Revue d'Histoire des Mathématiques

TEXTS & DOCUMENTS

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Tome 24 Fascicule 1 2018

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publiée avec le concours du Centre national de la recherche scientifique

REVUE D'HISTOIRE DES MATHÉMATIQUES

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Périodicité : La *Revue* publie deux fascicules par an, de 150 pages chacun environ.

Tarifs : Prix public Europe : 90 €; prix public hors Europe : 99 €;
prix au numéro : 43 €.
Des conditions spéciales sont accordées aux membres de la SMF.

Diffusion : SMF, Maison de la SMF, Case 916 - Luminy, 13288 Marseille Cedex 9
Hindustan Book Agency, O-131, The Shopping Mall, Arjun Marg, DLF Phase 1, Gurgaon 122002, Haryana, Inde

 Nº ISSN : 1262-022X

Maquette couverture : Armelle Stosskopf

Revue d'histoire des mathématiques
24 (2018), p. 41–106

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LOUIS BACHELIER'S 1938 MONOGRAPH ON THE CALCULUS OF SPECULATION: MATHEMATICAL FINANCE AND RANDOMNESS OF ASSET PRICES IN BACHELIER'S LATER WORK

Hichem Ben-El-Mechaiekh & Robert W. Dimand

Abstract. — Louis Bachelier's 1900 dissertation on the theory of speculation is now recognized as a landmark in the history of mathematical finance and stochastic processes, but his later work receives much less attention. Over the four decades following the defense of his dissertation on March 29, 1900, Bachelier repeatedly published new formulations of his theory of speculation: a more mathematically rigorous version in 1912, a less formal and more accessible chapter in 1914, and finally, in 1938, a monograph that was more concise and readable and more mathematically elegant than his earlier statements of the theory. That long-neglected monograph, Bachelier's final statement of his theory of speculation, is translated here into English for the first time, making it accessible to a larger audience.

Résumé (TEXTES & DOCUMENTS : La monographie de Louis Bachelier (1938) sur le calcul de spéculation : les mathématiques financières et l'aléatoire des prix des actifs dans l'IJuvre tardive de Bachelier)

Alors que la thèse de doctorat de Louis Bachelier sur la théorie de la spéculation est considérée à juste titre comme une fondation des mathématiques financières et des processus aléatoires, ses derniers travaux sont loin d'avoir reçus la même attention. Tout au long des quarante années qui suivirent la date

Texte reçu le 3 mai 2016, accepté le 29 juillet 2016, révisé le 27 juin 2017.

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2000 Mathematics Subject Classification : 01A60, 01A75, 60–03, 62–03, 62P05.

Key words and phrases : Louis Bachelier, financial mathematics, theory of speculation, Brownian motion, asset prices.

Mots clefs. — Louis Bachelier, mathématiques financières, théorie de la spéculation, mouvement brownien, prix des actifs.

de la publication de la thèse le 29 mars 1900, Bachelier s'employa a publier de nouvelles versions de sa théorie: une reformulation mathématiquement plus rigoureuse en 1912, une moins formelle et plus accessible en 1914, et finalement une exposition plus concise et mathématiquement plus élégante que les versions antérieures sous la forme d'une monographie en 1938. Ce compte-rendu final de la théorie de la spéculation de Bachelier, traduit ci-dessous en langue anglaise pour la première fois, est mis à la disposition d'un lectorat élargi.

INTRODUCTION

Louis Bachelier's doctoral dissertation *Théorie de la Spéculation*, defended on 29 March 1900 [Bachelier 1900], achieved renown after half a century of neglect. At first, mathematicians paid little attention to an introductory study of the analytical valuation of the pricing of options on government bonds not only as being outside the usual range of topics studied by contemporary mathematicians, but also because of over-simplifications that may appear natural to a physicist with hands-on experience in the Paris stock exchange, it lacked the mathematical rigor, formalism and depth [Bernstein 2005] despite its remarkable originality. On the other hand, the absence of any echo to Bachelier [1900] from those actively involved in speculation in bonds in early twentieth century Paris could only have resulted from the fact that either they have not heard of it, or simply did not have the mathematical background necessary to understand it.

While mathematics and finance have been in close relationship since antiquity, Bachelier's modeling of stock prices by equations for Brownian motion is heralded today as the birth of financial mathematics [Courtault et al. 2000]. But this area of human thought has only gained recognition as a scientific field after 1940 under the formidable impetus of the MIT modern finance pioneers such as Nobel laureates Paul Samuelson, Franco Modigliani, Myron Scholes, and Robert Merton. The limited early resonance of Bachelier's seminal ideas largely finds explanation in the absence of an "organized scientific community interested in his research" [Jovanovic 2010].

As pointed out by an anonymous referee—whom we wholeheartedly thank for insightful remarks—it is important to mitigate the myth of a disenchanted Bachelier who went totally unnoticed only to be incidentally rediscovered by Benoît Mandelbrot [1967] decades later (see e.g., [Weatherall 2013, 22–24]). As a matter of fact, Bachelier himself published various new accounts of his early ideas between 1906 and 1941, even though without novel inspirations or vitality after 1915. And his 1912 *Traité* was relatively well cited between 1913 and 1923 but enjoyed weak dissemination between 1924 and 1960 (see [Jovanovic 2012], for a detailed bibliometric analysis of the dissemination of Bachelier's work). The situation changed dramatically when A. James Boness's English translation of Bachelier [1900] was published in Paul Cootner's *The Random Character of Stock Market Prices* [Cootner 1964] and when William Feller [1966, 181], influenced by Lévy [1948], introduced the term Wiener-Bachelier process as a synonym for Brownian motion, an identification also made on the opening page of Itô & McKean Jr. [1965]. Benoît [Mandelbrot 1989, 86] holds that the subject of finance "has its Gregor Mendel in Louis Bachelier" (see also [Mandelbrot & Hudson 2004]), while Bernard Bru (in [Taqqu 2001] and in [Courtault et al. 2000, 98]) considers Bachelier [1900] as "the Newton of the Bourse" and Jules Regnault [1863] as "its Kepler" (see also [Dimand 1993], [Jovanovic & Le Gall 2001], and [Jovanovic 2006]).

Mark Davis and Alison Etheridge note that Bachelier "defined Brownian motion—predating Einstein by five years—and the Markov property, derived the Chapman-Kolmogorov equation and established the connection between Brownian motion and the heat equation [Davis & Etheridge 2006, xiii]. The purpose of all this was to give a theory for the valuation of financial options. Bachelier came up with a formula which, given his mathematical model of asset prices, is correct" (although, because of limited liability, which prevents asset prices from going below zero, later writers since Paul Samuelson and M. F. M. Osborne have assumed geometric Brownian motion of the logarithm of asset price, rather than of the level of asset prices). Peter Bernstein [1992], Paul Samuelson (in his foreword to [Davis & Etheridge 2006]), [Jovanovic 2000], [Dimand & Ben-El-Mechaiekh 2006], and [Read 2013] tell the remarkable story of the rediscovery of Bachelier [1900]. Bachelier [1900] is justly celebrated, with [Courtault & Kabanov 2002] publishing the proceedings of a symposium held on 29 March 2000 to mark the centenary of Bachelier's thesis defense.

