case. For a subclass of BD covering groups, we show in Theorem 6.6 that the L-group short exact sequence (1.1) has such a family of distinguished splittings; this completes the results of **[70]** where the cases for odd n and $n = 2$ were treated.

(iii) (Distinguished genuine characters) We show that the distinguished splittings of ${}^L\overline{G}$ are in natural bijection with a family of distinguished genuine characters of the center $Z(\overline{T})$ of the covering torus $\overline{T}$ (where T is a maximal F-split torus of G). These distinguished genuine characters of $Z(\overline{T})$ are "as close to being trivial characters as possible," and are invariant under the natural Weyl group action for *certain* $\overline{G}$ (see Theorem 6.8). Thus, they serve as natural base-points for the definition of principal series representations, extending the results of Savin **[57]** to general n and these $\overline{G}$. We also give an explicit construction of such distinguished genuine characters of $Z(\overline{T})$, using the Weil index and the n-th Hilbert symbol.

We point out, however, that for general $\overline{G}$, there may not exist distinguished splittings of ${}^L\overline{G}$ or Weyl-invariant genuine characters of $Z(\overline{T})$. For example, for the covering groups $\overline{G}_\eta$ discussed in (i), Weyl-invariant genuine characters exist if and only if $(\eta, -1)_2 = 1$.

(iv) (LLC) The fact that the extension ${}^L\overline{G}$ is split allows one to define the set of L-parameters for $\overline{G}$ as the set of splittings of ${}^L\overline{G}$ modulo the conjugation action of $\overline{G}^\vee$. In particular, one can formulate a conjectural (coarse) local Langland correspondence (LLC). This is done in §11. We verify this conjectural LLC in three cases:

- (LLC for covering tori) When $\mathbb{G} = \mathbb{T}$ is a split torus, the above passage between distinguished splittings of ${}^L\overline{T}$ and distinguished genuine characters of $Z(\overline{T})$ extends to give the local Langlands correspondence for covering tori, a problem first investigated in **[66]** and also treated in **[73]**. This is contained in Theorem 8.2.
- (Unramified LLC) For general split $\mathbb{G}$, a Satake isomorphism has been shown by McNamara **[44]**, W.-W. Li **[39]** and Weissman **[73]** (relative to a splitting s of $\overline{G}$ over K). In Theorem 9.4, we formulate the Satake isomorphism in terms of ${}^L\overline{G}$ and show in Theorem 9.8 that s-unramified representations of $\overline{G}$ are naturally parametrized by "s-unramified splittings" of ${}^L\overline{G}$. In particular, this gives the notion of Satake parameters (relative to s), which is the main ingredient in the definition of automorphic L-functions.
- (Metaplectic groups) When $\overline{G} = \mathrm{Mp}_{2n}(F)$ is the degree 2 cover of $\mathrm{Sp}_{2n}(F)$, we use the results of **[3, 25]** (established using the theory of theta correspondences and the LLC for odd special orthogonal groups) to deduce the LLC for Mp_{2n} in terms of the L-group ${}^L\overline{G}$ considered here. In particular, this LLC for $\mathrm{Mp}_{2n}(F)$ (see Theorem 11.1) is independent of the choice of a nontrivial additive character ψ of F.

Using the results of Ban-Jantzen [7] (on the Langlands classification for covering groups) and W.-W. Li [36] (on the theory of R-groups), one can reduce the coarse LLC to the discrete series case.

(v) (Enlagred L-group) In Section 10, we suggest a slightly different treatment of the L-group given in [70, 69, 73], by treating several closely related BD covering groups together. For example, the representation theory of all the groups $\overline{G}_\eta$ in (i) can clearly be treated together in terms of the representation theory of $\mathrm{GL}_2(F)$. Extending this instructive example, one can slightly enhance Weissman's construction in [69, 73] to give an enlarged L-group ${}^L\overline{G}^{\#}$:

$$1 \longrightarrow \overline{G}^{\vee} \longrightarrow {}^L\overline{G}^{\#} \longrightarrow W_F \times (T_{Q,n}^{\mathrm{sc}})^{\vee}[n] \longrightarrow 1 \tag{1.2}$$

where $(T_{Q,n}^{\mathrm{sc}})^{\vee}[n]$ is a finite group: it is the group of n-torsion points in the maximal split torus of the adjoint quotient of $\overline{G}^{\vee}$. One can write the above exact sequence as:

$$1 \longrightarrow \overline{G}^{\#} = \overline{G}^{\vee} \rtimes (T_{Q,n}^{\mathrm{sc}})^{\vee}[n] \longrightarrow {}^L\overline{G}^{\#} \longrightarrow W_F \longrightarrow 1 \tag{1.3}$$

where the action of $(T_{Q,n}^{\mathrm{sc}})^{\vee}[n]$ on $\overline{G}^{\vee}$ is via the canonical adjoint action of $(\overline{G}^{\vee})_{ad}$ on $\overline{G}^{\vee}$. If one pulls back (1.2) using the map

$$\mathrm{id} \times \chi_\eta : W_F \longrightarrow W_F \times \mu_2,$$

one recovers the L-group extension ${}^L\overline{G}_\eta$. Thus, ${}^L\overline{G}^{\#}$ is an amalgam of all ${}^L\overline{G}_\eta$ and it always has a distinguished splitting. Using this enlarged L-group and exploiting z-extensions (see §2.9), we can reduce the LLC for BD covering groups to a distinguished subclass of such groups, corresponding to $\eta = 1$.

As an example, for the groups $\overline{G}_\eta$ discussed in (i), (1.2) is:

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{C}) \longrightarrow W_F \times \mathrm{SL}_2(\mathbb{C})^{\pm} \xrightarrow{\det} W_F \times \mu_2 \longrightarrow 1,$$

whereas (1.3) is:

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{C})^{\pm} \longrightarrow W_F \times \mathrm{SL}_2(\mathbb{C})^{\pm} \longrightarrow W_F \longrightarrow 1.$$

(vi) (Automorphic L-functions) Using the above results, we define in §13 the notion of "automorphic L-functions" associated to an automorphic representation of a global BD covering group and a representation R of ${}^L\overline{G}$. How can one describe the representations R of ${}^L\overline{G}$? One way is to exploit the distinguished splittings of ${}^L\overline{G}$ constructed above.

Indeed, a distinguished splitting $s_0 : W_F \longrightarrow {}^L\overline{G}^{\#}$ gives rise to the notion of *L-parameters relative to* s_0: for any splitting $s : W_F \longrightarrow {}^L\overline{G}^{\#}$, one sets

$$\phi_s(w) = s(w)/s_0(w)^{-1} \in \overline{G}^{\#} = \overline{G}^{\vee} \rtimes (T_{Q,n}^{\mathrm{sc}})^{\vee}[n]$$

so that $\phi_s : W_F \longrightarrow \overline{G}^{\#}$. In particular, the Satake isomorphism furnishes the notion of *Satake parameters relative to* s_0. A distinguished splitting s_0 thus allows one to define the notion of the L-function (with respect to s_0) associated to a representation R of the (enlarged) dual group $\overline{G}^{\#}$ (whose representations are easy to write down in terms of those of the connected Lie groups $\overline{G}^{\vee}$). Together with the Satake isomorphism, we thus have the notion of the "automorphic L-function" associated to s_0 and R.

In fact, if R factors through the adjoint group of $\overline{G}^{\#}$, the resulting L-function is independent of the choice of s_0; in particular the adjoint L-function is canonically defined. Another instance is the Langlands-Shahidi type L-functions, as we explain in §13.4. In the PhD thesis [**26**] of the second author, it was shown that the constant terms of Eisenstein series on BD covering groups are expressed in terms of these Langlands-Shahidi type L-functions, which extends the results of Langlands' famous monograph [**33**] to the nonlinear setting of this paper. One would like to show, as in the linear case, that these automorphic L-functions are nice, i.e., have meromoprhic continuation and satisfy a functional equation of the usual type. The meromorphic continuation of the Langlands-Shahidi type L-functions has been shown by the second author [**26**], but is open at this moment in general.

(vii) (Anomalies) We explain how the L-group formalism explains certain anomalies which have been empirically observed in the representation theory of certain covering groups. One is the fact that representations in a given L-packet of $\mathrm{Mp}(2n)$ can have different central characters. Another is the observation that when restricting an irreducible genuine representation of a Kazhdan-Patterson cover of $\mathrm{GL}(n)$ to its cover over $\mathrm{SL}(n)$, the irreducible summands need not occur in the same L-packet. These phenomena can be neatly explained in terms of the L-group.

(viii) (Langlands Functoriality) Finally, we examine certain instances of Langlands functoriality. The L-group formalism already gives a suggestion of the "endoscopic groups" of a BD covering group $\overline{G}$. Indeed, if we let $G_{Q,n}$ be the quasi-split linear algebraic group whose L-group is isomorphic to the L-group of $\overline{G}$. Then it is natural to think of the endoscopic groups of $G_{Q,n}$ as the endoscopic groups of $\overline{G}$. This should be correct to a first approximation, though (as one sees from the work of W.-W. Li for $\mathrm{Mp}(2n)$) the notion of isomorphism of endoscopic data has to be modified. In any case, one then expects instances of endoscopic transfer, at least on the level of weak liftings. One supporting evidence is that one has an isomorphism of Iwahori-Hecke algebras of $\overline{G}$ and $G_{Q,n}$, as formulated in Theorem 15.1.

Another example we examine is the case of base change. It turns out that it is not so automatic to formulate this notion on the L-group side. We achieve this by appealing to a result of E. Bender [**8**]. We then show in Theorem 14.3 that, for an extension K/F of local fields, base change for covering tori is obtained by pulling back via a lifting of the usual norm map $\mathscr{N}_T : \mathbb{T}(K) \longrightarrow \mathbb{T}(F)$ to the level of covers, i.e., using a commutative diagram:

$$\begin{array}{ccc} \overline{T}_K & \xrightarrow{\mathscr{N}_{\overline{T}}} & \overline{T}_F \\ \big\downarrow & & \big\downarrow \\ \mathbb{T}(K) & \xrightarrow{\mathscr{N}_T} & \mathbb{T}(F), \end{array}$$

with the map $\mathscr{N}_{\overline{T}}$ being constructed by the results of Bender [8]. These topics are contained in §14 and §15.

We conclude the paper with a number of examples, illustrating the above constructions and results, as well as highlighting a few basic questions which we feel are crucial for carrying the theory forward.

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2. Brylinski-Deligne Extensions

Let F be a field and $\mathbb{G}$ a connected reductive linear algebraic group over F. We shall assume that $\mathbb{G}$ is split over F in this paper. Fix a maximal split torus $\mathbb{T}$ contained in a Borel subgroup $\mathbb{B}$ of $\mathbb{G}$. Let $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ be the cocharacter lattice and $X = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$ the character lattice. Then one has the set of roots $\Phi \subset X$ and the set of coroots $\Phi^\vee \subset Y$ of $(\mathbb{G}, \mathbb{T})$ respectively. The Borel subgroup $\mathbb{B}$ determines the set of simple roots $\Delta \subset \Phi$ and the set of simple coroots $\Delta^\vee \in \Phi^\vee$. For each $\alpha \in \Phi$, one has the associated root subgroup $\mathbb{U}_\alpha \subset \mathbb{G}$ which is normalized by $\mathbb{T}$. We shall fix an épinglage or pinning for $(\mathbb{G}, \mathbb{T})$, so that for each $\alpha \in \Phi$, one has an isomorphism $x_\alpha : \mathbb{G}_a \longrightarrow \mathbb{U}_\alpha$.

Hence our initial data for this paper is a pinned connected split reductive group $(\mathbb{G}, \mathbb{T}, \mathbb{B}, x_\alpha)$ over F.

2.1. Multiplicative K_2-torsors. — The algebraic group $\mathbb{G}$ defines a sheaf of groups on the big Zariski site on $\mathrm{Spec}(F)$. Let $\mathbb{K}_2$ denote the sheaf of groups on $\mathrm{Spec}(F)$ associated to the K_2-group in Quillen's K-theory. Then a multiplicative K_2-torsor is an extension

$$1 \longrightarrow \mathbb{K}_2 \longrightarrow \overline{\mathbb{G}} \longrightarrow \mathbb{G} \longrightarrow 1$$

of sheaves of groups on $\mathrm{Spec}(F)$. We consider the category $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ of such extensions where the morphisms between objects are given by morphisms of extensions. Given two such central extensions, one can form their Baer sum: this equips $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ with the structure of a commutative Picard category.

In **[15]**, Brylinski and Deligne made a deep study of $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ and obtained an elegant classification of this category when $\mathbb{G}$ is a connected reductive group. We recall their results briefly in the case when $\mathbb{G}$ is split.

2.2. Split torus. — Suppose $\mathbb{T}$ is a split torus, with cocharacter lattice $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$ and character lattice $X = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m)$. Then we have:

Proposition 2.1. — *Let $\mathbb{T}$ be a split torus over F. The category* $\mathbf{CExt}(\mathbb{T}, \mathbb{K}_2)$ *is equivalent as a commutative Picard category (by an explicit functor) to the category whose objects are pairs $(Q, \mathcal{E})$, where*

- *Q is a $\mathbb{Z}$-valued quadratic form on Y, with associated symmetric bilinear form $B_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2)$;*
- *$\mathcal{E}$ is a central extension of groups*

$$1 \longrightarrow F^\times \longrightarrow \mathcal{E} \longrightarrow Y \longrightarrow 1$$

whose associated commutator map $[-,-] : Y \times Y \to F^\times$ is given by

$$[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}.$$

The set of morphisms between $(Q, \mathcal{E})$ and $(Q', \mathcal{E}')$ is empty unless $Q = Q'$, in which case it is given by the set of isomorphisms of extensions from $\mathcal{E}$ to $\mathcal{E}'$. The Picard structure is defined by

$$(Q, \mathcal{E}) + (Q', \mathcal{E}') = (Q + Q', \textit{Baer sum of } \mathcal{E} \textit{ and } \mathcal{E}').$$

Observe that the isomorphism class of the extension $\mathcal{E}$ is completely determined by the commutator map and hence by the quadratic form Q. The extension $\mathcal{E}$ is obtained from $\overline{\mathbb{T}}$ as follows. Let $F(\!(\tau)\!)$ denote the field of Laurent series in the variable τ over F. Then one has

$$1 \longrightarrow \mathbb{K}_2(F(\!(\tau)\!)) \longrightarrow \overline{\mathbb{T}}(F(\!(\tau)\!)) \longrightarrow \mathbb{T}(F(\!(\tau)\!)) = Y \otimes_{\mathbb{Z}} F(\!(\tau)\!)^\times \longrightarrow 1.$$

The map $y \mapsto y(\tau)$ defines a group homomorphism $Y \longrightarrow \mathbb{T}(F(\!(\tau)\!))$. Pulling back by this morphism and pushing out by the residue map

$$\mathrm{Res} : \mathbb{K}_2(F(\!(\tau)\!)) \longrightarrow \mathbb{K}_1(F) = F^\times$$

defined by

$$\mathrm{Res}(f,g) = (-1)^{v(f)\cdot v(g)} \cdot \left(\frac{f^{v(g)}}{g^{v(f)}}(0)\right),$$

one obtains the desired extension $\mathcal{E}$.

2.3. Simply-connected groups. — Suppose now that $\mathbb{G}$ is a split simply-connected semisimple group over F and recall that we have fixed the épinglage $(\mathbb{T}, \mathbb{B}, x_\alpha)$. Let $W = N(\mathbb{T})/\mathbb{T}$ be the corresponding Weyl group. Since $\mathbb{G}$ is simply-connected, the coroot lattice is equal to Y, so that the set of simple coroots $\Delta^\vee$ is a basis for Y.

Now we have:

Proposition 2.2. — *The category* $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ *is equivalent (as commutative Picard categories) to the category whose objects are W-invariant $\mathbb{Z}$-valued quadratic form Q on Y, and whose only morphisms are the identity morphisms on each object.*

As a result of this proposition, whenever we are given a quadratic form Q on Y, Q gives rise to a multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}_Q$ on $\mathbb{G}$, unique up to unique isomorphism, which may be pulled back to a multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{T}}_Q$ on $\mathbb{T}$ and hence gives rise to an extension $\mathcal{E}_Q$ of Y by $F^\times$. The automorphism group of the extension $\mathcal{E}_Q$ is $\mathrm{Hom}(Y, F^\times)$. Following [**15**, §11], one can rigidify $\mathcal{E}_Q$ by giving it an extra structure, as we now explain.

2.4. Rigidifying $\mathcal{E}_Q$. — We continue to assume that $\mathbb{G}$ is simply-connected. We have already fixed the épinglage $\{x_\alpha : \alpha \in \Phi\}$ for $\mathbb{G}$, so that

$$x_\alpha : \mathbb{G}_a \longrightarrow \mathbb{U}_\alpha \subset \mathbb{G}.$$

Indeed, one has an embedding

$$i_\alpha : \mathrm{SL}_2 \hookrightarrow \mathbb{G}$$

which restricts to $x_{\pm\alpha}$ on the upper and lower triangular subgroup of unipotent matrices. By [**15**], one has a canonical lifting

$$\tilde{x}_\alpha : \mathbb{G}_a \longrightarrow \overline{\mathbb{U}}_\alpha \subset \overline{\mathbb{G}}.$$

For $t \in \mathbb{G}_m$, we set

$$n_\alpha(t) = x_\alpha(t) \cdot x_{-\alpha}(-t^{-1}) \cdot x_\alpha(t) = i_\alpha \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \in N(\mathbb{T}_Q),$$

and

$$\tilde{n}_\alpha(t) = \tilde{x}_\alpha(t) \cdot \tilde{x}_{-\alpha}(-t^{-1}) \cdot \tilde{x}_\alpha(t) \in \overline{\mathbb{G}}_Q.$$

Then one has a map

$$s_\alpha : \mathbb{T}_\alpha := \alpha^\vee(\mathbb{G}_m) \longrightarrow \overline{\mathbb{T}}_Q$$

given by

$$\alpha^\vee(t) \mapsto \tilde{n}_\alpha(t) \cdot \tilde{n}_\alpha(-1).$$

This is a section of $\overline{\mathbb{G}}_Q$ over $\mathbb{T}_\alpha$, which is trivial at the identity element. The section s_α is useful in describing the natural conjugation action of $N(\mathbb{T}_Q)$ on $\overline{\mathbb{T}}_Q$. By [**15**, Prop. 11.3], one has the nice formula:

$$\tilde{n}_\alpha(1) \cdot \tilde{t} \cdot \tilde{n}_\alpha(1)^{-1} = \tilde{t} \cdot s_\alpha(\alpha^\vee(\alpha(t)^{-1})). \tag{2.3}$$

Moreover, the collection of sections $\{s_\alpha : \alpha \in \Delta\}$ provides a collection of elements $s_\alpha(\alpha^\vee(a)) \in \overline{\mathbb{T}}_Q$ with $a \in \mathbb{G}_m$, and $\overline{\mathbb{T}}_Q$ is generated by K_2 and the collection of $s_\alpha(\alpha^\vee(a))$.

Taking points in $F(\!(\tau)\!)$, we have the element

$$s_\alpha(\alpha^\vee(\tau)) \in \overline{\mathbb{T}}_Q(F(\!(\tau)\!)),$$

which gives rise (via the construction of $\mathcal{E}_Q$) to an element

$$s_Q(\alpha^\vee) \in \mathcal{E}_Q.$$

Then we rigidify $\mathcal{E}_Q$ by equipping it with the set $\{s_Q(\alpha^\vee) : \alpha^\vee \in \Delta^\vee\}$: there is a unique automorphism of $\mathcal{E}_Q$ which fixes all these elements.

In the following, we shall fix a choice of the data $(\overline{\mathbb{G}}_Q, \overline{\mathbb{T}}_Q, \mathcal{E}_Q)$ for each W-invariant quadratic form Q on Y when $\mathbb{G}$ is split and simply-connected. This is not a real choice, because any two choices are isomorphic by a unique isomorphism. The section s_Q constructed above provides $\mathcal{E}_Q$ with a system of generators: $\mathcal{E}_Q$ is generated by $s_Q(\alpha^\vee) \in \Delta^\vee$ and $a \in F^\times$ subject to the relations:

- $a \in F^\times$ is central;
- $[s_Q(\alpha^\vee), s_Q(\beta^\vee)] = (-1)^{B_Q(\alpha^\vee, \beta^\vee)}$ for $\alpha^\vee, \beta^\vee \in \Delta^\vee$.

2.5. General reductive groups. — Now let $\mathbb{G}$ be a split connected reductive group over F, with fixed épinglage $(\mathbb{T}, \mathbb{B}, x_\alpha)$. Let

$$i_{\mathrm{sc}} : Y^{\mathrm{sc}} := \mathbb{Z}[\Delta^\vee] \subset Y$$

be the inclusion of the coroot lattice Y^{sc} into Y, and let $X^{\mathrm{sc}} \subset X \otimes_{\mathbb{Z}} \mathbb{Q}$ be the dual lattice of Y^{sc}. Then the quadruple $(X^{\mathrm{sc}}, \Delta, Y^{\mathrm{sc}}, \Delta^\vee)$ is the root datum of the simply-connected cover $\mathbb{G}^{\mathrm{sc}}$ of the derived group of $\mathbb{G}$, and one has a natural map

$$q : \mathbb{G}^{\mathrm{sc}} \to \mathbb{G}.$$

Let $\mathbb{T}^{\mathrm{sc}}$ be the preimage of $\mathbb{T}$ in $\mathbb{G}^{\mathrm{sc}}$, so that one has a commutative diagram

$$\begin{tikzcd} \mathbb{T}^{\mathrm{sc}} \arrow[r] \arrow[d] & \mathbb{G}^{\mathrm{sc}} \arrow[d, "q"] \\ \mathbb{T} \arrow[r] & \mathbb{G}. \end{tikzcd}$$

The classification of the multiplicative $\mathbb{K}_2$-torsors on $\mathbb{G}$ is an amalgam of the two Propositions above. Given a multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ of $\mathbb{G}$, the above commutative diagram induces by pullbacks a commutative digram of multiplicative $\mathbb{K}_2$-torsors:

$$\begin{tikzcd} \overline{\mathbb{T}}^{\mathrm{sc}} \arrow[r] \arrow[d] & \overline{\mathbb{G}}^{\mathrm{sc}} \arrow[d, "q"] \\ \overline{\mathbb{T}} \arrow[r] & \overline{\mathbb{G}}. \end{tikzcd}$$

Then

- the multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{T}}$ gives a pair $(Q, \mathcal{E})$ by Proposition 2.1;
- the multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}^{\mathrm{sc}}$ corresponds to a quadratic form Q^{sc}, and it was shown in **[15]** that Q^{sc} is simply the restriction of Q to Y^{sc}.
- we have fixed the data $(\overline{\mathbb{G}}_{Q^{\mathrm{sc}}}, \overline{\mathbb{T}}_{Q^{\mathrm{sc}}}, \mathcal{E}_{Q^{\mathrm{sc}}})$ associated to Q^{sc}. Thus we have a canonical isomorphism

  $$f : \overline{\mathbb{G}}_{Q^{\mathrm{sc}}} \longrightarrow \overline{\mathbb{G}}^{\mathrm{sc}},$$

  restricting to an isomorphism

  $$f : \overline{\mathbb{T}}_{Q^{\mathrm{sc}}} \longrightarrow \overline{\mathbb{T}}^{\mathrm{sc}}$$

  which then induces an isomorphism

  $$f : \mathcal{E}_{Q^{\mathrm{sc}}} \longrightarrow \mathcal{E}^{\mathrm{sc}} = q^*(\mathcal{E}).$$

  This isomorphism is characterised as the unique one which sends the elements $s_{Q^{\mathrm{sc}}}(\alpha^\vee) \in \mathcal{E}_{Q^{\mathrm{sc}}}$ (for $\alpha^\vee \in \Delta^\vee$) to the corresponding elements $s(\alpha^\vee) \in q^*(\mathcal{E})$. In particular, we have a commutative diagram

  $$\begin{tikzcd} 1 \arrow[r] & F^\times \arrow[r] \arrow[d, equal] & \mathcal{E}_{Q^{\mathrm{sc}}} \arrow[r] \arrow[d, "f"] & Y^{\mathrm{sc}} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & F^\times \arrow[r] & q^*(\mathcal{E}) \arrow[r] & Y^{\mathrm{sc}} \arrow[r] & 1. \end{tikzcd}$$

We have thus attached to a multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ a triple $(Q, \mathcal{E}, f)$. Now we have:

Theorem 2.4. — *The category* $\mathbf{CExt}(\mathbb{G}, \mathbb{K}_2)$ *is equivalent (via the above construction) to the category* $\mathbf{BD}_{\mathbb{G}}$ *whose objects are triples* $(Q, \mathcal{E}, f)$*, where*

- $Q : Y \to \mathbb{Z}$ *is a* W*-invariant quadratic form;*
- $\mathcal{E}$ *is an extension of* Y *by* $F^\times$ *with commutator map* $[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}$*;*
- $f : \mathcal{E}_{Q^{\mathrm{sc}}} \cong q^*(\mathcal{E})$ *is an isomorphism of extensions of* Y^{sc} *by* $F^\times$*.*

The set of morphisms from $(Q, \mathcal{E}, f)$ *to* $(Q', \mathcal{E}', f')$ *is empty unless* $Q = Q'$*, in which case it consists of isomorphisms of extensions* $\phi : \mathcal{E} \longrightarrow \mathcal{E}'$ *such that* $f = f' \circ q^*(\phi)$*. In particular, the automorphism group of an object is* $\mathrm{Hom}(Y/Y^{\mathrm{sc}}, F^\times)$*.*

2.6. Bisectors and Incarnation. — While the above results give a nice classification of multiplicative $\mathbb{K}_2$-torsors over split reductive groups $\mathbb{G}$ over F, it is sometimes useful and even necessary to work with explicit cocycles for computation. The paper [**15**] does provide a category of nice algebraic cocycles, as explicated in [**70**], and one may replace the category of triples $(Q, \mathcal{E}, f)$ by a slightly simpler category with more direct connections to cocycles.

Extending the treatment in [**70**], we consider (not necessarily symmetric) $\mathbb{Z}$-valued bilinear forms D on Y satisfying:

$$D(y, y) = Q(y) \text{ for all } y \in Y,$$

so that

$$B_Q(y_1, y_2) = D(y_1, y_2) + D(y_2, y_1).$$

Such a D is called a bisector of Q. As shown in [**69**], for any Q, there exists an associated bisector.

We consider a category

$$\mathbf{Bis}_{\mathbb{G}} = \bigcup_Q \mathbf{Bis}_{\mathbb{G},Q}$$

where the full subcategory $\mathbf{Bis}_{\mathbb{G},Q}$ consists of pairs (D, η) where D is a bisector of Q and

$$\eta : Y^{\mathrm{sc}} \longrightarrow F^{\times}$$

is a group homomorphism. Given two pairs (D_1, η_1) and (D_2, η_2), the set of morphisms is the set of functions $\xi : Y \longrightarrow F^{\times}$ such that

(a) $\xi(y_1 + y_2) \cdot \xi(y_1)^{-1} \cdot \xi(y_2)^{-1} = (-1)^{D_1(y_1,y_2) - D_2(y_1,y_2)}$;
(b) $\xi(\alpha^{\vee}) = \eta_2(\alpha^{\vee})/\eta_1(\alpha^{\vee})$ for all $\alpha^{\vee} \in \Delta^{\vee}$.

and where the composition of morphisms is given by multiplication: $\xi_1 \circ \xi_2(y) = \xi_1(y) \cdot \xi_2(y)$. Note that in (b), ξ may not be a group homomorphism when restricted to Y^{sc}, but we are only requiring the identity in (b) to hold as functions of sets when restricted to $\Delta^{\vee}$. Moreover, as shown in [**70**], given two bisectors D_1 and D_2 of Q, one can always find ξ such that (a) holds. Thus, up to isomorphism, there is no loss of generality in fixing D (for a fixed Q).

Then it was shown in [**70**, §2.2] that there is an incarnation functor

$$\mathbf{Inc}_{\mathbb{G}} : \mathbf{Bis}_{\mathbb{G}} \longrightarrow \mathbf{BD}_{\mathbb{G}} \longrightarrow \mathbf{CExt}(\mathbb{G}, \mathbb{K}_2).$$

The second functor is a quasi-inverse to the functor in Theorem 2.4. On the level of objects, the first functor sends the pair (D, η) in $\mathbf{Bis}_{\mathbb{G},Q}$ to the triple $(Q, \mathcal{E}, f)$ defined as follows:

- $\mathcal{E} = Y \times F^{\times}$ with group law

$$(y_1, a_1) \cdot (y_2, a_2) = (y_1 + y_2, a_1 a_2 (-1)^{D(y_1,y_2)}).$$

- $f : \mathcal{E}_{Q^{\mathrm{sc}}} \to \mathcal{E}$ is given by

$$f(s_{Q^{\mathrm{sc}}}(\alpha^{\vee})) = (\alpha^{\vee}, \eta(\alpha^{\vee})) \quad \text{for any } \alpha \in \Delta.$$

It was shown in [**70**] that this functor is an essentially surjective functor. In fact, our definition of morphisms in $\mathbf{Bis}_{\mathbb{G}}$ differs slightly from that of [**70**]. With our version here, this functor is fully faithful as well so that **Inc** is in fact an equivalence of categories.

Moreover, one can choose **Inc** so that if $\overline{\mathbb{G}}$ is the BD extension corresponding to (D,η), then $\overline{\mathbb{T}}$ can be described explicitly using D. Namely, if

$$D = \sum_i x_1^i \otimes x_2^i \in X \otimes X,$$

then one may regard $\overline{\mathbb{T}} = \mathbb{T} \times K_2$ with group law:

$$(t_1, 1) \cdot (t_2, 1) = (t_1 t_2, \prod_i \{x_1^i(t_1), x_2^i(t_2)\}).$$

The associated extension $\mathcal{E}$ is then described as above in terms of the bisector D. Further, we have the following explicit description for the section $q \circ s_\alpha$:

Proposition 2.5. — *In terms of the above realisation of $\overline{T}$ by D, the section*

$$q \circ s_\alpha : \alpha^\vee(\mathbb{G}_m) \longrightarrow \overline{\mathbb{T}} \quad \textit{for } \alpha \in \Delta$$

is given by

$$q \circ s(\alpha^\vee(a)) = (\alpha^\vee(a), \{\eta(\alpha^\vee), a\}).$$

Proof. — Suppose that $\overline{\mathbb{G}}$ is incarnated by (D,η) and corresponds to the triple $(Q, \mathcal{E}, f)$ where $\mathcal{E}$ and f are described by (D,η) as above. For fixed $\alpha \in \Delta$, one may write

$$q \circ s_\alpha(a) = (\alpha^\vee(a), \aleph_\alpha(a)),$$

where $\aleph_\alpha \in \mathrm{Hom}_{\mathrm{Zar}}(\mathbb{G}_m, \mathbb{K}_2)$ is a homomorphism of sheaves of abelian groups for the big Zariski site. By [**15**, §3.7-3.8], or in more details [**9**, Thm. 1.1], we have

$$F^\times = \mathbb{K}_1(F) \cong \mathrm{Hom}_{\mathrm{Zar}}(\mathbb{G}_m, \mathbb{K}_2)$$

where the isomorphism is given by

$$b \mapsto (a \mapsto \{b, a\}).$$

Hence, there exists some $\lambda_\alpha \in F^\times$ such that

$$\aleph_\alpha(a) = \{\lambda_\alpha, a\} \quad \text{for } a \in \mathbb{G}_m.$$

From this, it follows from the definition of f that

$$f(s_{Q^{\mathrm{sc}}}(\alpha^\vee)) = (\alpha^\vee, \lambda_\alpha) \in \mathcal{E}.$$

By hypothesis, however, we have:

$$f(s_{Q^{\mathrm{sc}}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee)).$$

Hence we deduce that

$$\lambda_\alpha = \eta(\alpha^\vee)$$

and thus

$$q \circ s_\alpha(a) = (\alpha^\vee(a), \aleph_\alpha(a)) = (\alpha^\vee(a), \{\eta(\alpha^\vee), a\}),$$

as desired. □

Thus, the category $\mathbf{Bis}_{\mathbb{G}}$ provides a particularly nice and explicit family of cocycles for BD extensions, and the essential surjectivity of **Inc** says that every BD extension possesses such a cocycle, at least on the maximal torus $\mathbb{T}$. This will be useful for computations.

2.7. Fair bisectors. — In [**70**], Weissman singled out a property of bisectors which he called *fair*. By definition, a bisector D is fair if it satisfies the following:

- for any $\alpha \in \Delta$ such that $Q(\alpha^\vee) \equiv 0 \mod 2$, $D(\alpha^\vee, y) \equiv D(y, \alpha^\vee) \equiv 0 \mod 2$ for all $y \in Y$.

Weissman showed that for any Q, $\mathbf{Bis}_{\mathbb{G},Q}$ contains a fair bisector. We shall henceforth fix a fair bisector for a given Q. The value of fairness will be apparent later on.

At the moment, we simply note that the objects $(D, 1) \in \mathbf{Bis}_{\mathbb{G},Q}$ with D fair and η the trivial homomorphism are quite special (as we shall see). Thus, we have a distinguished class of multiplicative $\mathbb{K}_2$-torsors with invariant Q.

When $Q = 0$, for example, the bisector $D = 0$ is fair, and $(D, 1)$ gives the split extension $\mathbb{G} \times \mathbb{K}_2$. In some sense, the $\mathbb{K}_2$-torsor with invariants $(D, 1)$ should be regarded as "closest to being a split extension among those with invariants (D, η)". As we shall illustrate in the rest of this section, the general BD extensions can often be described in terms of these distinguished BD extensions.

2.8. The case $Q = 0$. — Let us consider the example when $Q = 0$. Then we may take the bisector $D = 0$ and regard the objects of $\mathbf{Bis}_Q$ as the set of homomorphisms $\eta : Y^{\mathrm{sc}} \to F^\times$. Let $\overline{\mathbb{G}}_\eta$ be the corresponding multiplicative $\mathbb{K}_2$-torsor on $\mathbb{G}$. Then $\overline{\mathbb{G}}_{\eta_1}$ and $\overline{\mathbb{G}}_{\eta_2}$ are isomorphic precisely when η_1/η_2 can be extended to a homomorphism of Y to $F^\times$.

How can we characterize the distinguished BD-extension in $\mathbf{Bis}_Q$ using the BD data $(Q, \mathcal{E}, f)$ when $D = 0$? Since $D = 0$, $\mathcal{E}$ is an abelian group and hence is a split extension: $\mathcal{E} = Y \times F^\times$. Each $\eta \in \mathrm{Hom}(Y^{\mathrm{sc}}, F^\times)$ gives a map

$$f_\eta : \mathcal{E}_{Q^{\mathrm{sc}}} \longrightarrow \mathcal{E}$$

defined by

$$f_\eta(s_{Q^{\mathrm{sc}}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee)) \quad \text{for } \alpha \in \Delta.$$

Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, F^\times)$ to the short exact sequence $Y^{\mathrm{sc}} \to Y \to Y/Y^{\mathrm{sc}}$, we obtain as part of the long exact sequence:

$$\mathrm{Hom}_{\mathbb{Z}}(Y, F^\times) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(Y^{\mathrm{sc}}, F^\times) \xrightarrow{\ \delta\ } \mathrm{Ext}^1_{\mathbb{Z}}(Y/Y^{\mathrm{sc}}, F^\times) \longrightarrow 0.$$

For any $\eta \in \mathrm{Hom}(Y^{\mathrm{sc}}, F^\times)$, its image under δ is just the push-out of $Y^{\mathrm{sc}} \to Y \to Y/Y^{\mathrm{sc}}$ via η:

$$1 \longrightarrow F^\times \longrightarrow (Y \times F^\times)/f_\eta(Y^{\mathrm{sc}}) \longrightarrow Y/Y^{\mathrm{sc}} \longrightarrow 1.$$

The long exact sequence thus implies that this extension is split precisely when η is equivalent to 1. This gives a way of characterizing the distinguished isomorphism class in $\mathbf{Bis}_Q$ when $D = 0$.

How can we construct the other BD extensions in $\mathbf{Bis}_Q$ when $D = 0$? We assume further that $\mathbb{G}$ is semisimple and let $q : \mathbb{G}^{\mathrm{sc}} \longrightarrow \mathbb{G}$ be the natural isogeny with kernel $Z = \mathrm{Tor}_{\mathbb{Z}}(Y/Y^{\mathrm{sc}}, \mathbb{G}_m) \hookrightarrow \mathbb{T}^{\mathrm{sc}} = Y^{\mathrm{sc}} \otimes_{\mathbb{Z}} \mathbb{G}_m$. Then $q^*(\overline{\mathbb{G}}_\eta)$ is isomorphic to the split extension $\mathbb{G}^{\mathrm{sc}} \times \mathbb{K}_2$ by a unique isomorphism. We may thus construct $\overline{\mathbb{G}}_\eta$ by starting with $\mathbb{G}^{\mathrm{sc}} \times \mathbb{K}_2$ and then considering a quotient of this by a suitable embedding $Z \hookrightarrow \mathbb{G}^{\mathrm{sc}} \times \mathbb{K}_2$. For this, we note that η induces a map

$$i_\eta : Z = \mathrm{Tor}_{\mathbb{Z}}(Y/Y^{\mathrm{sc}}, \mathbb{G}_m) \hookrightarrow Y^{\mathrm{sc}} \otimes_{\mathbb{Z}} \mathbb{G}_m \longrightarrow F^\times \otimes_{\mathbb{Z}} \mathbb{G}_m \longrightarrow \mathbb{K}_2.$$

Then we have

$$\overline{\mathbb{G}}_\eta = (\overline{\mathbb{G}} \times \mathbb{K}_2)/\{(z, i_\eta(z)) : z \in Z\}.$$

2.9. z-extensions. — We consider another example which will play a crucial role later on, namely when Y/Y^{sc} is a free abelian group. In this case, for any $(D, \eta) \in \mathbf{Bis}_Q$, η can be extended to a homomorphism of Y, and so any two (D, η_1) and (D, η_2) are isomorphic. This means that there is a unique isomorphism class of objects in $\mathbf{Bis}_Q$, just like the case when $\mathbb{G}$ is simply-connected (where $Y = Y^{\mathrm{sc}}$). However, the automorphism group $\mathrm{Hom}(Y/Y^{\mathrm{sc}}, F^\times)$ of an object is not trivial (unless $Y = Y^{\mathrm{sc}}$).

As we shall see later, some questions about a BD extension can be reduced to the case when Y/Y^{sc} is free. This is achieved via the consideration of $z-$extensions. More precisely, it is not hard to see that given any $\mathbb{G}$, one can find a central extension of connected reductive groups:

$$1 \longrightarrow Z \longrightarrow \mathbb{H} \xrightarrow{\pi} \mathbb{G} \longrightarrow 1,$$

where

- $\mathbb{H}$ is such that $Y_{\mathbb{H}}/Y_{\mathbb{H}}^{\mathrm{sc}}$ is free;
- $\pi_* : Y_{\mathbb{H}}^{\mathrm{sc}} \longrightarrow Y_{\mathbb{G}}^{\mathrm{sc}}$ is an isomorphism;
- Z is a split torus which is central in $\mathbb{H}$.

Such an extension is called a z-extension.

Given such a z-extension, and a BD extension $\overline{G}_\eta$ with invariant (D, η), we obtain a BD extension $\mathbb{H}_\eta := \pi^*(\overline{G}_\eta)$ on $\mathbb{H}$ with BD invariant $(D \circ \pi, \eta \circ \pi)$, so that one has

$$1 \longrightarrow Z_\eta = Z \longrightarrow \overline{\mathbb{H}}_\eta \longrightarrow \overline{\mathbb{G}}_\eta \longrightarrow 1.$$

By our discussion above, we may choose an isomorphism $\xi : \overline{\mathbb{H}}_1 \cong \overline{\mathbb{H}}_\eta$. Thus, via ξ, we have

$$1 \longrightarrow \xi^{-1}(Z_\eta) \longrightarrow \overline{\mathbb{H}}_1 \longrightarrow \overline{\mathbb{G}}_\eta \longrightarrow 1.$$

This shows that for any given bisector D, all the BD extensions $\overline{\mathbb{G}}_\eta$ can be described as the quotient of a fixed BD extension $\overline{\mathbb{H}}_1$ (with $\eta = 1$) by a suitable splitting of the split torus Z.

2.10. Running example. — Let us illustrate the above discussion using a simple example, where

$$\mathbb{G} = \mathrm{PGL}_2, \quad Y = \mathbb{Z} \supset Y^{\mathrm{sc}} = 2\mathbb{Z} \quad \text{and} \quad D = 0.$$

Then we take the z-extension to be

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{GL}_2 \xrightarrow{\ \pi\ } \mathrm{PGL}_2 \longrightarrow 1.$$

The distinguished BD-extension with $\eta = 1$ is the split extension $\overline{\mathbb{G}}_1 = \mathrm{PGL}_2 \times \mathbb{K}_2$ and its pullback to GL_2 is the split extension $\overline{\mathbb{H}} = \pi^*(\overline{\mathbb{G}}_1) = \mathrm{GL}_2 \times \mathbb{K}_2$. For any $\eta \in \mathrm{Hom}(Y^{\mathrm{sc}}, F^\times) \cong F^\times$, the BD extension $\overline{\mathbb{G}}_\eta$ can then be described as

$$\overline{\mathbb{G}}_\eta = (\mathrm{GL}_2 \times \mathbb{K}_2)/\{(z, i_\eta(z)) : z \in \mathbb{G}_m\},$$

where

$$i_\eta(z) = \{\eta, z\} \in \mathbb{K}_2.$$

This is a rather trivial family of BD extensions since their pullback to $\mathbb{G}^{\mathrm{sc}} = \mathrm{SL}_2$ is split. Nonetheless, we shall use them as our running examples, as they already exhibit various properties we want to highlight in this paper.

3. Topological Covering Groups

In this section, we will pass from the algebro-geometric world of multiplicative $\mathbb{K}_2$-torsors to the world of topological central extensions. Let F be a local field. If F is nonarchimedean, let $\mathcal{O}$ denote its ring of integers with residue field κ.

3.1. BD covering groups. — Start with a multiplicative $\mathbb{K}_2$-torsor $\mathbb{K}_2 \to \overline{\mathbb{G}} \to \mathbb{G}$, with associated BD data $(Q, \mathcal{E}, f)$ or bisector data (D, η). By taking F-points, we obtain (since $H^1(F, \mathbb{K}_2) = 0$) a short exact sequence of abstract groups

$$1 \longrightarrow \mathbb{K}_2(F) \longrightarrow \overline{\mathbb{G}}(F) \longrightarrow G = \mathbb{G}(F) \longrightarrow 1.$$

Now let $\mu(F)$ denote the set of roots of unity contained in the local field $F \neq \mathbb{C}$; when $F = \mathbb{C}$, we let $\mu(F)$ be the trivial group. Then the Hilbert symbol gives a map

$$(-,-)_F : \mathbb{K}_2(F) \longrightarrow \mu(F).$$

For any n dividing $\#\mu(F)$, one has the n-th Hilbert symbol

$$(-,-)_n = (-,-)_F^{\#\mu(F)/n} : \mathbb{K}_2(F) \longrightarrow \mu_n(F).$$

Pushing the above exact sequence out by the n-th Hilbert symbol, one obtains a short exact sequence of locally compact topological groups

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{G} \longrightarrow G \longrightarrow 1.$$

We shall call this the BD covering group associated to the BD data $(Q, \mathcal{E}, f, n)$, or to the bisector data (D, η, n).

Since we are considering degree n covers, it will be useful to refine certain notions taking into account the extra data n:

– for a bisector data (D, η), we write η_n for the composite

$$\eta_n : Y^{\mathrm{sc}} \longrightarrow F^\times \longrightarrow F^\times / F^{\times n};$$

– with D fixed, we say that η_n and η'_n are equivalent if η_n/η'_n extends to a homomorphism $Y \longrightarrow F^\times/F^{\times n}$.

3.2. Canonical unipotent section. — Because a BD extension is uniquely split over any unipotent subgroup, one has unique splittings:

$$\tilde{x}_\alpha : F \longrightarrow \overline{U}_\alpha \quad \text{for each } \alpha \in \Phi.$$

Indeed, as shown in [**47**, Appendix I] and [**39**], there is a unique section

$$i : \{\text{all unipotent elements of } G\} \longrightarrow \overline{G},$$

satisfying:

– for each unipotent subgroup $\mathbb{U} \subset \mathbb{G}$, the restriction of i to $U = \mathbb{U}(F)$ is a group homomorphism;
– the map i is G-equivariant.

For example, for each $\alpha \in \Phi$, we have seen that there is a homomorphism

$$\tilde{x}_\alpha : \mathbb{G}_a \longrightarrow \overline{\mathbb{G}}$$

which induces a homomorphism

$$\tilde{x}_\alpha : F \longrightarrow \overline{G}$$

lifting the inclusion $x_\alpha : F \hookrightarrow G$. Then one has $\tilde{x}_\alpha = i \circ x_\alpha$.

3.3. Covering torus $\overline{T}$. — We may consider the pullback of $\overline{G}$ to the maximal split torus T:

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{T} \longrightarrow T \longrightarrow 1.$$

As we observed in the last section, the bisector D furnishes a cocycle for the multiplicative $\mathbb{K}_2$-torsor on $\mathbb{T}$. Thus the covering torus $\overline{T}$ also inherits a nice cocycle, giving us a rather concrete description of $\overline{T}$.

More precisely, $\overline{T} = T \times_D \mu_n(F)$ is generated by elements $\zeta \in \mu_n(F)$ and $y(a)$, for $y \in Y$ and $a \in F^\times$, subject to the relations:

– the elements ζ are central;
– $[y_1(a), y_2(b)] = (a,b)_n^{B_Q(y_1,y_2)}$ for all $y_1, y_2 \in Y$ and $a, b \in F^\times$;
– $y_1(a) \cdot y_2(a) = (y_1 + y_2)(a) \cdot (a,a)_n^{D(y_1,y_2)}$;
– $y(a) \cdot y(b) = y(ab) \cdot (a,b)_n^{Q(y)}$.

As the second relation shows, $\overline{T}$ is not necessarily an abelian group. Moreover, the sections $q \circ s_\alpha$ for $\alpha \in \Delta$ take the form

$$q \circ s_\alpha(\alpha^\vee(a)) = \alpha^\vee(a) \cdot (\eta(\alpha^\vee), a)_n.$$

3.4. The torus $T_{Q,n}$. — The center $Z(\overline{T})$ of $\overline{T}$ is generated by $\mu_n(F)$ and those elements $y(a)$ such that
$$B_Q(y,z) \in n\mathbb{Z} \quad \text{for all } z \in Y.$$
Thus, we define:
$$Y_{Q,n} := Y \cap nY^*,$$
where $Y^* \subset Y \otimes_{\mathbb{Z}} \mathbb{Q}$ is the dual lattice of Y relative to B_Q. Then the center of $\overline{T}$ is generated by $\mu_n(F)$ and the elements $y(a)$ for all $y \in Y_{Q,n}$ and $a \in F^\times$. It is clear that $nY \subset Y_{Q,n}$.

Let $\mathbb{T}_{Q,n}$ be the split torus with cocharacter group $Y_{Q,n}$ and $T_{Q,n} := \mathbb{T}_{Q,n}(F)$. The inclusion $Y_{Q,n} \hookrightarrow Y$ gives an isogeny of tori
$$h : T_{Q,n} \longrightarrow T.$$
We may pullback the covering $\overline{T}$ using h, thus obtaining a covering torus $\overline{T}_{Q,n}$:
$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{T}_{Q,n} \longrightarrow T_{Q,n} \longrightarrow 1.$$
Then $\overline{T}_{Q,n}$ is generated by $\zeta \in \mu_n(F)$ and elements $y(a)$ with $y \in Y_{Q,n}$ with the same relations as those for $\overline{T}$. However, the second relation now becomes
$$[y_1(a), y_2(b)] = 1, \quad \text{for all } y_1, y_2 \in Y_{Q,n} \text{ and } a, b \in F^\times.$$
Thus $\overline{T}_{Q,n}$ is an abelian group. Moreover, it follows from the definition of pullbacks that there is a canonical homomorphism $\mathrm{Ker}(h) \to \overline{T}_{Q,n}$, so that one has a short exact sequence of topological groups
$$1 \longrightarrow \mathrm{Ker}(h) \longrightarrow \overline{T}_{Q,n} \longrightarrow Z(\overline{T}) \longrightarrow 1.$$
In particular, to give a character of $Z(\overline{T})$ is to give a character of $\overline{T}_{Q,n}$ trivial on the subgroup $\mathrm{Ker}(h)$.

3.5. The kernel of h. — We need to have a better handle of $\mathrm{Ker}(h)$. The inclusions $nY \longrightarrow Y_{Q,n} \longrightarrow Y$ give rise to isogenies
$$T \xrightarrow{\ g\ } T_{Q,n} \xrightarrow{\ h\ } T$$
so that $h \circ g$ is the n-power map on T. We have:

Lemma 3.1. — *The kernel of h is contained in the image of g. Indeed,* $\mathrm{Ker}(h) = g(T[n])$.

Proof. — By the elementary divisor theorem, we may pick a basis $\{e_i\}$ of Y such that a basis of $Y_{Q,n}$ is given by $\{k_i e_i\}$ for some positive integers k_i. Such bases allow us to identify the maps
$$T = (F^\times)^r \xrightarrow{\ g\ } T_{Q,n} = (F^\times)^r \xrightarrow{\ h\ } T = (F^\times)^r$$
explicitly as
$$g(t_i) = (t_i^{n/k_i}) \quad \text{and} \quad h(t_i) = (t_i^{k_i}).$$
Thus,
$$\mathrm{Ker}(h) = \{(\zeta_i) : \zeta_i^{k_i} = 1\} = g(T[n]) = \{(\zeta_i^{n/k_i}) : \zeta_i^n = 1\}. \qquad \square$$

Using the generators and relations for $\overline{T}_{Q,n}$, it is easy to see that the map

$$y(a) \mapsto (ny)(a) \in \overline{T}_{Q,n}$$

gives a group homomorphism $\tilde{g} : T \longrightarrow \overline{T}_{Q,n}$ lifting g and whose image contains the canonical image of $\mathrm{Ker}(h)$ in $\overline{T}_{Q,n}$. The lemma implies that a character of $\overline{T}_{Q,n}$ trivial on the image of $\tilde{g}$ necessarily factors through to a character of $Z(\overline{T})$.

4. Tame Case

Suppose now that F is a p-adic field with residue field κ. Assume that p does not divide n: we shall call this the tame case. The main question we want to consider in this section is whether a tame BD cover $\overline{G}$ necessarily splits over a hyperspecial maximal compact subgroup K of G. Note that the $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ over F may not be defined over $\mathcal{O}_F$. Otherwise the splitting of K into $\overline{G}$ is canonical, as discussed in [**73**, §7].

Since we have fixed a Chevalley system of épinglage for $\mathbb{G}$, we have its associated maximal compact subgroup K generated by $x_\alpha(\mathcal{O}_F)$ for all $\alpha \in \Phi$ and the maximal compact subgroup $\mathbb{T}(\mathcal{O}) = Y \otimes_{\mathbb{Z}} \mathcal{O}^\times$ of T. In particular $K = \underline{\mathbb{G}}(\mathcal{O})$ for a smooth reductive group $\underline{\mathbb{G}}$ over $\mathcal{O}$. To ease notation, we shall simply write $\mathbb{G}$ for $\underline{\mathbb{G}}$ in what follows. Let $\mathbb{G}_\kappa$ denote the special fiber $\mathbb{G} \times_{\mathcal{O}} \kappa$ of $\mathbb{G}$. One has a natural reduction map

$$\mathbb{G}(\mathcal{O}) \longrightarrow G_\kappa := \mathbb{G}_\kappa(\kappa),$$

whose kernel is a pro-p group. Restricting the BD cover to K, one has a topological central extension

$$1 \longrightarrow \mu_n \longrightarrow \overline{\mathbb{G}}(\mathcal{O}) \longrightarrow \mathbb{G}(\mathcal{O}) \longrightarrow 1.$$

Here, observe that we have abused notation and write $\overline{\mathbb{G}}(\mathcal{O})$ for the inverse image of $\mathbb{G}(\mathcal{O})$ in $\overline{G}$. We would like to determine if this extension splits.

4.1. The tame extension. — All extensions in the tame case arise in the following way from the multiplicative $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$. The prime-to-p part of $\mu(F)$ is naturally isomorphic to $\kappa^\times$, and there is a tame symbol $\mathbb{K}_2(F) \longrightarrow \kappa^\times$ defined by

$$\{a, b\} \mapsto \text{ the image of } (-1)^{\mathrm{ord}(a)\cdot\mathrm{ord}(b)} \cdot \frac{a^{\mathrm{ord}(b)}}{b^{\mathrm{ord}(a)}} \text{ in } \kappa^\times.$$

Pushing out by this tame symbol gives the tame extension

$$1 \longrightarrow \kappa^\times \longrightarrow \overline{G}^{\mathrm{tame}} \longrightarrow G \longrightarrow 1.$$

Hence any degree n BD extension $\overline{G}$ with $(n, p) = 1$ is obtained as a pushout of $\overline{G}^{\mathrm{tame}}$.

4.2. Residual extension. — We shall consider first the case of the tame extension $\overline{G}^{\mathrm{tame}}$ so that $n = \#\kappa^{\times}$. It was shown in [**15**, §12] and [**67**] that there is an extension of reductive algebraic groups over κ:

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \tilde{\mathbb{G}}_{\kappa} \longrightarrow \mathbb{G}_{\kappa} \longrightarrow 1$$

with the following property:

– for any unramified extension F' of F with ring of integers $\mathcal{O}'$ and residue field κ', the tame extension

$$1 \longrightarrow \kappa'^{\times} \longrightarrow \overline{\mathbb{G}}(\mathcal{O}') \longrightarrow \underline{\mathbb{G}}(\mathcal{O}') \longrightarrow 1$$

is the pullback of the extension

$$1 \longrightarrow \mathbb{G}_m(\kappa') = \kappa'^{\times} \longrightarrow \tilde{G}_{\kappa'} = \tilde{\mathbb{G}}_{\kappa}(\kappa') \longrightarrow G_{\kappa'} = \mathbb{G}_{\kappa}(\kappa') \longrightarrow 1$$

with respect to the reduction map $\mathbb{G}(\mathcal{O}') \longrightarrow G_{\kappa'}$.

We call this extension of algebraic groups over κ the residual extension.

4.3. Classification. — In [**67**], algebraic extensions of $\mathbb{G}_{\kappa}$ by $\mathbb{G}_m$ were classified in terms of enhanced root theoretic data similar in spirit to (but simpler than) the BD data. We give a sketch in the case when $\mathbb{G}_{\kappa}$ is split. Then such extensions are classified by the category of pairs $(\mathcal{E}_{\kappa}, f_{\kappa})$ with

– $1 \to \mathbb{Z} \to \mathcal{E}_{\kappa} \to Y \to 1$ is an extension of free $\mathbb{Z}$-modules;
– $f_{\kappa} : Y^{\mathrm{sc}} \longrightarrow \mathcal{E}_{\kappa}$ is a splitting of $\mathcal{E}_{\kappa}$ over Y^{sc}.

Moreover, the extension $\tilde{\mathbb{G}}_{\kappa}$ in question is split if and only if the map f_{κ} can be extended to a splitting $Y \longrightarrow \mathcal{E}_{\kappa}$, or equivalently, if and only if the extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E}_{\kappa}/f_{\kappa}(Y^{\mathrm{sc}}) \longrightarrow Y/Y^{\mathrm{sc}} \longrightarrow 1$$

is split. This holds for example if Y/Y^{sc} is free. In particular, if $\mathbb{G} = \mathbb{G}^{\mathrm{sc}}$ is simply connected, then $\tilde{\mathbb{G}}_{\kappa}$ is split and the splitting is unique.

Given an extension $\tilde{\mathbb{G}}_{\kappa}$, one obtains the above two data as follows. If $\mathbb{T}_{\kappa}$ is a maximal split torus of $\mathbb{G}_{\kappa}$, with preimage $\tilde{\mathbb{T}}_{\kappa}$, then the cocharacter lattice of $\tilde{\mathbb{T}}_{\kappa}$ gives the extension $\mathcal{E}_{\kappa}$. The pullback of $\tilde{\mathbb{G}}_{\kappa}$ to $\mathbb{G}_{\kappa}^{\mathrm{sc}}$ is canonically split. On restricting this canonical splitting to the maximal split torus $\mathbb{T}_{\kappa}^{\mathrm{sc}}$ (which is the pullback of $\mathbb{T}_{\kappa}$), one obtains the splitting f_{κ} on the level of coroot lattices.

4.4. Splitting of $\overline{K}$. — Since the kernel of the reduction map is a pro-p group, the set of splittings of the topological extension $\overline{K} = \overline{\mathbb{G}}(\mathcal{O})$ is in bijection with those of the abstract extension $\tilde{G}_{\kappa}$. Further, a splitting of the residual extension $\tilde{\mathbb{G}}_{\kappa}$ gives rise to a splitting of $\tilde{G}_{\kappa}$ and thus of $\overline{\mathbb{G}}(\mathcal{O})$. We will investigate the existence of splittings for $\tilde{\mathbb{G}}_{\kappa}$: they give rise to splittings of $\overline{\mathbb{G}}(\mathcal{O})$ of "algebraic origin".

For example, when $\mathbb{G} = \mathbb{G}^{\mathrm{sc}}$ is simply-connected, the unique splitting of the residual extension $\tilde{\mathbb{G}}_{\kappa}$ gives rise to a unique compatible system of splittings of $\mathbb{G}(\mathcal{O}')$ for all unramified extensions $\mathcal{O}'$ of $\mathcal{O}$. Indeed, one has a natural bijection

$$\{\text{splittings of residual extension } \tilde{\mathbb{G}}_{\kappa}\} \longleftrightarrow \{\text{compatible system of splittings of } \mathbb{G}(\mathcal{O}')\}.$$

4.5. Determining $\tilde{\mathbb{G}}_\kappa$. — One can now figure out the residual extension $\tilde{\mathbb{G}}_\kappa$ obtained from a BD extension $\overline{\mathbb{G}}$ with associated BD data $(Q, \mathscr{E}, f)$.

Proposition 4.1. — *If $\overline{\mathbb{G}}$ has BD data $(Q, \mathscr{E}, f)$, then the associated data $(\mathscr{E}_\kappa, f_\kappa)$ for the residual extension $\tilde{\mathbb{G}}_\kappa$ is obtained as follows:*

- *$\mathscr{E}_\kappa$ is the pushout of $\mathscr{E}$ by $-\mathrm{ord} : F^\times \to \mathbb{Z}$, where* ord *is the valuation map;*
- *f_κ is deduced from the associated map $(-\mathrm{ord})_* \circ f : \mathscr{E}_{Q^{\mathrm{sc}}} \to \mathscr{E} \to \mathscr{E}_\kappa$:*

$$f_\kappa(\alpha^\vee) = (-\mathrm{ord})_*(f(s_{Q^{\mathrm{sc}}}(\alpha^\vee))) \quad \textit{for } \alpha \in \Delta.$$

Proof. — This question is systematically and more elegantly addressed in [**71**], and we shall give a more ad hoc argument here. We assume that $\overline{\mathbb{G}}$ is incarnated by (D, η) for concreteness. As we discussed above, for any unramified extension $\mathcal{O}'$ of $\mathcal{O}$, there is a commutative diagram of extensions:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \kappa'^\times & \longrightarrow & \overline{\mathbb{G}}(\mathcal{O}') & \longrightarrow & \mathbb{G}(\mathcal{O}') & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{G}_m(\kappa') & \longrightarrow & \tilde{\mathbb{G}}_\kappa(\kappa') & \longrightarrow & \mathbb{G}_\kappa(\kappa') & \longrightarrow & 1,
\end{array}$$

and our goal is to determine the invariants $(\mathscr{E}_\kappa, f_\kappa)$ for the extension $\tilde{\mathbb{G}}_\kappa$.

Now we note:

In terms of the bisector data (D, η), one has $\mathscr{E} = Y \times_D F^\times$ and $f(s_{Q^{\mathrm{sc}}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee))$ for $\alpha \in \Delta$. On pushing out by the valuation map $-\mathrm{ord}$, one sees that one needs to show:

$$\mathscr{E}_\kappa = Y \times \mathbb{Z} \quad \text{(direct product of groups)}$$

and

$$f_\kappa(\alpha^\vee) = (\alpha^\vee, -\mathrm{ord}(\eta(\alpha^\vee))) \quad \text{for } \alpha \in \Delta.$$

(a) From the construction in [**15**, §12.11], one has a commutative diagram of extensions and splittings (s_κ of $\tilde{\mathbb{G}}^{\mathrm{sc}}_\kappa$) over κ:

$$\begin{array}{ccccc}
\mathbb{T}^{\mathrm{sc}}_\kappa & \xrightarrow{s_\kappa} & \tilde{\mathbb{T}}^{\mathrm{sc}}_\kappa & \longrightarrow & \tilde{\mathbb{T}}_\kappa \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{G}^{\mathrm{sc}}_\kappa & \xrightarrow{s_\kappa} & \tilde{\mathbb{G}}^{\mathrm{sc}}_\kappa & \longrightarrow & \tilde{\mathbb{G}}_\kappa.
\end{array}$$

pulling back to a compatible system

$$\begin{array}{ccccc}
\mathbb{T}^{\mathrm{sc}}(\mathcal{O}') & \xrightarrow{s_{\mathcal{O}'}} & \overline{\mathbb{T}}^{\mathrm{sc}}(\mathcal{O}') & \longrightarrow & \overline{\mathbb{T}}(\mathcal{O}') \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{G}^{\mathrm{sc}}(\mathcal{O}') & \xrightarrow{s_{\mathcal{O}'}} & \overline{\mathbb{G}}^{\mathrm{sc}}(\mathcal{O}') & \longrightarrow & \overline{\mathbb{G}}(\mathcal{O}').
\end{array}$$

(b) Using the description of $\overline{\mathbb{T}}$ in terms of (D,η), one sees immediately that $\overline{\mathbb{T}}(\mathcal{O}') = \mathbb{T}(\mathcal{O}') \times \kappa'^{\times}$ (a direct product of groups), since the tame symbol is trivial on $\mathcal{O}'^{\times} \times \mathcal{O}'^{\times}$. Thus $\tilde{\mathbb{T}}_\kappa = \mathbb{T}_\kappa \times \mathbb{G}_m$. From this, one deduces that

$$\mathcal{E}_\kappa = Y \times \mathbb{Z} \quad \text{(as groups)}.$$

(c) The invariant f_κ is deduced from the composite map

$$\tilde{f}_\kappa : \mathbb{T}^{\mathrm{sc}}_\kappa \xrightarrow{\ s_\kappa\ } \tilde{\mathbb{T}}^{\mathrm{sc}}_\kappa \longrightarrow \tilde{\mathbb{T}}_\kappa = \mathbb{T}_\kappa \times \mathbb{G}_m$$

from (a). Suppose that

$$\tilde{f}_\kappa \circ \alpha^\vee : t \mapsto (\alpha^\vee(t), t^{m_\alpha}) \quad \text{with } m_\alpha \in \mathbb{Z} \text{ and for } t \in \mathbb{G}_m.$$

Then we need to show that

$$m_\alpha = -\mathrm{ord}(\eta(\alpha^\vee)) \quad \text{for all } \alpha \in \Delta.$$

(d) Now the splitting $s_\kappa : \mathbb{G}^{\mathrm{sc}}_\kappa \longrightarrow \tilde{\mathbb{G}}^{\mathrm{sc}}_\kappa$ from (a) is uniquely determined by its restriction to the root subgroups $\mathrm{U}_{\alpha,\kappa}$ for $\alpha \in \Delta$. Hence s is determined by $s_\kappa \circ x_\alpha$, where $x_\alpha : \mathbb{G}_a \to \mathrm{U}_\alpha$ is part of the fixed épinglage. Since

$$\alpha^\vee(t) = n_\alpha(t)\cdot n_\alpha(-1) \in \mathbb{T}^{\mathrm{sc}}_\kappa \subset \mathbb{T}_\kappa \quad \text{with} \quad n_\alpha(t) = x_\alpha(t)\cdot x_{-\alpha}(-t^{-1})\cdot x_\alpha(t),$$

this implies that

$$(\alpha^\vee(t), t^{m_\alpha}) = f_\kappa \circ \alpha^\vee(t) = \text{image of } n_\alpha(t)\cdot n_\alpha(-1) \text{ in } \tilde{\mathbb{T}}_\kappa.$$

(e) Likewise, the induced system of splittings $s_{\mathcal{O}'} : \mathbb{G}^{\mathrm{sc}}(\mathcal{O}') \to \overline{\mathbb{G}}^{\mathrm{sc}}(\mathcal{O}')$ is determined by the unique splitting

$$\tilde{x}_\alpha : F' \longrightarrow \overline{G}^{\mathrm{tame}}_{F'} \quad \text{for all } \alpha \in \Delta.$$

This implies that the composite

$$f_{\mathcal{O}'} : \mathbb{T}^{\mathrm{sc}}(\mathcal{O}') \xrightarrow{\ s_{\mathcal{O}'}\ } \overline{\mathbb{T}}^{\mathrm{sc}}(\mathcal{O}') \longrightarrow \overline{\mathbb{T}}(\mathcal{O}')$$

from (a) is given by

$$f_{\mathcal{O}'} \circ \alpha^\vee(t) = \text{image of } \tilde{n}_\alpha(t)\cdot\tilde{n}_\alpha(-1) \text{ in } \overline{\mathbb{T}}(\mathcal{O}').$$

The RHS is nothing but the section

$$s_\alpha(t) = (\alpha^\vee(t), (\eta(\alpha^\vee), t)_n) \in \overline{\mathbb{T}}(\mathcal{O}')$$

for $t \in \mathcal{O}'^{\times}$, whose image under the reduction map is

$$(\alpha^\vee(t), t^{-\mathrm{ord}(\eta(\alpha^\vee))}) \in \tilde{\mathbb{T}}_\kappa(\kappa') = \mathbb{T}_\kappa(\kappa') \times \kappa'^{\times}.$$

By (a),

$$f_\kappa \circ \alpha^\vee(t) = \text{the image of } f_{\mathcal{O}'} \circ \alpha^\vee(t) \text{ under reduction map.}$$

Hence, it follows that

$$m_\alpha = -\mathrm{ord}(\eta(\alpha^\vee))$$

for $\alpha \in \Delta$, as desired. □

In particular, one has:

Theorem 4.2. — *If Y/Y^{sc} is free or if η takes value in $\mathcal{O}_F^\times$, then the algebraic extension $\tilde{\mathbb{G}}_\kappa$ is split. Thus, the topological central extension $\overline{K}$ of K is also split.*

We have assumed that $n = \#\kappa^\times$ above. In general, when p does not divide n, the n-th Hilbert symbol map $\mathbb{K}_2(F) \longrightarrow \mu_n$ factors through $\mathbb{K}_2(F) \longrightarrow \kappa^\times \longrightarrow \mu_n$. So the degree n BD covering $\overline{\mathbb{G}}$ is obtained from the one of degree $\#\kappa^\times$ as a pushout. In particular, when the conditions of the above corollary holds, the degree n cover $\overline{K}$ is split as well. Indeed, whenever η takes value in $\mathcal{O}^\times \cdot F^{\times n}$, the cover $\overline{K}$ splits.

Note that we have merely given some simple sufficient conditions for $\overline{K}$ to be split. These conditions may not be necessary in a given case, but as we will see below, it is possible for $\overline{K}$ to be non-split when they fail. Moreover, note that the splitting of $\overline{K}$ is not necessarily unique (if it exists).

4.6. Running example. — We illustrate the discussion in this section with our running example: $\mathbb{G} = \mathrm{PGL}_2$, $D = 0$ and $n = 2$. Then we have the BD extensions

$$\overline{\mathbb{G}}_\eta = (\mathrm{GL}_2 \times \mathbb{K}_2)/\{(z, \{\eta, z\}) : z \in \mathbb{G}_m\}.$$

The associated BD covering groups are:

$$\overline{G}_\eta = (\mathrm{GL}_2(F) \times \mu_2)/\{(z, (\eta, z)_2) : z \in F^\times\}.$$

Let $\pi_\eta : \mathrm{GL}_2(F) \times \mu_2 \longrightarrow \overline{G}_\eta$ be the natural map, and let

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in F^\times \right\} \subset \mathrm{GL}_2(F).$$

The projection map identifies A with a maximal split torus T of $\mathrm{PGL}_2(F)$ and π_η identifies $A \times \mu_2$ with $\overline{T}$ of $\mathrm{PGL}_2(F)$. In this case, $\overline{T}$ is abelian (since $D = 0$) and so $\overline{T} = \overline{T}_{Q,2}$ and $f : T_{Q,2} \longrightarrow T$ is the identity map.

Now consider the issue of whether the covering splits over $K = \mathrm{PGL}_2(\mathcal{O})$. We have already seen from general arguments that it does when $\eta \in \mathcal{O}^\times \cdot F^{\times 2}$. When $\eta = \varpi$ is a uniformizer, we shall show now that the covering $\overline{K}_\eta$ is not split.

If a splitting $K \longrightarrow \overline{K}_\eta$ exists, we would have a group homomorphism

$$\phi : \mathrm{GL}_2(\mathcal{O}) \longrightarrow (\mathrm{GL}_2(F) \times \mu_2)/\{(z, (\eta, z)_2) : z \in F^\times\}$$

which is trivial on the center $Z(\mathcal{O})$ of $\mathrm{GL}_2(\mathcal{O})$. For $k \in \mathrm{GL}_2(\mathcal{O})$, we may write

$$\phi(k) = \text{the class of } (k, \mu(k))$$

for some $\mu(k) = \pm 1$. Now it is easy to check that $\mu : \mathrm{GL}_2(\mathcal{O}) \longrightarrow \mathcal{O}^\times/\mathcal{O}^{\times 2} = \{\pm 1\}$ is a group homomorphism and thus μ factors as

$$\mu : \mathrm{GL}_2(\mathcal{O}) \xrightarrow{\det} \mathcal{O}^\times \longrightarrow \kappa^\times \longrightarrow \pm 1.$$

If now $k = z \in Z(\mathcal{O})$ is a scalar matrix, then the fact that $\phi(z)$ is trivial means that

$$\mu(z) = (\varpi, z)_2.$$

Since μ factors through det, we see that $\mu(z) = 1$, but $(\varpi, z)_2$ is not 1 for some $z \in \mathcal{O}^\times$. With this contradiction, we see that the covering $\overline{K}_\eta$ is not split when $\eta = \varpi$ is a uniformizer.

5. Dual and L-Groups

In this section, we shall recall the definition of the L-group ${}^L\overline{G}$ of a BD covering $\overline{G}$ for a split $\mathbb{G}$ over a local field, following Weissman **[70, 69]**. The construction in **[70]** is quite involved, using a double twisting of the Hopf algebra of a candidate dual group. Moreover, it turns out to give the "correct" L-group only in the case (D, η) with η trivial. In a letter to Deligne **[69]**, Weissman gave a revision of his construction in **[70]**, using fully the BD data $(Q, \mathcal{E}, f, n)$ or the bisector data (D, η). We shall follow this more streamlined treatment in **[69]**. We note that the construction in **[69]** is subsequently extended to the case of quasi-split $\mathbb{G}$, using the language of étale gerbes, and we again caution the reader that these constructions may continue to evolve as the subject develops.

Since we will simply be presenting the construction of these objects in this section, the definition of the dual group or L-group of $\overline{G}$ may seem rather unmotivated at the end of the section. Whether they are the right objects or not will largely depend on whether they give the right framework to describe the representation theory of $\overline{G}$. In the subsequent sections, we will address these concerns.

5.1. Dual group à la Finkelberg-Lysenko-McNamara-Reich. — Let $\mathbb{G}$ be a split connected reductive group over F, with maximal split torus $\mathbb{T}$ and cocharacter lattice Y. Let $\Phi^\vee \subset Y$ be the set of coroots of $\mathbb{G}$ and let $Y^{sc} \subset Y$ be the sublattice generated by $\Phi^\vee$. Similarly, let $\Phi \subset X$ be the set of roots generating a sublattice X^{sc} in the character lattice X of $\mathbb{T}$.

Suppose that $\overline{\mathbb{G}}$ is a multiplicative $\mathbb{K}_2$-torsor with associated a BD data $(Q, \mathcal{E}, \phi)$. The data $(Q, \mathcal{E}, \phi, n)$ (with $|\mu_n(F)| = n$) then gives a central extension of locally compact groups

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{G} \longrightarrow G \longrightarrow 1.$$

Using the data $(Y, \Phi^\vee, Q, n)$, we may define a modified root datum as follows:

- we have already set
$$Y_{Q,n} = Y \cap nY^*$$
where $Y^* \subset Y \otimes_{\mathbb{Z}} \mathbb{Q}$ is the dual lattice of Y relative to B_Q. Let $X_{Q,n} \subset X \otimes_{\mathbb{Z}} \mathbb{Q}$ be the dual lattice to $Y_{Q,n}$;
- for each $\alpha^\vee \in \Phi^\vee$, set
$$n_\alpha = \frac{n}{\gcd(n, Q(\alpha^\vee))},$$
and
$$\alpha^\vee_{Q,n} = n_\alpha \cdot \alpha^\vee.$$

Denote by $\Phi^\vee_{Q,n}$ the set of such $\alpha^\vee_{Q,n}$'s and observe that

$$\Phi^\vee_{Q,n} \subset Y_{Q,n}.$$

We let $Y^{sc}_{Q,n}$ denote the sublattice of $Y_{Q,n}$ generated by $\Phi^\vee_{Q,n}$;

– likewise, for $\alpha \in \Phi$, set

$$\alpha_{Q,n} = n_\alpha^{-1} \cdot \alpha$$

and denote by $\Phi_{Q,n}$ the set of such $\alpha_{Q,n}$'s, so that $\Phi_{Q,n} \subset X_{Q,n}$.

Then it was shown in **[44]** and **[70]** that the quadruple $(Y_{Q,n}, \Phi^\vee_{Q,n}, X_{Q,n}, \Phi_{Q,n})$ is a root datum, and hence determine a split connected reductive group $\overline{G}^\vee$ over $\mathbb{C}$. *The group $\overline{G}^\vee$ is by definition the dual group of the BD extension $\overline{G}$.* Observe that it only depends on (Q, n) and is independent of the third ingredient f of a BD data $(Q, \mathcal{E}, f)$; equivalently, it only depends on (D, n) but not on η.

Let $Z(\overline{G}^\vee)$ be the center of $\overline{G}^\vee$. Then note that

$$Z(\overline{G}^\vee) = \mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbb{C}^\times).$$

5.2. L-group à la Weissman. — We can now describe Weissman's proposal for the L-group of $\overline{G}$ in **[69]**. This is done by defining an extension

$$1 \longrightarrow Z(\overline{G}^\vee) \longrightarrow E \longrightarrow F^\times/F^{\times n} \longrightarrow 1$$

followed by pushing out by the natural inclusion $Z(\overline{G}^\vee) \to \overline{G}^\vee$ and pulling back via the natural projection $W_F \to F^\times/F^{\times n}$. This results in an extension

$$1 \longrightarrow \overline{G}^\vee \longrightarrow {}^L\overline{G} \longrightarrow W_F \longrightarrow 1,$$

which we call Weissman's L-group extension. Observe that by construction, the extension ${}^L\overline{G}$ is equipped with a canonical splitting over the finite-index subgroup

$$W_{F,n} = \mathrm{Ker}(W_F \longrightarrow F^\times/F^{\times n}).$$

The construction of E is as a Baer sum $E_1 + E_2$ of two extensions E_1 and E_2. These are defined as follows:

– E_1 is defined explicitly using the cocycle

$$c_1 : F^\times/F^{\times n} \times F^\times/F^{\times n} \longrightarrow Z(\overline{G}^\vee) = \mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbb{C}^\times)$$

given by

$$c_1(a,b)(y) = (a,b)_n^{Q(y)}.$$

Since $2 \cdot Q(y) = B_Q(y,y) \in n\mathbb{Z}$ for $y \in Y_{Q,n}$, we see that this cocycle is trivial when n is odd, and is valued in the 2-torsion subgroup $Z(\overline{G}^\vee)[2]$ when n is even. Note that E_1 depends only on (Q, n).

Here is another description of E_1. Set

$$m = \begin{cases} n \text{ if } n \text{ is odd;} \\ n/2 \text{ if } n \text{ is even.} \end{cases}$$

Consider the extension

$$1 \longrightarrow \mu_2 \longrightarrow E_0 \longrightarrow F^\times/F^{\times n} \longrightarrow 1$$

defined by the cocycle

$$c(a,b) = (a,b)_n^m.$$

Let

$$j : \mu_2 \longrightarrow Z(\overline{G}^\vee) = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times)$$

be the homomorphism defined by

$$j(-1)(y) = e^{2\pi i \cdot \frac{Q(y)}{n}} = (-1)^{\frac{2}{n}Q(y)} \in \mathbb{C}^\times.$$

This is a homomorphism because the map $y \mapsto \frac{Q(y)}{n}$ is a group homomorphism $Y_{Q,n} \longrightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ and $Q(y) \in n\mathbb{Z}$ for all $y \in Y_{Q,n}^{\mathrm{sc}}$. Then E_1 is the pushout of E_0 by j;

- the construction of E_2 is slightly more involved and uses the full BD data $(Q, \mathcal{E}, f)$, where we recall that $\mathcal{E}$ is an extension

$$1 \longrightarrow F^\times \longrightarrow \mathcal{E} \longrightarrow Y \longrightarrow 1,$$

and

$$f : \mathcal{E}_{Q^{\mathrm{sc}}} \longrightarrow q^*(\mathcal{E})$$

is an isomorphism, with $q : Y^{\mathrm{sc}} \to Y$ the natural map.