The difficulties of Bachelier's academic career after his return from the First World War are well covered in the existing literature: a series of limited-term sabbatical replacement positions (as *chargé de cours*) in the provinces until he finally obtained a chair in mathematics in Besançon in 1927 at the age of fifty-seven, a mere decade before retirement (see details in [Taqqu 2001]). However, it seems to be much less well known that Bachelier offered new mathematical statements of his theory of speculation after 1900. For example, Jeremy Bernstein [2005, 395] remarks only that "Bachelier continued to publish occasionally, including a long paper in 1901, a treatise on the mathematical laws of chance underlying gambling." Bachelier devoted to the theory of speculation three chapters of the first volume of an uncompleted comprehensive treatise on probability [Bachelier 1912, Chapters XII-XIV, pp. 277–322], a chapter in a treatment of probability intended for a wider audience [Bachelier 1914, 176–213], and, near the close of his career, an entire short monograph of 49 pages [Bachelier 1938]. The last of these, Bachelier's final account of his theory of financial markets, is translated into English here for the first time. Bachelier [1938] is not even listed in the bibliography of [Davis & Etheridge 2006], although they do list a companion monograph on the calculus of probabilities, Bachelier [1939] (but not the other companion monograph, Bachelier [1937]).

As noted by Henri Poincaré in the opening sentence of his report on Bachelier's thesis,[1] the choice of topic was very unusual in a period marked by the supremacy of real, complex and functional analysis, mathematical physics and differential equations. This choice was presumably responsible for his dissertation's grade of *honorable* rather than *très honorable* [Taqqu 2001, 4]. Nonetheless, Poincaré appreciated Bachelier's stroke of genius in adopting methods of mathematical physics (such as radiation, reflection, and refraction) to the movement of probabilities in order to model a speculator's informal reasoning. Poincaré was much impressed by Bachelier's observation that if the probability density follows the diffusion equation, then "there can be only one probability law simplified by the principle of the mathematical expectation" [Bachelier 1938, §7], i.e., it must take the form of a normal distribution. It was five years later that Einstein rediscovered, in the case of Brownian particles, what Bachelier had known and applied before to the probability of prices: that the first moment (first integral of the density) imposes that the particles must remain on average stationary over time [Read 2013] and that the mean spread of Brownian motion is proportional to the square root of time [Bachelier 1938, §21]. Poincaré's report on the dissertation was generally favorable. The thesis was however considered, administratively, as a thesis in mathematical physics, although it dealt with the stock market, a topic certainly not recognized by the mathematical establishment. Probability theory itself was of little concern for the French

1 See [Taqqu 2001] for the complete French text of Poincaré's report on the thesis together with a short defense report signed by Paul Appell. An English translation was provided by Baftiri-Balazoski and Hausmann Courtault et al. [2000].

leading mathematicians of the late 19th century and had no champion (while Émile Borel started publishing in probability theory in 1905, he was not interested in Bachelier's work). As pointed out in Juliette Leloup's doctoral thesis [Leloup 2009, §6.1.1], in the early years of the 20th century the field of probabilities in France essentially reduced to two notorious names: Borel succeeding Poincaré as the leading French probabilist. Both of them did not see probability as a *bona fide* aera of pure mathematics. They were reticent in investing mathematical energy and powerful tools in the study of problems where hazard plays a role as it is the "antithesis of any law" (Poincaré [1907], quoting Joseph Bertrand). Poincaré's long held skepticism in the power of probabilities to have any significant impact on sciences in general and social sciences in particular transpires in his report on Bachelier's thesis when he wrote: *"In his introduction and further in the paragraph entitled "Probability in Stock Exchange Operations," he (Bachelier) strives to set limits within which one can legitimately apply this type of reasoning. He does not exaggerate the range of his results, and I do not think that he is deceived by his formulas."* [Taqqu 2001]. Borel's paper *Sur les principes de la théorie cinétique des gaz* [Borel 1906] was instrumental in Poincaré's radical reversal in the appreciation of the power of probability for studying non-deterministic problems. This reversal is expressed by Poincaré in his paper "*Le hasard*" that appeared in Borel's journal, *La Revue du mois* in 1907 (see also [Leloup 2009]). The intensive study of random processes in France started later under Borel's initial impulse continued by the engagement of Fréchet (who saw the calculus of probability as a branch of physics), Darmois and Paul Lévy at the end of the twenties-early thirties ([Mazliak 2007] and [Leloup 2009]). Mathematical physics, then, mostly meant partial differential equations, soon to turn its spotlights towards the developments of relativity and quantum mechanics. Bachelier [1938, Preface] recalled that Poincaré was responsible for the publication of Bachelier's books by Gauthier-Villars (which published all the leading French mathematicians of the time, including Poincaré, Borel, and later Lévy) and for Bachelier being able to teach at the Faculty of Sciences of the University of Paris, where from 1909 to 1914 Bachelier gave each year a series of twenty lectures on *"Probability Calculus with Applications to Financial Operations and Analogies with Certain Questions from Physics."* Poincaré presented one of Bachelier's papers to the Academy of Sciences for publication in its Memoirs in 1910, as did Paul Appell (another of the examiners of Bachelier's thesis) in 1913, the year after Poincaré's death. It was only in 1913 that the University gave Bachelier a regular salary and additional teaching in general mathematics, but such unsalaried lecturing, paid course by course (the University gave Bachelier a grant of 2400 or 2500 francs each academic year from 1910–11), was then not uncommon at the start of an academic career. In 1914, the Council of the University unanimously authorized the Dean of the Faculty of Sciences to seek funds to make Bachelier's appointment permanent (see [Courtault & Kabanov 2002] for the relevant documents).

Before 1914, Bachelier also attracted some attention from a few other Paris-based authors (see [Jovanovic 2000; 2006; 2012]). Robert de Montessus de Ballore [1908, 94–107], another "free lecturer" at the Faculty of Sciences with a doctorate in mathematics, gave an abridged account (with full acknowledgement) of Bachelier's theory of speculation in a volume of elementary lectures on the calculus of probability, giving the name "Bachelier's theorem" to the proposition that the speculator's mathematical expectation of gain is zero. Alfred Barriol [Barriol 1931, 374–384], subsequently the first professor of finance at the Institute of Statistics of the University of Paris, used Bachelier's equations and Henri Lefèvre's diagrams (without acknowledgement) in his only book, a textbook on the theory and practice of financial operations (although Barriol did cite [Regnault 1863]). Vincenz Bronzin, the Trieste professor who made a notable contribution to option pricing theory in 1908, may or may not have known Bachelier [1900] but did not cite him [Hafner & Zimmerman 2009; Zimmerman & Hafner 2006; 2007], just as Bachelier did not mention Regnault although Jovanovic & Le Gall [2001] (see also [Courtault & Kabanov 2002]) conclude that Bachelier may well have known Regnault [1863]. Maurice Gherardt [1910], a speculator on the Paris Bourse, cited both Bachelier [1900] and Regnault [1863], but offered statistical evidence to argue that, contrary to Regnault and Bachelier, it was possible to devise a mathematical formula to win at the Stock market.