Since we have the inclusion $Y_{Q,n}^{\mathrm{sc}} \to Y_{Q,n} \to Y$, we may pullback the extensions $\mathcal{E}_{Q^{\mathrm{sc}}}$ and $\mathcal{E}$ and pushout via $F^\times \longrightarrow F^\times/F^{\times n}$ to obtain

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \mathcal{E}_{Q^{\mathrm{sc}},n} & \longrightarrow & Y_{Q,n}^{\mathrm{sc}} & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 1 & \longrightarrow & F^\times/F^{\times n} & \longrightarrow & \mathcal{E}_{Q,n} & \longrightarrow & Y_{Q,n} & \longrightarrow & 1. \end{array}$$

Note that both $\mathcal{E}_{Q,n}$ and $\mathcal{E}_{Q^{\mathrm{sc}},n}$ are abelian groups.

For each $\alpha^\vee \in \Phi^\vee \subset Y^{\mathrm{sc}}$, we have defined before an element $s_{Q^{\mathrm{sc}}}(\alpha^\vee) \in \mathcal{E}_{Q^{\mathrm{sc}}}$ lying over $\alpha^\vee$. Indeed, $s_{Q^{\mathrm{sc}}}(\alpha^\vee)$ is the image of the element $s_\alpha(\alpha^\vee(\tau)) \in \overline{\mathbb{T}}^{\mathrm{sc}}(F(\!(\tau)\!))$ under pushout by the residue map $\mathrm{Res} : K_2(F(\!(\tau)\!)) \longrightarrow F^\times$. Analogously, we have the element $s_{Q^{\mathrm{sc}}}(n_\alpha \cdot \alpha^\vee) \in \mathcal{E}_{Q^{\mathrm{sc}}}$ which is the image of the element

$$s_\alpha((n_\alpha \cdot \alpha^\vee)(\tau)) = s_\alpha(\alpha^\vee(\tau^{n_\alpha})) \in \overline{\mathbb{T}}^{\mathrm{sc}}(F(\!(\tau)\!)).$$

It lies over $\alpha_{Q,n}^\vee = n_\alpha \alpha^\vee \in Y_{Q,n}^{\mathrm{sc}}$. Weissman showed that this induces a group homomorphism

$$s_{Q^{\mathrm{sc}}} : Y_{Q,n}^{\mathrm{sc}} \longrightarrow \mathcal{E}_{Q^{\mathrm{sc}},n}.$$

Composing this with f, one obtains

$$s_f = f \circ s_{Q^{\mathrm{sc}}} : Y_{Q,n}^{\mathrm{sc}} \longrightarrow \mathcal{E}_{Q,n}.$$

Viewing $Y_{Q,n}^{\mathrm{sc}}$ as a subgroup of $\mathcal{E}_{Q,n}$ by the splitting s_f, we inherit an extension

(5.1)
$$1 \longrightarrow F^\times/F^{\times n} \longrightarrow \overline{\mathcal{E}}_{Q,n} = \mathcal{E}_{Q,n}/s_f(Y_{Q,n}^{\mathrm{sc}}) \longrightarrow Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}} \longrightarrow 1.$$

Taking $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^\times)$ (which is exact, since $\mathbb{C}^\times$ is divisible and hence injective), we obtain the desired extension:

(5.2)
$$1 \longrightarrow Z(\overline{G}^\vee) \longrightarrow E_2 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F^\times/F^{\times n},\mathbb{C}^\times) \cong F^\times/F^{\times n} \longrightarrow 1$$

where the last isomorphism is via the n-th Hilbert symbol: $a \in F^\times/F^{\times n}$ giving rise to the character $\chi_a : b \mapsto (b,a)_n$.

5.3. Description using bisectors. — We may describe the construction of E_2 in terms of the bisector data (D,η). The bisector D allows us to realize the extension $\mathcal{E}$ as a set $Y \times F^\times$ with group law

$$(y_1,a)\cdot(y_2,b) = (y_1+y_2, ab\cdot(-1)^{D(y_1,y_2)}).$$

Pushing this out by $F^\times \to F^\times/F^{\times n}$ and pulling back to $Y_{Q,n}$ gives the extension $\mathcal{E}_{Q,n} = Y_{Q,n} \times F^\times/F^{\times n}$ with the same group law as above. In particular, if n is odd, $-1 \in F^{\times n}$ so that the cocycle $(-1)^{D(y_1,y_2)}$ is trivial.

The map $f : \mathcal{E}_{Q^{\mathrm{sc}}} \longrightarrow \mathcal{E}$ is defined by

$$f(s_{Q^{\mathrm{sc}}}(\alpha^\vee)) = (\alpha^\vee, \eta(\alpha^\vee)) \in Y \times F^\times, \quad \text{for } \alpha \in \Delta.$$

Then the splitting s_f is given by

$$s_f(\alpha^\vee_{Q,n}) = (\alpha^\vee_{Q,n}, \eta(\alpha^\vee_{Q,n})) \in Y_{Q,n} \times F^\times, \quad \text{for } \alpha \in \Delta.$$

It is instructive to note that the above constructions are functorial in nature. Given any isomorphism $\xi : (D,\eta) \longrightarrow (D',\eta')$, ξ carries the map s_f corresponding to (D,η) to the map $s_{f'}$ corresponding to (D',η').

5.4. Running example. — Again, we illustrate the discussion in this section using our running example: $G = \mathrm{PGL}_2(F)$, $Q = 0$ and $n = 2$. In this case, $Y = \mathbb{Z} = Y_{Q,n}$ and $Y^{\mathrm{sc}} = 2\mathbb{Z} = Y^{\mathrm{sc}}_{Q,n}$. So

$$Z(\overline{G}^\vee_\eta) = \mu_2 \subset \overline{G}^\vee_\eta = \mathrm{SL}_2(\mathbb{C}) \quad \text{for any } \eta.$$

Moreover, E_1^η is the split extension $Z(\overline{G}^\vee_\eta) \times F^\times/F^{\times 2}$ and $\mathcal{E}_\eta = Y \times F^\times/F^{\times 2}$ is split. Hence

$$\overline{\mathcal{E}}_\eta = (\mathbb{Z} \times F^\times/F^{\times 2})/\{(2y,\eta^y) : y \in \mathbb{Z}\}$$

and

$$E_1^\eta + E_2^\eta = \mathrm{Hom}(\overline{\mathcal{E}}_\eta, \mathbb{C}^\times) = \{(t,a) \in \mathbb{C}^\times \times F^\times/F^{\times 2} : t^2 = (\eta,a)_2\}.$$

This contains $Z(\overline{G}^\vee_\eta) = \mu_2$ as the subgroup of elements $(\pm 1, 1)$, and the associated quotient is via the second projection to $F^\times/F^{\times 2}$.

Observe that when $F = \mathbb{R}$ and $\eta = -1 \in \mathbb{R}^\times$, then $E_1^\eta + E_2^\eta$ is the cyclic group μ_4 and so the above extension is non-split! However, when we push out via the natural map $Z(\overline{G}^\vee_\eta) = \mu_2 \hookrightarrow \overline{T}^\vee_\eta = \mathbb{C}^\times$, then the pushout sequence is split. Indeed the sequence splits once we pushout by $\mu_2 \hookrightarrow \mu_4$.

For general local field F, one sees that when one pushes $\overline{\mathcal{E}}_\eta$ out by $\mu_2 \hookrightarrow \mathrm{SL}_2(\mathbb{C})$, one obtains the short exact sequence:

$$\mathrm{SL}_2(\mathbb{C}) \longrightarrow \{(g,a) \in \mathrm{GL}_2(\mathbb{C}) \times F^\times/F^{\times 2} : \det(g) = (\eta, a)_2\} \longrightarrow F^\times/F^{\times 2}.$$

Pulling back to W_F, one obtains:

$${}^L\overline{G}_\eta = \{(g,w) \in \mathrm{GL}_2(\mathbb{C}) \times W_F : \det(g) = \chi_\eta(w)\} \cong \mathrm{SL}_2(\mathbb{C}) \rtimes_\eta W_F$$

where $w \in W_F$ acts on $\mathrm{SL}_2(\mathbb{C})$ by the conjugation action of the diagonal matrix $\mathrm{diag}(\chi_\eta(w), 1)$. Thus, while the L-group extension is always split, ${}^L\overline{G}_\eta$ is not isomorphic to the direct product $\mathrm{SL}_2(\mathbb{C}) \times W_F$ for general η.

5.5. Functoriality for Levi subgroups. — Suppose that $\mathbb{M} \subset \mathbb{G}$ is a proper Levi subgroup, then a BD-covering $\overline{G}$ restricts to one on M. If the bisector data for $\overline{G}$ is (D, η, n), then that for $\overline{M}$ is $(D, \eta|_{Y^{\mathrm{sc}}_{\mathbb{M}}}, n)$, where we have restricted η to the sublattice $Y^{\mathrm{sc}}_{\mathbb{M}}$ generated by the simple coroots of M in Y. The above construction produces L-groups extensions ${}^L\overline{M}$ and ${}^L\overline{G}$. An examination of the construction shows:

Lemma 5.3. — *This is a natural commutative diagram of short exact sequences:*

$$\begin{tikzcd}
1 \arrow[r] & Z(\overline{G}^\vee) \arrow[r] \arrow[d] & E_G \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & Z(\overline{M}^\vee) \arrow[r] \arrow[d] & E_M \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \overline{M}^\vee \arrow[r] \arrow[d] & {}^L\overline{M} \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \overline{G}^\vee \arrow[r] & {}^L\overline{G} \arrow[r] & W_F \arrow[r] & 1.
\end{tikzcd}$$

Proof. — It suffices to exhibit a natural map from the extension E_G to E_M, which gives:

$$E_M \cong \frac{Z(\overline{M}^\vee) \times E_G}{\Delta Z(\overline{G}^\vee)}$$

and thus induces a map

$${}^L\overline{M} := \frac{\overline{M}^\vee \times E_M}{\Delta Z(\overline{M}^\vee)} \cong \frac{\overline{M}^\vee \times E_G}{\Delta Z(\overline{G}^\vee)} \longrightarrow \frac{\overline{G}^\vee \times E_G}{\Delta Z(\overline{G}^\vee)} =: {}^L\overline{G}$$

making the above diagram commute.

Write $E_G = E_1 + E_2$ and $E_M = E_{M,1} + E_{M,2}$ as Baer sums. Let $Y^{\mathrm{sc}}_{M,Q,n} \subseteq Y^{\mathrm{sc}}_{\mathbb{M}} \cap Y^{\mathrm{sc}}_{Q,n}$ be the sublattice of $Y^{\mathrm{sc}}_{\mathbb{M}}$ generated by $\alpha^\vee_{Q,n}$ for $\alpha^\vee \in Y^{\mathrm{sc}}_{\mathbb{M}}$. The cocycle defining E_1 takes value in $\mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}/Y^{\mathrm{sc}}_{Q,n}, \mathbb{C}^\times)$, and the cocycle for $E_{M,1}$ takes the same formula and is valued in $\mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}/Y^{\mathrm{sc}}_{M,Q,n}, \mathbb{C}^\times)$. Thus there is a natural map from E_1 to $E_{M,1}$, which in fact is canonically isomorphic to the push-out of E_1.

Consider E_2 and $E_{M,2}$. The pull-back of $\mathscr{E}_{Q^{\mathrm{sc}},n}$ via $Y^{\mathrm{sc}}_{M,Q,n} \subseteq Y^{\mathrm{sc}}_{Q,n}$ gives an extension

$$1 \longrightarrow F^\times/F^{\times n} \longrightarrow \mathscr{E}_{M,Q^{\mathrm{sc}},n} \longrightarrow Y^{\mathrm{sc}}_{M,Q,n} \longrightarrow 1$$

which has the splitting $s_{M,Q^{\mathrm{sc}}}$ from the restriction of $s_{Q^{\mathrm{sc}}}$ to $Y^{\mathrm{sc}}_{M,Q,n}$. Let $s_{M,f} := f \circ s_{M,Q^{\mathrm{sc}}}$ and $\overline{\mathscr{E}}_{M,Q,n} := \mathscr{E}_{M,Q^{\mathrm{sc}},n}/s_{M,f}(Y^{\mathrm{sc}}_{M,Q,n})$. Then one obtains the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & F^\times/F^{\times n} \arrow[r] & \overline{\mathscr{E}}_{Q,n} \arrow[r] & Y_{Q,n}/Y^{\mathrm{sc}}_{Q,n} \arrow[r] & 1 \\
1 \arrow[r] & F^\times/F^{\times n} \arrow[r] \arrow[u] & \overline{\mathscr{E}}_{M,Q,n} \arrow[r] \arrow[u] & Y_{Q,n}/Y^{\mathrm{sc}}_{M,Q,n} \arrow[r] \arrow[u] & 1.
\end{tikzcd}$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^\times)$ and followed by the identification $\mathrm{Hom}_{\mathbb{Z}}(F^\times/F^{\times n},\mathbb{C}^\times) \simeq F^\times/F^{\times n}$ via the n-th Hilbert symbol, we obtain a natural map from E_2 to $E_{M,2}$ as in

$$\begin{tikzcd}
1 \arrow[r] & Z(\overline{G}^\vee) \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & F^\times/F^{\times n} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & Z(\overline{M}^\vee) \arrow[r] & E_{M,2} \arrow[r] & F^\times/F^{\times n} \arrow[r] & 1.
\end{tikzcd}$$

Combining the two extensions for the Baer sum, we see that there is a natural map from E_G to E_M. This gives a natural isomorphism

$$Z(\overline{M}^\vee) \times E_G/\Delta Z(\overline{G}^\vee) \longrightarrow E_M. \qquad \square$$

5.6. Functoriality for z-extensions. — If

$$1 \longrightarrow Z \longrightarrow \mathbb{H} \longrightarrow \mathbb{G} \longrightarrow 1$$

is a z-extension, and $\overline{G}$ is a BD covering with bisector data (D,η), then we obtain a BD covering $\overline{H}$ with essentially the same bisector data, such that

$$1 \longrightarrow Z \xrightarrow{\ i\ } \overline{H} \longrightarrow \overline{G} \longrightarrow 1.$$

From the construction of the L-group extension, one deduces:

Lemma 5.4. — *There is a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \overline{G}^\vee \arrow[r] \arrow[d] & {}^L\overline{G} \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \overline{H}^\vee \arrow[r] \arrow[d] & {}^L\overline{H} \arrow[r] \arrow[d] & W_F \arrow[r] & 1 \\
 & Z^\vee \arrow[r, equal] & Z^\vee. & &
\end{tikzcd}$$

6. Distinguished Splittings of L-Groups

In this section, we study the L-group extension proposed in the previous section. More precisely, we will investigate whether this extension actually splits. Since the L-group extension is defined from the extension $E_1 + E_2$, we have seen that it has a canonical splitting over the finite index subgroup $W_{F,n} = \mathrm{Ker}(W_F \longrightarrow F^\times/F^{\times n})$ and it is natural to ask if this canonical splitting extends to W_F. This amounts to asking whether $E_1 + E_2$ splits: this is the question addressed in this section. We have seen in the last section that it does not in general, but we would like to understand where the obstruction lies. We shall assume that $\overline{\mathbb{G}}$ is defined by a pair (D, η) where D is a *fair* bisector.

6.1. Splittings of $E_1 + E_2$. — What does it mean to give a splitting of $E_1 + E_2$? We first observe that when $Z(\overline{G}^\vee)$ is a finite group with order relatively prime to that of $F^\times/F^{\times n}$, then the abelian extensions E_1 and E_2 split uniquely over $F^\times/F^{\times n}$. Therefore, in this case $E_1 + E_2$ splits and one has ${}^L\overline{G} \simeq \overline{G}^\vee \times W_F$.

In general, since E_1 is given by an explicit cocycle c_1 (valued in $Z(\overline{G}^\vee)[2]$), the issue of equipping $E_1 + E_2$ with a splitting is equivalent to finding a set theoretic section of E_2 whose associated cocycle is equal to c_1. In addressing this question, we shall work explicitly using a bisector data (D, η).

In the construction of E_2 above, the extension (5.1) is equipped with a cocycle inherited from the fair bisector D. However, in taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to obtain (5.2), we have partially lost the data of a cocycle. To obtain a cocycle, we need to choose a set theoretic section of (5.2). Hence, for each $a \in F^\times/F^{\times n}$, we need to extend the character

$$\chi_a \in \mathrm{Hom}_{\mathbb{Z}}(F^\times/F^{\times n}, \mathbb{C}^\times)$$

to a character $\tilde{\chi}_a$ of $\mathcal{E}_{Q,n}$. Another way of expressing this is to consider the pushout of $\mathcal{E}_{Q,n}$ by χ_a:

$$\begin{tikzcd}
1 \arrow[r] & F^\times/F^{\times n} \arrow[r] \arrow[d, "\chi_a"] & \mathcal{E}_{Q,n} \arrow[r] \arrow[d] & Y_{Q,n} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_n \arrow[r] & \tilde{\mathcal{E}}_{Q,n} \arrow[r] & Y_{Q,n} \arrow[r] & 1.
\end{tikzcd}$$

Then extending χ_a amounts to finding a genuine character of $\tilde{\mathcal{E}}_{Q,n}$ which is trivial when precomposed with $f \circ s_{Q^{sc}}$.

Note that $\tilde{\mathcal{E}}_{Q,n} = Y_{Q,n} \times \mu_n$ has group law

$$(y_1, 1) \cdot (y_2, 1) = (y_1 + y_2, \chi_a(-1)^{D(y_1, y_2)}) = (y_1 + y_2, (a, a)_n^{D(y_1, y_2)}).$$

Thus to give a genuine character $\tilde{\chi}_a$ of $\tilde{\mathcal{E}}_{Q,n}$ is to give a function

$$\tilde{\chi}_a : Y_{Q,n} \longrightarrow \mathbb{C}^\times$$

satisfying

$$\tilde{\chi}_a(y_1) \cdot \tilde{\chi}_a(y_2) = \tilde{\chi}_a(y_1 + y_2) \cdot (a, a)_n^{D(y_1, y_2)}.$$

For the composite $\tilde{\chi}_a \circ f \circ s_{Q^{\mathrm{sc}}}$ to be trivial, one needs to require that

$$\tilde{\chi}_a(\alpha^\vee_{Q,n}) = (a, \eta(\alpha^\vee_{Q,n}))_n \quad \text{for all } \alpha \in \Delta.$$

Let us fix such a genuine character $\tilde{\chi}_a$ for each $a \in F^\times/F^{\times n}$. Using this as a set theoretic section for (5.2), we may write

$$E_2 = F^\times/F^{\times n} \times \mathrm{Hom}_{\mathbb{Z}}(Y_{Q,n}, \mathbb{C}^\times)$$

with cocycle

$$c_2(a,b)(y) = \tilde{\chi}_a(y)\tilde{\chi}_b(y)\tilde{\chi}_{ab}(y)^{-1} \quad \text{for all } y \in Y_{Q,n}.$$

To impose the condition that $c_2 = c_1$ means:

$$\tilde{\chi}_a(y)\tilde{\chi}_b(y) = \tilde{\chi}_{ab}(y) \cdot (a,b)_n^{Q(y)}$$

for all $y \in Y_{Q,n}$.

To summarize, we have shown:

Lemma 6.1. — *Giving a splitting of $E_1 + E_2$ is equivalent to giving a function*

$$\chi : F^\times \times Y_{Q,n} \longrightarrow \mathbb{C}^\times$$

such that

(a) $$\chi(ab^n, y) = \chi(a, y) \quad \text{for } a, b \in F^\times \text{ and } y \in Y_{Q,n},$$

(b) $$\chi(a, y_1) \cdot \chi(a, y_2) = \chi(a, y_1 + y_2) \cdot (a,a)_n^{D(y_1,y_2)},$$

(c) $$\chi(a, y) \cdot \chi(b, y) = \chi(ab, y) \cdot (a,b)_n^{Q(y)},$$

(d) $$\chi(a, \alpha^\vee_{Q,n}) = (a, \eta(\alpha^\vee_{Q,n}))_n \quad \text{for } a \in F^\times \text{ and } \alpha \in \Delta.$$

In §3.3 and §3.4, we have described the covering torus $\overline{T}_{Q,n}$ by generators and relations, using the bisector D. It follows immediately from these that to give a splitting of $E_1 + E_2$ is equivalent to giving a genuine character χ of $\overline{T}_{Q,n}$ (by properties (b) and (c)) satisfying some properties (dictated by properties (a) and (d)). More precisely, properties (a) and (d) can be rephrased as:

(a′) the inclusion $nY_{Q,n} \hookrightarrow Y_{Q,n}$ gives the n-power isogeny $n : T_{Q,n} \to T_{Q,n}$ and this lifts to a group homomorphism

$$i_n : T_{Q,n} \longrightarrow \overline{T}_{Q,n}$$

given by

$$i_n(y(a)) = (ny)(a) = y(a^n) \quad y \in Y_{Q,n}.$$

Then property (a) says that $\chi \circ i_n$ is trivial.

(d′) the inclusion $Y^{\mathrm{sc}}_{Q,n} \hookrightarrow Y_{Q,n}$ induces an isogeny $T^{\mathrm{sc}}_{Q,n} \to T_{Q,n}$ and the map

$$\alpha^\vee_{Q,n}(a) \mapsto \alpha^\vee_{Q,n}(a) \cdot (\eta(\alpha^\vee_{Q,n}), a)_n \in \overline{T}_{Q,n}, \quad \text{for all } \alpha \in \Delta,$$

defines a splitting of this isogeny to give

$$s_\eta : T^{\mathrm{sc}}_{Q,n} \longrightarrow \overline{T}_{Q,n}.$$

Then property (d) says that $\chi \circ s_\eta$ is trivial. Because D is fair, we have in fact, for any $y \in Y_{Q,n}^{\mathrm{sc}}$,

$$s_\eta(y(a)) = y(a) \cdot (\eta(y), a)_n \in \overline{T}_{Q,n}. \tag{6.2}$$

6.2. Obstruction. — The question is thus: does there exist genuine characters of $\overline{T}_{Q,n}$ such that (a′) and (d′) are satisfied? We shall see that there will be some obstructions. More precisely, suppose that $y(a) \in T_{Q,n}^{\mathrm{sc}}$ ($y \in Y_{Q,n}^{\mathrm{sc}}$) belongs to

$$\mathrm{Ker}(T_{Q,n}^{\mathrm{sc}} \to T_{Q,n}) = \mathrm{Tor}_1(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, F^\times).$$

Note that this kernel is generated by such pure tensors $y(a)$. Then (d′) requires χ to be trivial on the element $(\eta(y), a)_n \in \mu_n(F)$. But χ is supposed to be a genuine character. So we have our first obstruction:

Obstruction 1. — A genuine character χ satisfying (d′) exists if and only if

$$(\eta(y), a)_n = 1 \quad \text{whenever } y \otimes a = 0 \text{ in } Y_{Q,n} \otimes F^\times \text{ with } y \in Y_{Q,n}^{\mathrm{sc}}. \tag{6.3}$$

This condition does not hold automatically. It does hold, however, if $\eta_n|_{Y_{Q,n}^{\mathrm{sc}}}$ can be extended to a homomorphism $Y_{Q,n} \longrightarrow F^\times/F^{\times n}$.

Another obstruction is the following. Suppose that $y \in nY_{Q,n} \cap Y_{Q,n}^{\mathrm{sc}}$. Then properties (a′) and (d′) require that

$$\chi(y(a)) = 1 = \chi(y(a)) \cdot (\eta(y), a)_n \quad \text{for any } a \in F^\times.$$

Thus we have our second obstruction:

Obstruction 2. — A genuine character χ satisfying (a′) and (d′) exists if and only if (6.3) holds and

$$(a, \eta(y))_n = 1 \quad \text{for any } y \in nY_{Q,n} \cap Y_{Q,n}^{\mathrm{sc}} \text{ and any } a \in F^\times. \tag{6.4}$$

Again, this condition does not hold automatically, but it does hold if $\eta_n|_{Y_{Q,n}^{\mathrm{sc}}}$ is extendable to $Y_{Q,n}$.

6.3. Existence of splitting. — To summarize, we have shown:

Proposition 6.5. — (i) *To give a splitting of $E_1 + E_2$ is equivalent to giving a genuine character of $\overline{T}_{Q,n}$ satisfying conditions* (a′) *and* (d′) *above. Any such genuine character is of finite order.*

(ii) *If $\eta_n|_{Y_{Q,n}^{\mathrm{sc}}}$ is extendable to a homomorphism $Y_{Q,n} \to F^\times/F^{\times n}$, then such genuine characters as in* (i) *exist, so that the sequence $E_1 + E_2$ is split.*

(iii) *Under the hypothesis in* (ii)*, the set of such splittings is a torsor under the group*

$$\mathrm{Hom}(F^\times/F^{\times n}, Z(\overline{G}^\vee)) = \mathrm{Hom}(W_F, Z(\overline{G}^\vee)[n]).$$

Proof. — We have already shown (i). For (ii), the existence of such genuine characters follows by Pontryagin duality, noting that $\overline{T}_{Q,n}$ is abelian. Finally (iii) is clear. □

We shall see later that these obstructions do occur in our running example.

6.4. Distinguished splitting,— The definition of E_1+E_2 uses essentially only the data $(n, Y_{Q,n}, \Phi^\vee_{Q,n}, Q|_{Y_{Q,n}}, \mathscr{E}_{Q,n})$. In particular, it does not make use of the full information available in (Y,Q). Recall that there is a short exact sequence

$$1 \longrightarrow \mathrm{Ker}(h) \longrightarrow \overline{T}_{Q,n} \longrightarrow Z(\overline{T}) \longrightarrow 1.$$

Since we are after all defining the L-group of $\overline{G}$, it is natural to consider those splittings of E_1+E_2 which corresponds to those genuine characters of $\overline{T}_{Q,n}$ which are trivial on $\mathrm{Ker}(h)$ and thus factors to give characters of $Z(\overline{T})$. For this, it follows by Lemma 3.1 that it suffices to consider those genuine characters which are trivial on the image of $\tilde{g}: T \to \overline{T}_{Q,n}$, with

$$\tilde{g}: y(a) \mapsto (ny)(a) \in \overline{T}_{Q,n}.$$

Then we have:

(a″) a genuine character χ of $\overline{T}_{Q,n}$ factors to $Z(\overline{T})$ if $\chi \circ \tilde{g} = 1$.

Note that $\tilde{g}$ agrees with i_n when pulled back to $T_{Q,n}$, so that this requirement subsumes condition (a′). It is now natural to make the following definition:

Definition. — We call a genuine character of $\overline{T}_{Q,n}$ which satisfies (a″) and (d′) a *distinguished genuine character*. We call the corresponding splitting of E_1+E_2 a *distinguished splitting*.

Assume that we have a genuine character satisfying (a′) and (d′) already. As before, we see that there is an obstruction to (a″). Namely,

Obstruction 3. — A genuine character satisfying (a″) and (d′) exists if and only if (6.3) holds and

$$(a, \eta(y))_n = 1 \quad \text{for any } y \in nY \cap Y^{\mathrm{sc}}_{Q,n} \text{ and any } a \in F^\times.$$

This condition is not automatic, but is satisfied when $\eta_n : Y^{\mathrm{sc}} \to F^\times/F^{\times n}$ can be extended to a homomorphism of Y.

We have thus shown:

Theorem 6.6. — *Assume that $\eta_n : Y^{\mathrm{sc}} \to F^\times/F^{\times n}$ can be extended to a homomorphism of Y, and let $J = nY + Y^{\mathrm{sc}}_{Q,n}$.*

(i) *The set of distinguished genuine characters of $\overline{T}_{Q,n}$ is nonempty.*

(ii) *Consider the subgroup*

$$Z^\heartsuit(\overline{G}^\vee) := \mathrm{Hom}(Y_{Q,n}/J, \mathbb{C}^\times) \subset Z(\overline{G}^\vee).$$

Then the set of distinguished splittings of E_1+E_2 is a torsor under

$$\mathrm{Hom}(F^\times/F^{\times n}, Z^\heartsuit(\overline{G}^\vee)) = \mathrm{Hom}(W_F, Z^\heartsuit(\overline{G}^\vee)[n]).$$

(iii) *Each distinguished splitting of E_1+E_2 gives rise to a splitting $s : W_F \longrightarrow {}^L\overline{G}$ which agrees with the canonical splitting over $W_{F,n}$ and induces an isomorphism*

$$\overline{G}^\vee \times W_F \cong {}^L\overline{G}.$$

Henceforth, when $\eta_n : Y^{\mathrm{sc}} \to F^\times/F^{\times n}$ can be extended to a homomorphism of Y, we shall consider the L-group of a degree n BD covering $\overline{G}$ as the extension

$$1 \longrightarrow \overline{G}^\vee \longrightarrow {}^L\overline{G} \longrightarrow W_F \longrightarrow 1$$

equipped with the set of distinguished splittings. In particular, when $\mathbb{G} = \mathbb{T}$ is a split torus, then the set of distinguished splittings is nonempty since $Y^{\mathrm{sc}} = 0$.

6.5. Weyl invariance. — Under the hypothesis of Theorem 6.6, we have distinguished genuine characters of $\overline{T}_{Q,n}$. These distinguished characters possess another desirable property. Namely, the action of $N(\mathbb{T})$ on $\overline{\mathbb{T}}$ gives rise to an action of $N(\mathbb{T})$ on $\overline{T}$ which preserves the center $Z(\overline{T})$. Moreover, the action on $Z(\overline{T})$ factors through the Weyl group $W = N(\mathbb{T})/\mathbb{T}$. Since $\overline{T}_{Q,n}$ is the pullback of $Z(\overline{T})$ using the map $T_{Q,n} \longrightarrow T$, it inherits an action of W as well. Hence it makes sense to ask if a genuine character of $Z(\overline{T})$ or $\overline{T}_{Q,n}$ is W-invariant.

For $\alpha \in \Phi$, let w_α denote the element of W corresponding to the element $q(n_\alpha(1))$. Consider $\mathrm{Int}(w_\alpha)(\tilde{t}) := w_\alpha \cdot \tilde{t} \cdot w_\alpha^{-1}$. Then we have seen in (2.3) that for $\tilde{t} \in Z(\overline{T})$,

$$\mathrm{Int}(w_\alpha)(\tilde{t}) = \tilde{t} \cdot q(s_\alpha(\alpha^\vee(\alpha(t)^{-1}))).$$

One has an analogous formula for the W-action on $\overline{T}_{Q,n}$. More precisely, if $t = y(a) \in T_{Q,n}$, with $y \in Y_{Q,n}$, then

$$\mathrm{Int}(w_\alpha)(\tilde{t}) = \tilde{t} \cdot s_\eta(-\langle \alpha, y\rangle \cdot \alpha^\vee(a)). \tag{6.7}$$

The following proposition shows that $\langle \alpha, y\rangle \cdot \alpha^\vee \in Y_{Q,n}^{\mathrm{sc}}$ so that the right hand side of the above formula is well-defined.

Theorem 6.8. — *For $y \in Y_{Q,n}$, n_α divides $\langle \alpha, y\rangle$ for each $\alpha \in \Phi$. Hence, χ is W-invariant if χ satisfies (d′). In particular, distinguished genuine characters of $\overline{T}_{Q,n}$ gives rise to W-invariant genuine characters of $Z(\overline{T})$.*

Proof. — Since Q is W-invariant, we have

$$Q(y) = Q(w_\alpha y) = Q(y - \langle y, \alpha\rangle \alpha^\vee) = Q(y) - \langle y, \alpha\rangle \cdot B(y, \alpha^\vee) + \langle y, \alpha\rangle^2 Q(\alpha^\vee).$$

This implies that

$$\langle \alpha, y\rangle = 0 \quad \text{or} \quad B(y, \alpha^\vee) = \langle \alpha, y\rangle \cdot Q(\alpha^\vee).$$

In the latter case, note that $B(y, \alpha^\vee)$ is divisible by n if $y \in Y_{Q,n}$, in which case n_α divides $\langle \alpha, y\rangle$ as desired. □

6.6. Splitting of ${}^L\overline{G}$. — We have so far considered splittings of the fundamental extension $E := E_1 + E_2$ of $F^\times/F^{\times n}$ by $Z(\overline{G}^\vee)$ with good properties. Since the L-group of $\overline{G}$ is obtained from this fundamental extension by a combination of pushout and pullback, one may consider splittings of the extensions derived from these operations. Of course, these extensions will possess more splittings than the fundamental extension from which they are derived.

For example, one may pull back the fundamental extension to obtain

$$1 \longrightarrow Z(\overline{G}^{\vee}) \longrightarrow \tilde{E} \longrightarrow W_F \longrightarrow 1$$

and one may ask for distinguished splittings of this central extension. This amounts to dropping condition (a′) above, so that one is finding genuine characters of $\overline{T}_{Q,n}$ which satisfy only (d′). To ensure that this character factors through to $Z(\overline{T})$, it would not be reasonable to require the condition (a″) (since (a′) is not assumed); one would simply require the character to be trivial on $\mathrm{Ker}(h)$.

Now recall from §5.5 that the L-group construction is functorial with respect to inclusion of Levi subgroups, so that one has an embedding ${}^L\overline{T} \hookrightarrow {}^L\overline{G}$. Since we know that the extension ${}^L\overline{T}$ splits over W_F, we deduce:

Proposition 6.9. — *The extension ${}^L\overline{G}$ splits over W_F, so that ${}^L\overline{G}$ is abstractly a semi-direct product.*

As we have seen in our running example, it is not a direct product in general. Rather, it is the type of split extensions which one typically encounters in the usual theory of endoscopy (as we explained in the introduction).

Further, suppose we fix a fair bisector D and consider all BD-extensions with BD invariants (D, η). All these BD covering groups $\overline{G}_\eta$ have isomorphic covering tori $\overline{T} = T \times_D \mu_n$. When $\eta_n = 1$ is trivial, we have seen that the set of distinguished splittings is nonempty. If we fixed a distinguished splitting of E for $\eta_n = 1$, we would have fixed a splitting of ${}^L\overline{T}$ and hence for ${}^L\overline{G}_\eta$ for all η. If χ is the associated genuine character of $Z(\overline{T})$, note that χ is Weyl-invariant for the Weyl action associated to $\eta_n = 1$. But this χ need not be Weyl-invariant for general η, since the Weyl action on $Z(\overline{T})$ depends on η.

6.7. Running example. — We consider our running example to illustrate the discussion of this section. Recall that we have:

$$1 \longrightarrow \mu_2 \longrightarrow \overline{G}_\eta = (\mathrm{GL}_2(F) \times \mu_2)/\{(z, (\eta, z)_2) : z \in F^\times\} \longrightarrow \mathrm{PGL}_2(F) \longrightarrow 1$$

and

$$E_1^\eta + E_2^\eta = \{(t, a) \in \mathbb{C}^\times \times F^\times/F^{\times 2} : t^2 = (\eta, a)_2\}.$$

Moreover, $Y_{Q,2} = Y \supset Y_{Q,n}^{\mathrm{sc}} = Y^{\mathrm{sc}} = 2Y$. In this case, Obstruction 1 says that

$$(\eta, -1)_2 = 1.$$

Clearly, this may fail if $-1 \notin F^{\times 2}$. However, it does hold if $\eta \in F^{\times 2}$, or equivalently, if η_2 is trivial. Obstruction 2 says that

$$(\eta, a)_2 = 1 \quad \text{for all } a \in F^\times.$$

This is clearly a stronger condition than the one above, and it holds if and only if η_2 is trivial. Thus, we see that if $\eta \notin F^{\times 2}$, then the sequence $E_1^\eta + E_2^\eta$ does not split.

When η_2 is trivial, however, the above two obstructions, as well as Obstruction 3, are all absent and so a distinguished character of $\overline{T}_\eta = Z(\overline{T}_\eta)$ exists. If we identify $\overline{T}_\eta$ with $T \times \mu_2$, so that genuine characters of $\overline{T}_\eta$ is in natural bijection with the characters of T, then the distinguished characters correspond naturally to quadratic characters of T.

The Weyl group action on $\overline{T}_\eta$ is given by

$$\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}, 1\right) \mapsto \left(\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix}, (\eta, a)_2\right),$$

so that the Weyl action on genuine characters is given by

$$\chi \mapsto \chi^{-1} \cdot (\eta, -)_2.$$

In particular, we see that χ is W-invariant if and only if

$$\chi^2 = (\eta, -)_2,$$

i.e., if χ is a square root of the quadratic character $(\eta, -)_2$. Such a square root exists if and only if

$$(\eta, -1)_2 = 1,$$

i.e., if and only if Obstruction 1 is absent.

7. Construction of Distinguished Genuine Characters

It is useful in practice to have an explicit construction of the distinguished genuine characters (when they exist). Note that once one constructs one such distinguished character, the others are obtained by twisting it by a character of the finite group $T_{Q,n}/T_J$ (see §7.1 below for the definition of T_J). We shall see that a distinguished genuine character can be constructed using the Hilbert symbol $(-,-)_n$ and the Weil index γ_ψ (cf. [**53**]) associated to a nontrivial additive character ψ of F. Here,

$$\gamma_\psi : F^\times \longrightarrow \mu_4 \subset \mathbb{C}^\times$$

satisfies

$$\gamma_\psi(a) \cdot \gamma_\psi(b) = \gamma_\psi(ab) \cdot (a, b)_2$$

for all $a, b \in F^\times$. Moreover, $\gamma_{\psi_{a^2}} = \gamma_\psi$. Such Weil indices play an important role in the classical theory of the metaplectic groups $\mathrm{Mp}_{2n}(F)$.