But the Great War, which ended or altered so many lives, disrupted Bachelier's modestly promising career. When he returned from the war, he no longer had a teaching position in Paris, and his mentor Poincaré, who could have supported him for another appointment, had died in 1912. The second volume of Bachelier's 1912 treatise never appeared. Montessus de Ballore, who had expounded Bachelier's theory of speculation, changed fields, and thereafter published on meteorology. Barriol published further editions of his textbook, but no new works. Bachelier found temporary positions, replacing professors on leave in Besançon (1919–22) and Dijon (1922–25), then an associate professorship in Rennes. Bachelier married in 1920, but was soon widowed. He was rejected for a chair in Dijon in 1926 when Paul Lévy of the École polytechnique confirmed an apparent error that a Dijon professor of mechanics had found in a 1913 paper by Bachelier. This error resulted from Bachelier's overly simplified account of the model of Brownian motion he had described much earlier in his thesis ([Courtault et al. 2000], as reprinted in [Courtault & Kabanov 2002, 11–13] and [Taqqu 2001, 16–17]). By then some damage to Bachelier's reputation as a mathematician had been done. It is worth noting that the prolific 1880–1910 period for the French school of mathematics with stars like Hermite, Poincaré, Picard, Elie Cartan, Jordan, Borel, Lebesgue, Darboux, Baire, Hadamard and Lévy was followed by a lull in creativity during the second decade and the first half of the third of the 20th century. The stars of the golden period 1890–1910 have either passed away or retired into old age. By then, the French have lost their supremacy, not only in mathematics but in other sciences, to the English and the Germans and to rising schools in Russia, Poland, Scandinavia, Hungary, and America. [2] A reason for this decline is that unlike the UK, Germany, or Russia who elected not to send their scientists to the front, egalitarianism was the credo in France. Cohorts of graduates from the grandes écoles and universities were wiped out in the trenches (see [Aubin & Goldstein 2014] for the effects of the war on the global development of mathematics). Promising pupils from the oligarchic grandes écoles—notably "Rue d'Ulm" (École normale supérieure) or "l'X" (École polytechnique) would probably not have chosen probability for their dissertations' topics ([Mazliak 2007] and [Leloup 2009, 59]), much less applied to the fluctuations of stock market prices! Not only Bachelier was not one of them, but leading mathematicians were not interested in his work. These serious handicaps, aggravated by a less than stellar academic performance at the Sorbonne would be enough to be snubbed by Professor Lévy. Lévy was particularly critical of Bachelier for having used the equation of Brownian motion in 1913 as if it was his own discovery—not realizing that indeed it had been Bachelier's own precursor discovery in 1900. Finally Bachelier received a professorship in Besançon in 1927. Illness and teaching kept him from publishing for the next ten years, but upon his retirement Gauthier-Villars published three monographs, from 36 to 69 pages each, restating his probability theory [Bachelier 1937; 1938; 1939], the second

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2 While Bourbaki's monumental encyclopedic achievement gave impetus to amazing mathematical discoveries during and after the second world war, it started—in the second half of the thirties—as an effort to reformulate selected areas of mathematics on the solid grounds of rigor, completeness, and abstraction with the aim of reinjecting some vitality into French mathematics.

of which, on speculation and the calculus of probabilities, is translated here.

Lévy and Bachelier eventually reconciled, after Lévy discovered that, notwithstanding the unclearly defined symbols in Bachelier [1913], Bachelier had gone on to derive many valid and important results, although even then Lévy complained in his notebook about "trop question de Bourse" in Bachelier's work [Courtault et al. 2000, 346], thus outlining the fundamental reason for the mathematician's indifference to Bachelier's work. Lévy [1948] made an important acknowledgement of Bachelier's work on Brownian motion. In a letter to his former student Benoît Mandelbrot on 25 January 1964 (in [Courtault & Kabanov 2002, 66]), Lévy wrote that the reconciliation followed his noticing a reference to "Der Bacheliersche Fall" (the Bachelier case) in Kolmogorov [1931, §16], which led him to reread Bachelier [1913] and read Bachelier [1912]. Bernard Bru (interviewed in [Taqqu 2001]) drew on a 1943 letter from Lévy to Maurice Fréchet to argue that the reconciliation followed Lévy's reading of Bachelier [1941], at a time when Lévy, as a Jew, was barred from publishing by the racial laws of the Vichy regime and then, when the Germans occupied the Vichy zone, lived in semi-clandestinity under an assumed name in Grenoble. A ban that Lévy managed to defy, not without temerity, by publishing not less than a dozen papers under his real name [Audin 2009, §3.2]. Lévy's articles on Brownian motion reveal that his discovery of Bachelier preceded Bachelier [1941] but was considerably later than Kolmogorov [1931]. Lévy [1939] made no mention of Bachelier, but the first footnote of Lévy [1940, 487], which was received by the editors on 17 October 1939, warmly recognized Bachelier's priority even though expressing well-founded reservations about the details of Bachelier's execution of his grand project: [3] "At this date, Bachelier appears as a precursor. While the way in which problems where time plays the role of a continuous variable leaves to be desired, it remains that it is in this work that one finds for the first time the idea that Gauss' law necessarily derives as a consequence of the continuity of an additive process and the

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[3] "À cette date, Bachelier apparaît comme un précurseur. Si la manière dont sont introduits les problèmes où le temps joue le rôle d'une variable continue laisse à désirer, il n'en reste pas moins que c'est dans cet ouvrage que l'on trouve pour la première fois l'idée que la loi de Gauss s'introduit nécessairement comme conséquence de la continuité d'un processus additif, et la relation entre ce processus et l'équation de la chaleur. Il faut aussi signaler plusieurs formules relatives à l'écart maximum, et peut-être la formule (que j'ai cherchée en vain dans un grand nombre d'ouvrages antérieurs) qui, dans le cas des lois absolument continues, définit la loi dont dépend la somme de deux variables aléatoires indépendantes."

relationship between this process and the heat equation. One has to also point out several formulas relative to the maximum spread, and perhaps the formula (that I searched in vain in numerous previous work) that, in the case of absolutely continuous laws, defines the law on which depends the sum of two independent random variables."

At about the time of reevaluation of Bachelier's work by Lévy [1939], the recently-retired Bachelier offered final, definitive restatements of his contributions to the theory of probability and of the pioneering principles and methods for understanding speculation transactions [Bachelier 1937; 1938; 1939]. Bachelier's 1912 presentation of his theory of speculation was cleaner than that in his 1900 thesis, his 1914 chapter was less technical and more accessible, and his 1938 monograph more concise and readable.

Despite these differences in presentation, Bachelier's central message was the same in 1900, 1912, 1914, and 1938: working out the implications for the diffusion of probability of the fundamental economic principle that in an efficient market, the absence of unexploited profit opportunities means that the mathematical expectation of the speculator's gain is zero (in the absence of systematic mistakes by speculators implying that they have rational expectations). The current price of an asset is the best predictor of its future price, and incorporates all available information. Regnault [1863] had analyzed these relationships verbally, but it was Bachelier who presented the mathematical analysis, and showed that the probability function for price changes conforms to the diffusion equation for heat (see [Jovanovic 2000; 2006; 2012], [Jovanovic & Le Gall 2001]). The greatest missed opportunity for an earlier discovery by economists of Bachelier on the efficient market hypothesis, rational expectations, and stochastic processes came when the reviewer of Bachelier [1912] in the *Journal of the Royal Statistical Society* was so averse to the whole concept of continuous probabilities that he failed even to notice the existence of Bachelier's three substantial chapters on the theory of speculation. The reviewer was John Maynard Keynes [1912]. Another missed opportunity came in 1911 when Poincaré and Einstein both attended the Solvay Congress, where Brownian motion was one of the topics. Jeremy Bernstein [2005, 398] remarks that "Poincaré did not use the occasion to mention Bachelier's work ... Einstein was rather disappointed with Poincaré's views on the quantum theory and did not spend much time talking to him."

It is important to be reminded that Bachelier became aware of the limitations of the Gaussian random walk model as noted by Mandelbrot [1967, 394]: "Bachelier 1914 made no mention of his earlier claims of the empirical evidence in favour of Brownian motion ... Bachelier noted that his original model contradicts the evidence in at least two ways: Firstly, the sample variance of $L(t,T)$ varies in time ... Second, Bachelier noted that no reasonable mixture of Gaussian distributions could account for the sizes of the very largest price changes, and he treated them as "contaminators" or "outliers". Thus, he pioneered not only in discovering the Gaussian random-walk model, but also in noting its major weakness."

The 1937 memoir restates (without proofs) results from Chapters VI, VII, VIII and XVIII of the 1912 *Traité*. It describes approximating asymptotic formulas for large numbers of independent Bernoulli trials with complementary constant or variable probabilities. The 1939 memoir essentially summarizes parts of Chapters VI, IX, X, XI, XVIII, XX, and XXI of Bachelier [1912]. It also contains material from the memoir published in the *Annales scientifiques de l'École Normale Supérieure* [Bachelier 1910a], as well as from the paper presented by Poincaré to the *Comptes Rendus* [Bachelier 1910b]. The 1938 monograph is a condensed version of the thesis [Bachelier 1900] and of the *Traité* [Bachelier 1912]. The content corresponds to that of chapters XII to XV of 1912 with the notable differences of a more elaborate exposition on transactions including purchase or sales of futures against (multiple) options at the end of chapter II on one hand, and a concise summary in chapter III of the results on the maximum spread, the second mean spread, and the average time contained in Chapter XV of [Bachelier 1912] (where the full derivations were provided) on the other.

While Bachelier [1938] can be seen as a swan song without new material beyond what was already in the 1900 thesis, its exposition is more focused and concise (94 short paragraphs) reminiscent of a financial mathematics practitioner guide with ready-to-use formulas and a comprehensive description of transactions operations. Bachelier seems to have resolutely opted for steering the course of practicality, away from the fancy philosophical considerations debated by French and European mathematicians studying probability in the first decade of the twentieth century as well as from the formidable mathematical probability edifice founded on measure theory that was taking shape. Did Louis Jean-Baptiste Alphonse Bachelier want to leave to posterity a last account of his work and reaffirm ownership of what Bernard Bru calls the "revelation and fascination that never left him" [Taqqu 2001]? The preface and the last two paragraphs (93 and 94) of Bachelier [1938] unequivocally announce his motivations. Firstly, to reaffirm a paternity for the first theory of continuous time stochastic processes which he saw as a revival, after nearly a century, of a tradition set by Laplace (the last sentence of Bachelier [1938]). Bachelier explicitly reclaims antecedence for the concepts of stochastic processes (which he calls "probabilités connexes") and hints to possible mathematical generalizations denying the efficient market hypothesis: "By solely adopting a mathematical point of view completely removed from any idea of rapprochement with reality, one could attempt to build a theory more general than the one presented here. Such a theory … would loose contact with reality and, mostly, would draw its interest from its difficulty." [Bachelier 1938, §93]. Quite remarkably, Bachelier reiterates "the superiority" of his theory: "an expression of reality" since calculated and observed data coincide [Bachelier 1938, §94]. In this, he was a precursor (with Cournot in 1843) to Kolmogorov's belief that to be complete, probability theory had to be meaningfully grounded in the real world. A philosophy of probability expressed by Kolmogorov in a 1939 letter to Fréchet: "You are also right in attributing to me the opinion that the formal axiomatization should be accompanied by an analysis of its real meaning" [Shafer & Vovk 2006].

One can only be perplexed by Bachelier's choice not to refer to works by others. This was noted very early by Keynes [1912, 571] protesting against Bachelier [1912] "giving no reference to other writers, even when he is borrowing from them. Save that on one occasion he calls Bernoulli's theorem after Bernoulli, there is, I think, no single reference throughout the book to any other author." Bachelier's letter of protestation against Paul Lévy's stance on his 1913 paper, clearly indicates that he was aware of what was being done in France and England. Much as in earlier work, the 1938 memoir is disconnected from both pre-1900 and post-1912 contributions. Bachelier took Poincaré's course on probability, used the 1st 1896 edition of Poincaré's book and was certainly familiar with Bertrand's 1888 *Calcul des Probabilités* to which he does not refer. Nor does he refer to Regnault [1863] on applications to stock options (particularly the law on the square root of time increase of the deviation of prices), or to Jevons [1878] and Brown [1828] concerning Brownian motion. There is no indication of Bachelier being aware of Poincaré's adaptation in 1890 of Boltzmann's statistical interpretation of the second law of thermodynamics (1870's) to celestial mechanics. Save for referring to Laplace's work, a cursory mention of Ampère's analysis of the mathematical theory of games, and an expression of heartfelt thanks to Poincaré, he does not refer to the 1843 principle of Cournot, nor to any of his predecessors, such as Bresson in 1830, Lefèvre in 1870, Castelli in 1877, Edgeworth in 1886 and 1888, etc. (see [Girlich 2002]). While mentionning "la loi de Gauss" in his thesis—but not in the 1938 memoir—Bachelier does not refer to the prince of mathematics, Carl Friedrich Gauss himself. Bachelier's objection to the "unfortunate" and "illogical" use of the term "probabilités formant chaîne" in the second last paragraph of the memoir shows that he was aware of some of the work done on stochastic processes before 1937. [4] Yet, Bachelier ignores even in his later work the remarkable contemporary developments in the theory of probability, stochastic and diffusion processes, further isolating himself. It is fair to assume that Bachelier could not have been aware of Markov's 1906 and 1915 papers on discrete time series as they were written in Russian. It is also well established that new advances were very slow reaching provincial Besançon or Dijon. Significant advances were indeed taking place closer to him in England and continental Europe (see [Straja 1997]). Einstein's celebrated paper on Brownian motion appeared in 1905. Independently and concurrently, Polish physicist Marian Smoluchowski published a series of paper on the kinetic theory of matter and Brownian motion [von Smoluchowski 1906; 1912; 1915; 1916]. In his 1913 thesis and two subsequent papers, Adriaan Daniel Fokker derived, along with Max Planck, the celebrated Fokker-Planck-Kolmogorov equation [Fokker 1913; 1914; Planck 1917]. By the end of the twendies, Markov chains were becoming a hot area of research—consecrated in the 1928 Bologna International Congress [Bru 2003]—with the rediscovery in 1928 by Lévy and Hostinský of Markov's ergodic theorem on the convergence of positive probabilities for chains' transitions and the publication of the first proof of a general ergodic principle by Birkhoff [1931]. Kolmogorov's *Grundbegriffe* [Kolmogorov 1933] outlined in great detail the axiomatic foundation of probability theory building on earlier works by Bernstein, Borel, Cantelli, Chebychev, Chuprov, Daniell, Fréchet, Khinchin, Lebesgue, Lévy, Łomnicki, Markov, Slutsky, Steinhaus, Ulam, and von Mises (see [Shafer & Vovk 2006], for a lively description of the advances in probability theory that led to the *Grundbegriffe*). Kolmogorov was indeed the first to acknowledge the influence of Bachelier [1912] as the precursor of continuous time stochastic processes [Kolmogorov 1931]. By the mid-thirties, the *Institut Henri Poincaré* was becoming a focal point for research in probability with Borel, Fréchet, Hadamard, Darmois and such brilliant visitors as von Mises (the mathematician), Hostinský, Onicescu, and Bernstein. Wolfgang Doeblin, then a young pupil of Fréchet working on Markov processes, solved in 1936 Lévy's conjecture on the dispersion of sums of random variables and began thesis work on the stochastic solvability of the