7.1. Reduction to rank 1 case. — By the elementary divisor theorem, we may pick a basis $\{y_i\}$ of $Y_{Q,n}$ such that $\{k_i y_i\}$ is a basis for the lattice $J = nY + Y_{Q,n}^{\mathrm{sc}}$ for some $k_i \in \mathbb{Z}$. Let $\mathbb{T}_J$ be the split torus associated to J and let T_J be its F-rational points. This gives a decomposition

$$J = k_1 Y_1 \oplus \cdots \oplus k_r Y_r \subset Y_1 \oplus \cdots \oplus Y_r = Y$$

and a map

$$T_J = \prod_i (k_i Y_i) \otimes_{\mathbb{Z}} F^\times \longrightarrow \prod_i Y_i \otimes_{\mathbb{Z}} F^\times$$

which is the k_i-power map on the i-th coordinate. Write $T_{Q,n,i}$ for the 1-dimensional torus corresponding to Y_i and $T_{J,i}$ for that corresponding to k_iY_i. Now, because $\overline{T}_{Q,n}$ is abelian, we see that

$$\overline{T}_{Q,n} \cong \overline{T}_1 \times \cdots \times \overline{T}_r / Z$$

where

$$Z = \{(\epsilon_i) \in \prod_{i=1}^{r} \mu_n : \prod_i \epsilon_i = 1\}.$$

Moreover, the group law on $\overline{T}_{Q,n,i} = (Y_i \otimes F^\times) \times \mu_n$ is given by

$$y_i(a) \cdot y_i(b) = y_i(ab) \cdot (a,b)_n^{Q(y_i)}.$$

It follows that the map $T_{J,i} \to T_{Q,n,i}$ splits naturally to give

$$T_{J,i} \longrightarrow \overline{T}_{Q,n,i}.$$

Thus, to construct a distinguished genuine character on $\overline{T}_{Q,n}$, it suffices to construct a genuine character of $\overline{T}_{Q,n,i}$ which is trivial on the image of $T_{J,i}$; the product of these characters will then be a distinguished genuine character of $\overline{T}_{Q,n}$. We are thus reduced to constructing genuine characters of 1-dimensional covering tori.

7.2. The definition. — We set

$$\chi((y_i(a)) = \gamma_\psi(a)^{f_i}$$

for some $f_i \in \mathbb{Z}$ to be determined. Now we need to check various requirements:

- we first need to check the relation for $\overline{T}_{Q,n,i}$:

  $$\chi(y_i(a)) \cdot \chi(y_i(b)) = \chi(y_i(ab)) \cdot (a,b)_n^{Q(y_i)}.$$

  This amounts to the requirement that

  $$f_i \equiv A_i := \frac{2}{n} \cdot Q(y_i) \mod 2;$$

- next we need to ensure that χ is trivial on the image of $T_{J,i}$, i.e., trivial on $y_i(a^{k_i})$. But a short computation gives:

  $$\chi(y_i(a^{k_i})) = \gamma_\psi(a)^{k_i f_i + k_i(k_i-1)\cdot A_i}.$$

  Thus we need

  $$k_i \cdot (f_i + (k_i - 1)A_i) \equiv 0 \mod 4.$$

  To ensure this, we shall simply take

  $$f_i := -(k_i - 1)A_i.$$

  Then when k_i is even, it is automatic that $f_i \equiv A_i \mod 2$. We need to ensure that this continues to hold when k_i is odd.

  For this, we need to show that $A_i = 0 \mod 2$ when k_i is odd; equivalently, we need to show that $Q(y_i) \equiv 0 \mod n$. Since we already know that $Q(y_i)$ is divisible by $n/2$, it remains to show that if 2^e divides n, then 2^e divides $Q(y_i)$.

As $k_i y_i \in J = Y_{Q,n}^{\mathrm{sc}} + nY$, we know that $Q(k_i y_i) \equiv 0 \mod n$. So 2^e divides $k_i^2 \cdot Q(y_i)$. Since k_i is odd, we see that 2^e does divide $Q(y_i)$, as desired.

We have completed the construction of a distinguished genuine character χ_ψ. A formula for χ_ψ can be given as follows. For $y = \sum_i n_i y_i \in Y_{Q,n}$ and $a \in F^\times$,

$$\chi_\psi(y(a)) = \prod_i \gamma_\psi(a^{n_i})^{f_i} \cdot (a,a)_n^{\sum_{i<j} n_i n_j D(y_i, y_j)},$$

with

$$f_i = -(k_i - 1) \cdot \frac{2}{n} \cdot Q(y_i).$$

Though explicit, a slightly unsatisfactory aspect of this formula is that one first needs to find compatible bases for the lattice chain $J \subset Y_{Q,n}$.

8. LLC for Covering Tori

We are going to specialize the investigation of §6 to several examples. In this section, we assume that $\mathbb{G} = \mathbb{T}$ is a split torus.

8.1. LLC for $\overline{T}_{Q,n}$. — When $\mathbb{G} = \mathbb{T}$, one has $Y^{\mathrm{sc}} = 0$, so that $\eta = 1$, and the extension $E_1 + E_2$ is

$$1 \longrightarrow \overline{T}_{Q,n}^{\vee} \longrightarrow E_1 + E_2 \longrightarrow F^\times / F^{\times n} \longrightarrow 1.$$

In the previous section, we have seen that to give a splitting of this sequence is equivalent to giving a genuine character on $\overline{T}_{Q,n}$ satisfying conditions (a′) and (d′). Since $Y^{\mathrm{sc}} = 0$, (d′) is vacuous, and since $\eta = 1$, (a′) holds. So we obtain a bijection between the set of splittings of $E_1 + E_2$ and the set of genuine characters of $\overline{T}_{Q,n}$ satisfying (a′).

On the other hand, to obtain the L-group of $\overline{T}$, we pull $E_1 + E_2$ back by $W_F \longrightarrow F^\times \longrightarrow F^\times / F^{\times n}$. The same considerations show that to give a splitting of ${}^L\overline{T}$ is equivalent to giving a genuine character of $\overline{T}_{Q,n}$, where we don't insist on condition (a′) anymore. Hence, we have obtained a natural bijection

$$\{\text{Splittings of } {}^L\overline{T}\} \longleftrightarrow \{\text{genuine characters of } \overline{T}_{Q,n}\}.$$

This is a classification of the genuine characters of the abelian group $\overline{T}_{Q,n}$.

We can explicate the bijection above by tracing through the discussion in §6. Let $\chi \in \mathrm{Hom}(\overline{T}_{Q,n}, \mathbb{C}^\times)$ be a genuine character of $\overline{T}_{Q,n}$ and write ρ_χ for the splitting of ${}^L\overline{T}$ given by the LLC. Note that splittings of ${}^L\overline{T}$ over W_F are in bijection with splittings of the $E_1 + E_2$ extension (pulled-back to be) over $F^\times$. Recall that we have

$$E_1 = \overline{T}^{\vee} \times_{c_1} F^\times \quad \text{defined using the cocycle } c_1,$$

and

$$E_2 = \mathrm{Hom}(\mathscr{E}_{Q,n}, \mathbb{C}^\times)$$

with

$$\mathscr{E}_{Q,n} = Y_{Q,n} \times_D F^\times \quad \text{defined using a cocycle determined by the bisector } D.$$

With this identification, one has

$$\rho_\chi(a) = (1,a) +_{\mathrm{Baer}} \tilde{\chi}_a \in E_1 + E_2 \tag{8.1}$$

for $a \in F^\times$, where

- $(1,a) \in E_1 = \overline{T}^\vee \times_{c_1} F^\times$;
- $\tilde{\chi}_a \in E_2 = \mathrm{Hom}(\mathscr{E}_{Q,n}, \mathbb{C}^\times)$ is the character of $\mathscr{E}_{Q,n}$ given by

$$\tilde{\chi}_a(y,b) = (b,a)_n \cdot \chi(y(a)) \quad \text{for } (y,b) \in Y_{Q,n} \times_D F^\times,$$

  noting that $y(a) \in \overline{T}_{Q,n}$.

8.2. Construction for $\overline{T}$. — The covering torus $\overline{T}$ is a Heisenberg type group (cf. [**28, 44, 66**]), and one has a natural bijection

$$\{\text{genuine characters of } Z(\overline{T})\} \longleftrightarrow \{\text{irreducible genuine representations of } \overline{T}\}.$$

This bijection is defined as follows. Choose and fix a maximal abelian subgroup H of $\overline{T}$ containing $Z(\overline{T})$. Given a genuine character χ of $Z(\overline{T})$, extend χ arbitrarily to a character χ_H of H. Then the induced representation

$$i(\chi) = \mathrm{ind}_H^{\overline{T}} \chi_H$$

is irreducible and independent of the choice of (H, χ_H). By the analog of the Stone-von-Neumann theorem, it is characterized as the unique irreducible genuine representation of $\overline{T}$ which has central character χ.

8.3. LLC for $\overline{T}$. — Combining these, we obtain the following result which is the LLC for covering (split) tori:

Theorem 8.2. — *There is a natural injective map*

$$\mathscr{L}_{\overline{T}} : \{\textit{irreducible genuine representations of } \overline{T}\} \hookrightarrow \{\textit{Splittings of } {}^L\overline{T}\}.$$

The image of this injection can be described as follows. Tensoring the short exact sequence

$$0 \longrightarrow X \longrightarrow X_{Q,n} \longrightarrow X_{Q,n}/X \longrightarrow 0$$

with $\mathbb{C}^\times$, the associated long exact sequence gives:

$$0 \longrightarrow \mathrm{Tor}_1(X_{Q,n}/X, \mathbb{C}^\times) \longrightarrow X \otimes \mathbb{C}^\times = T^\vee \longrightarrow X_{Q,n} \otimes \mathbb{C}^\times = \overline{T}^\vee \longrightarrow 0.$$

This is a short exact sequence of W_F-modules and gives an exact sequence

$$H^1(W_F, T^\vee) \xrightarrow{f_*} H^1(W_F, \overline{T}^\vee) \xrightarrow{\delta} H^2(W_F, \mathrm{Tor}_1(X_{Q,n}/X, \mathbb{C}^\times)).$$

If we fix a distinguished splitting s_0 of ${}^L\overline{T}$, which corresponds to a genuine character χ_0 of $Z(\overline{T})$, then all other splittings of ${}^L\overline{T}$ are of the form $s = s_0 \cdot \rho$ with $\rho \in H^1(W_F, \overline{T}^\vee)$. This gives an identification

$$H^1(W_F, \overline{T}^\vee) = \{\text{Splittings of } {}^L\overline{T}\}.$$

Then one sees that a splitting s of ${}^L\overline{T}$ is in the image of the map $\mathscr{L}_{\overline{T}}$ if and only if s lies in the image of f_*, or equivalently

$$\delta(s) = 0 \in H^2(W_F, \mathrm{Tor}_1(X_{Q,n}/X, \mathbb{C}^\times)).$$

Note that if $X_{Q,n}/X \cong \prod_i \mathbb{Z}/n_i\mathbb{Z}$, then

$$\mathrm{Tor}_1(X_{Q,n}/X, \mathbb{C}^\times)) \cong \prod_i \mu_{n_i}(\mathbb{C}).$$

9. LLC for Unramified Representations

We consider the tame case in this section, so that p does not divide n. We assume that there is a splitting

$$s : K = \mathbb{G}(\mathcal{O}) \longrightarrow \overline{G}.$$

Then one may consider the s-unramified genuine representations of $\overline{G}$, i.e, those with nonzero $s(K)$-fixed vectors. We would like to obtain an LLC for such s-unramified genuine representations.

9.1. Torus case. — We first consider the case when $\mathbb{G} = \mathbb{T}$ is a split torus. Suppose that $\overline{\mathbb{T}}$ has bisector data D, so that

$$\overline{T} = T \times_D \mu_n.$$

In the tame case, the trivial section $y(a) \mapsto (y(a), 1) \in T \times_D \mu_n$ is a splitting over $\mathbb{T}(\mathcal{O})$ and any splitting $s : \mathbb{T}(\mathcal{O}) \longrightarrow \overline{T}$ is given by

$$s(t) = (t, \mu_s(t)) \quad \text{for } t \in \mathbb{T}(\mathcal{O})$$

where $\mu_s : \mathbb{T}(\mathcal{O}) \longrightarrow \mu_n$ is a group homomorphism. Such an s will give rise to a splitting of $\mathbb{T}_{Q,n}(\mathcal{O})$, denoted by s as well, via pulling back.

We call a genuine representation $i(\chi)$ (see §8.2) of $\overline{T}$ s-unramified if $i(\chi)$ has a nonzero vector fixed by $s(\mathbb{T}(\mathcal{O}))$. In this tame case, one can check that $H = Z(\overline{T}) \cdot s(\mathbb{T}(\mathcal{O}))$ is a maximal abelian subgroup of $\overline{T}$. From this, one deduces:

Lemma 9.1. — *A representation $i(\chi)$ is s-unramified if and only if χ is trivial when restricted to $Z(\overline{T}) \cap s(\mathbb{T}(\mathcal{O}))$. In this case, the space of $s(\mathbb{T}(\mathcal{O}))$-fixed vectors is 1-dimensional.*

We say that a genuine character of $Z(\overline{T})$ or $\overline{T}_{Q,n}$ is s-unramified if it is trivial on the image of s. Thus, the above lemma says that $i(\chi)$ is s-unramified if and only if χ is s-unramified. Such a χ will pullback to an s-unramified character of $\overline{T}_{Q,n}$. Conversely, observe that an s-unramified genuine character of $\overline{T}_{Q,n}$ automatically factors through to a genuine character of $Z(\overline{T})$ since $\mathrm{Ker}(h) \subset s(\mathbb{T}_{Q,n}(\mathcal{O}))$. Under the LLC for $\overline{T}$ defined in the last section, the s-unramified genuine characters correspond to a subset of splittings of ${}^L\overline{T}$. We would like to explicate this subset.

Lemma 9.2. — *The L-parameters $\rho_\chi : W_F \longrightarrow {}^L\overline{T}$ of the s-unramified characters χ have the same restriction ρ_s to the inertia group I_F.*

Proof. — We need to show that, as maps from $W_F^{ab} = F^\times$ to ${}^L\overline{T}$, all these ρ_χ's have the same restriction ρ_s on $\mathcal{O}^\times$. This follows from an examination of the construction of the LLC for $\overline{T}$. In particular, this restriction ρ_s can be described explicitly as follows.

Suppose an s-unramified χ is given. By (8.1), we see that for $a \in \mathcal{O}^\times$, $\rho_\chi(a) \in {}^L\overline{T}$ is determined by the character $\tilde{\chi}_a : \mathcal{E}_{Q,n} \longrightarrow \mathbb{C}^\times$ described there. Using the notations in (8.1), the s-unramified condition says that for $(y,b) \in \mathcal{E}_{Q,n}$,

$$\tilde{\chi}_a(y,b) = (b,a)_n \cdot \mu_s(y(a))^{-1},$$

which is independent of χ. □

We shall call a splitting ρ of ${}^L\overline{T}$ *s-unramified* if $\rho|_{I_F} = \rho_s$. Dividing ${}^L\overline{T}$ by $\rho_s(I_F)$, we obtain a short exact sequence

$$1 \longrightarrow \overline{T}^\vee \longrightarrow {}^L\overline{T}_s \xrightarrow{p} \mathbb{Z} \cdot \mathrm{Frob}_s \longrightarrow 1.$$

To summarize, we have shown:

Proposition 9.3. — *Under the LLC for covering tori, one has a bijection*

$$\{\textit{irreducible } s\textit{-unramified genuine representations of } \overline{T}\}$$
$$\updownarrow$$
$$\{\overline{T}^\vee\textit{-conjugacy classes of } s\textit{-unramified splittings } \rho \textit{ of } {}^L\overline{T}\ \}$$
$$\updownarrow$$
$$\{\overline{T}^\vee\textit{-orbits in } p^{-1}(\mathrm{Frob}_s)\}.$$

9.2. Satake isomorphism. — We now consider the case of general $\mathbb{G}$. The key step in understanding the s-unramified representations of $\overline{G}$ is the Satake isomorphism. More precisely, let $\mathcal{H}(\overline{G}, s)$ be the $\mathbb{C}$-algebra of anti-genuine locally constant, compactly supported functions on $\overline{G}$ which are bi-invariant under $s(\mathbb{G}(\mathcal{O}))$. Let $\mathcal{H}(\overline{T}, s)$ denote the analogous $\mathbb{C}$-algebra for $\overline{T}$. One can check that for an element $f \in \mathcal{H}(\overline{T}, s)$, the support of f is contained in $Z(\overline{T}) \cdot s(\mathbb{T}(\mathcal{O}))$, and thus f is completely determined by its restriction to $Z(\overline{T})$. Moreover, the Weyl group W acts naturally on $Z(\overline{T}) \cdot s(\mathbb{T}(\mathcal{O}))$ and thus on $\mathcal{H}(\overline{T}, s)$.

One has an explicit $\mathbb{C}$-algebra morphism

$$\mathcal{S} : \mathcal{H}(\overline{G}, s) \longrightarrow \mathcal{H}(\overline{T}, s)$$

given by

$$\mathcal{S}(f)(t) = \delta(t)^{1/2} \int_U f(tu)du \text{ for all } f \in \mathcal{H}(\overline{G}, s).$$

The following is shown in [**44, 39, 73**]:

Theorem 9.4. — *The Satake map $\mathcal{S}$ induces an isomorphism of $\mathbb{C}$-algebras:*

$$\mathcal{H}(\overline{G}, s) \cong \mathcal{H}(\overline{T}, s)^W.$$

As a consequence of this, one deduces a bijection

$$\{\text{irreducible } s\text{-unramified genuine representations of } \overline{G}\}$$
$$\updownarrow$$
$$\{\text{irreducible modules of } \mathscr{H}(\overline{G}, s)\}$$
$$\updownarrow$$
$$\{W\text{-orbits of } s\text{-unramified genuine characters of } Z(\overline{T})\}$$
$$\updownarrow$$
$$\{W\text{-orbits of } s\text{-unramified genuine characters of } \overline{T}_{Q,n}\}.$$

Explicitly, the bijection [**44**] is constructed as in the linear case. Namely, given a s-unramified genuine character χ of $\overline{T}_{Q,n}$, we saw in §9.1 and Lemma 9.1 that χ descends to a genuine character of $Z(\overline{T})$ and gives an irreducible s-unramified genuine representation $i(\chi)$ of $\overline{T}$. Now we set

$$I(\chi) = \mathrm{Ind}_{\overline{B}}^{\overline{G}} i(\chi),$$

where the induction is normalized. Then $I(\chi)$ has a 1-dimensional space of $s(K)$-fixed vectors and thus a unique irreducible constituent which is unramified. The action of $\mathscr{H}(\overline{G}, s)$ on this 1-dimensional space of $s(K)$-fixed vectors is easily calculated to be via the character:

$$f \mapsto \int_{\overline{T}} \chi(t) \cdot \mathscr{S}(f)(t)\, dt.$$

Note that this character of $\mathscr{H}(\overline{G}, s)$ depends only on the W-orbit of χ since $\mathscr{S}(f)$ is W-invariant. This shows that the $s(K)$-unramified representation associated to χ is the unique $s(K)$-unramified constituent of $I(\chi)$.

9.3. W-equivariance of LLC. — We would like to give an interpretation of spherical Hecke algebra in terms of the L-group ${}^L\overline{G}$. In view of the natural inclusion ${}^L\overline{T} \hookrightarrow {}^L\overline{G}$, it is natural to apply the LLC for $\overline{T}$ (or $\overline{T}_{Q,n}$) at this point. However, we first need to check the following proposition:

Theorem 9.5. — *The local Langlands correspondence for $\overline{T}$ is equivariant with respect to the action of the Weyl group W on both sides.*

Proof. — Let χ be a genuine character of $\overline{T}_{Q,n}$, and let ρ_χ be the associated splitting of the L-group given by LLC. It suffices to show that for any simple reflection $w = w_\alpha$, with $\alpha \in \Delta$,

$${}^{w}(\rho_\chi) = \rho_{{}^w\chi}.$$

The action of $N(\mathbb{T})$ on $\overline{\mathbb{T}}$ is given by formula (2.3). It induces actions of the Weyl group W on the topological cover $\overline{T}_{Q,n}$ (given by (6.7)), as well as the central extension $\mathscr{E}_{Q,n}$. The latter gives rise to an inherited action on E_2, and thus on ${}^L\overline{T}$. Note that the Weyl group acts trivially on E_1.

Fix $a \in F^\times$, by the explicit form of ρ_χ given in (8.1), it is enough show

$${}^w\tilde{\chi}_a = (\widetilde{{}^w\chi})_a \in \mathrm{Hom}(\mathscr{E}_{Q,n}, \mathbb{C}^\times).$$

Using the notations in (8.1), we need to verify that

$$
{}^{w}\tilde{\chi}_a(y,b) = (\widetilde{{}^{w}\chi})_a(y,b) \tag{9.6}
$$

for all $(y,b) \in \mathscr{E}_{Q,n}$.

For ease of notation, set

$$
y_\alpha = -\langle \alpha, y\rangle \cdot \alpha^\vee \in Y^{\mathrm{sc}}_{Q,n} \quad \text{since } n_\alpha \text{ divides } \langle \alpha, y\rangle
$$

and write y for $(y,1) \in \mathscr{E}_{Q,n}$. The left hand side of the desired identity (9.6) is

$$
\tilde{\chi}_a\big((y,b)\cdot s_{Q^{\mathrm{sc}}}(y_\alpha)\big) = (b,a)_n \cdot \chi(y(a)) \cdot \tilde{\chi}_a\big(s_{Q^{\mathrm{sc}}}(y_\alpha)\big).
$$

On the other hand, the right hand side of the desired identity (9.6) is

$$
(b,a)_n \cdot ({}^{w}\chi)\big(y(a)\big) = (b,a)_n \cdot \chi\big(y(a)\cdot s_\eta(y_\alpha(a))\big),
$$

where $s_\eta : T^{\mathrm{sc}}_{Q,n} \longrightarrow \overline{T}_{Q,n}$ was introduced in (6.2).

Thus, to obtain the equality (9.6), it now suffices to show

$$
\tilde{\chi}_a\big(\underbrace{s_{Q^{\mathrm{sc}}}(y_\alpha)}_{\in \mathscr{E}_{Q,n}}\big) = \chi\big(\underbrace{s_\eta(y_\alpha(a))}_{\in \overline{T}_{Q,n}}\big). \tag{9.7}
$$

However, on the LHS of (9.7),

$$
s_{Q^{\mathrm{sc}}}(y_\alpha) = (y_\alpha, \eta(y_\alpha)) \in \mathscr{E}_{Q,n},
$$

whereas on the RHS of (9.7),

$$
s_\eta(y_\alpha(a)) = \big(\eta(y_\alpha), a\big)_n \cdot y_\alpha(a) \in \overline{T}_{Q,n}.
$$

Thus, both sides of the desired identity (9.7) are equal to

$$
\big(\eta(y_\alpha), a\big)_n \cdot \chi\left(y_\alpha(a)\right).
$$

The proof of the theorem is thus completed. □

9.4. Passing to dual side. — Consequently, there is a bijection between

$$
\{W\text{-orbits of } s\text{-unramified genuine characters of } \overline{T}_{Q,n}\}
$$
$$
\updownarrow
$$
$$
\{W\text{-orbits of } s\text{-unramified splittings of } {}^L\overline{T}\}.
$$

We know from Lemma 9.2 that there is a splitting $\rho_s : I_F \hookrightarrow {}^L\overline{T}$ such that all s-unramified splittings of ${}^L\overline{T}$ restricts to ρ_s on I_F. Dividing out by $\rho_s(I_F)$, one has a commutative diagram of short exact sequences

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{T}^\vee & \longrightarrow & {}^L\overline{T}_s & \xrightarrow{p} & \mathbb{Z}\cdot \mathrm{Frob}_s & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_s & \xrightarrow{p} & \mathbb{Z}\cdot \mathrm{Frob}_s & \longrightarrow & 1.
\end{array}
$$

Hence, we have bijections

$$
\{W\text{-orbits of } s\text{-unramified splittings of } {}^L\overline{T}\}
$$

$$\updownarrow$$
$$\{W\text{-orbits of splittings of } {}^L\overline{T}_s\}$$
$$\updownarrow$$
$$\{\overline{G}^\vee\text{-orbits of splittings of } {}^L\overline{G}_s\}.$$

Here the last bijection can be seen by the arguments in [**10**, §6.3–§6.7].

9.5. Representation ring. — Finally, we note that the bijections above are induced by algebra isomorphisms

$$\mathscr{H}(\overline{T}, s)^W \simeq \mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s)^W \simeq \mathrm{Rep}({}^L\overline{G}_s, \mathrm{Frob}_s),$$

where $\mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s)$ denotes the algebra of functions on the preimage of Frob_s in ${}^L\overline{T}_s$ obtained as the restriction of the representation ring of ${}^L\overline{T}_s$ and $\mathrm{Rep}({}^L\overline{G}_s, \mathrm{Frob}_s)$ is analogously defined. Here the second isomorphism can be shown following [**10**, §6.3–§6.7] and the first can be reduced to the linear algebraic case as follows.

(1) We claim that there exists a W-invariant s-unramified genuine character χ of $\overline{T}_{Q,n}$, i.e., $\chi \circ s$ is trivial and $\chi\big(s_\eta((\langle \alpha, y\rangle \cdot \alpha^\vee)(a))\big) = 1$ for all $a \in F^\times, y \in Y_{Q,n}$ and $\alpha \in \Delta$. However, for $a \in \mathscr{O}_F^\times$, one has

$$s\big((\langle \alpha, y\rangle \cdot \alpha^\vee)(a)\big) = s_\eta\big((\langle \alpha, y\rangle \cdot \alpha^\vee)(a)\big), \quad \text{with } (\langle \alpha, y\rangle \cdot \alpha^\vee)(a) \in T^{\mathrm{sc}}_{Q,n}(\mathscr{O}),$$

since s_η is by definition given by the unique splitting on unipotent subgroups on which it agrees with s. Thus the two conditions are compatible, and by Pontryagin duality, there exists such unramified Weyl-invariant genuine character.

(2) Fix a χ as in (1). Dividing by χ gives a Weyl-equivariant algebra isomorphism

$$\mathscr{H}(\overline{T}, s) \cong \mathbb{C}[Y_{Q,n}].$$

The unramified character χ gives rise to an element $t_\chi \in p^{-1}(\mathrm{Frob}_s)$ under the LLC for $\overline{T}_{Q,n}$. By Theorem 9.5, t_χ is Weyl-invariant. Therefore, it gives a Weyl-equivariant algebra isomorphism

$$\mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s) \cong \mathrm{Rep}(\overline{T}^\vee).$$

It follows that there is a natural isomorphism

$$\mathscr{H}(\overline{T}, s)^W \simeq \mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s)^W,$$

which can be checked to be independent of the choice of χ.

To summarize, we have shown:

Theorem 9.8. — *The Satake isomorphism gives isomorphisms*

$$\mathscr{H}(\overline{G}, s) \simeq \mathscr{H}(\overline{T}, s)^W \simeq \mathrm{Rep}({}^L\overline{T}_s, \mathrm{Frob}_s)^W \simeq \mathrm{Rep}({}^L\overline{G}_s, \mathrm{Frob}_s),$$

which induces bijections

$$\{\textit{irreducible } s\textit{-unramified genuine representations of } \overline{G}\}$$
$$\updownarrow$$

$$\{W\text{-}orbits\ of\ s\text{-}unramified\ splittings\ of\ {}^L\overline{T}\}$$
$$\updownarrow$$
$$\{\overline{G}^\vee\text{-}orbits\ of\ s\text{-}unramified\ splittings\ of\ {}^L\overline{G}\}$$
$$\updownarrow$$
$$\{\overline{G}^\vee\text{-}orbits\ of\ semisimple\ elements\ in\ p^{-1}(\mathrm{Frob}_s)\}.$$

10. L-Groups: Second Take

After the discussion of the previous sections and a study of our running example, we may draw the following conclusions:

(i) For a fixed *fair* bisector D, and among all BD covering groups (of degree n) with bisector data (D, η), those with $\eta_n = 1$ are most nicely behaved. For example, their maximal covering tori $\overline{T}$ have certain distinguished Weyl-invariant genuine representations and $\overline{G}$ splits over the hyperspecial maximal compact subgroup $G(\mathcal{O})$. Moreover, their L-groups are isomorphic to a direct product $\overline{G}^\vee \times W_F$.

(ii) The BD covering groups $\overline{G}_\eta$ for a fixed bisector data are closely related, and it may be useful to consider them together, both structurally as well as from the point of view of representation theory. For example, they all have the same dual group $\overline{G}_Q^\vee$.

In this section, we would like to suggest a slightly different take on the L-group extension, so as to treat the closely related groups $\overline{G}_\eta$ together.

10.1. The case $Q = 0$. — To guide our efforts, we shall consider the genuine representation theory of the covering groups in the case when $Q = 0$. This expands upon our running example and will provide a clue about the modifications needed.

When $Q = 0 = D$, the objects in $\mathbf{Bis}_{\mathbb{G},Q}$ are simply homomorphisms $\eta : Y^{\mathrm{sc}} \to F^\times$. Choose a z-extension

$$1 \longrightarrow Z \longrightarrow \mathbb{H} \longrightarrow \mathbb{G} \longrightarrow 1,$$

so that $Y_{\mathbb{H}}^{\mathrm{sc}} = Y^{\mathrm{sc}}$ and $Y_{\mathbb{H}}/Y^{\mathrm{sc}}$ is free. For any η, we have a corresponding short exact sequence

$$1 \longrightarrow Z = Z_\eta \longrightarrow \overline{\mathbb{H}}_\eta \longrightarrow \overline{\mathbb{G}}_\eta \longrightarrow 1.$$

Since all $\eta : Y_{\mathbb{H}}^{\mathrm{sc}} = Y^{\mathrm{sc}} \to F^\times$ are equivalent to the trivial homomorphism 1 as objects of $\mathbf{Bis}_{\mathbb{H},Q}$, we may choose an isomorphism

$$\xi : \overline{\mathbb{H}}_1 = \mathbb{H} \times \mathbb{K}_2 \longrightarrow \overline{\mathbb{H}}_\eta.$$

After taking F-points, and noting that $H^1_{Zar}(F, Z) = 0$, we then have

$$\overline{H}_1 = H \times \mu_n \xrightarrow{\ \xi\ } \overline{H}_\eta \longrightarrow \overline{G}_\eta,$$

and the kernel of this map is the subgroup

$$\xi^{-1}(Z_\eta) = \{(z, \chi_{\eta,\xi}(z)^{-1}) : z \in Z\} \subset H \times \mu_n,$$

where $\chi_{\eta,\xi}$ is the map

$$Z = Y_Z \otimes F^\times \xrightarrow{\ \xi\ } F^\times \otimes F^\times \xrightarrow{(-,-)_n} \mu_n.$$

Hence, the set of genuine representations of $\overline{G}_\eta$ can be identified (by pulling back) with a subset of the genuine representations of the split extension $H \times \mu_n$ whose restriction to the central subgroup $Z \subset H$ is the character $\chi_{\eta,\xi}$.

Now the L-group of $\overline{H} = H \times \mu_n$ is a short exact sequence

$$H^\vee \longrightarrow {}^L\overline{H} \longrightarrow W_F$$

which is equipped with a finite set of distinguished splittings. For example one may take the distinguished splitting s_0 which corresponds to the trivial character of the maximal torus T_H of H. Then we may identify the set of all splittings (modulo conjugacy by $H^\vee$) with the set of L-parameters

$$W_F \longrightarrow H^\vee$$

of H. Thus, if the LLC holds for the linear group H, there is a finite-to-one map

$$\bigcup_\eta \mathrm{Irr}_{\mathrm{gen}}(\overline{G}_\eta) \longrightarrow \{\text{L-parameters for } H\}$$

which may be construed as a (weak) LLC for the family of covering groups $\overline{G}_\eta$ (as η varies). Moreover, the image of $\mathrm{Irr}_{\mathrm{gen}}(\overline{G}_\eta)$ for a particular η can be described as follows. By Lemma 5.4, there is a natural short exact sequence:

$$1 \longrightarrow G^\vee \longrightarrow H^\vee \xrightarrow{\ \rho\ } Z^\vee = \mathrm{Hom}(Y_Z, \mathbb{C}^\times) \longrightarrow 1.$$

Then $\mathrm{Irr}_{\mathrm{gen}}(\overline{G}_\eta)$ corresponds to the set of L-parameters ϕ of H such that $\rho\circ\phi : W_F \longrightarrow Z^\vee$ is the L-parameter of the character $\chi_{\eta,\xi}$. Observe that the genuine representations of $\overline{G}_1 = G \times \mu_n$ is then parametrized by L-parameters of H which factors through $G^\vee$.

There is an obvious notion of a z-extension H dominating another H', and one can easily check that the above classification of the genuine representations of $\overline{G}_\eta$ behave functorially with respect to dominance. This suggests that it is possible (and certainly desirable) to formulate the LLC for $\overline{G}_\eta$ without reference to the z-extension H. Such use of z-extensions is similar to the use of z-extensions in the usual theory of endoscopy for linear groups.

10.2. Modification of L-group. — Motivated by the above discussion, we can revisit the L-group construction in the general setting. Fix the quadratic form Q on Y. The crucial E_2 construction starts with

$$1 \longrightarrow F^\times/F^{\times n} \longrightarrow \mathcal{E}_{Q,n} \longrightarrow Y_{Q,n} \longrightarrow 1$$

and then use the section

$$s_\eta : Y_{Q,n}^{\mathrm{sc}} \longrightarrow \mathcal{E}_{Q,n}$$

to form the quotient

$$1 \longrightarrow F^\times/F^{\times n} \longrightarrow \mathcal{E}_{Q,n}/s_\eta(Y_{Q,n}^{\mathrm{sc}}) \longrightarrow Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}} \longrightarrow 1,$$

before applying $\mathrm{Hom}(-,\mathbb{C}^\times)$. To incorporate all η's together, we observe that the section s_η is independent of η when restricted to the sublattice $nY_{Q,n}^{\mathrm{sc}}$. Then one has the commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & F^\times/F^{\times n} \arrow[r] \arrow[d, equal] & \mathcal{E}_{Q,n}/s_\eta(nY_{Q,n}^{\mathrm{sc}}) \arrow[r] \arrow[d] & Y_{Q,n}/nY_{Q,n}^{\mathrm{sc}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F^\times/F^{\times n} \arrow[r] & \mathcal{E}_{Q,n}/s_1(Y_{Q,n}^{\mathrm{sc}}) \arrow[r] & Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}} \arrow[r] & 1.
\end{tikzcd}$$

Taking $\mathrm{Hom}(-,\mathbb{C}^\times)$, one obtains the commutative diagram with exact rows and columns, which defines the modification $\tilde{E}_2$ of E_2:

$$\begin{tikzcd}
Z(\overline{G}_1^\vee) = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}},\mathbb{C}^\times) \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & F^\times/F^{\times n} \arrow[d, equal] \\
\mathrm{Hom}(Y_{Q,n}/nY_{Q,n}^{\mathrm{sc}},\mathbb{C}^\times) \arrow[r] \arrow[d] & \tilde{E}_2 \arrow[r] \arrow[d] & F^\times/F^{\times n} \arrow[d] \\
(T_{Q,n}^{\mathrm{sc}})^\vee[n] = \mathrm{Hom}(Y_{Q,n}^{\mathrm{sc}}/nY_{Q,n}^{\mathrm{sc}},\mathbb{C}^\times) \arrow[r, equal] & (T_{Q,n}^{\mathrm{sc}})^\vee[n] \arrow[r] & 1.
\end{tikzcd}$$

The cocycle defining E_1 also defines an extension

$$1 \longrightarrow \mathrm{Hom}(Y_{Q,n}/nY_{Q,n}^{\mathrm{sc}},\mathbb{C}^\times) \longrightarrow \tilde{E}_1 \longrightarrow F^\times/F^{\times n} \longrightarrow 1.$$

Then we can form the Baer sum and obtain

$$\begin{tikzcd}
Z(\overline{G}_1^\vee) \arrow[r] \arrow[d] & E = E_1 + E_2 \arrow[r] \arrow[d] & F^\times/F^{\times n} \arrow[d, equal] \\
\mathrm{Hom}(Y_{Q,n}/nY_{Q,n}^{\mathrm{sc}},\mathbb{C}^\times) \arrow[r] \arrow[d] & \tilde{E} = \tilde{E}_1 + \tilde{E}_2 \arrow[r] \arrow[d] & F^\times/F^{\times n} \arrow[d] \\
(T_{Q,n}^{\mathrm{sc}})^\vee[n] \arrow[r, equal] & (T_{Q,n}^{\mathrm{sc}})^\vee[n] \arrow[r] & 1.
\end{tikzcd}$$

From this, we infer the short exact sequence:

$$1 \longrightarrow Z(\overline{G}_1^\vee) \longrightarrow \tilde{E} \longrightarrow F^\times/F^{\times n} \times (T_{Q,n}^{\mathrm{sc}})^\vee[n] \longrightarrow 1.$$

This is our enlarged fundamental extension. Pushing this out by $Z(\overline{G}_1^\vee) \hookrightarrow \overline{G}^\vee$ and pulling back to W_F, one obtains:

$$1 \longrightarrow \overline{G}_Q^\vee \longrightarrow {}^L\overline{G}_Q^{\#} \longrightarrow W_F \times (T_{Q,n}^{\mathrm{sc}})^\vee[n] \longrightarrow 1,$$

which is our enlarged L-group extension for the family of BD covers with fixed BD-invariant Q. Here we also use the notation $\overline{G}_Q^\vee$ for $\overline{G}^\vee$.

Note that $(T_{Q,n}^{\mathrm{sc}})^\vee$ is a maximal torus in the adjoint quotient $(\overline{G}^\vee)_{ad}$ of $\overline{G}^\vee$, so that $(T_{Q,n}^{\mathrm{sc}})^\vee[n]$ is its n-torsion subgroup.

10.3. Relation with $^L\overline{G}_\eta$. — How can one recover the L-group of $\overline{G}_\eta$, as previously defined, from the enlarged L-group defined here? Given an $\eta : Y^{\mathrm{sc}}_{Q,n} \longrightarrow F^\times$, one obtains a natural map

$$\varphi_\eta : W_F \longrightarrow W_F^{ab} = F^\times \longrightarrow (T^{\mathrm{sc}}_{Q,n})^\vee[n] = \mathrm{Hom}(Y^{\mathrm{sc}}_{Q,n}, \mu_n)$$

given by

$$\varphi_\eta(a)(y) = (\eta(y), a)_n \quad \text{for all } a \in F^\times \text{ and } y \in Y^{\mathrm{sc}}_{Q,n}.$$

Pulling back the enlarged L-group extension by the diagonal map

$$\mathrm{id} \times \varphi_\eta : W_F \longrightarrow W_F \times (T^{\mathrm{sc}}_{Q,n})^\vee[n],$$

one obtains the L-group extension $^L\overline{G}_\eta$.