4 The terminology was used by Markov in a 1907 note to the Imperial Academy of Sciences of Saint Petersburg [Markov 1908].

Chapman-Kolmogorov equation [Mazliak 2007]. Surely, Bachelier's thesis precedes Schrödinger [1915] in providing the first passage distribution for diffusion processes in the absence of drift, resulting in the stock prices following a Gaussian distribution. Bachelier's framework yields both to Wiener processes (constant drift, [Wiener 1921a;b; 1923]) and to the Ornstein-Uhlenbeck processes (whereby the drift varies depending on the current value of the process compared to the mean [Uhlenbeck & Ornstein 1930]). Isolated in Besançon, it was only in 1936 that Bachelier discovered Hostinský's work on Markov chains on continuous state space published in 1931–32. This late discovery prompted Bachelier to write an angry letter to the Čech probabilist rightfully claiming antecendence some two decades earlier [Mazliak 2007]. Although it also precedes Chapman [1916; 1917], Wiener [1921a;b; 1923], and Kolmogorov [1931], Bachelier's work—including [Bachelier 1937; 1938; 1939]—does not have the mathematical rigor found in Kolmogorov's derivation of the Chapman-Kolmogorov's equation, nor the depth seen in later foundational works leading to the axiomatization of the theory of probability [Bernstein 2005]. It also lacks the explicit consideration of mathematical extensions as rightfully pointed out in Lévy [1940]. Published on the eve of war and invasion, the 1938 monograph attracted even less attention than Bachelier's 1900 thesis had won. Far short in mathematical sophistication and so remote from the phenomenal production shaping the foundations of an emerging theory of probability, Bachelier 1938 appears to be aimed at a public, conspicuous by its absence, of mathematically inclined stock market speculators. It is now translated to offer readers Bachelier's final and most concise exposition of his theory of speculation.

Acknowledgments

We thank Stephen Buser, Franck Jovanovic, Perry Mehrling, Geoffrey Poitras, and other participants in the American Finance Association/History of Economics Society joint session "What Was/Is Financial Economics?" at the 2008 AFA meeting for helpful comments. The manuscript has benefited from a pointed and expert reading by Norbert Schappacher to whom we are much indebted for insightful editorial comments. We are very grateful to Dunod Éditeur (as successor firm to Gauthier-Villars) for permission to translate into English Louis Bachelier's *La Spéculation et le Calcul des Probabilités* (Gauthier-Villars: Paris, 1938). The footnotes to the translation have been added by its authors.

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SPECULATION

AND

THE CALCULUS OF PROBABILITY

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(Translation of the 1938 Monograph)

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LIBRAIRE DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE
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1938

PREFACE

From the mathematical point of view, the theory of speculation is an exact science far superior to the classical calculus of probability whose fundamental problems, tainted by approximations and asymptotisms, have resisted progress for so long. [5]

The theory of speculation allowed those problems to have clear, precise, rigorous and lively formulations. It gave birth to conceptions for the treatment of questions seemingly innaccessible by their difficulty, such as the general case of the law of large numbers for example.

By the necessary introduction of the notions of time and absolute continuity, it provided the idea of movement of probabilities, their radiation, reflection, and refraction. Not only did it expand the limits of our knowledge, but it allowed for extreme simplifications as in the case of the famous analysis of Ampère on the mathematical theory of games.

The advantages I am alluding to did not elude my illustrious teacher Henri Poincaré. [6] He is the one who got interested in my work and had it published; it is thanks to him that I presented it over a number of years at the Faculty of Sciences of Paris.

One can readily conceive the advantage of using infinitesimal calculus, much simpler and handier than the usage of finite quantities. The possibility of making usage of this calculus would be sufficient to appreciate, from the point of view of mathematical science, the interest of the theory of speculation.

From the point of view of applications, this theory is very useful as the results obtained from the examination of stock options are perfectly consistent with the theoretical results provided by the calculus.

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5 Here, Bachelier may be just grumbling in general about the state of probability theory before him, not thinking of a particular author or work.

6 Bachelier was the first student to pass Poincaré's exam in mathematical physics in 1897 at the Sorbonne (the course did not have a final examination prior to new regulations resulting from the law of 1896 on French universities). Bachelier took Poincaré's course on probability and Joseph Boussinesq's course on the theory of heat. The jury for Bachelier's thesis defense consisted of Poincaré (reporting), Paul Appell, and Boussinesq. See [Taqqu 2001].

This correspondence between the theory and the observations is equally interesting on philosophical grounds: it shows that the market indeed behaves randomly.

This result was indeed predictable: constantly subjected to infinitely many varying influences acting along diverse directions, such a market must ultimately behave as if no single cause came to play but as if randomness acted alone.

The results of the theory would be contradicted only if a single cause would be constantly acting in the same direction; in general, the diversity of causes allows their elimination; the incoherence of the market is itself its method; and it is because it does not obey any law that it fatally follows the law of randomness.

The theory of speculation was first presented in my Doctorat ès sciences mathématiques thesis which was published as a treatise. [7] Complements have appeared the following year in the *Annales scientifiques de l'École normale supérieure.* [8]

This theory was presented in its definite and mathematically rigorous from in my *Traité du calcul des probabilités,* and was popularized in my treatise entitled *Le jeu, la chance et le hasard* published in the Bibliothèque de Philosophie Scientifique of which more than seven thousand copies were printed. [9]

The current book provides a complete treatment of the topic, but it is written in a descriptive form; it contains no proofs nor developments in mathematical analysis.

I have included at the end of this book few lines on the nowadays fashionable so-called *theory of probabilities in chains* which I earlier named *theory of connected probabilities.* [10]

Let me take the liberty to point out that, safe for special cases considered by Laplace, it was me who initiated studying these notions. To convince oneself, it suffices to read Chapter IX of my treatise, in particular

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7 [Bachelier 1900].

8 [Bachelier 1901].

9 [Bachelier 1912; 1914] respectively.

10 While Bachelier repeatedly uses the terminologie "*probabilités connexes*" in his traité *Calcul des Probabilités* [Bachelier 1912], the terminology "*probabiltés en chaînes*" was neither used in his thesis [Bachelier 1900] nor in the 1912 traité. The latter became however widely used in the thirties with important papers on the topic by Fréchet [1933] and later by Lévy, Doeblin, Kolmogorov, and others (see the extensive bibliography in [Bru 1993]. The term "*chain*" was of course used by Markov as early as 1906 to designate a sequence of pairwise dependent random variables.

paragraph 314. [11] This same study has appeared six years earlier in a Memoir of the *Journal de Mathématiques.* [12]

Coming back to the theory of speculation, starting point for numerous advances, one can say that its knowledge is indispensable to a real understanding of the most vivid aspects of the calculus of probability.

11 [Bachelier 1912].

12 [Bachelier 1906].

CHAPTER I

THE LAW OF PROBABILITY

1. — In the theory of speculation, we consider the variations of the price of a given security in a large market. To fix ideas, we shall call this security a government bond.

We assume that these price variations are random and the fundamental problem consists in seeking the probability that at a given time, the price differs from the present price by a given quantity.

By saying that the price variations are random, we wish to express the fact that due to the excessive complexity of the causes of these variations, everything happens as if chance was acting alone.

Therefore, we are careful not to undertake the analysis of the causes of the fluctuations; such an analysis would be vain and would only lead to errors. It is precisely because we choose to ignore everything that it becomes possible to know. It is precisely because this study appears inextricably complicated that it is, in reality, very simple.

2. Variation and instability. — We consider a certain security which we shall call a government bond.

Let us place ourselves at a determined instant. The price quoted at that instant has as many purchasers as sellers; purchasers believe in a rise and sellers in a drop of that price. The *market*, that is the collection of all speculators, does not believe in either rise nor drop, since for the quoted price there are as many sellers as there are purchasers. It thus considers the quoted price as representing the real value of the bond under consideration

But although the market does not believe in a rise or a drop, it can assume fluctuations of a given amplitude as more or less probable.

The amplitude of these movements (as acknowledged by the market) is measured, at each time and for a given value, by a single quantity, a single parameter which we call *coefficient of instability*.

If this coefficient is large, the market acknowledges that strong upward or downward fluctuations are probable. If it small, the understanding is that fluctuations are likely to be weak.

We have placed ourselves at a certain instant; a minute later, for reasons we do not wish to analyze, another price is quoted and the market acknowledges another coefficient of instability.

A moment later, another price is quoted and the market acknowledges a new coefficient, and so on.

At each instant, one must therefore consider the actual price and the coefficient of instability measuring the upward or downward amplitudes of future variations.

(This concerns obviously the price of a forward-date bond [13] and one must adjust quoted prices as a result of coupons and contangoes. I can only refer to my book on the *Théorie de la spéculation.* [14])

The notion of instability will be made precise later; we shall see how this instability is characterized, at each instant, by a unique coefficient.

3. General notions on probability. — Let us restate the fundamental problem to be solved.

We are looking for the probability that, at time t (that is after a period t), the price differs by a given quantity from the actual quoted price $(t = 0)$.

Consider the actual price as zero; the price x is therefore a relative price representing the *spread* from the actual price.

For example, if the bond's actual price is 75^{fr}, the price $75^{\text{fr}}.50$ is equivalent to a spread of $0^{\text{fr}}.50$; the price $74^{\text{fr}}.75$ equals a spread of $-0^{\text{fr}}.25$.

In the first case, we say that the price x is $0^{\text{fr}}.50$ and in the second case, the price x is $-0^{\text{fr}}.25$. The price is positive when it corresponds to a rise, and negative for a fall.

The probability for a price x to be quoted at a time t is, in reality, the probability that this price is between x and $x + dx$; we consider it as a function $f(x,t)dx$ of x and t called *elementary probability* (x and t being continuous variables).

The probability that, at a given time t, the price is between two limiting values $x = a$ and $x = b$, called the *total probability* on the interval a, b, is obtained by integrating the function f.

13 Rentes are government bonds. "Operations à termes" are forward-date transactions.

14 [Bachelier 1900] for a study of these corrections.

A priori, this function is arbitrary provided it is positive and its values for all possible x add up to one.

This function can be represented by a curve with positive ordinates and total area equals to one as this area corresponds to the sum of all probabilities.

We shall see that this function, which is a priori arbitrary, has in reality a unique and absolutely determined form.

4. Independence principle. — The price variations taking place at an arbitrarily given instant are independent of previous variations and of the price quoted at that instant.

One has to understand the true significance of this principle: it is obvious that in reality there is no independence, but because of the extreme complexity of the causes coming into play, one can assume that there is indeed independence.

If the price is z at time t_1, the probability of a new spread y during a new time period t_2, is independent of z, but depends only on y, t_1, and t_2.

5. Mathematical expectation. — The *mathematical expectation* of an eventual gain is the product of that gain by the probability of its occurring.

The mathematical expectation is therefore negative when it corresponds to a loss.

The *total mathematical expectation* for a player is the sum of the products of all uncertain gains by the corresponding probabilities of their occurrence.

Obviously a player is neither advantaged nor disadvantaged when his total mathematical expectation vanishes. In which case, the game is said to be *fair.*

If a game has several rounds, the total mathematical expectation is the sum of the mathematical expectations relative to each round, given that these rounds are indeed completed.

In particular, if the game consists of identical rounds, then the total expectation equals the product of the relative expectation of a round by the number of rounds in the game.

If the game is fair at each round, it is fair as a whole. Hence, a fair game cannot be made advantageous or disadvantageous by any possible combination.

6. Principle of mathematical expectation. — Transactions of the exchange are subject to the law of supply and demand and as every speculator is free to perform a given transaction or its inverse, we cannot allow that a speculation transaction a priori favors or penalizes one of the parties. A transaction that would systematically favor one of the parties would not meet a counterpart.

A transaction cannot be, *a priori,* neither advantageous nor disadvantageous; this is expressed by saying:

the mathematical expectation for any transaction vanishes.

One has to realize the generality of this principle. It applies not only to forward-date transactions and to options that will expire at a predetermined instant but also to any transaction, regardless of its complexity, as long as it is based on subsequent fluctuations in price.

This is indeed obvious, for if the speculator adopts a system whereby a given transaction would be performed in response to a given price fluctuation, then each such transaction would be fair when performed. The system would overall be fair.

7. Probability law. — It follows from the independence principle that there can only be one probability law simplified by the principle of the mathematical expectation.