10.4. The modified dual group. — Consider the kernel $\overline{G}^\#_Q \subseteq {}^L\overline{G}^\#_Q$ of the following composition of surjections:

$$^L\overline{G}^\#_Q \longrightarrow W_F \times (T^{\mathrm{sc}}_{Q,n})^\vee[n] \longrightarrow W_F,$$

where the second map is the projection on the first component. By the definition of $^L\overline{G}^\#_Q$, the group $\overline{G}^\#_Q$ lies in the exact sequence

$$1 \longrightarrow \overline{G}^\vee_Q \longrightarrow \overline{G}^\#_Q \longrightarrow (T^{\mathrm{sc}}_{Q,n})^\vee[n] \longrightarrow 1, \tag{10.1}$$

which is the push out of

$$1 \longrightarrow Z(\overline{G}^\vee_Q) \longrightarrow \mathrm{Hom}(Y_{Q,n}/nY^{\mathrm{sc}}_{Q,n}, \mathbb{C}^\times) \longrightarrow (T^{\mathrm{sc}}_{Q,n})^\vee[n] \longrightarrow 1. \tag{10.2}$$

Since (10.1) can be obtained from (10.2) by first pushing out via $Z(\overline{G}^\vee_Q) \to \overline{T}^\vee$ and there is a natural map $\mathrm{Hom}(Y_{Q,n}/nY^{\mathrm{sc}}_{Q,n}, \mathbb{C}^\times) \to \overline{T}^\vee$, there is a canonical splitting of (10.1) which gives an isomorphism

$$\overline{G}^\#_Q \simeq \overline{G}^\vee_Q \rtimes (T^{\mathrm{sc}}_{Q,n})^\vee[n].$$

Moreover, the action of $(T^{\mathrm{sc}}_{Q,n})^\vee[n]$ on $\overline{G}^\vee$ is identity on its maximal torus $\overline{T}^\vee = X_{Q,n} \otimes \mathbb{C}^\times$ and preserves the maximal unipotent subgroup corresponding to the set of simple roots $\Delta^\vee_{Q,n}$. This shows that every irreducible representation of $\overline{G}^\vee$ is invariant under the action of $(T^{\mathrm{sc}}_{Q,n})^\vee[n]$, and thus extends (non-canonically) to $\overline{G}^\#_Q$. In other words, the representation theory of the disconnected group $\overline{G}^\#_Q$ is not more complicated than that of $\overline{G}^\vee$.

10.5. Running example. — Consider the case $\mathbb{G} = \mathrm{PGL}_2$ and $D = 0$, $n = 2$. The dual group is $\overline{G}_Q^\vee = \mathrm{SL}_2(\mathbb{C})$ and the exact sequence (10.2) is

$$1 \longrightarrow \mu_2 \longrightarrow \mu_4 \longrightarrow \mu_2 \longrightarrow 1.$$

We obtain $\overline{G}_Q^\# \simeq \mathrm{SL}_2(\mathbb{C}) \rtimes \mu_2$. The action of the nontrivial element $\epsilon \in \mu_2$ on $\mathrm{SL}_2(\mathbb{C})$ is given by

$$\epsilon : g \mapsto \begin{pmatrix} \xi & \\ & \xi^{-1} \end{pmatrix} g \begin{pmatrix} \xi & \\ & \xi^{-1} \end{pmatrix}^{-1},$$

where $\xi \in \mu_4$ is any square root of ϵ. As there is no splitting of $\overline{G}_Q^\#$ over μ_2 valued in the center $Z(\overline{G}_Q^\vee)$, the group $\overline{G}_Q^\#$ is not isomorphic to the direct product $\mathrm{SL}_2(\mathbb{C}) \times \mu_2$.

This example shows that in general $\overline{G}_Q^\# \simeq \overline{G}_Q^\vee \rtimes (T_{Q,n}^{\mathrm{sc}})^\vee[n]$ is not a direct product of the two groups.

11. The LLC

After the discussion in the previous sections, we can now formulate the LLC for BD covering groups.

11.1. L-parameters. — After introducing the L-group extension, one now has the following notions:

- an L-parameter for the covering group $\overline{G}_\eta$ is a splitting $\phi : W_F \longrightarrow {}^L\overline{G}_\eta$ of the extension ${}^L\overline{G}_\eta$, taken up to conjugacy by $\overline{G}^\vee$. Equivalently, it is a splitting $\phi : W_F \longrightarrow {}^L\overline{G}_Q^\#$ of the enlarged L-group extension such that
$$p \circ \phi = \varphi_\eta,$$
where $p : {}^L\overline{G}_Q^\# \longrightarrow (T_{Q,n}^{\mathrm{sc}})^\vee[n]$ is the natural projection;
- we have demonstrated the existence of a finite set of distinguished splittings for ${}^L\overline{G}_1$ and thus for ${}^L\overline{G}_Q^\#$. If we fix one such splitting ϕ_0, then all splittings of ${}^L\overline{G}_Q^\#$ are of the form $\phi_0 \cdot \phi$ where
$$\phi : W_F \longrightarrow \overline{G}_Q^\# = \overline{G}_Q^\vee \rtimes (T_{Q,n}^{\mathrm{sc}})^\vee[n].$$
We call such ϕ's the L-parameters relative to the distinguished splitting ϕ_0.

11.2. Local L-factors. — Given a representation

$$R : {}^L\overline{G}_\eta \longrightarrow \mathrm{GL}(V)$$

where V is a complex finite-dimensional vector space over $\mathbb{C}$, and a splitting ϕ of ${}^L\overline{G}_\eta$, one obtains a complex representation $R \circ \phi$ of W_F and hence an Artin L-factor $L(s, \phi, R)$. Alternatively, if $\phi : W_F \longrightarrow \overline{G}^\#$ is an L-parameter relative to a distinguished splitting ϕ_0 of ${}^L\overline{G}^\#$ over W_F, and $R : \overline{G}^\# \longrightarrow \mathrm{GL}(V)$ is a representation, then one has an associated L-factor $L_{\phi_0}(s, \phi, R)$. As we noted before, the irreducible representations of $\overline{G}^{\#}$ are simply extensions of those of $\overline{G}^{\vee}$. We shall give a more detailed treatment of this in the next section, where we introduce automorphic L-functions.

11.3. The LLC. — In view of the unramified LLC discussed in Section 9, one is tempted to conjecture the existence of a finite-to-one map giving rise to a commutative diagram:

$$\begin{array}{ccc} \mathcal{L}_\eta : \mathrm{Irr}\overline{G}_\eta & \xrightarrow{\mathcal{L}_\eta} & \{\text{splittings of } {}^L\overline{G}_\eta\} \\ \downarrow & & \downarrow \\ \bigcup_\eta \mathrm{Irr}\overline{G}_\eta & \xrightarrow{\mathcal{L}} & \{\text{splittings of } {}^L\overline{G}_Q^{\#}\}. \end{array}$$

This is a weak LLC. As shown in the case when $\mathbb{G} = \mathbb{T}$ is a split torus, one should not expect this map to be surjective. Thus, one would also like to have a conjectural parametrization of the fibers of this map. This would be a strong LLC.

11.4. Reduction to $\eta = 1$. — We shall show that this weak LLC for general $\overline{G}_\eta$ can be reduced to the case of trivial η. This is similar to the discussion at the beginning of the last section and relies on the consideration of z-extensions.

More precisely, we choose a z-extension $1 \longrightarrow Z \longrightarrow \mathbb{H} \longrightarrow \mathbb{G} \longrightarrow 1$ giving rise to

$$1 \longrightarrow Z \longrightarrow \overline{\mathbb{H}}_\eta \xrightarrow{p} \overline{\mathbb{G}}_\eta \longrightarrow 1.$$

Choose an isomorphism $\xi : \overline{\mathbb{H}}_1 \longrightarrow \overline{\mathbb{H}}_\eta$, which realises

$$\overline{\mathbb{G}}_\eta \cong \overline{\mathbb{H}}_1/\xi^{-1}(Z).$$

Thus one has an injection

$$\xi^* \circ p^* : \mathrm{Irr}\overline{G}_\eta \hookrightarrow \mathrm{Irr}\overline{H}_1$$

whose image consists of those irreducible genuine representations of $\overline{H}_1$ whose restriction to Z is a prescribed character χ_ξ. If the LLC holds for the case of trivial η, then one would have a map

$$\mathcal{L} \circ \xi^* \circ p^* : \mathrm{Irr}\overline{G}_\eta \longrightarrow \{\text{splittings of } {}^L\overline{H}_1\}.$$

Now recall that by Lemma 5.4, there is a natural map

$$p : {}^L\overline{H}_1 \longrightarrow Z^\vee.$$

If one assumes the (weak) LLC for $\overline{H}_1$ satisfies the natural property that the restriction of the central character of $\pi \in \mathrm{Irr}(\overline{H}_1)$ to Z corresponds to the parameter $p \circ \mathcal{L}(\pi)$ under the usual LLC for the (linear) torus Z, then one sees that

$$\mathrm{Irr}\overline{G}_\eta \longrightarrow \{\text{splittings } \phi \text{ of } {}^L\overline{H}_1\text{: } p \circ \phi \text{ corresponds to } \chi_\xi\} = \{\text{splittings of } {}^L\overline{G}_\eta\}.$$

The relation with the enlarged L-group ${}^L\overline{G}^{\#}$ is as follows. One has a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \overline{G}^{\#} \arrow[r] \arrow[d] & {}^L\overline{G}^{\#} \arrow[r] \arrow[d] & W_F \arrow[r] \arrow[d, "id"] & 1 \\ 1 \arrow[r] & \overline{H}_1^{\vee} \arrow[r] & {}^L\overline{H}_1 \arrow[r] & W_F \arrow[r] & 1, \end{tikzcd}$$

and the L-parameters of $\overline{H}_1$ which intervene in the description of the LLC for $\overline{G}_\eta$ (as η varies) are those which factors through ${}^L\overline{G}^{\#}$.

11.5. Reduction to discrete series. — The existence of the (weak) LLC map $\mathscr{L}$ can be reduced to the case of (quasi)-discrete series representation, much like the case of linear reductive groups. More precisely,

- Ban and Jantzen [**7**] have established the analog of the Langlands classification for general covering groups over p-adic fields; the case of covers of real groups is also known [**11**]. This says that every irreducible representation is uniquely expressed as the unique irreducible quotient of a standard module. As in the linear case, this reduces the definition of $\mathscr{L}$ to (quasi)-tempered representations.
- As shown in the work of W.-W. Li [**36**], any tempered representation is contained in a representation parabolically induced from (quasi-)discrete series representations; moreover the decomposition of this induced representation is governed by an R-group [**36**]. This reduces the definition of $\mathscr{L}$ to the case of (quasi-)discrete series representations.

11.6. The example of Mp_{2n}**. —** The only nontrivial case where one has rather complete results for the LLC is the classical case of the metaplectic double covering group $\overline{G} = \mathrm{Mp}_{2n}(F)$ of $\mathrm{Sp}_{2n}(F)$. In this case, the L-group ${}^L\overline{G}$ is isomorphic (non canonically) to the direct product $\mathrm{Sp}_{2n}(\mathbb{C}) \times W_F$, and such an isomorphism is given by the choice of a distinguished genuine character of the covering torus $\overline{T} \subset \mathrm{Mp}_{2n}$. In §16.1.2, we show that such a distinguished genuine character of $\overline{T}$ is simply a genuine character χ_ψ defined using the Weil index associated to a nontrivial additive character ψ of F. Thus the choice of ψ gives a bijection

$$\mathscr{S}_\psi : \{\text{splittings of } {}^L\overline{G} \text{ over } WD_F\} \longleftrightarrow \{\text{homomorphisms } WD_F \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})\}.$$

On the other hand, using the theory of theta correspondence, it was shown in [**3**] (for archimedean F) and [**25**] (for nonarchimedean F) that there is a bijection

$$\Theta_\psi : \mathrm{Irr}_{\mathrm{gen}}(\mathrm{Mp}_{2n}(F)) \longleftrightarrow \bigcup_{V_n} \mathrm{Irr}(\mathrm{SO}(V_n))$$

where V_n runs over all isomorphism classes of quadratic spaces of dimension $2n+1$ and trivial discriminant. Combining this with the LLC for odd special orthogonal groups (due to Arthur [**6**] and Moeglin [**46**]), one obtains an LLC map

$$\mathscr{L}_\psi : \mathrm{Irr}_{\mathrm{gen}}(\mathrm{Mp}_{2n}(F)) \longrightarrow \{\text{homomorphisms } WD_F \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})\}.$$

We thus have a surjective map

$$\mathcal{S}_\psi^{-1} \circ \mathcal{L}_\psi : \mathrm{Irr}_{\mathrm{gen}}(\mathrm{Mp}_{2n}(F)) \longrightarrow \{\text{L-parameters for } \mathrm{Mp}_{2n}\}.$$

The main observation we want to make is:

Theorem 11.1. — *The composite $\mathcal{S}_\psi^{-1} \circ \mathcal{L}_\psi$ is independent of the choice of ψ. In particular, assuming the LLC for odd special orthogonal groups, one has an LLC for $\mathrm{Mp}_{2n}(F)$ in terms of the L-group considered in this paper.*

Proof. — All nontrivial characters of F are of the form $\psi_a(x) = \psi(ax)$ for some $a \in F^\times$. The corresponding distinguished characters of $\overline{T}$ are related by

$$\chi_{\psi_a} = \chi_\psi \cdot \chi_a$$

where $\chi_a = (a, -)_2$. Thus, the distinguished splittings s_ψ and s_{ψ_a} differ by the quadratic character χ_a (regarded as a map $W_F \longrightarrow \mu_2 \subset \overline{T}^\vee$), and the bijections $\mathcal{S}_\psi$ and $\mathcal{S}_{\psi_a}$ differ by twisting by χ_a. On the other hand, it was shown in **[3, 25]** that $\mathcal{L}_{\psi_a}$ and $\mathcal{L}_\psi$ are also related by twisting by χ_a. It follows that the two dependence on a cancel and $\mathcal{S}_\psi^{-1} \circ \mathcal{L}_\psi$ is independent of the choice of ψ. ☐

12. Desiderata and Anomalies

In this section, we shall explain how the L-group formalism developed thus far allows one to explain (at least conjecturally) certain anomalies which have been empirically observed in the genuine representation theory of covering groups.

12.1. Central characters. — In the LLC for Mp_{2n} discussed at the end of the previous section, it is known that the central character is not an invariant of an L-packet. This is in contrast to the case of linear groups, where all representations in a given L-packet have the same central character. Let us see how this anomaly is explained by the L-group formalism.

For a BD covering group $\overline{G}$, we have attached an L-group ${}^L\overline{G}$, which is abstractly isomorphic to a semi-direct product. Let us consider the case when ${}^L\overline{G}$ has a distinguished splitting, so that ${}^L\overline{G} \cong \overline{G}^\vee \times W_F$. Let $\mathbb{G}_{Q,n}$ be the split linear algebraic group over F, with dual group $\overline{G}^\vee$ (this $\mathbb{G}_{Q,n}$ should be the principal endoscopic group of $\overline{G}$, as we discuss later on). Then by construction $G_{Q,n} := \mathbb{G}_{Q,n}(F)$ contains $T_{Q,n}$ as maximal split torus. Now an L-parameter ϕ of $\overline{G}$ can be regarded as one for $G_{Q,n}$ (relative to the choice of a distinguished splitting). Such an L-parameter certainly encodes the central characters of representations of $G_{Q,n}$ in the associated L-packet.

Let us compare the centres of $G_{Q,n}$ and $\overline{G}$, which are contained in $T_{Q,n}$ and $\overline{T}$ respectively. One has

$$Z(G_{Q,n}) \cong \mathrm{Hom}(X_{Q,n}/X_{Q,n}^{\mathrm{sc}}, F^\times) \subset T_{Q,n}$$

and

$$p(Z(\overline{G})) = Z(G) \cap p(Z(\overline{T})) \subset \mathrm{Hom}(X/X^{\mathrm{sc}}, F^\times) \subset T,$$

where $p : \overline{G} \longrightarrow G$ is the natural projection. Moreover, recall that there is an isogeny

$$i : T_{Q,n} \longrightarrow T,$$

associated to the natural embedding $Y_{Q,n} \hookrightarrow Y$ and such that

$$i(T_{Q,n}) = p(Z(\overline{T})).$$

It is easy to check that

$$i(Z(G_{Q,n})) \subset p(Z(\overline{G})).$$

This leads us to the following:

Speculation. — under the LLC for $\overline{G}$, for an L-parameter ϕ for $\overline{G}$ with associated L-packet Π_ϕ, all representations in Π_ϕ transform by the same character (determined by ϕ) when restricted to the preimage in $\overline{G}$ of the central subgroup $i(Z(G_{Q,n}))$. In particular, this leaves open the possibility for the representations in Π_ϕ to transform by different characters under the whole center $Z(\overline{G})$.

If we apply this to the case of $\overline{G} = \mathrm{Mp}_{2n}$, then $\mathbb{G}_{Q,n} = \mathrm{SO}_{2n+1}$ so that $Z(G_{Q,n})$ is trivial, whereas $p(Z(\overline{G}))$ is $\mu_2(F)$. Thus, both genuine central characters are allowed in a given L-packet, as has been observed.

12.2. Twisting by characters. — Suppose that π is an irreducible genuine representation of $\overline{G}$ with L-parameter $\phi : W_F \longrightarrow {}^L\overline{G}$. Let $\chi : G \longrightarrow \mathbb{C}^\times$ be a 1-dimensional character. Then we may consider the irreducible genuine representation $\pi \otimes \chi$. What should be the L-parameter of $\pi \otimes \chi$?

Let $\phi_\chi : W_F \longrightarrow G^\vee$ be the L-parameter of χ, with $G^\vee$ the Langlands dual group of G. One would like to twist the L-parameter ϕ of π by ϕ_χ, but the two parameters take value in different groups. However, one knows that ϕ_χ factors through the center $Z(G^\vee)$ of $G^\vee$:

$$\phi_\chi : W_F \longrightarrow Z(G^\vee) \longrightarrow G^\vee.$$

On the other hand, one has a natural map

$$\delta : Z(G^\vee) = \mathrm{Hom}(Y/Y^{\mathrm{sc}}, \mathbb{C}^\times) \longrightarrow \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times)$$

given by restricting from Y to $Y_{Q,n}$. Then one has the composite

$$\delta \circ \phi_\chi : W_F \longrightarrow Z(G^\vee) \longrightarrow Z(\overline{G}^\vee) \subset \overline{G}^\vee.$$

Now it is natural to have the following expectation:

Speculation. — If ϕ is the L-parameter of the irreducible genuine representation π of $\overline{G}$, then the L-parameter of $\pi \otimes \chi$ is given by $\phi \otimes (\delta \circ \phi_\chi)$.

12.3. Restriction to derived subgroups. — Another anomaly concerns the restriction of representations of $\overline{G}$ to, for example, $\overline{G}^{\mathrm{der}}$ or $\overline{G}^{\mathrm{sc}}$. For example, in the linear case, all irreducible summands in the restriction of an irreducible representations of $\mathrm{GL}_2(F)$ to $\mathrm{SL}_2(F)$ belongs to the same L-packet. However, if one takes the degree 2 Kazhdan-Patterson cover $\overline{\mathrm{GL}}_2$ of $\mathrm{GL}_2(F)$, then the restriction of an irreducible genuine representation σ of $\overline{\mathrm{GL}}_2$ to $\overline{\mathrm{SL}}_2 = \mathrm{Mp}_2$ may contain constituents belonging to different L-packets of Mp_2. More precisely, one has

$$\sigma|_{\mathrm{Mp}_2} \cong \bigoplus_{\chi} \pi_\chi$$

where the sum runs over quadratic characters χ of $F^\times$. Moreover, if the L-parameter of π_1 is ϕ, then that of π_χ is $\phi \otimes \chi$. How can this be explained by the L-group formalism?

Let $\mathbb{G}$ be a linear reductive group over F and

$$\mathbb{G}^{\mathrm{sc}} \longrightarrow \mathbb{G}^{\mathrm{der}} \longrightarrow \mathbb{G}$$

the natural sequence of maps. In forming the L-group of $\overline{G}$, we consider the lattices

$$\begin{array}{ccc} Y^{\mathrm{sc}}_{Q,n} & \longrightarrow & Y^{\mathrm{sc}} \\ \big\downarrow & & \big\downarrow \\ Y_{Q,n} & \longrightarrow & Y. \end{array}$$

Here the arrows are inclusions, and the first column is used to define the root datum for the dual group of $\overline{G}$. Now we may pullback the BD cover to G^{sc} and construct the L-group of $\overline{G}^{\mathrm{sc}}$; this gives the diagram, writing $Z = Y^{\mathrm{sc}}$ for readability:

$$\begin{array}{ccc} Z^{\mathrm{sc}}_{Q,n} & \longrightarrow & Z \\ \big\downarrow & & \big\| \\ Z_{Q,n} & \longrightarrow & Z. \end{array}$$

Here the first column is used to form the dual group of $\overline{G}^{\mathrm{sc}}$.

Let's examine how these two diagrams interact. One has:

$$\begin{array}{ccccc} Y_{Q,n} & \longleftarrow & Y_{Q,n} \cap Z & \longleftarrow & Y^{\mathrm{sc}}_{Q,n} \\ & & \big\| & & \big\| \\ (Y^{\mathrm{sc}})_{Q,n} = Z_{Q,n} & \longleftarrow & Y_{Q,n} \cap Z & \longleftarrow & Z^{\mathrm{sc}}_{Q,n} \end{array}$$

and the point is that there is an inclusion

$$Z_{Q,n} \supset Y_{Q,n} \cap Z = Y_{Q,n} \cap Y^{\mathrm{sc}}$$

which is not necessarily an equality (it would be if $n = 1$).

This means that one has the following diagram of dual groups:

$$\overline{G}^\vee \longrightarrow H^\vee \longleftarrow (\overline{G}^{\mathrm{sc}})^\vee$$

where $H^\vee$ is the connected reductive group with root datum

$$(Y_{Q,n} \cap Z, \Delta^\vee_{Q,n}, \text{dual lattice of } Y_{Q,n} \cap Z, \Delta_{Q,n}),$$

and the second arrow is an isogeny. Now suppose one is given an L-parameter for $\overline{G}$, i.e.,

$$\phi : W_F \longrightarrow \overline{G}^\vee.$$

Let

$$\overline{\phi} : W_F \longrightarrow \overline{G}^\vee \longrightarrow H^\vee.$$

This leads to the following speculation:

Speculation. — If one takes a representation π in the associated L-packet Π_ϕ, then the pullback of π to $\overline{G}^{\mathrm{sc}}$ will decompose into irreducible summands. The L-parameters of these summands are given by those

$$\phi' : W_F \longrightarrow (\overline{G}^{\mathrm{sc}})^\vee$$

such that

$$\overline{\phi} = \overline{\phi}' \quad \text{i.e., equality when both } \phi \text{ and } \phi' \text{ are projected to } H^\vee.$$

Since the projection from $(\overline{G}^{\mathrm{sc}})^\vee$ to $H^\vee$ is not an isomorphism in general, it is possible for several ϕ' to arise. This explains the above phenomenon in the examples mentioned above.

13. Automorphic L-functions

While we have considered only the case of local fields for most of this paper, we shall now consider the global setting, so that k is a number field with ring of adeles $\mathbb{A}$. We shall briefly explain how the construction of the L-group extension extends to the global situation, referring to **[73]** for the details. The goal of the section is to give a definition of the notion of *automorphic L-functions.*

13.1. Adelic BD covering. — Starting with a BD extension $\overline{\mathbb{G}}$ over $\mathrm{Spec}(k)$ and a positive integer n such that $|\mu_n(k)| = n$, Brylinski and Deligne showed using results of Moore **[48]** that one inherits the following data:

- for each place v of k, a local BD covering group $\overline{G}_v$ of degree n;
- for almost all v, a splitting $s_v : \mathbb{G}(\mathcal{O}_v) \longrightarrow \overline{G}_v$;
- a restricted direct product $\prod'_v \overline{G}_v$ with respect to the family of subgroups $s_v(\mathbb{G}(\mathcal{O}_v))$, from which one can define:

$$\overline{G}(\mathbb{A}) := \prod_v{}' \ \overline{G}_v / \{(\zeta_v) \in \oplus_v \mu_n(k_v) : \prod_v \zeta_v = 1\},$$

  which gives a topological central extension

$$1 \longrightarrow \mu_n(k) \longrightarrow \overline{G}(\mathbb{A}) \longrightarrow \mathbb{G}(\mathbb{A}) \longrightarrow 1,$$

  called the adelic or global BD covering group;

- a natural inclusion

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n(k_v) & \longrightarrow & \overline{G}_v & \longrightarrow & \mathbb{G}(k_v) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_n(k) & \longrightarrow & \overline{G}(\mathbb{A}) & \longrightarrow & \mathbb{G}(\mathbb{A}) & \longrightarrow & 1 \end{array}$$

  for each place v of k;
- a natural splitting

$$i : \mathbb{G}(k) \longrightarrow \overline{G}(\mathbb{A}),$$

  which allows one to consider the space of automorphic forms on $\overline{G}(\mathbb{A})$.

13.2. Global L-group extension. — One may define the L-group extension for the adelic BD cover $\overline{G}(\mathbb{A})$ following the same procedure as in the local setting. We briefly summarize the process, highlighting the differences. Suppose that $\overline{\mathbb{G}}$ has BD invariant $(Q, \mathscr{E}, f)$ or bisector data (D, η). Then one has:

- The dual group of $\overline{G}(\mathbb{A})$ is defined in exactly the same way as in the local setting. Namely, one may define the lattice $Y_{Q,n}$ and the modified coroot lattice $Y_{Q,n}^{\mathrm{sc}}$ in the same way. This gives the dual group $\overline{G}^{\vee}$. Indeed, since $\mathbb{G}$ is split, the definition of these objects works over any k or k_v and gives the same complex dual group $\overline{G}^{\vee}$.
- The role of $F^{\times}/F^{\times n}$ in the local setting is replaced by $\mathbb{A}^{\times}/k^{\times}\mathbb{A}^{\times n}$. More precisely, with $(-,-)_n$ denoting the global n-th Hilbert symbol, the 2-cocycle

$$c_1(a,b)(y) = (a,b)_n^{Q(y)}$$

  for $a, b \in \mathbb{A}^{\times}/\mathbb{A}^{\times n}$ defines an extension

$$1 \longrightarrow \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^{\times}) \longrightarrow E_1' \longrightarrow \mathbb{A}^{\times}/\mathbb{A}^{\times n} \longrightarrow 1.$$

  Since c_1 is trivial on $k^{\times} \times k^{\times}$, this sequence splits canonically over the image of $k^{\times}$ in $\mathbb{A}^{\times}/\mathbb{A}^{\times n}$. Dividing out by the image of $k^{\times}$ under the splitting gives the extension E_1:

$$1 \longrightarrow \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^{\times}) \longrightarrow E_1 \longrightarrow \mathbb{A}^{\times}/k^{\times}\mathbb{A}^{\times n} \longrightarrow 1.$$

- The extension E_2 is defined in the same way, applying $\mathrm{Hom}(-, \mathbb{C}^{\times})$ to the sequence

$$1 \longrightarrow k^{\times}/k^{\times n} \longrightarrow \mathscr{E}/s_f(Y_{Q,n}^{\mathrm{sc}}) \longrightarrow Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}} \longrightarrow 1,$$

  and pulling back by

$$\mathbb{A}^{\times}/k^{\times}\mathbb{A}^{\times n} \longrightarrow \mathrm{Hom}(k^{\times}/k^{\times n}, \mathbb{C}^{\times}).$$

  Forming Baer sum with E_1 gives the global fundamental sequence:

$$1 \longrightarrow Z(\overline{G}^{\vee}) \longrightarrow E \longrightarrow \mathbb{A}^{\times}/k^{\times}\mathbb{A}^{\times n} \longrightarrow 1.$$

Pulling back to the global Weil group W_k and pushing out to $\overline{G}^\vee$ gives the global L-group extension, which fits into a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_v & \longrightarrow & W_{k_v} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_{\mathbb{A}} & \longrightarrow & W_k & \longrightarrow & 1 \end{array}$$

for each place v of k.

- This global L-group extension satisfies the same functoriality with respect to Levi subgroups and z-extensions as in the local case.

13.3. Distinguished splittings. — One may examine the question of splitting of the global L-group extension. By construction, it has a canonical splitting over the subgroup

$$W_{k,n} = \mathrm{Ker}(W_k \longrightarrow k^\times \mathbb{A}^{\times n} \backslash \mathbb{A}^\times).$$

As in the local case, the problem of extending this canonical splitting to one over W_k is equivalent to finding a splitting of the global fundamental sequence. One can define the notion of distinguished splitting of ${}^L\overline{G}_{\mathbb{A}}$ analogously as in the local case, and this amounts to finding a genuine automorphic character

$$\chi : \mathbb{T}_{Q,n}(k) \cdot \mathbb{T}_J(\mathbb{A}) \backslash \overline{T}_{Q,n}(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

where $J = Y^{\mathrm{sc}}_{Q,n} + nY \subset Y_{Q,n}$. Such a character exists when $\eta_n = 1$ is trivial, and thus a distinguished splitting of the fundamental sequence E exists in this case. If we fix such an automorphic character $\chi = \prod_v \chi_v$, then each χ_v corresponds to a distinguished splitting of the local fundamental sequence. Moreover, χ is invariant under the Weyl group $W(k) = N(\mathbb{T})(k)/\mathbb{T}(k)$. The explicit construction of a distinguished genuine character given in §7 produces an automorphic character.

We deduce from the above discussion that the L-group extension is always split, and thus is abstractly a semi-direct product, but it may not be a direct product in general.

13.4. Automorphic L-functions. — We now have all the ingredients to define the notion of (partial) automorphic L-functions. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\overline{G}_{\mathbb{A}}$. For almost all v, π_v is s_v-unramified. By the unramified LLC, π_v gives rise to an s_v-unramified splitting

$$\rho_{\pi,v} : W_{k_v} \longrightarrow {}^L\overline{G}_v \subset {}^L\overline{G}_{\mathbb{A}}.$$

Let $R : {}^L\overline{G}_{\mathbb{A}} \longrightarrow \mathrm{GL}(V)$ be a continuous finite dimensional representation which is trivial on the subgroup

$$W_{k,n} = \mathrm{Ker}(W_k \longrightarrow k^\times \mathbb{A}^{\times n} \backslash \mathbb{A}^\times) \subset {}^L\overline{G}_{\mathbb{A}}.$$

Recall here that the subgroup $W_{k,n} \subset W_k$ admits a canonical splitting into ${}^L\overline{G}_{\mathbb{A}}$ and observe that $W_k/W_{k,n}$ has exponent n. We may form the local Artin L-factor for the representation $R \circ \rho_{\pi,v} : W_{k_v} \longrightarrow \mathrm{GL}(V)$:

$$L(s, \pi_v, R) = \frac{1}{\det(1 - \rho_{\pi,v}(\mathrm{Frob}_v) \cdot q_v^{-s} | V^{I_v})}$$

where q_v denotes the size of the residue field of k_v. Then we may form the partial global L-function relative to R:

$$L^S(s, \pi, R) = \prod_{v \notin S} L(s, \pi_v, R)$$

for a sufficiently large finite set S of places of k including all archimedean ones.

Theorem 13.1. — *For a cuspidal representation π, the Euler product*

$$L^S(s, \pi, R) = \prod_{v \notin S} L(s, \pi_v, R)$$

converges uniformly when $\mathrm{Re}(s)$ *is sufficiently large.*

Proof. — The proof is essentially the same as that of Langlands' in [**33**] and we shall give a sketch, following Borel's exposition in [**10**, §13] closely. Fixing a splitting $\phi : W_k \longrightarrow {}^L\overline{T}_{\mathbb{A}} \subset {}^L\overline{G}_{\mathbb{A}}$ which agrees with the canonical splitting on $W_{k,n}$, we may write ${}^L\overline{G}_{\mathbb{A}}$ as a semidirect product $\overline{G}^\vee \rtimes W_k$. Moreover, the conjugation action of W_k fixes $\overline{T}^\vee$ pointwise and normalizes the Borel subgroup $\overline{B}^\vee$. For each place v, the restriction of ϕ to W_{k_v} corresponds under LLC to a genuine character χ_v of $\overline{T}_{Q,n,v}$ which has finite order by Proposition 6.5(i), so that the positive-valued character $|\chi_v|$ is trivial.

Now the representation R is thus pulled back from a representation (still denoted by R) of $\overline{G}^\vee \rtimes W_k/W_{k,n}$. We need to show that there exists $A > 0$ such that for almost all places v, any eigenvalue α of $R(\rho_{\pi,v}(\mathrm{Frob}_v))$ satisfies

$$|\alpha| \leq q_v^A.$$

Let us write:

$$\rho_{\pi,v}(\mathrm{Frob}_v) = (t_v, \mathrm{Frob}_v) \in \overline{T}^\vee \rtimes W_k/W_{k,n}.$$

Since $W_k/W_{k,n}$ has exponent n and t_v commutes with Frob_v, one has:

$$(t_v, \mathrm{Frob}_v)^n = (t_v, 1)^n.$$

It suffices to show that there exists $A > 0$ such that for almost all places v, any eigenvalue α of $R(t_v) \in \mathrm{GL}(V)$ satisfies

$$|\alpha| \leq q_v^A.$$

Since π is cuspidal, we may assume without loss of generality that π has unitary central character and hence is unitary. For almost all v, π_v is a unitary s_v-unramified representation, which is associated to an unramified genuine character χ_v of $\overline{T}_{Q,n,v}$. The character

$$\alpha(\chi_v) : \overline{T}_{Q,n,v} \xrightarrow{\chi_v} \mathbb{C}^\times \xrightarrow{|-|} \mathbb{R}^\times \xrightarrow{\log_{q_v}} \mathbb{R}$$

factors through $T_{Q,n,v}/T_{Q,n,v} \cap K_v \cong Y_{Q,n}$ and thus can be identified with an element of $X(\mathbb{T}_{Q,n}) \otimes \mathbb{R} = X(\mathbb{T}) \otimes \mathbb{R}$ (it is called the real logarithm of χ_v in [**10**, §13]). We may assume that $\alpha(\chi_v)$ lies in the closure of the positive Weyl chamber associated to a Borel subgroup $\mathbb{B}$. Then the principal series representation $I(\chi)$ is a direct sum of standard modules, exactly one of which is s_v-unramfied. Then π_v is the unique Langlands quotient of this s_v-unramfied standard module. This follows from the Gindikin-Karpelevich formula shown in [**26**], which shows that the standard intertwining operator associated to this standard module is nonzero on the spherical vector.

Now the spherical matrix coefficient f_{π_v} of π_v is a bounded genuine function on $\overline{G}_v$. On the other hand, the asymptotic of f_{π_v} is governed by the Jacquet module of π_v along the Borel subgroup $\overline{B}_v$, i.e., by the central exponents of π_v; this is a result of Casselman for linear algebraic groups but has been extended to the covering case by Ban-Jantzen; see [**7**, §3.2], especially the proof of [**7**, Theorem 3.4]. By the discussion of the previous paragraph, the normalized Jacquet module of π_v contains the character χ_v^{-1} as a submodule. From this, one deduces as in the linear case that for f_{π_v} to be bounded, $\alpha(\chi_v) \in X(\mathbb{T}) \otimes \mathbb{R}$ must satisfy the following. With

$$\rho = \frac{1}{2} \cdot \sum_{\alpha \in \Phi} \alpha,$$

one has

$$\langle \alpha^\vee, \alpha(\chi_v) \rangle \leq \langle \alpha^\vee, \rho \rangle = 1 \quad \text{for all } \alpha^\vee \in \Delta^\vee.$$

For any weight λ of $\overline{T}^\vee$ in the representation R, one may write λ as a $\mathbb{Q}$-linear combination of $\alpha^\vee \in \Delta^\vee$ and elements in the character group $X(Z(\overline{G}^\vee))$ of $Z(\overline{G}^\vee)$:

$$\lambda = \sum_{\alpha \in \Delta} \lambda_\alpha \cdot \alpha^\vee \qquad \text{mod } X(Z(\overline{G}^\vee)) \otimes \mathbb{Q}.$$

Setting

$$\lambda^{abs} = \sum_{\alpha \in \Delta} |\lambda_\alpha| \cdot \alpha^\vee \qquad \text{mod } X(Z(\overline{G}^\vee)) \otimes \mathbb{Q},$$

we then infer from the above inequality that

$$|\lambda(t_v)| = q_v^{\langle \lambda, \alpha(\chi_v) \rangle} \leq q_v^{\langle \lambda^{abs}, \rho \rangle}.$$

Since there are only finitely many such λ's and hence λ^{abs}'s (as $\dim V$ is finite), we deduce the desired upper bound on the eigenvalues of $R(t_v)$ for almost all v. □

We want to highlight some instances where one can write down these automorphic L-functions.

(i) (L-functions relative to a distinguished splitting) If ${}^L\overline{G}_{\mathbb{A}}$ possesses a distinguished splitting ρ_0 (e.g., if $\eta_n = 1$), then ρ_0 is s_v-unramified for almost all v, and so we have an unramified homomorphism

$$\rho_{\pi,v}/\rho_{0,v} : W_{k_v} \longrightarrow k_v^\times \longrightarrow \mathbb{Z} \longrightarrow \overline{G}^\vee.$$

In other words, for almost all v, we have a Satake parameter $s_{\pi_v} \in \overline{G}^\vee$, well-defined up to conjugacy, and depending on $\rho_{0,v}$.

In this setting, if one has a representation $R : \overline{G}^\vee \longrightarrow \mathrm{GL}(V)$, one can form the partial L-function

$$L^S(s,\pi,R,\rho_0) := \prod_{v \notin S} \frac{1}{\det(1 - s_{\pi_v} \cdot q_v^{-s}|V)}.$$

We call this the (R,ρ_0) L-function of π.

More generally, if one fixes a distinguish splitting ρ_0 of the enlarged L-group ${}^L\overline{G}^{\#}$ (which always exists), one has the notion of unramified L-parameters relative to ρ_0:

$$\rho_{\pi,v} : W_{k_v} \to k_v^\times \to \mathbb{Z} \to \overline{G}^{\#},$$

which gives rise to a Satake parameter $s_{\pi_v} \in \overline{G}^{\#}$. If one extends R above to the disconnected group $\overline{G}^{\#}$, one can define the partial L-function $L^S(s,\pi,R,\rho_0)$ as above.

(ii) (Adjoint type L-functions) If $\overline{G}^\vee_{ad} := \overline{G}^\vee / Z(\overline{G}^\vee)$ denotes the adjoint quotient of $\overline{G}^\vee$, there is a natural commutative diagram of extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{G}^\vee & \longrightarrow & {}^L\overline{G}_{\mathbb{A}} & \longrightarrow & W_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow p & & \downarrow & & \\
1 & \longrightarrow & \overline{G}^\vee_{ad} & \longrightarrow & \overline{G}^\vee_{ad} \times W_k & \longrightarrow & W_k & \longrightarrow & 1.
\end{array}$$

Thus, if $R : \overline{G}^\vee_{ad} \longrightarrow \mathrm{GL}(V)$ is any representation, we may pull it back to ${}^L\overline{G}_{\mathbb{A}}$ and obtain a partial L-function $L^S(s,\pi,R)$.