The probability for a price x to be quoted at a time t, i.e. that it is included between x and $x + dx$, is expressed by the formula

$$\frac{e^{-\frac{x^2}{\varphi(t)}}}{\sqrt{\pi}\sqrt{\varphi(t)}}dx,$$

$\varphi(t)$ being a function of t, a priori arbitrary, but positive and increasing, called *instability function* for reasons to be made precise later.

Unless stated otherwise, our problems are relative to a unique period of time t.

For these problems, $\varphi(t)$ can be assumed to merely be a given constant.

8. — This kind of formula is well known since Laplace who was the first to use it for analogous problems.

The idea that a multitude of small causes acting independently in various directions must lead to a single law is to be credited to Laplace.

In other theories, this principle is referred to as the hypothesis of infinitesimal errors.

Although the formula is well-known, it has a special property here:

It is exact.

Whereas it is approximated and asymptotic in other applications. Being exact and rigorously continuous in both variables x and t, it served as a starting point for a vast study, whereas it was ulteriorly an end in itself.

9. — The prices $+x$ and $-x$ have the same probability; this follows from the fact that the principle of the mathematical expectation does not only hold for a time period t, but generally at all times and all time intervals.

10. Probability curve. — The function

$$y = \frac{e^{-\frac{x^2}{\varphi(t)}}}{\sqrt{\pi}\sqrt{\varphi(t)}}$$

can be represented by a curve with maximum ordinate at the origin and with two inflection points at

$$x = \frac{\pm\sqrt{\varphi(t)}}{\sqrt{2}}.$$

The probability of the price x is a function of t increasing up to a certain instant then decreasing; this function is maximum when the price x corresponds to the inflection point of the probability curve.

11. Mathematical expectation. — It is the positive mathematical expectations of a speculator buying a bond at the current price in order to sell it at time t.

This expectation has value

$$\int_0^{\infty} \frac{xe^{-\frac{x^2}{\varphi(t)}}}{\sqrt{\pi}\sqrt{\varphi(t)}} dx = \frac{\sqrt{\varphi(t)}}{2\sqrt{\pi}};$$

it is proportional to the square root of the instability function.

We shall denote the quantity $\frac{\sqrt{\varphi(t)}}{2\sqrt{\pi}}$ by the letter a and, in several questions, we shall express the price variations using a as the unit.

Under these conditions, the probability of the price x is expressed by the formula

$$\frac{e^{-\frac{x^2}{4\pi a^2}}}{2\pi a} dx.$$

12. Probability in a given interval. — The previous formula expresses the *elementary probability*.

The probability for the price to be between 0 and x is expressed by

$$\int_0^x \frac{e^{-\frac{x^2}{\varphi(t)}}}{\sqrt{\pi}\sqrt{\varphi(t)}}dx$$

or, after putting $\frac{x^2}{\varphi(t)} = \lambda^2$,

$$\frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{\varphi(t)}}} e^{-\lambda^2} d\lambda = \frac{1}{2}\Theta[\frac{x}{\sqrt{\varphi(t)}}],$$

where $\Theta(y)$ denotes

$$\Theta(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-y^2} dy.$$

There are tables of this function Θ, known as *Kramp's tables.* Those included at the end of my book have seven decimals.

13. — The probability for the price to be included in the interval $-x, +x$ has value

$$\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{\varphi(t)}}} e^{-\lambda^2} d\lambda = \Theta[\frac{x}{\sqrt{\varphi(t)}}].$$

As t tends to infinity, this probability tends to zero.

The probability in the interval $-x_1, +x_2$ is

$$\frac{1}{2}\Theta[\frac{x_1}{\sqrt{\varphi(t)}}] + \frac{1}{2}\Theta[\frac{x_2}{\sqrt{\varphi(t)}}].$$

This expression tends to zero whenever t grows without bound.

14. — The probability

$$\mathcal{P} = \frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{\varphi(t)}}} e^{-\lambda^2} d\lambda = \frac{1}{2} - \frac{1}{2}\Theta[\frac{x}{\sqrt{\varphi(t)}}]$$

for the price to be greater than x at time t is said to be the total probability of the first kind; it increases steadily with time. If t is infinite, it would equal $\frac{1}{2}$, which is obvious.

15. — Instead of relying on tables of the function Θ, it is generally simpler to make use of the following table which gives directly the probability $\mathcal{P}$ corresponding to the price x expressed by taking $a = \frac{\sqrt{\varphi(t)}}{2\sqrt{\pi}}$ as a unit.

Spread	Probability $\mathcal{P}$	Spread	Probability $\mathcal{P}$
$0.0a$	0.500	$2.3a$	0.179
$0.1a$	0.484	$2.4a$	0.169
$0.2a$	0.469	$2.5a$	0.159
$0.3a$	0.453	$2.6a$	0.150
$0.4a$	0.437	$2.7a$	0.141
$0.5a$	0.422	$2.8a$	0.132
$0.6a$	0.404	$2.9a$	0.124
$0.7a$	0.390	$3.0a$	0.116
$0.8a$	0.374	$3.1a$	0.108
$0.9a$	0.360	$3.2a$	0.101
$1.0a$	0.345	$3.3a$	0.094
$1.1a$	0.331	$3.4a$	0.087
$1.2a$	0.316	$3.5a$	0.080
$1.3a$	0.302	$3.6a$	0.075
$1.4a$	0.289	$3.7a$	0.070
$1.5a$	0.275	$3.8a$	0.065
$1.6a$	0.262	$3.9a$	0.060
$1.7a$	0.249	$4.0a$	0.055
$1.8a$	0.237	$4.5a$	0.037
$1.9a$	0.225	$5a$	0.023
$2.0a$	0.213	$5.5a$	0.015
$2.1a$	0.202	$6a$	0.009
$2.2a$	0.190	$7a$	0.003

The various probabilities can be expressed easily in terms of the function $\mathcal{P}$; for instance, the probability for the price to be included in the interval $-x_1, +x_2$ has value $1 - \mathcal{P}_{x_1} - \mathcal{P}_{x_2}$.

One should not forget that prices are expressed by using $a = \frac{\sqrt{\varphi(t)}}{2\sqrt{\pi}}$ as a unit.

16. Mean spread. — The mean spread is by definition the mathematical expectation of a speculator who should receive an amount equal to the absolute value of the spread at time t. It is therefore the quantity

$$2\int_0^\infty \frac{xe^{-\frac{x^2}{\varphi(t)}}}{\sqrt{\pi}\sqrt{\varphi(t)}}dx = \frac{\sqrt{\varphi(t)}}{\sqrt{\pi}} = 2a.$$

The mean spread is proportional to the square root of the instability function.

The probability that the mean spread is exceeded in one direction is, according to the table in paragraph 15, 0.214. The probability that the mean spread is exceeded in either direction is 0.428.

17. Probable spread. — We thus call the spread $\pm\alpha$ such that at time t, the price would have one chance in two to be included in the time interval in question. The quantity α is determined by the equation $\mathcal{P}_\alpha = \frac{1}{4}$. We deduce that

$$\alpha = 1.688\ldots\ a = 1.688\ldots\ \frac{\sqrt{\varphi(t)}}{2\sqrt{\pi}}.$$

The probable spread is proportional to the square root of the instability function. It equals the mean spread times the number 0.844...