(iii) (Langlands-Shahidi L-functions) More generally, suppose that $\mathbb{P} = \mathbb{M}\mathbb{N} \subset \mathbb{G}$ is a parabolic subgroup, and π is an automorphic representation of the BD covering $\overline{M}_{\mathbb{A}}$. By functoriality of the L-group construction, one has inclusions

$$E_{\overline{G}} \hookrightarrow {}^L\overline{M}_{\mathbb{A}} \hookrightarrow {}^L\overline{G}_{\mathbb{A}}$$

where $E_{\overline{G}}$ is the fundamental sequence for $\overline{G}_{\mathbb{A}}$. As in (ii) above, one has a natural commutative diagram of extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{M}^\vee & \longrightarrow & {}^L\overline{M}_{\mathbb{A}} & \longrightarrow & W_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \overline{M}^\vee / Z(\overline{G}^\vee) & \longrightarrow & \overline{M}^\vee / Z(\overline{G}^\vee) \times W_k & \longrightarrow & W_k & \longrightarrow & 1,
\end{array}$$

i.e., a canonically split extension. Then any representation of $\overline{M}^\vee$ which is trivial on $Z(\overline{G}^\vee)$ pulls back to a representation of ${}^L\overline{M}_{\mathbb{A}}$.

A source of such representations is the adjoint action of ${}^L\overline{M}_{\mathbb{A}}$ on $\mathrm{Lie}(N^\vee)$. Let R be an irreducible summand, so that R is trivial on $Z(\overline{G}^\vee)$. Then we obtain a partial automorphic L-function $L^S(s,\pi,R)$. As shown in the PhD thesis **[26]** of the second author, these Langlands-Shahidi type L-functions appear in the constant term of the Eisenstein series on $\overline{G}_{\mathbb{A}}$, as in the case of linear groups.

A basic open question is whether such automorphic L-functions associated to automorphic representations of BD covering groups have the usual nice properties such as meromorphic continuation and functional equations. In [**26**], the second author has shown that the Langlands-Shahidi L-functions for BD covers have meromorphic continuation. A related question is whether such an automorphic L-function agrees with one for a linear reductive group. We shall examine this question in §15.

14. Langlands Functoriality: Base Change

Besides giving the definition of automorphic L-functions, the L-group formalism allows one to define the notion of "Langlands functoriality". In this section, we return to the local setting and examine an instance of Langlands functoriality, namely the notion of base change. Hence, F is again a local field in this section.

14.1. Base change. — For linear groups, the notion of base change can be directly defined in terms of character identities (in the theory of twisted endoscopy) or defined on the L-group side as the restriction of L-parameters from WD_F to WD_K for a field extension K/F. We adopt the second approach. Thus, given a BD extension $\overline{\mathbb{G}}$ over F, a positive integer n such that $|\mu_n(F)| = n$ and a Galois extension K of F, we have the topological degree n covering groups $\overline{G}_F$ and $\overline{G}_K$ and their associated L-groups ${}^L\overline{G}_F$ and ${}^L\overline{G}_K$. Observe that the dual groups of $\overline{G}_F$ and $\overline{G}_K$ are identical by definition; we shall simply denote this dual group by $\overline{G}^{\vee}$. Then we would like to define a natural commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{G}^{\vee} & \longrightarrow & {}^L\overline{G}_K & \longrightarrow & W_K & \longrightarrow & 1 \\ & & {\scriptstyle id}\big\downarrow & & {\scriptstyle \eta}\big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \overline{G}^{\vee} & \longrightarrow & {}^L\overline{G}_F & \xrightarrow{\pi} & W_F & \longrightarrow & 1. \end{array}$$

Such a diagram would induce an isomorphism

$${}^L\overline{G}_K \cong \pi^{-1}(W_K) \subset {}^L\overline{G}_F.$$

Then any splitting of ${}^L\overline{G}_F$ (i.e., any L-parameter of $\overline{G}_F$) will give by restriction to W_K a splitting of ${}^L\overline{G}_K$ (i.e., an L-parameter for $\overline{G}_K$), thus defining the notion of base change on the L-group side. Moreover, by the construction of the L-group extension, it suffices to construct a natural commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(\overline{G}^{\vee}) & \longrightarrow & E(\overline{G}_K) & \longrightarrow & K^{\times} & \longrightarrow & 1 \\ & & {\scriptstyle id}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{N}_{K/F}} & & \\ 1 & \longrightarrow & Z(\overline{G}^{\vee}) & \longrightarrow & E(\overline{G}_F) & \longrightarrow & F^{\times} & \longrightarrow & 1. \end{array}$$

14.2. An example. — Before going on, let us consider an example: the case of Mp_2 or even more pertinently, the covering split torus $\overline{T}$ in Mp_2. Let's convince ourselves that there is a canonical base change in this case.

We have $\overline{T}_F = T(F) \times \mu_2$ with the group law given by the quadratic Hilbert symbol of F:

$$(t_1, \epsilon_1) \cdot (t_2, \epsilon_2) = (t_1 t_2, \epsilon_1 \epsilon_2 \cdot (t_1, t_2)_F).$$

For each nontrivial additive character ψ of F, we have a genuine character χ_ψ determined by the Weil index. Having fixed ψ, all other genuine characters are of the form $\chi_\psi \chi$ for χ a character of $F^\times$. Now suppose K/F is a Galois extension and consider the covering torus $\overline{T}_K$ defined analogously by the quadratic Hilbert symbol of K. The character ψ of F gives rise to an additive character $\psi_K = \psi \circ \mathrm{Tr}_{K/F}$ of K which is $\mathrm{Gal}(K/F)$-invariant and hence a genuine character χ_{ψ_K}. One would imagine that "base change" should be the map

$$\chi_\psi \cdot \chi \mapsto \chi_{\psi_K} \cdot (\chi \circ \mathrm{N}_{K/F}).$$

In particular, it carries χ_ψ to χ_{ψ_K}. Let us call this the ψ-base change map for the moment.

Now let's observe that the ψ-base change map is independent of the choice of ψ. Any other additive character is of the form ψ_a for $a \in F^\times$ and we have

$$\chi_\psi \cdot \chi = \chi_{\psi_a} \cdot (a, -)_F \cdot \chi \quad \text{as characters of } \overline{T}_F.$$

Then the ψ_a-base change of the RHS is, by definition,

$$\chi_{\psi_{a,K}} \cdot ((a, -)_F \circ \mathrm{N}_{K/F}) \cdot (\chi \circ \mathrm{N}_{K/F}).$$

Since

$$\chi_{\psi_{a,K}} = \chi_{\psi_K} \cdot (a, -)_K \quad \text{and} \quad (a, -)_F \circ \mathrm{N}_{K/F} = (a, -)_K,$$

we see that the ψ_a-base change of RHS is the same as the ψ-base change of the LHS. Thus we see that there should be a canonical base change map for $\overline{T}$.

14.3. Base change map. — The next question is whether it is given by a map of the L-group-extensions. Recall that the L-group of $\overline{T}_F$ is defined as the Baer sum $E_1 + E_2$ of two extensions E_i of $F^\times / F^{\times n}$ by $\overline{T}^\vee$ (pulled back via $W_F \to F^\times \to F^\times / F^{\times n}$). It is easy to see that the construction of E_2 is functorial with respect to base change, in the sense that there is a natural map

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{T}^\vee \cong \mathbb{C}^\times & \longrightarrow & E_{2,K} & \longrightarrow & K^\times & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{N}_{K/F}} & & \\
1 & \longrightarrow & \overline{T}^\vee \cong \mathbb{C}^\times & \longrightarrow & E_{2,F} & \longrightarrow & F^\times & \longrightarrow & 1.
\end{array}$$

One would like an analogous diagram for E_1, but the construction of E_1 does not immediately lead to such a diagram.

To be more precise, one would like to define a *natural* map of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & E_{1,K} & \longrightarrow & K^\times & \longrightarrow & 1 \\ & & {\scriptstyle id}\big\downarrow & & \big\downarrow{\scriptstyle \eta_f} & & \big\downarrow{\scriptstyle \mathrm{N}_{K/F}} & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & E_{1,F} & \longrightarrow & F^\times & \longrightarrow & 1 \end{array}$$

where the top row is defined by the cocycle $(-,-)_K$ whereas the bottom is defined by $(-,-)_F$. Writing N for $\mathrm{N}_{K/F}$ for simplicity, the map η_f has the form

$$\eta_f(a,z) = (\mathrm{N}(a), z\cdot f(a)) \quad \text{with } z\in\mathbb{C}^\times \text{ and } a\in K^\times,$$

for some $f: K^\times \longrightarrow \mathbb{C}^\times$ satisfying:

$$f(ab)f(a)^{-1}f(b)^{-1} = (a,b)_K \cdot (\mathrm{N}(a),\mathrm{N}(b))_F \quad \text{for all } a,b\in K^\times.$$

Does such an f exist? The following theorem was proved by Bender [**8**]:

Theorem 14.1. — *Define $f : K^\times \longrightarrow \{\pm 1\} \subset \mathbb{C}^\times$ by:*

$$f(a) = \frac{\textit{(Hasse-Witt invariant of the quadratic form } Tr_{K/F}(ax^2))}{\textit{(Hasse-Witt -invariant of the quadratic form } Tr_{K/F}(x^2))}.$$

Then

$$f(ab)f(a)^{-1}f(b)^{-1} = (a,b)_K\cdot(N(a),N(b))_F \quad \textit{for all } a,b\in K^\times.$$

The Hasse-Witt invariant of the trace quadratic forms in the theorem has been studied by Serre [**58**]. Thus we have a completely natural function (valued in μ_2) which depends only on the arithmetic of K/F and which induces a natural homomorphism ${}^L\overline{T}_K \longrightarrow {}^L\overline{T}_F$. This induces a map from the set of L-parameters (i.e., splitting of L-group) of $\overline{T}_F$ to $\overline{T}_K$.

Explicitly, this map of L-parameters is given as follows. Since the L-group is the Baer sum of two extensions E_1 and E_2, an L-parameter is a section of E_2 whose associated 2-cocycle is equal to that of E_1. Suppose ϕ_F is an L-parameter for $\overline{T}_F$, thought of as a section of E_2 (i.e., as a function $\phi_F : F^\times \longrightarrow \mathbb{C}^\times$) such that

$$\phi_F(ab) = \phi_F(a)\phi_F(b)\cdot(a,b)_F, \quad \text{with } a,b\in F^\times \cong W_F^{ab}.$$

Then ϕ_F is mapped to the L-parameter ϕ_K of $\overline{T}_K$, thought of as a section of E_2 (i.e., a function $K^\times \longrightarrow \mathbb{C}^\times$), given by:

$$\phi_K(a) = \phi_F(\mathrm{N}(a))\cdot f(a).$$

Via the LLC for covering tori, we can work out the base change of genuine characters of $\overline{T}_F$ to $\overline{T}_K$. The map

$$\mathscr{N}_{\overline{T}} : \overline{T}_K \longrightarrow \overline{T}_F$$

given by

$$\mathscr{N}_{\overline{T}}(t,\epsilon) = (\mathrm{N}_{K/F}(t), \epsilon\cdot f(t))$$

is easily checked to be a group homomorphism; we call it the norm map for the covering torus $\overline{T}$ as it lifts the usual norm map of the linear torus T so that one has a commutative diagram:

$$\begin{array}{ccc} \overline{T}_K & \xrightarrow{\mathcal{N}_{\overline{T}}} & \overline{T}_F \\ \downarrow & & \downarrow \\ \mathbb{T}(K) & \xrightarrow{\mathcal{N}_T} & \mathbb{T}(F). \end{array}$$

Then the base change of genuine characters corresponding to the base change of L-parameters is simply the pullback by the norm map:

$$\mathrm{BC}(\chi)(t,\epsilon) = \chi(\mathcal{N}_{\overline{T}}(t,\epsilon))$$

for any genuine character χ of $\overline{T}_F$.

14.4. Consistency. — Of course, the question is whether the functoriality implied by this homomorphim of L-groups for $\overline{T}$ agrees with the base change constructed above. In particular, if ϕ_F corresponds to the genuine character χ_ψ, we would like to see that ϕ_K corresponds to the character $\chi_{\psi_K} = \chi_{\psi\circ\mathrm{Tr}}$. This means that we need to check:

Proposition 14.2. — *We have*

$$\chi_{\psi_K}(a) = \chi_\psi(N(a)) \cdot f(a) \quad \textit{for } a \in K^\times.$$

Proof. — We shall verify this proposition by a computation. Since both sides of the identity are roots of unity, it suffices to verify the identity in $\mathbb{C}^\times/\mathbb{R}^\times_{>0}$. Recall that

$$\chi_{\psi_K}(a) = \frac{\gamma_{\psi_K}}{\gamma_{\psi_{K,a}}}$$

where the factors on the RHS are the Weil indices defined by the equation (of distributions):

$$(*) \quad \int_K \psi_K(ax^2)\cdot\psi_K(xy)\,dx = \gamma_{\psi_{K,a}}\cdot|a|_K^{-1/2}\cdot\psi_K(a^{-1}y^2).$$

Now we may compute the LHS as follows. Replacing x by x/a, we get

$$\mathrm{LHS} = |a|_K^{-1}\cdot\int_K \psi(\mathrm{Tr}(a^{-1}x^2))\cdot\psi(\mathrm{Tr}(a^{-1}xy))\,dx.$$

Let $q_{a^{-1}}$ denote the quadratic form $x \mapsto \mathrm{Tr}(a^{-1}x^2)$: it is a quadratic form on the F-vector space K. We may find an F-basis $\{\alpha_i\}$ of K such that

$$q_{a^{-1}}(x) = \sum_i a_i x_i^2 \quad (\text{with } x = \sum_i x_i\alpha_i).$$

Then the integral over K factorizes into $[K:F]$ integrals over F to give:

$$\begin{aligned}\text{LHS} &= \prod_i \int_F \psi(a_i x_i^2)\cdot\psi(a_i x_i y_i)\,dx_i \mod \mathbb{R}^\times_{>0}\\ &= \prod_i \gamma_{\psi_{a_i}}\cdot\psi(a_i y^2) \mod \mathbb{R}^\times_{>0}.\end{aligned}$$

Comparing this with the RHS of (*), we deduce that

$$\gamma_{\psi_{K,a}}^{-1} = \prod_i \gamma_{\psi_{a_i}}^{-1} = \cdots = \gamma_\psi^{-[K:F]}\cdot\chi_\psi(\det(q_{a^{-1}}))\cdot \mathrm{HW}(q_{a^{-1}}),$$

where

$$\det(q_{a^{-1}}) = \prod_i a_i \quad\text{and}\quad \mathrm{HW}(q_{a^{-1}}) = \prod_{i<j}(a_i,a_j)_F.$$

Thus,

$$\chi_{\psi_K}(a) = \frac{\chi_\psi(\det(q_a))}{\chi_\psi(\det(q_1))}\cdot\frac{\mathrm{HW}(q_a)}{\mathrm{HW}(q_1)}.$$

(there is no harm replacing in $q_{a^{-1}}$ by q_a here). On the other hand, we have (see [**58**, Pg. 668])

$$\det(q_a) = \mathrm{N}(a)\cdot\mathrm{disc}(K/F)\in F^\times/F^{\times 2},$$

so that

$$\det(q_a) = \det(q_1)\cdot\mathrm{N}(a).$$

Note that in [**58**], Serre has written $\mathrm{disc}(q_a)$ in place of our $\det(q_a)$ here. From this, we deduce that

$$\chi_{\psi_K}(a) = \chi_\psi(\mathrm{N}(a))\cdot\frac{\mathrm{HW}(q_a)}{\mathrm{HW}(q_1)},$$

as desired. Here, note that we have used $(\mathrm{N}(a),\det(q_1))_F = (\mathrm{N}(a),\mathrm{disc}(K/F))_F = 1$. The proposition is proved. □

14.5. General case. — We have focused exclusively on a very simple covering torus $\overline{T}$ above, but the discussion in fact applies generally. The point is that, for any BD cover $\overline{G}$, the miraculous function f allows us to define a natural map of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(\overline{G}^\vee) & \longrightarrow & E_1(\overline{G}_K) & \longrightarrow & K^\times & \longrightarrow & 1\\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \eta_f} & & \Big\downarrow{\scriptstyle \mathrm{N}_{K/F}} & & \\ 1 & \longrightarrow & Z(\overline{G}^\vee) & \longrightarrow & E_1(\overline{G}_F) & \longrightarrow & F^\times & \longrightarrow & 1\end{array}$$

by the same formula and using the natural map $j:\mu_2\longrightarrow Z(\overline{G}^\vee)$ used in the definition of E_1 in §5.2 (recall that E_1 is a pushout of an extension E_0 of $F^\times$ by μ_2 under the map j). Since the construction of E_2 is functorial with respect to base change, one has an analogous diagram for E_2 and taking the Baer sum (together with pushout and pullback), we obtain a natural diagram

$$\begin{tikzcd} 1 \arrow[r] & \overline{G}^\vee \arrow[r] \arrow[d, "id"'] & {}^L\overline{G}_K \arrow[r] \arrow[d, "\eta_f"'] & W_K \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \overline{G}^\vee \arrow[r] & {}^L\overline{G}_F \arrow[r, "\pi"] & W_F \arrow[r] & 1. \end{tikzcd}$$

It is this diagram that allows one to define the notion of base change for $\overline{G}$ on the side of the L-groups.

14.6. Example of Mp_{2n}. — Since the LLC is known for Mp_{2n}, one does have base change of L-packets from $\mathrm{Mp}_{2n}(F)$ to $\mathrm{Mp}_{2n}(K)$. Recall that the LLC for Mp_{2n} is defined via theta correspondence with SO_{2n+1}. Thus, Proposition 14.2 allows us to describe this base change more concretely, via the theta correspondence and base change for odd special orthogonal groups.

More precisely, given an L-packet Π_{ϕ_F} of $\mathrm{Mp}_{2n}(F)$, the theta correspondence gives the corresponding L-packet $\Pi_{\phi'_F,\psi}$ on $\mathrm{SO}_{2n+1}(F)$ (which depends on an additive character ψ) and its base change $\Pi_{\phi'_K,\psi}$ on $\mathrm{SO}_{2n+1}(K)$; here ϕ'_K is simply the restriction of ϕ'_F from WD_F to WD_K. Theta correspondence then gives a packet $\Theta_{\psi_K}(\Pi_{\phi'_K,\psi})$ on $\mathrm{Mp}_{2n}(K)$. Similar to our treatment of base change for the covering torus $\overline{T}$, it is easy to see that this packet on $\mathrm{Mp}_{2n}(K)$ is independent of ψ and its L-parameter is that obtained from ϕ_F by the base change commutative diagram constructed above (using the map f provided by Bender).

14.7. Example of covering tori. — Another case we should check is the case of general covering (split) tori $\overline{T}$. We may assume that $\overline{T}$ is defined by a bisector D, so that it is described by generators and relations as in §3.3. Then the covering torus $\overline{T}_{Q,n}$ is presented analogously as we explained in §3.4. For ease of reference, let us recall this presentation here. The covering torus $\overline{T}_{Q,n,F}$ is generated by $\epsilon \in \mu_n$ and $y(a)$ with $y \in Y_{Q,n}$ and $a \in F^\times$, subject to:

- $\overline{T}_{Q,n}$ is abelian;
- $y_1(a) \cdot y_2(a) = (y_1 + y_2)(a) \cdot (-1, a)_{F,n}^{D(y_1,y_2)}$ for $y_1, y_2 \in Y_{Q,n}$ and $a \in F^\times$, and where $(-,-)_{n,F}$ denotes the n-th Hilbert symbol for F;
- $y(a) \cdot y(b) = y(ab) \cdot (a,b)_{F,n}^{Q(y)}$ for $y \in Y_{Q,n}$ and $a, b \in F^\times$.

One has the analogous presentation for $\overline{T}_{Q,n,K}$. We also recall that for $y \in Y_{Q,n}$, $\frac{2}{n} \cdot Q(y) \in \mathbb{Z}$.

Now the base change map of L-parameters actually implies a base change of genuine characters of $\overline{T}_{Q,n,F}$ to those of $\overline{T}_{Q,n,K}$ and this is given by the following theorem:

Theorem 14.3. — (i) *The map* $\mathcal{N}_{\overline{T}} : \overline{T}_{Q,n,K} \longrightarrow \overline{T}_{Q,n,F}$ *given by*

$$\mathcal{N}_{\overline{T}}(y(a)) = y(N_{K/F}(a)) \cdot f(a)^{\frac{2}{n}\cdot Q(y)} \quad \textit{for } y \in Y_{Q,n} \textit{ and } a \in K^\times,$$

and

$$\mathscr{N}_{\overline{T}}(\epsilon) = \epsilon \quad \textit{for } \epsilon \in \mu_n,$$

is a group homomorphism, and there is a commutative diagram

$$\begin{array}{ccc} \overline{T}_{Q,n,K} & \xrightarrow{\mathscr{N}_{\overline{T}}} & \overline{T}_{Q,n,F} \\ \big\downarrow & & \big\downarrow \\ \mathbb{T}(K) & \xrightarrow{\mathscr{N}_{T}} & \mathbb{T}(F). \end{array}$$

Moreover, the map $\mathscr{N}_{\overline{T}}$ *descends to give a homomorphism*

$$\mathscr{N}_{\overline{T}} : Z(\overline{T}_K) \longrightarrow Z(\overline{T}_F).$$

(ii) *The base change map*

$$\mathrm{BC} : \mathrm{Irr}\,\overline{T}_{Q,n,F} \longrightarrow \mathrm{Irr}\,\overline{T}_{Q,n,K}$$

given by the base change morphism of L-groups is given by the pullback of genuine characters defined by $\mathscr{N}_{\overline{T}}$*. Moreover,* BC *restricts to give a map*

$$\mathrm{BC} : \mathrm{Irr}\,\overline{T}_F \longrightarrow \mathrm{Irr}\,\overline{T}_K.$$

Proof. — (i) To show that $\mathscr{N}_{\overline{T}}$ is a group homomorphism, we need to verify that it respects the defining relations of the covering tori. The first relation to check is that

$$\mathscr{N}_{\overline{T}}(y_1(a)) \cdot \mathscr{N}_{\overline{T}}(y_2(a)) = \mathscr{N}_{\overline{T}}\big((y_1+y_2)(a) \cdot (-1,a)_{K,n}^{D(y_1,y_2)}\big).$$

On the left, one has by definition:

$$y_1(\mathrm{N}_{K/F}(a)) \cdot y_2(\mathrm{N}_{K/F}(a)) \cdot f(a)^{\frac{2}{n}\cdot(Q(y_1)+Q(y_2))},$$

whereas on the right, one has

$$(y_1+y_2)(\mathrm{N}_{K/F}(a)) \cdot f(a)^{\frac{2}{n}\cdot Q(y_1+y_2)} \cdot (-1,a)_{K,n}^{D(y_1,y_2)}.$$

Since

$$(y_1+y_2)(\mathrm{N}_{K/F}(a)) = y_1(\mathrm{N}_{K/F}(a)) \cdot y_2(\mathrm{N}_{K/F}(a)) \cdot (-1, \mathrm{N}_{K/F}(a))_{F,n}^{D(y_1,y_2)},$$

the first relation then follows from

$$(-1,a)_{K,n} = (-1, \mathrm{N}_{K/F}(a))_{F,n}$$

and

$$f(a)^{\frac{2}{n}\cdot(Q(y_1+y_2)-Q(y_1)-Q(y_2))} = f(a)^{\frac{2}{n}\cdot B_Q(y_1,y_2)} = 1$$

since $B_Q(y_1,y_2) \equiv 0 \mod n$ for $y_1 \in Y_{Q,n}$.

The second relation to verify is the identity

$$\mathscr{N}_{\overline{T}}(y(a)) \cdot \mathscr{N}_{\overline{T}}(y(a)) = \mathscr{N}_{\overline{T}}(y(ab)) \cdot (a,b)_{K,n}^{Q(y)}.$$

On the left, one has

$$\begin{aligned} &y(\mathrm{N}_{K/F}(a)) \cdot y(\mathrm{N}_{K/F}(b)) \cdot f(a)^{\frac{2}{n}\cdot Q(y)} \cdot f(b)^{\frac{2}{n}\cdot Q(y)} \\ &\qquad = y(\mathrm{N}_{K/F}(ab)) \cdot (\mathrm{N}_{K/F}(a), \mathrm{N}_{K/F}(b))_{F,n}^{Q(y)} \cdot f(a)^{\frac{2}{n}\cdot Q(y)} \cdot f(b)^{\frac{2}{n}\cdot Q(y)}, \end{aligned}$$

whereas on the right, one has

$$y(\mathrm{N}_{K/F}(ab)) \cdot (a,b)_{K,n}^{Q(y)} \cdot f(ab)^{\frac{2}{n}\cdot Q(y)}.$$

The desired identity follows by noting that

$$(a,b)_{K,n}^{Q(y)} = (a,b)_{K,2}^{\frac{2}{n}\cdot Q(y)} \quad \text{and} \quad (\mathrm{N}_{K/F}(a), \mathrm{N}_{K/F}(b))_{F,n}^{Q(y)} = (\mathrm{N}_{K/F}(a), \mathrm{N}_{K/F}(b))_{F,2}^{\frac{2}{n}\cdot Q(y)}$$

and applying Theorem 14.1.

To prove the last assertion of (i), it suffices to show that $\mathscr{N}_{\overline{T}}$ sends $\mathrm{Ker}(\overline{T}_{Q,n,K} \longrightarrow \overline{T}_K)$ to $\mathrm{Ker}(\overline{T}_{Q,n,F} \longrightarrow \overline{T}_F)$. By Lemma 3.1, one has

$$\mathrm{Ker}(\overline{T}_{Q,n,K} \longrightarrow \overline{T}_K) = \tilde{g}(T_K[n]))$$

where $\tilde{g}(y(a)) = (ny)(a) \in \overline{T}_{Q,n,K}$. From the proof of Lemma 3.1, we see that this kernel is generated by certain elements of the form $(ny)(\zeta)$, with $\zeta \in \mu_n(K) = \mu_n(F)$. Then

$$\mathscr{N}_{\overline{T}}((ny)(\zeta)) = (ny)(\zeta) \cdot f(\zeta)^{\frac{2}{n}\cdot Q(ny)} = (ny)(\zeta),$$

which lies in $\mathrm{Ker}(\overline{T}_{Q,n,F} \longrightarrow \overline{T}_F)$ by Lemma 3.1 again.

(ii) If $\chi \in \mathrm{Irr}(\overline{T}_{Q,n,F})$ has L-parameter ϕ, then by Lemma 6.1, ϕ is a map

$$\phi : F^{\times} \times Y_{Q,n} \longrightarrow \mathbb{C}^{\times}$$

satisfying the conditions (b) and (c) in Lemma 6.1. As we remarked after Lemma 6.1, the presentation of $\overline{T}_{Q,n,F}$ shows that this map ϕ defines a genuine character of $\overline{T}_{Q,n,F}$, and this was how the LLC for $\overline{T}$ was shown.

By the base change homomorphism of L-group, the L-parameter ϕ gives rise to an L-parameter ϕ_K for $\overline{T}_{Q,n,K}$. Regarding ϕ_K as a map from $K^{\times} \times Y_{Q,n} \longrightarrow \mathbb{C}^{\times}$, the map ϕ_K is given by

$$\phi_K(a,y) = \phi(\mathrm{N}_{K/F}(a), y) \cdot f(a)^{\frac{2}{n}\cdot Q(y)}.$$

From this and the construction of the LLC, we deduce that the genuine character defined by ϕ_K is the pullback of χ by $\mathscr{N}_{\overline{T}}$. If χ is trivial on $\mathrm{Ker}(\overline{T}_{Q,n,F} \longrightarrow \overline{T}_F)$, then the last part of (i) implies that $\chi \circ \mathscr{N}_{\overline{T}}$ is trivial on $\mathrm{Ker}(\overline{T}_{Q,n,K} \longrightarrow \overline{T}_K)$; this proves the last assertion of (ii). □

14.8. Principal series. — Having described the base change for covering (split) tori, we can obtain the base change of principal series (on the level of L-packets) for a BD covering group $\overline{G}$ (attached to the invariant (D, η)). Let $T \subset G$ be a maximal split torus. In this case, the Weyl group W acts naturally on $\overline{T}_{Q,n,F}$ and $\overline{T}_{Q,n,K}$. We note the following lemma:

Lemma 14.4. — *The norm map* $\mathscr{N}_{\overline{T}} : \overline{T}_{Q,n,K} \longrightarrow \overline{T}_{Q,n,F}$ *is* W*-equivariant.*

Proof. — The W-action on $\overline{T}_{Q,n,F}$ is given by (6.7): for a root α and with $w_\alpha \in W$ the image of $q(n_\alpha(1))$, one has:

$$\mathrm{Int}(w_\alpha)(y(a)) = y(a) \cdot s_{\eta,F}(-\langle \alpha, y\rangle \cdot \alpha^{\vee}(a)) \in \overline{T}_{Q,n,F},$$

where for $z \in Y_{Q,n}^{\mathrm{sc}}$ (such as $z = -\langle \alpha, y\rangle \cdot \alpha^\vee$),

$$s_{\eta,F}(z(a)) = z(a) \cdot (\eta(z), a)_{F,n}.$$

One has the analogous formula for the action of w_α on $\overline{T}_{Q,n,K}$.

Now for $y \in Y_{Q,n}$ and $a \in K^\times$, we have

$$\begin{aligned}
&\mathscr{N}_{\overline{T}}(\mathrm{Int}(w_\alpha)(y(a))) \\
&\quad = \mathscr{N}_{\overline{T}}(y(a)) \cdot \mathscr{N}_{\overline{T}}(s_{\eta,K}(-\langle \alpha, y\rangle \cdot \alpha^\vee(a))) \\
&\quad = y(\mathrm{N}_{K/F}(a)) \cdot s_{\eta,F}(-\langle \alpha, y\rangle \cdot \alpha^\vee(\mathrm{N}_{K/F}(a))) \cdot f(a)^{\frac{2}{n}Q(y)} \cdot f(a)^{\frac{2}{n}\cdot\langle \alpha, y\rangle^2 \cdot Q(\alpha^\vee)} \\
&\quad = y(\mathrm{N}_{K/F}(a)) \cdot s_{\eta,F}(-\langle \alpha, y\rangle \cdot \alpha^\vee(\mathrm{N}_{K/F}(a))) \cdot f(a)^{\frac{2}{n}Q(y)}.
\end{aligned}$$

Here the second equality holds because

$$(\eta(z), a)_{K,n} = (\eta(z), \mathrm{N}_{K/F}(a))_{F,n} \quad \text{for } z \in Y_{Q,n}^{\mathrm{sc}} \text{ and } a \in K^\times,$$

as $\eta(z) \in F^\times$. Further, the last equality holds since

$$\frac{2}{n} \cdot \langle \alpha, y\rangle^2 \cdot Q(\alpha^\vee) = 0 \mod 2$$

by the proof of Theorem 6.8 (which shows that either $\langle \alpha, y\rangle = 0$ or $B(y, \alpha^\vee) = \langle \alpha, y\rangle \cdot Q(\alpha^\vee)$).

On the other hand,

$$\begin{aligned}
\mathrm{Int}(w_\alpha)\left(\mathscr{N}_{\overline{T}}(y(a))\right) &= \mathrm{Int}(w_\alpha)\left(y(\mathrm{N}_{K/F}(a)) \cdot f(a)^{\frac{2}{n}Q(y)}\right) \\
&= y(\mathrm{N}_{K/F}(a)) \cdot s_{\eta,F}\left(-\langle \alpha, y\rangle \cdot \alpha^\vee(\mathrm{N}_{K/F}(a))\right) \cdot f(a)^{\frac{2}{n}Q(y)}.
\end{aligned}$$

This proves the desired Weyl-equivariance. □

Corollary 14.5. — (i) *The base change of a W-invariant distinguished character of $\overline{T}_{Q,n,F}$ with respect to K/F is a W-invariant character of $\overline{T}_{Q,n,K}$.*

(ii) *The base change of a distinguished character of $\overline{T}_{Q,n,F}$ with respect to K/F is a distinguished character of $\overline{T}_{Q,n,K}$.*

(iii) *The base change of a tempered principal series representation $I_F(\chi)$ of $\overline{G}_F$ with respect to K/F is the tempered principal series representation $I_K(BC(\chi))$ of $\overline{G}_K$.*

15. Endoscopy

Another instance of Langlands functoriality is the theory of endoscopy. The L-group formalism should lead naturally to the notions of "endoscopic groups" or "endoscopic datum" $(H, \mathscr{H}, s, \xi)$ for the BD covering groups. Since the LLC can be reduced to the case of trivial η, we shall assume $\eta = 1$ for the rest of this section.

15.1. Endoscopic groups. — Given a BD covering group $\overline{G}$, we have associated with it a dual group $\overline{G}^\vee$ which is defined using a based root datum

$$(Y_{Q,n}, \Delta^\vee_{Q,n}, X_{Q,n}, \Delta_{Q,n}),$$

and an L-group extension ${}^L\overline{G}$ which we have shown to be a split extension. Since $\eta = 1$ by hypothesis, the L-group extension has a distinguished splitting. Any such distinguished splitting gives an isomorphism ${}^L\overline{G} \cong \overline{G}^\vee \times W_F$. Let $\mathbb{G}_{Q,n}$ be the split linear algebraic group over F whose dual group is isomorphic to $\overline{G}^\vee$, which comes equipped with a maximally split torus $\mathbb{T}_{Q,n}$. Recall that one has a natural isogeny $i : T_{Q,n} \longrightarrow T \subset G$.

It is natural to regard $\mathbb{G}_{Q,n}$ as the principal endoscopy group of $\overline{G}$ and the elliptic endoscopic groups $\mathbb{H}_{Q,n}$ of $\mathbb{G}_{Q,n}$ as the elliptic endoscopic groups of $\overline{G}$ relative to a distinguished splitting. Indeed, it follows by definition that there is a natural map of dual groups

$$H^\vee_{Q,n} \longrightarrow \overline{G}^\vee.$$

The choice of a distinguished splitting of ${}^L\overline{G}$ then gives rise to a map of L-groups

$${}^LH_{Q,n} \longrightarrow {}^L\overline{G}.$$

15.2. Speculations. — We do not have anything substantive to say beyond this, but content ourselves with a few highly speculative remarks.

- For any such endoscopic group $\mathbb{H}_{Q,n}$, the L-group formalism leads to a matching of stable conjugacy classes of regular semisimple elements for $H_{Q,n}$ and G. This matching is obtained by a twist of the isogeny $i : T_{Q,n} \longrightarrow T$ as explained by Langlands-Shelstad [**34**]. In particular, the stable conjugacy classes of G which occur in the orbit-matching are all "good," in the sense that they can support genuine invariant distributions.
- The notion of isomorphisms of endoscopic data in the covering case should be slightly different from that in the linear case. This is already evident from the example of Mp_{2n}, where a theory of endoscopy was developed by W.-W. Li [**35**]. Here the endoscopic groups of Mp_{2n} (relative to the choice of a distinguished splitting) are $\mathrm{SO}_{2a+1} \times \mathrm{SO}_{2b+1}$ where (a, b) are ordered pairs of non-negative integers such that $a + b = n$. These are also the endoscopic groups of SO_{2n+1} except that the pairs (a, b) are unordered.

  We suspect that the notion of equivalence of endoscopic data $(H, \mathscr{H}, s, \xi)$ should be modified as follows. In the linear algebraic case, the semisimple element s in the dual group is taken modulo the center $Z(G^\vee)$ of the dual group $G^\vee$. In the covering case, we suspect that it should be taken modulo a smaller group. Namely, the isogeny $i : T_{Q,n} \longrightarrow T$ gives on the dual side a map

  $$i^* : T^\vee \longrightarrow T^\vee_{Q,n}.$$

  The center $Z(G^\vee)$ of $G^\vee$ is a subgroup of $T^\vee$ and its image $i^*(Z(G^\vee))$ in $T^\vee_{Q,n}$ lies in the center $Z(\overline{G}^\vee) = Z(G^\vee_{Q,n})$ of $\overline{G}^\vee$. We speculate that in the definition of endoscopic datum $(H, \mathscr{H}, s, \xi)$ of a BD covering group, the semisimple element s should be taken modulo the group $i^*(Z(G^\vee)) \subset Z(G^\vee)$. This is a wild speculation at the moment, but it is related to the anomaly discussed in §12 about central characters being non-constant in an L-packet in the covering case.

- After the orbit-matching, one needs to define the transfer factors for endoscopic transfers. This is a function $\Delta_{H_{Q,n},\overline{G}}$ which is supported only on matching pairs of elements in $H_{Q,n} \times \overline{G}$ and which satisfies certain properties.
- These transfer factors should allow one to transfer orbital integrals on $\overline{G}$ to stable orbital integrals on $H_{Q,n}$. One would imagine that such a transfer can be proved as a consequence of the companion fundamental lemma. One would further imagine that the fundamental lemma itself could be proved by using Harish-Chandra descent to reduce it to a fundamental lemma on the Lie algebra. We believe that on the level of Lie algebra, this fundamental lemma should be a consequence of the so-called nonstandard fundamental lemma formulated by Waldspurger [**65**] and established by Ngo [**49**]. In other words, we believe that once the definitions of the transfer factors are set up correctly, the ultimate ingredients for proving it should already be available by the work of Waldspurger [**65**] and Ngo [**49**].

15.3. Principal series. — The isogeny $i : T_{Q,n} \longrightarrow T$ allows one to transfer principal series representations of $G_{Q,n}$ to those on $\overline{G}$, subject to picking a distinguished splitting of ${}^L\overline{G}$, or equivalently a Weyl-invariant genuine character χ_0 of $\overline{T}_{Q,n}$ which factors to $Z(\overline{T})$. Given any character χ of $T_{Q,n}$ which factors through i, one then obtains a genuine character $\chi_0 \cdot \chi$ of $Z(\overline{T})$, from which one obtains an irreducible genuine character $i(\chi_0\chi)$ of $\overline{T}$ and a principal series representation $I_{\overline{G}}(\chi_0\chi)$ of $\overline{G}$ by parabolic induction. Since the association $\chi \mapsto \chi_0\chi$ is W-equivariant (because χ_0 is Weyl-invariant), this gives a well-defined lifting

$$I_{G_{Q,n}}(\chi) \mapsto I_{\overline{G}}(\chi_0\chi)$$

which depends on the choice of χ_0 and for those principal series representations $I_{G_{Q,n}}(\chi)$ for χ trivial on $\mathrm{Ker}(i)$.

15.4. Iwahori-Hecke algebra isomorphisms. — When F is p-adic and $\mathbb{G}$ is simply-connected, Savin has studied the Iwahori-Hecke algebra of covers of G and established Iwahori-Hecke algebra isomorphisms with those of linear reductive groups. One can show the same results in the generality of this paper.

For this subsection, we resume the notations of §4. In particular, we have $\gcd(p, n) = 1$ and $K = \mathbb{G}(\mathscr{O})$ for a smooth reductive group scheme $\mathbb{G}$ over $\mathscr{O}$. Consider the natural reduction map $\mathbb{G}(\mathscr{O}) \to \mathbb{G}_\kappa(\kappa)$, and let I be the standard Iwahori subgroup, defined to the inverse image of $\mathbb{B}_\kappa(\kappa) \subset \mathbb{G}_\kappa(\kappa)$ with respect to the reduction map. We fix a splitting of K into $\overline{G}$ which gives a splitting of I. One may consider the Iwahori Hecke algebra $\mathscr{H}_\epsilon(\overline{G}, I)$; it is the algebra of anti-genuine I-biinvariant locally constant and compactly supported functions on $\overline{G}$.

In this subsection, we only consider those $\overline{G}$ for which $Z(\overline{T})$ possesses distinguished unramified genuine characters. In this case, ${}^L\overline{G}$ is isomorphic to $\overline{G}^\vee \times W_F$ (relative to the choice of a distinguished character) and thus we have the split "principal endoscopic group" $G_{Q,n}$ of $\overline{G}$ and its Iwahori-Hecke algebra $\mathscr{H}(G_{Q,n}, I_{Q,n})$.

***Theorem 15.1* (Savin).** — *Consider a BD covering group $\overline{G}$, for which there exist distinguished unramified genuine characters of $Z(\overline{T})$ (for example when $\eta = 1$). Then, its Iwahori-Hecke algebra $\mathscr{H}_\epsilon(\overline{G}, I)$ has the following description:*

$$\mathscr{H}_\epsilon(\overline{G}, I) = \left\langle T_y, E_{w_\alpha} : \ y \in Y_{Q,n}, \alpha^\vee \in \Delta^\vee \right\rangle$$

with relations given by

- $(E_{w_\alpha} - q)(E_{w_\alpha} + 1) = 0.$
- $(E_{w_\alpha} E_{w_\beta})^r = (E_{w_\beta} E_{w_\alpha})^r$ *if* $w_\alpha w_\beta$ *is of order* $2r$.
- $(E_{w_\alpha} E_{w_\beta})^r E_{w_\alpha} = (E_{w_\beta} E_{w_\alpha})^r E_{w_\beta}$ *if* $w_\alpha w_\beta$ *is of order* $2r+1$.
- $T_y \cdot T_{y'} = T_{y+y'}$.
- *Write* $\langle y, \alpha\rangle = m n_\alpha$. *Then*

$$E_{w_\alpha} \cdot T_y = \begin{cases} T_{y^{w_\alpha}} \cdot E_{w_\alpha} + (q-1)\sum_{k=0}^{m-1} T_{y - k n_\alpha \alpha^\vee} & \text{if } m > 0, \\ T_y \cdot E_{w_\alpha} & \text{if } m = 0, \\ T_{y^{w_\alpha}} \cdot E_{w_\alpha} - (q-1)\sum_{k=0}^{1-m} T_{y + k n_\alpha \alpha^\vee} & \text{if } m < 0. \end{cases}$$

The Iwahori-Hecke algebra $\mathscr{H}(G_{Q,n}, I_{Q,n})$ has the same description by generators and relations. Consequently, one has an isomorphism $\mathscr{H}_\epsilon(\overline{G}, I) \cong \mathscr{H}(G_{Q,n}, I_{Q,n})$ depending on the choice of the distinguished genuine character of $Z(\overline{T})$.

Proof. — The proof can be taken almost verbatim from [**57**], by noting the following. First, the argument in [**57**] relies crucially on the existence of a Weyl-invariant unramified character of $Z(\overline{T})$ (c.f. [**57**, Lemma 4.5]). Our assumption on $\overline{G}$ yields such existence as discussed in §6.5. Second, to obtain the above explicit relations between the generators of $\mathscr{H}(\overline{G}, I)$, one makes use of the property (d′) for any distinguished character χ, namely that

$$\chi \circ s_\eta\big(\alpha_{Q,n}^\vee(a)\big) = 1, \quad \text{for } \alpha \in \Delta \text{ and } a \in F^\times.$$

This is the result generalizing [**57**, §4], used in the proof of [**57**, Proposition 7.2] there. Besides these, Savin's argument could be carried out in our setting word for word. □

16. Examples

In this section, we give a number of examples to illustrate some of the topics treated in this paper. These examples are the ones which have been studied in the literature. As these groups arise as the cover $\overline{\mathbb{G}}$ of $\mathbb{G}$ which has simply-connected derived group, we may assume that $\overline{\mathbb{G}}$ is incarnated by a fair $(D, 1)$ without loss of generalities. For fixed $n \in \mathbb{N}$, we have the associated degree n cover $\overline{G}$.

We have seen that there always exist distinguished splittings of ${}^L\overline{G}$ for such $\overline{G}$, with respect to which ${}^L\overline{G} \simeq \overline{G} \times W_F$. In this section, we use χ_ψ to denote a distinguished character constructed in Section 7. It will be shown explicitly that our construction in the simply-connected simply-laced case agrees with the one given by Savin [**57**]. It is also compatible with the one for the classical double cover $\overline{\mathrm{Sp}}_{2r}$, as in [**31, 53**].

The computation of the bilinear form B_Q below uses crucially the identity

$$B_Q(\alpha^\vee, y) = Q(\alpha^\vee) \cdot \langle \alpha, y \rangle,$$

where $\alpha \in \Phi^\vee$ is any coroot and $y \in Y$.

16.1. Simply-connected case. — Consider a simply-connected simple group $\mathbb{G}$ of arbitrary type. There is up to unique isomorphism a $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ associated to a Weyl-invariant quadratic form on $Y^{\mathrm{sc}} = Y$. Consider $\overline{\mathbb{G}}$ incarnated by (D, η). As indicated above, there is no loss of generality in assuming D fair and $\eta = 1$, and we will do so in the following.

For simplicity we assume $n = 2$ except for the case of the exceptional G_2 where the computation is very simple for general n. We also assume that Q is the unique Weyl-invariant quadratic form which takes value 1 on the short coroots of $\mathbb{G}$. The general case of n and Q follows from similar computations.

Note that whenever we have assumed $n = 2$, we will write $Y_{Q,2}$ and $Y_{Q,2}^{\mathrm{sc}}$ for the lattices $Y_{Q,n}$ and $Y_{Q,n}^{\mathrm{sc}}$ which are of interest. We also have $J = 2Y + Y_{Q,2}^{\mathrm{sc}} = Y_{Q,2}^{\mathrm{sc}}$ since $Y = Y^{\mathrm{sc}}$.

16.1.1. *The simply-laced case A_r, D_r, E_6, E_7, E_8 and compatibility.* — Now let $\mathbb{G}$ be a simply-laced simply-connected group of type A_r for $r \geq 1$, D_r for $r \geq 3$, and E_6, E_7, E_8. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a fixed set of simple roots of $\mathbb{G}$. Let $\overline{\mathbb{G}}$ be the extension of $\mathbb{G}$ determined by the quadratic form Q with $Q(\alpha_i^\vee) = 1$ for all coroots $\alpha_i^\vee$. We obtain the two-fold cover $\overline{G}$ of G.

Clearly we have $n_\alpha = 2$ for all $\alpha \in \Phi$ in this case. Let $\alpha_i^\vee \in \Delta^\vee$ for $i = 1, \ldots, r$ be the simple coroots of $\mathbb{G}$. It is easy to compute the bilinear form B_Q associated with Q:

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} -1 & \text{if } \alpha_i \text{ and } \alpha_j \text{ connected in the Dynkin diagram,} \\ 0 & \text{otherwise.} \end{cases} \tag{16.1}$$

In order to show compatibility with Savin, we may further assume that $\overline{G}$ is incarnated by the following fair bisector D associated with B_Q as given in [**57**],

$$D(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 0 & \text{if } i < j, \\ Q(\alpha_i^\vee) & \text{if } i = j, \\ B_Q(\alpha_i^\vee, \alpha_j^\vee) & \text{if } i > j. \end{cases} \tag{16.2}$$

The following lemma is in [**57**] and reproduced here for convenience. The stated result can also be checked by straightforward computation.

Lemma 16.3. — *Let Ω be a subset of the vertices in the Dynkin diagram of $\mathbb{G}$ satisfying:*

(i) *no two vertices in* Ω *are adjacent;*
(ii) *every vertex not in* Ω *is adjacent to an even number of vertices in* Ω.

Then the map given by $\Omega \longmapsto e_\Omega$ *with* $e_\Omega := \sum_{\alpha_i \in \Omega} \alpha_i^\vee$ *gives a well-defined correspondence between such sets* Ω *and the cosets of* $Y_{Q,2}/J$. *In particular, the empty set corresponds to the trivial coset* J.

By properties of B_Q and (i) of Ω above, it follows that

$$Q(e_\Omega) = |\Omega|.$$

We now give a brief case by case discussion.

The A_r *case.* — There are two situations according to the parity of r.

Case 1. — r is even. As an illustration, we first do the straightforward computation. Let $\sum_i k_i \alpha_i^\vee \in Y_{Q,2}$ for proper $k_i \in \mathbb{Z}$. Then $B_Q(\sum_i k_i \alpha_i^\vee, \alpha_j^\vee) \in 2\mathbb{Z}$ for all $1 \le j \le r$ by the definition of $Y_{Q,2}$. In view of (16.1), it is equivalent to

$$\begin{cases} 2k_1 + (-1)k_2 & \in 2\mathbb{Z}, \\ (-1)k_1 + 2k_2 + (-1)k_3 & \in 2\mathbb{Z}, \\ (-1)k_2 + 2k_3 + (-1)k_4 & \in 2\mathbb{Z}, \\ \vdots & \\ (-1)k_{r-2} + 2k_{r-1} + (-1)k_r & \in 2\mathbb{Z}, \\ (-1)k_{r-1} + 2k_r & \in 2\mathbb{Z}. \end{cases} \tag{16.4}$$

It follows that k_2 is even and so are the successive $k_4, \ldots, k_r$ (we have assumed r to be even). Similarly, k_{r-1} is even and therefore all $k_{r-3}, \ldots, k_1$ are also even. This gives $Y_{Q,2} = J$.

Note that we could simply apply the lemma to get $Y_{Q,2} = J$, which corresponds to the fact that only the empty set satisfies properties (i) and (ii). There is nothing to check in this case, and the character χ such that $\chi \circ s_\eta$ is trivial will be a distinguished character.

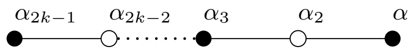
Case 2. — $r = 2k - 1$ is odd. Recall the notation

$$\alpha_{i,Q,2}^\vee := n_{\alpha_i} \cdot \alpha_i^\vee.$$

The consideration as in (16.4) works. However, for convenience we will apply the lemma to get $[Y_{Q,2}, J] = 2$ with a basis of $Y_{Q,2}$ given by

$$\{\alpha_{r,Q,2}^\vee, \alpha_{r-1,Q,2}^\vee, \ldots, \alpha_{2,Q,2}^\vee, e_\Omega = \sum_{m=1}^{k} \alpha_{2m-1}^\vee\}.$$

The nontrivial coset corresponds to the set Ω indicated by alternating bold circles below



A basis for $J = 2Y^{\mathrm{sc}}$ is given by

$$\{\alpha^\vee_{r,Q,2}, \alpha^\vee_{r-1,Q,2}, \ldots, \alpha^\vee_{2,Q,2}, 2e_\Omega\}.$$

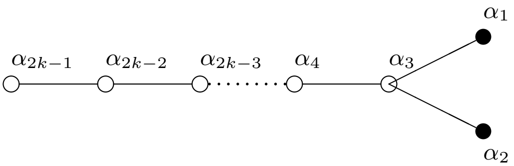
Now the distinguished character χ_ψ we constructed in §7 is determined by

$$\begin{cases} \chi_\psi(\alpha^\vee_{i,Q,2}(a)) = 1, \ 2 \le i \le r, \\ \chi_\psi(e_\Omega(a)) = \gamma_\psi(a)^{(2-1)Q(e_\Omega)} = \gamma_\psi(a)^{|\Omega|}. \end{cases} \tag{16.5}$$

This agrees with the formula in [**57**] when we substitute $a = \varpi$ in $\gamma_\psi(a)$ for ψ of conductor $\mathcal{O}_F$. See [**57**, pg 118].

The D_r case. — We also have two cases.

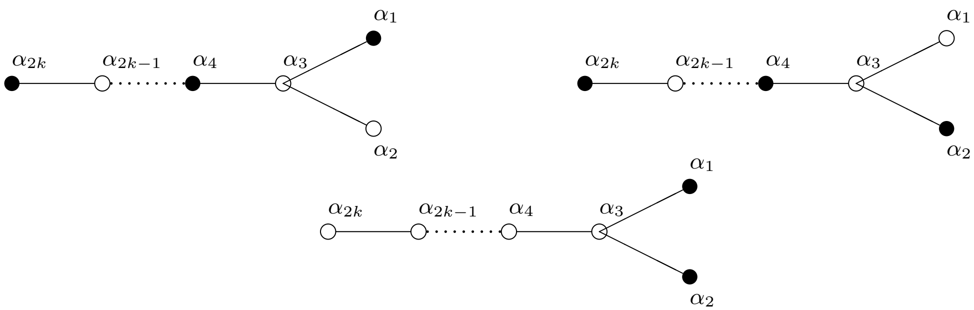
Case 1. — $r = 2k-1$ is odd with $k \ge 2$. Then $[Y_{Q,2}, J] = 2$ with the nontrivial $\Omega = \{\alpha_1, \alpha_2\}$:



Consider the basis $\{\alpha^\vee_{i,Q,2} : 2 \le i \le r\} \cup \{e_\Omega\}$ of $Y_{Q,2}$, then the construction of distinguished χ_ψ in Section 7 is determined by

$$\chi_\psi(e_\Omega(a)) = \gamma_\psi(a)^{(2-1)Q(e_\Omega)} = \gamma_\psi(a)^{|\Omega|}.$$

Case 2. — $r = 2k$ is even. Then $[Y_{Q,2} : J] = 4$. There are three nontrivial sets Ω_i for $i = 1, 2, 3$ as indicated by the bold circles below.



That is, $\Omega_1 = \{\alpha_1\} \cup \{\alpha_{2m} : 2 \le m \le k\}$, $\Omega_2 = \{\alpha_2\} \cup \{\alpha_{2m} : 2 \le m \le k\}$ and $\Omega_3 = \{\alpha_1, \alpha_2\}$. Note $|\Omega_1| = |\Omega_2| = k$ and $|\Omega_2| = 2$.

A basis of $Y_{Q,2}$ is given by

$$\{\alpha^\vee_{i,Q,2} : 3 \le i \le 2k-1\} \cup \{e_{\Omega_1}, e_{\Omega_2}, e_{\Omega_3}\}.$$

However, the construction of distinguished characters utilizes the elementary divisor theorem. Thus we have to provide bases for $Y_{Q,2}$ and J aligned in a proper way. To achieve this, consider the alternative basis of $Y_{Q,2}$ given by

$$\{\alpha_{i,Q,2}^{\vee} : 3 \le i \le 2k-1\} \cup \{e_{\Omega_1} + e_{\Omega_2} + e_{\Omega_3}, e_{\Omega_2} + e_{\Omega_3}, e_{\Omega_3}\}.$$

Then it is easy to check that the set

$$\{\alpha_{i,Q,2}^{\vee} : 3 \le i \le 2k-1\} \cup \{e_{\Omega_1} + e_{\Omega_2} + e_{\Omega_3}, 2(e_{\Omega_2} + e_{\Omega_3}), 2e_{\Omega_3}\}$$

is a basis for J. Note

$$Q(e_{\Omega_2} + e_{\Omega_3}) = |\Omega_2| + Q(2\alpha^{\vee}) = |\Omega_2| + 4.$$

Thus a distinguished character could be determined by

$$\begin{cases} \chi_{\psi}(\alpha_{i,Q,2}^{\vee}(a)) = 1, \ 3 \le i \le 2k-1, \\ \chi_{\psi}((e_{\Omega_1} + e_{\Omega_2} + e_{\Omega_3})(a)) = 1, \\ \chi_{\psi}((e_{\Omega_2} + e_{\Omega_3})(a)) = \gamma_{\psi}(a)^{Q(e_{\Omega_2}+e_{\Omega_3})} = \gamma_{\psi}(a)^{|\Omega_2|}, \\ \chi_{\psi}(e_{\Omega_3}(a)) = \gamma_{\psi}(a)^{|\Omega_3|}. \end{cases}$$

However, since we have assumed that D takes the special form given by (16.2), we have

$$\begin{aligned} D(e_{\Omega_1}, e_{\Omega_2} + e_{\Omega_3}) &= |\Omega_1| \\ D(e_{\Omega_2}, e_{\Omega_3}) &= Q(\alpha_i^{\vee}) = 1. \end{aligned}$$

Thus

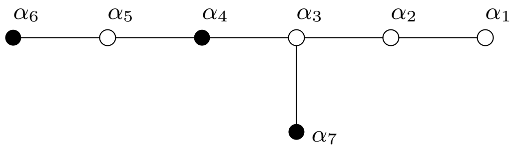
$$\begin{aligned} \chi_{\psi}(e_{\Omega_1}(a)) \cdot \chi_{\psi}((e_{\Omega_2} + e_{\Omega_3})(a)) &= (a,a)_2^{|\Omega_1|} \cdot \chi_{\psi}((e_{\Omega_1} + e_{\Omega_2} + e_{\Omega_3})(a)) \\ \chi_{\psi}((e_{\Omega_2}(a)) \cdot \chi_{\psi}(e_{\Omega_3}(a)) &= (a,a)_2 \cdot \chi_{\psi}((e_{\Omega_2} + e_{\Omega_3})(a)). \end{aligned}$$

Recall $\gamma_{\psi}(a)^2 = (a,a)_2$. This combined with the above results gives

$$\begin{cases} \chi_{\psi}(e_{\Omega_1}(a)) = \gamma_{\psi}(a)^{|\Omega_1|}, \\ \chi_{\psi}(e_{\Omega_2}(a)) = \gamma_{\psi}(a)^{|\Omega_2|}, \\ \chi_{\psi}(e_{\Omega_3}(a)) = \gamma_{\psi}(a)^{|\Omega_3|}. \end{cases}$$

It agrees with the genuine character given by Savin.

The E_6, E_7, E_8 case. — For E_6 and E_8, $Y_{Q,2} = J$ and so the situation is trivial. Consider E_7, then $[Y_{Q,n} : J] = 2$. The nontrivial Ω is given by $\Omega = \{\alpha_4, \alpha_6, \alpha_7\}$.



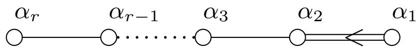
The set $\{\alpha_{i,Q,2}^{\vee} : 1 \le i \le 6\} \cup \{e_{\Omega}\}$ is a basis of $Y_{Q,2}$, while $\{\alpha_{i,Q,2}^{\vee} : 1 \le i \le 6\} \cup \{2e_{\Omega}\}$ a basis for J.

Our distinguished character is determined by

$$\chi_{\psi}(e_{\Omega}(a)) = \gamma_{\psi}(a)^{|\Omega|}.$$

This agrees with Savin also.

16.1.2. *The case C_r.* — Let Sp_{2r} be the simply-connected simple group with Dynkin diagram:



Let $\{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee\}$ be the set of simple coroots with $\alpha_1^\vee$ the short one. Let $n = 2$. Here $\overline{\mathrm{Sp}}_{2r}$ is determined by the unique Weyl-invariant quadratic form Q on Y with $Q(\alpha_1^\vee) = 1$.

It follows $n_{\alpha_1} = 2$. Also $Q(\alpha_i^\vee) = 2$ and $n_{\alpha_i} = 1$ for $2 \le i \le r$. Moreover, a basis of $Y_{Q,2} = Y^{\mathrm{sc}}$ is given by

$$\{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee\},$$

while a basis for $Y_{Q,2}^{\mathrm{sc}}$ is

$$\{2\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_r^\vee\}.$$

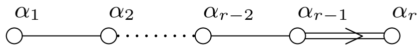
Since $J = Y_{Q,2}^{\mathrm{sc}}$, by the construction of distinguished character χ_ψ, it is determined by

$$\begin{cases} \chi_\psi(\alpha_i^\vee(a)) = 1, \text{ if } i = 2, 3, \dots, r; \\ \chi_\psi(\alpha_1^\vee(a)) = \gamma_\psi(a)^{(2-1)Q(\alpha_1^\vee)} = \gamma_\psi(a). \end{cases} \tag{16.6}$$

This uniquely determined a genuine character of $\overline{T}$ which is abelian. It can be checked that this agrees with the classical one (cf. [**31, 53**] for example).

16.1.3. *The B_r, F_4 and G_2 case.* — For completeness, we also give the explicit form of the distinguished character constructed in previous section for the double cover $\overline{G}$ of the simply connected group G of type B_r, F_4 and G_2. Recall that when $n = 2$ we have $J = Y_{Q,2}^{\mathrm{sc}}$.

The B_r case. — Consider the Dynkin diagram of B_r:



Let Q be the unique Weyl-invariant quadratic form with $Q(\alpha_i^\vee) = 1$ for $1 \le i \le r-1$. It gives $Q(\alpha_r^\vee) = 2$. We have also assumed that the double cover $\overline{G}$ is incarnated by a fair bisector D. The discussion now will be split into two cases according to the parity of r.

Case 1. — r is odd. Direct computation gives $Y_{Q,2}^{\mathrm{sc}} = Y_{Q,n}$ and therefore this case is trivial.

Case 2. — r is even. It is not difficult to compute the index $[Y_{Q,2} : Y_{Q,2}^{\mathrm{sc}}] = 2$. In fact, a basis of $Y_{Q,2}$ is given by

$$\{\alpha_1^\vee + \alpha_3^\vee + \dots + \alpha_{r-1}^\vee\} \cup \{2\alpha_i^\vee : 2 \le i \le r-1\} \cup \{\alpha_r^\vee\}.$$

This gives a basis of $J = Y_{Q,2}^{\mathrm{sc}}$:

$$\{2(\alpha_1^\vee + \alpha_3^\vee + \dots + \alpha_{r-1}^\vee)\} \cup \{2\alpha_i^\vee : 2 \le i \le r-1\} \cup \{\alpha_r^\vee\}.$$

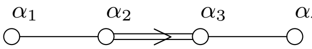
We have

$$Q(\alpha_1^\vee + \alpha_3^\vee + \cdots + \alpha_{r-1}^\vee) = r/2.$$

By the construction of distinguished character χ_ψ, it is determined by

$$\begin{cases} \chi_\psi\big((\alpha_1^\vee + \alpha_3^\vee + \cdots + \alpha_{r-1}^\vee)(a)\big) = \gamma_\psi(a)^{r/2}; \\ \chi_\psi\big((2\alpha_i^\vee)(a)\big) = 1, \text{ for } 2 \le i \le r-1; \\ \chi_\psi\big(\alpha_r^\vee(a)\big) = 1. \end{cases} \tag{16.7}$$

The F_4 case. — Consider the Dynkim diagram of F_4:



Let Q be such that $Q(\alpha_i^\vee) = 1$ for $i = 1, 2$. It implies $Q(\alpha_i^\vee) = 2$ for $i = 3, 4$. Clearly $n_{\alpha_i} = 2$ for $i = 1, 2$ and $n_{\alpha_i} = 1$ for $i = 3, 4$. We can compute

$$B_Q(\alpha_1^\vee, \alpha_2^\vee) = -1, \quad B_Q(\alpha_2^\vee, \alpha_3^\vee) = -2, \quad B_Q(\alpha_3^\vee, \alpha_4^\vee) = -2.$$

Also $B_Q(\alpha_i^\vee, \alpha_j^\vee) = 0$ if α_i and α_j are not adjacent in the Dynkin diagram.

Moreover, any $\sum_i k_i \alpha_i^\vee \in Y^{\mathrm{sc}}$ with certain $k_i \in \mathbb{Z}$ belongs to $Y_{Q,2}$ if and only if

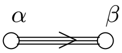
$$B_Q(\sum_i k_i \alpha_i^\vee, \alpha_j^\vee) \in 2\mathbb{Z} \text{ for all } 1 \le j \le 4,$$

which explicitly is given by

$$\begin{cases} 2k_1 + (-1)k_2 \in 2\mathbb{Z}, \\ (-1)k_1 + 2k_2 + (-2)k_3 \in 2\mathbb{Z}, \\ (-2)k_2 + 4k_3 + (-2)k_4 \in 2\mathbb{Z}, \\ (-2)k_3 + 4k_4 \in 2\mathbb{Z}. \end{cases}$$

Equivalently, $k_1, k_2 \in 2\mathbb{Z}$. This shows $Y_{Q,2} = Y_{Q,2}^{\mathrm{sc}}$, and thus the situation is trivial.

The G_2 case. — Consider the Dynkin diagram of G_2:



Let Q be such that $Q(\alpha^\vee) = 1$. This determines $Q(\beta^\vee) = 3$. Note $B_Q(\alpha^\vee, \beta^\vee) = -Q(\alpha^\vee) = -3$.

Since the computation is straightforward, we may assume $n \in \mathbb{N}_{\ge 1}$ is general instead of 2. It follows $n_\alpha = n$ and $n_\beta = n/\gcd(n, 3)$. Then $k_1\alpha^\vee + k_2\beta^\vee$ lies in $Y_{Q,n}$ if and only if

$$\begin{cases} 2k_1 - 3k_2 \in n\mathbb{Z}, \\ -3k_1 + 6k_2 \in n\mathbb{Z}. \end{cases}$$

Equivalently, $k_1 \in n\mathbb{Z}$ and k_2 divisible by $n/\gcd(n, 3)$. This exactly shows $Y_{Q,n} = Y_{Q,n}^{\mathrm{sc}}$ for arbitrary n. Also in this case, it is trivial to define the distinguished character for the fair D.

16.2. Kazhdan-Patterson coverings $\overline{\mathrm{GL}}_r$ [28, 29]. — We consider the group GL_r with root data $(X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee)$. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis for the cocharacter lattice Y of GL_r. Let $\Delta^\vee = \{\alpha_i^\vee := e_i - e_{i+1} : 1 \le i \le r\}$ denote a set of simple coroots of GL_r.

Consider the Weyl-invariant bilinear form on Y determined by

$$B(e_i, e_j) = \begin{cases} 2c & \text{if } i = j, \\ 2c+1 & \text{otherwise,} \end{cases}$$

where $c \in \mathbb{Z}$ is an integer. It follows that $Q(\alpha_i^\vee) = -1$ for any $\alpha_i^\vee$. The covering groups $\overline{\mathrm{GL}}_r$ arising are exactly those studied by Kazhdan-Patterson, and c is the twisting parameter in [**28**].

Write $c_{n,r} := n/\gcd(2cr + r - 1, n)$. It follows

$$Y_{Q,n} = \left\{ \sum_{i=1}^r m_i e_i : m_i \equiv m_j \bmod n, \text{ and } c_{n,r} | m_i \text{ for all } i, j \right\}.$$

In particular, a basis for $Y_{Q,n}$ is given by

$$\{ne_i : 1 \le i \le r-1\} \cup \left\{ c_{n,r} \cdot \left(\sum_{i=1}^r e_i \right) \right\}.$$

On the other hand, $Y_{Q,n}^{\mathrm{sc}}$ is spanned by $\{n \cdot \alpha_i^\vee : 1 \le i \le r-1\}$. It follows that $J = Y_{Q,n}^{\mathrm{sc}} + nY$ has a basis given by

$$\{ne_i : 1 \le i \le r-1\} \cup \left\{ n \cdot \left(\sum_{i=1}^r e_i \right) \right\}.$$

A distinguished character χ_ψ is thus determined by

$$\begin{cases} \chi_\psi\big(ne_i(a)\big) = 1 \text{ for all } 1 \le i \le r-1, \\ \chi_\psi\big((\sum_{i=1}^r c_{n,r} e_i)(a)\big) = \gamma_\psi(a)^{-\frac{r(2cr+r-1)c_{n,r}}{n}(n - c_{n,r})}. \end{cases} \tag{16.8}$$

In fact, this distinguished character is basically the genuine character of $Z(\overline{T})$ given in [**18**, Lemma 2], with associated parameter $s = 0$ in the notation of loc. cit. More precisely, the first equality in (16.8) corresponds to $s = 0$ in [**18**, Lemma 2], and the second equality in (16.8) corresponds to the equality (3.5) in the paper of Chinta and Offen.

An examination of the root datum shows that

$$\overline{G}^\vee \cong \{(g, \lambda) \in \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \det(g) = \lambda^{\gcd(2cr+r-1,n)}\} \subset \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}).$$

Thus, in general, the dual group $\overline{G}^\vee$ may not be GL_r. However, if $\gcd(2cr + r - 1, n) = 1$, then $\overline{G}^\vee = \mathrm{GL}_r$ in this case. In particular, for $r = 2$ and $c = 0$, the untwisted n-fold covering groups of GL_2 studied by Flicker in [**23**] belong to this class.

For more examples, we consider the embedding of $\mathrm{GL}_r \hookrightarrow \mathrm{SL}_{r+1}$ given by $g \mapsto (g, \det(g)^{-1})$. Let $\overline{\mathrm{SL}}_{r+1}$ be the degree n covering associated with (n, Q) where $Q(\alpha_i^\vee) = -1$ for all i. Then the pull-back covering of GL_r will be the covering associated to the quadratic form above with $c = -1$. Consider $n = 2$. If r is even, then

$$\overline{\mathrm{GL}}_r^\vee = \mathrm{GL}_r.$$

If r is odd, then $\overline{\mathrm{GL}}_r^\vee \subseteq \mathrm{GL}_r \times \mathrm{GL}_1$ and is given by

$$\overline{\mathrm{GL}}_r^\vee = \{(g, a) \in \mathrm{GL}_r \times \mathrm{GL}_1 : \det(g) = a^2\}.$$

When $r = 1$ or $r = 3$ there is an isomorphism $\overline{\mathrm{GL}}_r^\vee \simeq \mathrm{GL}_r$ given by $(g, a) \mapsto ga^{-1}$. However, for odd $r \geq 5$, there exists an isogeny $\overline{\mathrm{GL}}_r^\vee \to \mathrm{GL}_r$ of degree two given by $(g, a) \mapsto g$.

16.3. The cover $\overline{\mathrm{GSp}}_{2r}$. — Let $\mathbb{G}$ be the group GSp_{2r} of similitudes of symplectic type, and let $(X, \Delta, Y, \Delta^\vee)$ be its root data given as follows. The character group $X \simeq \mathbb{Z}^{r+1}$ has a standard basis $\{e_i^* : 1 \leq i \leq r\} \cup \{e_0^*\}$, and the roots are given by

$$\Delta = \{e_i^* - e_{i+1}^* : 1 \leq i \leq r - 1\} \cup \{2e_r^* - e_0^*\}.$$

The cocharacter group $Y \simeq \mathbb{Z}^{r+1}$ is given with a basis $\{e_i : 1 \leq i \leq r\} \cup \{e_0\}$. The coroots are

$$\Delta^\vee = \{e_i - e_{i+1} : 1 \leq i \leq r - 1\} \cup \{e_r\}.$$

Write $\alpha_i = e_i^* - e_{i+1}^*$, $\alpha_i^\vee = e_i - e_{i+1}$ for $1 \leq i \leq r - 1$, and also $\alpha_r = 2e_r^* - e_0^*$, $\alpha_r^\vee = e_r$. Consider a covering $\overline{\mathbb{G}}$ incarnated by $(D, 1)$. We are interested in those $\overline{\mathbb{G}}$ whose restricted to Sp_{2r} is the one with $Q(\alpha_r^\vee) = 1$. That is, we assume

$$Q(\alpha_i^\vee) = 2 \text{ for } 1 \leq i \leq r - 1, \quad Q(\alpha_r^\vee) = 1.$$

Since $\Delta^\vee \cup \{e_0\}$ gives a basis for Y, to determine Q it suffices to specify $Q(e_0)$. Let $n = 2$, and we obtain a double cover $\overline{\mathrm{GSp}}_{2r}$ which restricts to the classical metaplectic double cover $\overline{\mathrm{Sp}}_{2r}$. Note also the number $Q(e_0) \in \mathbb{Z}/2\mathbb{Z}$ determines whether the similitude factor $F^\times$ corresponding to the cocharacter e_0 splits into $\overline{\mathrm{GSp}}_{2r}$ or not. To recover the classical double cover of GSp_{2r}, we should take $Q(e_0)$ to be even.

Back to the case of $n = 2$ and general $Q(e_0)$. We compute the root data for the complex dual group $\overline{\mathrm{GSp}}_{2r}^\vee$. We have

$$Y_{Q,2} = \left\{ \sum_{i=1}^{r} k_i \alpha_i^\vee + k e_0 \in Y : k_i \in \mathbb{Z} \text{ for } 1 \leq i \leq r - 1, k_r, k \in 2\mathbb{Z} \right\}$$

and the sublattice $Y_{Q,2}^{\mathrm{sc}}$ is spanned by $\{\alpha_{i,Q,2}^\vee\}_{1 \leq i \leq r}$, i.e.,

$$\{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_{r-1}^\vee, 2\alpha_r^\vee\}.$$

The lattice $J = Y_{Q,2}^{\mathrm{sc}} + 2Y$ is thus equal to $Y_{Q,2}$. In this case, distinguished character χ_ψ will be determined by the condition (since we have assumed $(D, 1)$ to be fair)

$$\chi_\psi\big(y(a)\big) = 1 \text{ for all } y \in Y_{Q,2}, \ a \in F^\times.$$

An examination of the root datum gives:

$$\overline{\mathrm{GSp}}_{2r}^\vee = \begin{cases} \mathrm{GSp}_{2r}(\mathbb{C}), \ \text{if } r \text{ is odd;} \\ \mathrm{PGSp}_{2r}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}), \ \text{if } r \text{ is even.} \end{cases}$$

This explains the difference in the representation theory of $\overline{\mathrm{GSp}}_{2r}$ for even and odd r observed in the work of Szpruch [**63**].

17. Problems and Questions

In this final section, we highlight a few problems and questions which we feel are important to carry the program forward:

(a) (Real Groups) When $F = \mathbb{R}$, Harish-Chandra's classification of discrete series representation works equally well for BD covering groups. Might one be able to formulate this classification in terms of L-parameters in the spirit of this paper, analogous to what Langlands accomplished for linear real reductive groups? Moreover, can the results of the recent papers [**5, 4**] be formulated in the framework of this paper? In the paper [**73**], Weissman has shown that the classification of discrete series of $\overline{G}$ (given by Harish-Chandra) can be formulated in terms of L-parameters; this gives strong supporting evidence for the notion of L-groups introduced in [**73**] and described here.

(b) (Hecke algebra isomorphisms) In Theorem 15.1, we saw that there is an isomorphism of Iwahori-Hecke algebras for the covering group $\overline{G}$ and a linear algebraic group $G_{Q,n}$ in the tame case. There should perhaps be such Hecke algebra isomorphisms for other Bernstein components. One also expects some Hecke algebra isomorphisms outside the tame case, i.e., when the residue characteristic p divides the degree n of the covering. For Mp_{2n}, such results were obtained in [**74**] and [**64**].

(c) (Supercuspidal representations) Depth zero supercuspidal representations of BD covering groups have been studied by Howard-Weissman [**27**]. One would expect that the construction of supercuspidal representations of J.K. Yu [**75**] and S. Stevens [**60**] for linear reductive groups can be extended to BD covering groups. This is an ongoing investigation of the first author with J.L.Kim. One expects that Kim's proof of the exhaustion of Yu's construction [**75**] can be extended to give exhaustion in the covering case. For this, a better understanding of the Bruhat-Tits theory of BD covering groups is probably needed; a preliminary study has been conducted by Weissman [**67**].

(d) (Harmonic Analysis) For covering groups, any conjugacy-invariant function or distribution is necessarily supported on the subset of "good" or "relevant" elements. These are elements $g \in \overline{G}$ such that g is not conjugate to $g \cdot \epsilon$ for any $\epsilon \in \mu_n$ (for a degree n cover). It is clear that to have a better understanding of invariant harmonic analysis, a better understanding of such "good" or "relevant" elements is necessary. One might ask if the BD structure theory is robust enough to give one a classification of such elements.

W.-W. Li has extended many foundational results in harmonic analysis to the covering case in [**36**], such as the Plancherel theorem, the basic properties of invariant distributions and the properties of the standard intertwining operators. There are many further questions to investigate in invariant harmonic analysis for covering groups. One is to have a better understanding of the theory of R-groups for covering groups initiated in [**36**]. Another is to extend foundational results such as the Howe conjecture (proved by Clozel [**19**] in the linear case).

(e) (Automorphic L-functions) We have defined the global partial automoprhic L-function associated to an automorphic representation of a BD covering group. One would like to have a definition of the local L-factors at the remaining set of places so as to obtain a complete L-function. One would also like to show that the partial L-function has a meromorphic continuation, and the complete L-function satisfies a standard functional equation relative to $s \leftrightarrow 1-s$. For this, one may ask if Langlands-Shahidi theory can be extended to the covering case, but this is not clear since uniqueness of Whittaker models is false in general for covering groups. The only such success is the thesis work [**61, 62**] of D. Szpruch where the Langlands-Shahidi theory was extended to the group Mp_{2n}. One might also ask if the myriad of Rankin-Selberg integrals for various L-functions of linear groups have counterparts in the covering case. The recent preprint [**17**] of Cai-Friedberg-Ginzburg-Kaplan is a very exciting and promising work in this direction.

(f) (Functoriality) More generally, one would like to show that this class of automorphic L-functions from BD covering groups belong to the class of automorphic L-functions of linear reductive groups. In the context of (b), one might expect that there is a functorial transfer of automorphic representations from $\overline{G}$ to $G_{Q,n}$ which respects (partial) automorphic L-functions. One might imagine comparing the trace formula for these two groups. For this, we note that the work of W.-W. Li [**39, 36, 38, 40**] has carried the theory of the trace formula for covering groups to the point where one has the invariant trace formula. The earlier works of Flicker [**23**] and Flicker-Kazhdan [**24**] undertook such a comparison of trace formula for the Kazhdan-Patterson coverings.

(g) (Endoscopy) The next step in the theory of the trace formula for covering groups is undoubtedly the stable trace formula. For this, one needs to develop the theory of endoscopy for covering groups. This includes the definition of stable conjugation, the definition of endoscopic groups, the definition of correspondence of stable classes between a covering group and its endoscopic groups and the definition of the transfer factors. Since the theory of endoscopy for linear reductive groups is essentially of arithmetic origin and content, one might expect that a nice theory exists for the BD covering groups since these are of algebraic origin. The only covering group for which a theory of endoscopy exists is the group Mp_{2n}, where the theory is due to Adams [**1**], Renard [**55, 56**] and W.-W. Li [**35, 37**]. The recent preprint [**41**] of W.-W. Li has taken the first step towards the stabilization of the trace formula for Mp_{2n}.

(h) (Automorphic Discrete Spectrum) Naturally, one hopes to have an analog of the Arthur's conjecture for BD covering groups, including an analog of the Arthur multiplicity formula. This will very much depend on the shape of the theory of endoscopy. The only BD covering group for which a precise conjecture exists is Mp_{2n}, beyond the work of Flicker [**23**] and Flicker-Kazhdan [**24**].

(i) (General Covering Tori) The various questions highlighted above are already highly non-trivial and interesting when $\mathbb{G} = \mathbb{T}$ is a (not necessarily split) torus. The ongoing work of Weissman [**68, 73**] and Hiraga-Ikeda aim to understand this case completely but many mysteries remain.

(j) (Applications) The impetus for a program naturally depends on its potential applications. The motivation for our investigations is simply in the naive hope of including the representation theory and automorphic forms of BD covering groups in the framework of the Langlands philosophy. This is reasonable enough (if naive) for the point of view of representation theory. But what about from the point of view of number theory? Automorphic forms of covering groups have traditionally found applications in analytic number theory, such as in the work of Bump-Friedberg-Hoffstein [**16**]. It is reasonable to demand concrete arithmetic applications of this potential theory. The only thought we have to offer is perhaps in various branching or period problems, such as the arithmetic information contained in Fourier coefficients or in the analogs of the Gross-Prasad conjecture.

(k) (Geometric Counterpart) The definition of the dual group of a BD cover first appeared in the context of the Geometric Langlands Program, through the work of Finkelberg-Lysenko [**22**]. One might expect the geometric theory to offer more evidence for this program. From the geometric side, quantum groups seem to play an important role in the theory. This is also reflected to some extent in the work of Brubaker-Bump-Freidberg (c.f. [**13, 14, 12**]) on the Whittaker-Fourier coefficients of metaplectic Eisenstein series and the work of Chinta-Offen [**18**] and McNamara [**43, 45**] the metaplectic Casselman-Shalika formula. However, quantum groups are conspicuously missing from the framework developed in this article, and one may wonder if and how they should be incorporated.

(l) (Function Fields) On the other hand, one can consider classical function fields (of curves over finite fields) and ask whether V. Lafforgue's recent construction [**32**] of the global Langlands correspondence for arbitrary linear reductive groups could be extended to the case of BD covering groups: this is a very tantalizing problem whose resolution should shed much light on the Langlands-Weissman program and [**32**] has suggested that this should follow from the methods there.

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A COMPARISON OF L-GROUPS FOR COVERS OF SPLIT REDUCTIVE GROUPS

by

Martin H. Weissman

Abstract. — In one article, the author has defined an L-group associated to a cover of a quasisplit reductive group over a local or global field. In another article, Wee Teck Gan and Fan Gao define (following an unpublished letter of the author) an L-group associated to a cover of a pinned split reductive group over a local or global field. In this short note, we give an isomorphism between these L-groups. In this way, the results and conjectures discussed by Gan and Gao are compatible with those of the author. Both support the same Langlands-type conjectures for covering groups.

Résumé **(Une comparaison des L-groupes pour les revêtements de groupes réductifs déployés)**
Dans un article, l'auteur a défini un L-groupe associé à un revêtement de groupes réductifs quasi-déployés sur un corps local ou global. Dans un autre article, Wee Teck Gan et Fan Gao définissent (suite à une lettre inédite de l'auteur) un L-groupe associé à un revêtement de groupes réductifs quasi-déployés sur un corps local ou global. Dans cette courte note, nous donnons un isomorphisme entre ces L-groupes. De cette manière, les résultats et les conjectures discutés par Gan et Gao sont compatibles avec ceux de l'auteur. Les deux soutiennent les mêmes conjectures de type Langlands pour les revêtements des groupes.

Summary of two constructions

Let $\mathbf{G}$ be a split reductive group over a local or global field F. Choose a Borel subgroup $\mathbf{B} = \mathbf{T}\mathbf{U}$ containing a split maximal torus $\mathbf{T}$ in $\mathbf{G}$. Let $X = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ be the character lattice, and $Y = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$ be the cocharacter lattice of $\mathbf{T}$. Let $\Phi \subset X$ be the set of roots and Δ the subset of simple roots. For each root $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the associated root subgroup. Let $\Phi^\vee$ and $\Delta^\vee$ be the associated coroots and simple coroots. The root datum of $\mathbf{G} \supset \mathbf{B} \supset \mathbf{T}$ is

$$\Psi = (X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee).$$

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Fix a pinning (épinglage) of $\mathbf{G}$ as well – a system of isomorphisms $x_\alpha \colon \mathbf{G}_a \to \mathbf{U}_\alpha$ for every root α.

The following notions of covering groups and their dual groups match those in [**3**]. Let $\tilde{\mathbf{G}} = (\mathbf{G}', n)$ be a degree n cover of $\mathbf{G}$ over F; in particular, $\#\mu_n(F) = n$. Here $\mathbf{G}'$ is a central extension of $\mathbf{G}$ by $\mathbf{K}_2$ in the sense of [**1**], and write $(Q, \mathscr{D}, f)$ for the three Brylinski-Deligne invariants of $\mathbf{G}'$. Assume that if n is odd, then $Q \colon Y \to \mathbb{Z}$ takes only even values (this is [**3**, Assumption 3.1]).

Let $\tilde{G}^\vee \supset \tilde{B}^\vee \supset \tilde{T}^\vee$ be the dual group of $\tilde{\mathbf{G}}$, and let $\tilde{Z}^\vee$ be the center of $\tilde{G}^\vee$. The group $\tilde{G}^\vee$ is a pinned complex reductive group, associated to the root datum

$$(Y_{Q,n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, X_{Q,n}, \tilde{\Phi}, \tilde{\Delta}).$$

Here $Y_{Q,n} \subset Y$ is a sublattice containing nY. For each coroot $\alpha^\vee \in \Phi^\vee$, there is an associated positive integer n_α dividing n and a "modified coroot" $\tilde{\alpha}^\vee = n_\alpha \alpha^\vee \in \tilde{\Phi}^\vee$. The set $\tilde{\Phi}^\vee$ consists of the modified coroots, and $\tilde{\Delta}^\vee$ the modified simple coroots. Define $Y_{Q,n}^{\mathrm{sc}}$ to be the sublattice of $Y_{Q,n}$ generated by the modified coroots. Then

$$\tilde{T}^\vee = \mathrm{Hom}(Y_{Q,n}, \mathbb{C}^\times) \text{ and } \tilde{Z}^\vee = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \mathbb{C}^\times).$$

Let $\bar{F}/F$ be a separable algebraic closure, and $\mathrm{Gal}_F = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group. Fix an injective character $\epsilon \colon \mu_n(F) \hookrightarrow \mathbb{C}^\times$. From this data, the constructions of [**3**] and [**2**] both yield an L-group of $\tilde{\mathbf{G}}$ via a Baer sum of two extensions. In both papers, an extension

$$\tilde{Z}^\vee \hookrightarrow E_1 \twoheadrightarrow \mathrm{Gal}_F \tag{First twist}$$

is described in essentially the same way. When F is local, this "first twist" E_1 is defined via a $\tilde{Z}^\vee$-valued 2-cocycle on Gal_F. See [**2**, §5.2] and [**3**, §5.4] (in the latter, E_1 is denoted $(\tau_Q)_* \widetilde{\mathrm{Gal}}_F$). Over global fields, the construction follows from the local construction and Hilbert reciprocity.

Both papers include a "second twist". Gan and Gao [**2**, §5.2] describe an extension

$$\tilde{Z}^\vee \hookrightarrow E_2 \twoheadrightarrow \mathrm{Gal}_F, \tag{Second twist}$$

following an unpublished letter (June, 2012) from the author to Deligne. In [**3**], the second twist is the fundamental group of a gerbe, denoted $\pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{s})$. In this article $\bar{s} = \mathrm{Spec}(\bar{F})$, and so we write $\pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{F})$ instead.

Both papers proceed by taking the Baer sum of these two extensions, $E = E_1 \dotplus E_2$, to form an extension $\tilde{Z}^\vee \hookrightarrow E \twoheadrightarrow \mathrm{Gal}_F$. The extension E is denoted ${}^{\mathsf{L}}\tilde{Z}$ in [**3**, §5.4]. Then, one pushes out the extension E via $\tilde{Z}^\vee \hookrightarrow \tilde{G}^\vee$, to define the L-group

$$\tilde{G}^\vee \hookrightarrow {}^{\mathsf{L}}\tilde{G} \twoheadrightarrow \mathrm{Gal}_F\,. \tag{L-group}$$

The two constructions of the L-group, from [**2**] and [**3**] are the same, except for insignificant linguistic differences, and a significant difference between the "second twists". In this short note, by giving an isomorphism,

$$\pi_1^{\text{ét}}(\mathsf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{F}) \text{ (described by the author) } \xrightarrow{\sim} E_2 \text{ (described by Gan and Gao)}$$

we will demonstrate that the second twists, and thus the L-groups, of both papers are isomorphic. Therefore, the work of Gan and Gao in [**2**] supports the broader conjectures of [**3**].

Remark 0.1. — Among the "insignificant linguistic differences," we note that Gan and Gao use extensions of $F^\times / F^{\times n}$ (for local fields) or the Weil group $\mathscr{W}_F$ rather than Gal_F. But pulling back via the reciprocity map of class field theory yields extensions of Gal_F by $\tilde{Z}^\vee$ as above.

1. Computations in the gerbe

1.1. Convenient base points. — Let $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$ be the gerbe constructed in [**3**, §3]. Rather than using the language of étale sheaves over F, we work with $\bar{F}$-points and trace through the Gal_F-action. Let $\hat{T} = \mathrm{Hom}(Y_{Q,n}, \bar{F}^\times)$ and $\hat{T}_{\mathrm{sc}} = \mathrm{Hom}(Y_{Q,n}^{\mathrm{sc}}, \bar{F}^\times)$. Let $p \colon \hat{T} \to \hat{T}_{\mathrm{sc}}$ be the surjective Gal_F-equivariant homomorphism dual to the inclusion $Y_{Q,n}^{\mathrm{sc}} \hookrightarrow Y_{Q,n}$. Define

$$\hat{Z} = \mathrm{Ker}(p) = \mathrm{Hom}(Y_{Q,n}/Y_{Q,n}^{\mathrm{sc}}, \bar{F}^\times).$$

The reader is warned not to confuse $\hat{T}, \hat{T}_{\mathrm{sc}}, \hat{Z}$ with $\tilde{T}^\vee, \tilde{T}_{\mathrm{sc}}^\vee, \tilde{Z}^\vee$; the former are nontrivial Gal_F-modules (Homs into $\bar{F}^\times$) and the latter are trivial Gal_F-modules (Homs into $\mathbb{C}^\times$ as a trivial Gal_F-module).

Write $\bar{D} = \mathscr{D}(\bar{F})$ and $D = \mathscr{D}(F)$, where we recall $\mathscr{D}$ is the second Brylinski-Deligne invariant of the cover $\tilde{\mathbf{G}}$. We have a Gal_F-equivariant short exact sequence,

$$\bar{F}^\times \hookrightarrow \bar{D} \twoheadrightarrow Y.$$

By Hilbert's Theorem 90, the Gal_F-fixed points give a short exact sequence,

$$F^\times \hookrightarrow D \twoheadrightarrow Y.$$

Let $\bar{D}_{Q,n}$ and $\bar{D}_{Q,n}^{\mathrm{sc}}$ denote the preimages of $Y_{Q,n}$ and $Y_{Q,n}^{\mathrm{sc}}$ in $\bar{D}$. These are *abelian* groups, fitting into a commutative diagram with exact rows.

$$\begin{tikzcd}
\bar{F}^\times \arrow[r, hook] \arrow[d, "="] & \bar{D}_{Q,n}^{\mathrm{sc}} \arrow[r, two heads] \arrow[d, hook] & Y_{Q,n}^{\mathrm{sc}} \arrow[d, hook] \\
\bar{F}^\times \arrow[r, hook] & \bar{D}_{Q,n} \arrow[r, two heads] & Y_{Q,n}.
\end{tikzcd}$$

Let $\mathrm{Spl}(\bar{D}_{Q,n})$ be the $\hat{T}$-torsor of splittings of $\bar{D}_{Q,n}$, and similarly let $\mathrm{Spl}(\bar{D}_{Q,n}^{\mathrm{sc}})$ be the $\hat{T}_{\mathrm{sc}}$-torsor of splittings of $\bar{D}_{Q,n}^{\mathrm{sc}}$.

Let $\overline{\mathrm{Whit}}$ denote the $\hat{T}_{\mathrm{sc}}$-torsor of nondegenerate characters of $\mathbf{U}(\bar{F})$. An element of $\overline{\mathrm{Whit}}$ is a homomorphism (defined over $\bar{F}$) from $\mathbf{U}$ to $\mathbf{G}_a$ which is nontrivial on every simple root subgroup $\mathbf{U}_\alpha$. Gal_F acts on $\overline{\mathrm{Whit}}$, and the fixed points $\mathrm{Whit} = \overline{\mathrm{Whit}}^{\mathrm{Gal}_F}$ are those homomorphisms from $\mathbf{U}$ to $\mathbf{G}_a$ which are defined over F. The $\hat{T}_{\mathrm{sc}}$-action on $\overline{\mathrm{Whit}}$ is described in [**3**, §3.3].

The pinning $\{x_\alpha : \alpha \in \Phi\}$ of $\mathbf{G}$ gives an element $\psi \in \mathrm{Whit}$. Namely, let ψ be the unique nondegenerate character of $\mathbf{U}$ which satisfies

$$\psi(x_\alpha(1)) = 1 \text{ for all } \alpha \in \Delta.$$

In [**3**, §3.3], we define an surjective homomorphism $\mu\colon \hat{T}_{\mathrm{sc}} \to \hat{T}_{\mathrm{sc}}$, and a Gal_F-equivariant isomorphism of $\hat{T}_{\mathrm{sc}}$-torsors,

$$\bar{\omega}\colon \mu_* \overline{\mathrm{Whit}} \to \mathrm{Spl}(D^{\mathrm{sc}}_{Q,n}).$$

The isomorphism $\bar{\omega}$ sends ψ to the unique splitting $s_\psi \in \mathrm{Spl}(D^{\mathrm{sc}}_{Q,n})$ which satisfies

$$s_\psi(\tilde{\alpha}^\vee) = r_\alpha \cdot [e_\alpha]^{n_\alpha}, \text{ with } r_\alpha = (-1)^{Q(\alpha^\vee)\cdot \frac{n_\alpha(n_\alpha - 1)}{2}}.$$

We describe the element $[e_\alpha] \in D$ concisely here, based on [**1**, §11] and [**2**, §2.4]. Let $F(\!(v)\!)$ be the field of Laurent series with coefficients in F. The extension $\mathbf{K}_2 \hookrightarrow \mathbf{G}' \twoheadrightarrow \mathbf{G}$ splits over any unipotent subgroup, and so the pinning homomorphisms $x_\alpha\colon F(\!(v)\!) \to \mathbf{U}_\alpha(F(\!(v)\!))$ lift to homomorphisms

$$\tilde{x}_\alpha\colon F(\!(v)\!) \to \mathbf{U}'_\alpha(F(\!(v)\!)).$$

Define, for any $u \in F(\!(v)\!)^\times$,

$$\tilde{n}_\alpha(u) = \tilde{x}_\alpha(u)\tilde{x}_{-\alpha}(-u^{-1})\tilde{x}_\alpha(u).$$

This yields an element

$$\tilde{t}_\alpha = \tilde{n}_\alpha(v) \cdot \tilde{n}_\alpha(-1) \in \mathbf{T}'(F(\!(v)\!)).$$

Then t_α lies over $\alpha^\vee(v) \in \mathbf{T}(F(\!(v)\!))$. Its pushout via the tame symbol $\mathbf{K}_2(F(\!(v)\!)) \xrightarrow{\partial}\!\!\!\!\rightarrow F^\times$ is the element we call $[e_\alpha] \in D$.

Remark 1.1. — The element $s_\psi(\tilde{\alpha}^\vee) = r_\alpha \cdot [e_\alpha]^{n_\alpha}$ coincides with what Gan and Gao call $s_{Q^{\mathrm{sc}}}(\tilde{\alpha}^\vee)$ in [**2**, §5.2]; the sign r_α arises from the formulae of [**1**, §11.1.4, 11.1.5].

Let $j_0\colon \hat{T}_{\mathrm{sc}} \to \mu_*\overline{\mathrm{Whit}}$ be the unique isomorphism of $\hat{T}_{\mathrm{sc}}$-torsors which sends 1 to ψ (or rather the image of ψ via $\overline{\mathrm{Whit}} \to \mu_*\overline{\mathrm{Whit}}$). Since $\psi \in \mathrm{Whit}$ is Gal_F-invariant, this isomorphism j_0 is also Gal_F-invariant.

Finally, let $s \in \mathrm{Spl}(\bar{D}_{Q,n})$ be a splitting which restricts to s_ψ on $Y^{\mathrm{sc}}_{Q,n}$. Such a splitting s exists, since the map $\mathrm{Spl}(\bar{D}_{Q,n}) \to \mathrm{Spl}(\bar{D}^{\mathrm{sc}}_{Q,n})$ is surjective (since the map $\hat{T} \to \hat{T}_{\mathrm{sc}}$ is surjective). Note that s is not necessarily Gal_F-invariant (and often cannot be).

Let $h\colon \hat{T} \to \mathrm{Spl}(\bar{D}_{Q,n})$ be the function given by

$$h(x) = x^n * s \text{ for all } x \in \hat{T}.$$

The triple $\bar{z} = (\hat{T}, h, j_0)$ is an $\bar{F}$-object (i.e., a geometric base point) of the gerbe $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$. Note that the construction of $\bar{z}$ depends on two choices: a pinning of $\mathbf{G}$ (to obtain $\psi \in \mathrm{Whit}$) and a splitting s of $\bar{D}_{Q,n}$ extending s_ψ. We call such a triple $\bar{z}$ a ***convenient base point*** for the gerbe $\mathsf{E}_\epsilon(\tilde{\mathbf{G}})$.

1.2. The fundamental group. — For a convenient base point $\bar{z}$ associated to s, we consider the fundamental group

$$\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}) = \bigsqcup_{\gamma \in \mathrm{Gal}_F} \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z}).$$

This fundamental group fits into a short exact sequence

$$\tilde{Z}^\vee \hookrightarrow \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}) \twoheadrightarrow \mathrm{Gal}_F,$$

where the fiber over $\gamma \in \mathrm{Gal}_F$ is $\mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$. Thus to describe the fundamental group, it suffices to describe each fiber (as a $\tilde{Z}^\vee$-torsor), and the multiplication maps among fibers.

The base point ${}^\gamma \bar{z}$ is the triple $({}^\gamma \hat{T}, \gamma \circ h, \gamma \circ j_0)$, where ${}^\gamma \hat{T}$ is the $\hat{T}$-torsor with underlying set $\hat{T}$ and twisted action

$$u *_\gamma x = \gamma^{-1}(u) \cdot x.$$

To give an element $f \in \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ is the same as giving an element $\zeta \in \tilde{Z}^\vee$ and a map of $\hat{T}$-torsors $f_0 \colon \hat{T} \to {}^\gamma \hat{T}$ satisfying

$$(\gamma \circ h) \circ f_0 = h \text{ and } (\gamma \circ j_0) \circ p_* f_0 = j_0.$$

Any such map of $\hat{T}$-torsors is uniquely determined by the element $\tau \in \hat{T}$ satisfying $f_0(1) = \tau$. The two conditions above are equivalent to the two conditions

$$\tau^n = \gamma^{-1} s / s \text{ and } \tau \in \hat{Z}. \tag{1.1}$$

Thus, to give an element $f \in \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ is the same as giving a pair $(\tau, \zeta) \in \hat{T} \times \tilde{Z}^\vee$, where τ satisfies the two conditions above. Therefore, in what follows, we write $(\tau, \zeta) \in \mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ to indicate that τ satisfies the two conditions above, and to refer to the corresponding morphism in the gerbe $\mathbf{E}_\epsilon(\tilde{\mathbf{G}})$ in concrete terms.

We use $\epsilon \colon \mu_n(F) \xrightarrow{\sim} \mu_n(\mathbb{C})$ to identify $\hat{Z}_{[n]}$ with $\tilde{Z}^\vee_{[n]}$. Two pairs (τ, ζ) and (τ', ζ') are identified in $\mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ if and only if there exists $\xi \in \hat{Z}_{[n]}$ such that

$$\tau' = \xi \cdot \tau \text{ and } \zeta' = \epsilon(\xi)^{-1} \cdot \zeta.$$

The structure of $\mathrm{Hom}(\bar{z}, {}^\gamma \bar{z})$ as a $\tilde{Z}^\vee$-torsor is by scaling the second factor in $(\tau, \zeta) \in \hat{T} \times \tilde{Z}^\vee$. To describe the fundamental group completely, it remains to describe the multiplication maps among fibers. If $\gamma_1, \gamma_2 \in \mathrm{Gal}_F$, and

$$(\tau_1, \zeta_1) \in \mathrm{Hom}(\bar{z}, {}^{\gamma_1} \bar{z}) \text{ and } (\tau_2, \zeta_2) \in \mathrm{Hom}(\bar{z}, {}^{\gamma_2} \bar{z}),$$

then their composition in $\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z})$ is given by

$$(\tau_1, \zeta_1) \circ (\tau_2, \zeta_2) = (\gamma_2^{-1}(\tau_1) \cdot \tau_2, \zeta_1 \zeta_2).$$

Observe that

$$(\gamma_2^{-1}(\tau_1)\tau_2)^n = \gamma_2^{-1}\left(\gamma_1^{-1} s/s\right) \cdot \left(\gamma_2^{-1} s/s\right) = (\gamma_1\gamma_2)^{-1} s/s.$$

Therefore $(\gamma_2^{-1}(\tau_1) \cdot \tau_2, \zeta_1\zeta_2) \in \mathrm{Hom}(\bar{z}, {}^{\gamma_1\gamma_2} \bar{z})$ as required.

2. Comparison to the second twist

2.1. The second twist. — The construction of the second twist in [2] does not rely on gerbes at all, at the expense of some generality; it seems difficult to extend the construction there to nonsplit groups. But for split groups, the construction of [2] offers significant simplifications over [3]. The starting point in [2] is the same short exact sequence of abelian groups as in the previous section,

$$F^\times \hookrightarrow D_{Q,n} \twoheadrightarrow Y_{Q,n}.$$

And as before, we utilize the splitting $s_\psi \colon Y_{Q,n}^{\mathrm{sc}} \hookrightarrow D_{Q,n}^{\mathrm{sc}}$. Taking the quotient by $s_\psi(Y_{Q,n}^{\mathrm{sc}})$, we obtain a short exact sequence

$$F^\times \hookrightarrow \frac{D_{Q,n}}{s_\psi(Y_{Q,n}^{\mathrm{sc}})} \twoheadrightarrow \frac{Y_{Q,n}}{Y_{Q,n}^{\mathrm{sc}}}.$$

Apply $\mathrm{Hom}(\bullet, \mathbb{C}^\times)$ (and note $\mathbb{C}^\times$ is divisible) to obtain a short exact sequence,

$$\tilde{Z}^\vee \hookrightarrow \mathrm{Hom}\left(\frac{D_{Q,n}}{s_\psi(Y_{Q,n}^{\mathrm{sc}})}, \mathbb{C}^\times \right) \twoheadrightarrow \mathrm{Hom}(F^\times, \mathbb{C}^\times).$$

Define a homomorphism $\mathrm{Gal}_F \to \mathrm{Hom}(F^\times, \mathbb{C}^\times)$ by the Artin symbol,

$$\gamma \mapsto \left(u \mapsto \epsilon \left(\frac{\gamma^{-1}(\sqrt[n]{u})}{\sqrt[n]{u}} \right) \right).$$

Pulling back the previous short exact sequence by this homomorphism yields a short exact sequence

$$\tilde{Z}^\vee \hookrightarrow E_2 \twoheadrightarrow \mathrm{Gal}_F\,.$$

This E_2 is the second twist described in [2].

Remark 2.1. — There is an insignificant difference here—at the last step, over a local field F, Gan and Gao pull back to $F^\times / F^{\times n}$ via the Hilbert symbol whereas we pull further back to Gal_F via the Artin symbol.

Write $E_{2,\gamma}$ for the fiber of E_2 over any $\gamma \in \mathrm{Gal}_F$. Again, to understand the extension E_2, it suffices to understand these fibers (as $\tilde{Z}^\vee$-torsors), and to understand the multiplication maps among them. The steps above yield the following (somewhat) concise description of $E_{2,\gamma}$.

$E_{2,\gamma}$ is the set of homomorphisms $\chi \colon D_{Q,n} \to \mathbb{C}^\times$ such that

- χ is trivial on the image of $Y_{Q,n}^{\mathrm{sc}}$ via the splitting s_ψ,
- For every $u \in F^\times$, $\chi(u) = \epsilon(\gamma^{-1} \sqrt[n]{u} / \sqrt[n]{u})$.

Multiplication among fibers is given by usual multiplication, $\chi_1, \chi_2 \mapsto \chi_1 \chi_2$. The $\tilde{Z}^\vee$-torsor structure on the fibers is given as follows: if $\eta \in \tilde{Z}^\vee$, then

$$[\eta * \chi](d) = \eta(y) \cdot \chi(d) \text{ for all } d \in D_{Q,n} \text{ lying over } y \in Y_{Q,n}.$$

2.2. Comparison. — Now we describe a map from $\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z})$ to E_2, fiberwise over Gal_F. From the splitting s (used to define $\bar{z}$ and restricting to s_ψ on $Y_{Q,n}^{\mathrm{sc}}$), every element of $\bar{D}_{Q,n}$ can be written uniquely as $s(y) \cdot u$ for some $y \in Y_{Q,n}$ and some $u \in \bar{F}^\times$. Such an element $s(y) \cdot u$ is Gal_F-invariant if and only if

$$\gamma(s(y))\gamma(u) = s(y)u, \text{ or equivalently } \frac{\gamma^{-1}u}{u} \cdot \frac{\gamma^{-1}s}{s}(y) = 1, \text{ for al } \gamma \in \mathrm{Gal}_F\,.$$

Suppose that $\gamma \in \mathrm{Gal}_F$ and $(\tau, 1) \in \mathrm{Hom}(\bar{z}, {}^\gamma\bar{z})$. Define $\chi\colon D_{Q,n} \to \mu_n(\mathbb{C})$ by

$$\chi(s(y) \cdot u) = \epsilon\left(\gamma^{-1}\sqrt[n]{u}/\sqrt[n]{u} \cdot \tau(y)\right).$$

This makes sense, because Gal_F-invariance of $s(y) \cdot u$ implies

$$\left(\frac{\gamma^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau(y)\right)^n = \frac{\gamma^{-1}u}{u} \cdot \frac{\gamma^{-1}s}{s}(y) = 1.$$

To see that $\chi \in E_{2,\gamma}$, observe that

- χ is a homomorphism (a straightforward computation),
- if $y \in Y_{Q,n}^{\mathrm{sc}}$ then $\chi(s(y)) = \tau(y) = 1$ since $\tau \in \hat{Z}$,
- if $u \in F^\times$ then $\chi(u) = \epsilon(\gamma^{-1}\sqrt[n]{u}/\sqrt[n]{u})$ by definition.

Lemma 2.2. — *The map sending $(\tau, 1)$ to χ, described above, extends uniquely to an isomorphism of $\tilde{Z}^\vee$-torsors from* $\mathrm{Hom}(\bar{z}, {}^\gamma\bar{z})$ *to* $E_{2,\gamma}$.

Proof. — If this map extends to an isomorphism of $\tilde{Z}^\vee$-torsors as claimed, the map must send an element $(\tau, \zeta) \in \mathrm{Hom}(\bar{z}, {}^\gamma\bar{z})$ to the element $\zeta * \chi \in E_{2,\gamma}$. To demonstrate that the map extends to an isomorphism of $\tilde{Z}^\vee$-torsors, it must only be checked that

$$(\xi \cdot \tau, 1) \text{ and } (\tau, \epsilon(\xi))$$

map to the same element of $E_{2,\gamma}$, for all $\xi \in \hat{Z}_{[n]}$. For this, we observe that $(\xi \cdot \tau, 1)$ maps to the character χ' given by

$$\chi'(s(y) \cdot u) = \epsilon\left(\frac{\gamma^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\xi(y)\tau(y)\right) = \epsilon(\xi(y)) \cdot \epsilon\left(\frac{\gamma^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\tau(y)\right) = \epsilon(\xi(y)) \cdot \chi(s(y) \cdot u).$$

Thus $\chi' = \epsilon(\xi) * \chi$ and this demonstrates the lemma. □

From this lemma, we have a well-defined "comparison" isomorphism of $\tilde{Z}^\vee$-torsors,

$$C_\gamma\colon \mathrm{Hom}(\bar{z}, {}^\gamma z) \to E_{2,\gamma},$$

$$\text{(Comparison)} \qquad C_\gamma(\tau, \zeta)(s(y) \cdot u) = \epsilon\left(\frac{\gamma^{-1}\sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau(y)\right) \cdot \zeta(y).$$

Checking compatibility with multiplication yields the following.

Lemma 2.3. — *The isomorphisms C_γ are compatible with the multiplication maps, yielding an isomorphism of extensions of* Gal_F *by $\tilde{Z}^\vee$,*

$$C = C_{\bar{z}}\colon \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}) \to E_2.$$

Proof. — Suppose that $(\tau_1,\zeta_1)\in \mathrm{Hom}(\bar z, {}^{\gamma_1}\bar z)$ and $(\tau_2,\zeta_2)\in \mathrm{Hom}(\bar z, {}^{\gamma_2}\bar z)$. Their product in $\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}),\bar z)$ is $(\gamma_2^{-1}(\tau_1)\tau_2, \zeta_1\zeta_2)$. We compute

$$\begin{aligned}
C_{\gamma_1\gamma_2}(\tau_1\gamma^{-1}(\tau_2),\zeta_1\zeta_2)(s(y)\cdot u) &= \epsilon\left(\frac{(\gamma_1\gamma_2)^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\cdot\gamma_2^{-1}(\tau_1(y))\tau_2(y)\right)\cdot\zeta_1(y)\zeta_2(y)\\
&= \epsilon\left(\frac{\gamma_2^{-1}\gamma_1^{-1}\sqrt[n]{u}}{\gamma_2^{-1}\sqrt[n]{u}}\cdot\frac{\gamma_2^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\cdot\gamma_2^{-1}(\tau_1(y))\tau_2(y)\right)\\
&\quad\cdot\zeta_1(y)\zeta_2(y)\\
&= \epsilon\left(\gamma_2^{-1}\left(\frac{\gamma_1^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\cdot\tau_1(y)\right)\cdot\frac{\gamma_2^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\cdot\tau_2(y)\right)\\
&\quad\cdot\zeta_1(y)\zeta_2(y)\\
&= \epsilon\left(\frac{\gamma_1^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\cdot\tau_1(y)\right)\zeta_1(y)\\
&\quad\cdot\epsilon\left(\frac{\gamma_2^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\cdot\tau_2(y)\right)\zeta_2(y)\\
&= C_{\gamma_1}(\tau_1,\zeta_1)(s(y)\cdot u)\cdot C_{\gamma_2}(\tau_2,\zeta_2)(s(y)\cdot u).
\end{aligned}$$

In the middle step, we use the fact that $\left(\frac{\gamma_1^{-1}\sqrt[n]{u}}{\sqrt[n]{u}}\cdot\tau_1(y)\right)$ is an element of $\mu_n(F)$, and hence is Gal_F-invariant. This computation demonstrates compatibility of the isomorphisms C_γ with multiplication maps, and hence the lemma is proven. □

2.3. Independence of base point. — Lastly, we demonstrate that the comparison isomorphisms

$$C_{\bar z}\colon \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}),\bar z)\to E_2$$

depend naturally on the choice of convenient base point. With the pinned split group $\mathbf{G}$ fixed, choosing a convenient base point is the same as choosing a splitting of $\bar D_{Q,n}$ which restricts to s_ψ.

So consider two convenient base points $\bar z_1$ and $\bar z_2$, arising from splittings s_1, s_2 of $\bar D_{Q,n}$ which restrict to s_ψ on $Y_{Q,n}^{\mathrm{sc}}$. From [**3**, Theorem 19.6], any isomorphism ι from $\bar z_1$ to $\bar z_2$ in the gerbe $\mathbf{E}_\epsilon(\tilde{\mathbf{G}})$ defines an isomorphism

$$\iota\colon \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}),\bar z_1)\to \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}),\bar z_2),$$

and since $\tilde Z^\vee$ is contained in the center of $\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}),\bar z_1)$, the isomorphism of fundamental groups does not depend on the choice of isomorphism from $\bar z_1$ to $\bar z_2$; thus one may define a "Platonic" fundamental group

$$\pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}),\bar F)$$

without reference to an object of the gerbe.

Theorem 2.4. — *For any two convenient base points $\bar z_1,\bar z_2$, and any isomorphism $\iota\colon \bar z_1\to\bar z_2$, we have $C_{\bar z_2}\circ\iota = C_{\bar z_1}$. Thus E_2 is isomorphic to the fundamental group $\pi_1(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}),\bar F)$, as defined in* [**3**, Theorem A.7, Remark A.8].

Proof. — Choose any isomorphism from $\bar{z}_1 = (\hat{T}, h_1, j_0)$ to $\bar{z}_2 = (\hat{T}, h_2, j_0)$ in the gerbe $\mathbf{E}_\epsilon(\tilde{\mathbf{G}})$. Here $h_1(1) = s_1$ and $h_2(1) = s_2$, and $j_0(1) = s_\psi$. Such an isomorphism $\bar{z}_1 \xrightarrow{\sim} \bar{z}_2$ is given by an isomorphism $\iota\colon \hat{T} \to \hat{T}$ of $\hat{T}$-torsors satisfying the two conditions

$$h_2 \circ \iota = h_1 \text{ and } j_0 \circ p_* \iota = j_0.$$

Such an ι is determined by the element $b = \iota(1) \in \hat{T}$. The two conditions above are equivalent to the two conditions

$$b^n = s_1/s_2 \text{ and } b \in \hat{Z}.$$

The isomorphism $\bar{z}_1 \xrightarrow{\sim} \bar{z}_2$ determined by such a $b \in \hat{T}$ yields an isomorphism ${}^\gamma\iota\colon {}^\gamma\bar{z}_1 \to {}^\gamma\bar{z}_2$, for any $\gamma \in \mathrm{Gal}_F$. The isomorphism ${}^\gamma\iota$ is given by the isomorphism of $\hat{T}$-torsors from ${}^\gamma\hat{T}$ to ${}^\gamma\hat{T}$, which sends 1 to $\gamma(b)$.

This allows us to describe the isomorphism

$$\iota\colon \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_1) \to \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_2)$$

fibrewise over Gal_F. Namely, for any $\gamma \in \mathrm{Gal}_F$, and any $f \in \mathrm{Hom}(\bar{z}_1, {}^\gamma\bar{z}_1)$, we find a unique element $\iota(f) \in \mathrm{Hom}(\bar{z}_2, {}^\gamma\bar{z}_2)$ which makes the following diagram commute:

$$\begin{tikzcd} \bar{z}_1 \arrow[r, "f"] \arrow[d, "\iota"] & {}^\gamma\bar{z}_1 \arrow[d, "{}^\gamma\iota"] \\ \bar{z}_2 \arrow[r, "\iota(f)"] & {}^\gamma\bar{z}_2. \end{tikzcd}$$

If $f = (\tau, 1)$, then $\iota(f) = (\tau b/\gamma^{-1}b, 1)$. Indeed, when $\tau^n = \gamma^{-1}s_1/s_1$, we have

$$\left(\frac{\tau b}{\gamma^{-1}b}\right)^n = \frac{\gamma^{-1}s_1}{s_1}\frac{b^n}{\gamma^{-1}b^n} = \frac{\gamma^{-1}s_1}{s_1}\frac{s_1}{s_2}\frac{\gamma^{-1}s_2}{\gamma^{-1}s_1} = \frac{\gamma^{-1}s_2}{s_2}.$$

Thus $\iota(f) \in \mathrm{Hom}(\bar{z}_2, {}^\gamma\bar{z}_2)$ as required. In this way,

$$\iota\colon \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_1) \to \pi_1^{\text{ét}}(\mathbf{E}_\epsilon(\tilde{\mathbf{G}}), \bar{z}_2),$$

is given concretely on each fiber over $\gamma \in \mathrm{Gal}_F$ by

$$\iota(\tau, \zeta) = \left(\tau \cdot \frac{b}{\gamma^{-1}b}, \zeta\right).$$

Note that the conditions $b^n = s_1/s_2$ and $b \in \hat{Z}$ uniquely determine b up to multiplication by $\hat{Z}_{[n]}$. Since $\hat{Z}_{[n]}$ is a trivial Gal_F-module, the isomorphism ι of fundamental groups is independent of b. Finally, we compute, for any $y \in Y_{Q,n}, u \in \bar{F}^\times$ such that

$s_1(y) \cdot u \in D_{Q,n}$, and any $(\tau, \zeta) \in \mathrm{Hom}(\bar{z}_1, {}^\gamma \bar{z}_1)$,

$$\begin{aligned}
[C_{\bar{z}_2} \circ \iota](\tau, \zeta)(s_1(y) \cdot u) &= C_{\bar{z}_2}(\tau b/\gamma^{-1} b, \zeta)(s_1(y) \cdot u) \\
&= C_{\bar{z}_2}(\tau \gamma(b)/b, \zeta)(s_2(y) \cdot b^n(y) u) \\
&= \epsilon \left(\frac{\gamma^{-1} \sqrt[n]{b^n(y) u}}{\sqrt[n]{b^n(y) u}} \cdot \tau(y) \cdot \frac{b(y)}{\gamma^{-1}(b(y))} \right) \cdot \zeta(y) \\
&= \epsilon \left(\frac{\gamma^{-1} \sqrt[n]{u}}{\sqrt[n]{u}} \cdot \tau(y) \right) \cdot \zeta(u) \\
&= C_{\bar{z}_1}(\tau, \zeta)(s_1(y) \cdot u). \qquad \square
\end{aligned}$$

As noted in the introduction, this demonstrates compatibility between two approaches to the L-group.

Corollary 2.5. — *The L-group defined in* [**3**] *is isomorphic to the L-group defined in* [**2**], *for all pinned split reductive groups over local or global fields.*

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