18. Isoprobable spreads. — More generally, consider the spread $\pm\beta$ such that the probability that, at period t, the price would be included in this interval equals a given quantity u; we have

$$\int_0^\beta \frac{e^{-\frac{x^2}{\varphi(t)}}}{\sqrt{\pi}\sqrt{\varphi(t)}}dx = \frac{u}{2}$$

or

$$\Theta[\frac{\beta}{\sqrt{\varphi(t)}}] = u.$$

This interval β, when u is constant, varies proportionally to $\sqrt{\varphi(t)}$. *Spreads increase proportionately to the square root of the instability function.* The instability function owes its name to this very property

19. Uniformity Principle. — Up to this point we have not made any assumption on the function $\varphi(t)$ besides being positive and increasing; we can notice that some results do not depend on this function: for instance the ratio of the probable spread to the mean spread is 0.844, for any $\varphi(t)$. The probability that one of these spreads is exceeded is also independent of $\varphi(t)$.

It is evident that in most cases, the market has no reason to assume that the probability of a spread y in the interval $0, t_1$ is different from the probability of the same spread in the interval $t_1, 2t_1$ or $2t_1, 3t_1, \ldots$ The market therefore generally supposes that there is uniformity, that is that instability is identical for all time elements. In other terms, it supposes that $\varphi(t)$ is proportional to time and that we consequently have

$$\varphi(t) = 4\pi k^2 t.$$

We have denoted the constant by $4\pi k^2$ so that the mathematical expectation (§11)

$$a = \frac{\sqrt{\varphi(t)}}{2\sqrt{\pi}} = k\sqrt{t},$$

which is proportional to the square root of time reduces to a coefficient k whenever $t = 1$. k is *the coefficient of instability.*

If we assume uniformity, *the probability of price x at time t is*

$$\frac{e^{-\frac{x^2}{4\pi k^2 t}}}{2\pi k\sqrt{t}}dx.$$

20. — The notion of uniformity in game theory could be misleading. When we say that a game is uniform, we generally understand that it will be uniform in reality. To the contrary, in the theory of speculation uniformity cannot be real: at the actual time $t = 0$, the market considers the probabilities for future periods as uniform and characterized by the coefficient of instability k. At time Δt it assumes a uniformity characterized by another coefficient k'. At time $2\Delta t$ it assumes a uniformity characterized by yet another coefficient k'', etc.

Since we compute the values of the probabilities at time $t = 0$, the formulas are the same, whether uniformity is assumed or real.

The market departs very little from the law of uniformity even in speculation on commodities, whereby at some time periods, prices must a priori be instable.

If, ordinarily, we do not assume uniformity, it is for the sake of maximum generality; the interest being mostly theoretical.

21. — In order to obtain, in the previous theory, the formulas when uniformity is present, it suffices to replace the function $\varphi(t)$ by its particular form $4\pi k^2 t$.

Certain results are interestingly simple.

The mean spread $2k\sqrt{t}$ is proportional to the square root of time, the same holds true for the probable spread $1.688\ldots\ k\sqrt{t}$.

Consideration of isoprobable spreads leads to the following general proposition:

Spreads increase proportionally to the square root of time.

CHAPTER II

SPECULATION TRANSACTIONS

22. — In order to make use of our formulas, it is necessary to determine the instability function $\varphi(t)$ or simply, if we assume uniformity, the instability coefficient k.

In all rigor, and from the practical point of view, our formulas should take into consideration a second coefficient. The price considered by the market as the most probable at a given period m is not the current price; it is referred to as *the true price relative to the period* t; the second coefficient depends on the difference of the two prices, this difference being the result of what is called in the speculation jargon *contangoe.*

We shall not consider this second coefficient which is generally close to zero without inferring that its influence is always negligible.

In my treatise on the *Théorie de la spéculation*[15] this coefficient is considered in the case of some transactions.

The same treatise includes a comparative study between the theoretical results and the actual variation of prices. This study establishes a perfect match between theory and observation.

The same conclusion is reached after examination of statistics of a very different kind whose methodology is described on page 212 of my book on the *Le jeu, la chance et le hasard.*[16]

23. Forwards[17]. — There are two main types of speculation transactions: forwards and options.

These transactions can be combined infinitely many times, since one often treats several kinds of options.

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15 [Bachelier 1900].

16 [Bachelier 1914].

17 Modern terminology for *operations fermes* (fixed contracts).

The purchase of a forward is analogous to a spot purchase, [18] but is performed with a speculative aim.

Given a forward date, one can sell without having precedingly purchased, this transaction is known as *"vente à decouvert"*.

The purchase and sale of a forward are inverse but analogous transactions.

The buyer of a forward limits neither his gain nor loss; he gains the difference between the buying price and the selling price at the liquidation date. In other terms, he gains the value of the gap when this gap is positive and loses it when it is negative.

The inverse takes place for the seller of a forward.

The formulas that we have established express the probabilities for the buyer who wishes to end his transaction at a period t known as the liquidation date. These are probabilities of the first kind.

Instead of fixing a period t for the completion of a transaction, the buyer could fix in advance a price x. The search for the probability to complete the transaction at different periods t (probability of the second kind) constitutes the second problem of the theory of speculation.

24. Geometric representation. — A speculation transaction can be represented by a geometric construction.

The x-axis corresponds to the different prices and the y-axis to the corresponding profits.

Losses being negative profits are represented by negative $y's$.

For a purchase of a forward, to any price x corresponds a profit x; the operation is therefore represented by the line $y = x$.

Similarly, the sale of a forward is represented by the line $y = -x$.

25. Simple options. — Options allow to speculate with a risk limited in advance to a certain sum which is the amount, or the value, or the importance of the premium.

Simple options are used abroad as well as in France in speculations on commodities.

Foreseeing an increase for a given period t, speculator A acquires an option on the upside for that same period t in order not to expose himself to unlimited loss.

He pays first a certain sum, called a simple option, to a speculator B who believes in a price drop.

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18 Achat au comptant.

The payment of the option provides speculator A with the advantages of the buyer of a forward without incurring its risks.

At the liquidation date, as for the case of forwards, he makes a profit in case of a price increase but does not lose anything in case of a drop.

The maximum loss for a buyer of a simple option is the value of the premium; his risk is limited to this amount but his gain could be unlimited.

Options on the downside are treated in the same manner; their values being obviously equal to that of options on the upside with identical liquidation date.

26. — Denote by a the value of the simple option. The geometric representation for the purchase of this option on the upside (the case of speculator A) is as follows:

The line $y = -a$ for negative values of x and the line $y = x - a$ for positive values of x.

27. — In order to determine the value of the simple option, it suffices to apply the mathematical expectation principle: the market follows the law of supply and demand; moreover, every one is free to buy or sell options, perform any transaction or its inverse. Consequently, the purchases or sales of options cannot be a priori advantageous nor disadvantageous.

The mathematical expectation of every speculation vanishes.

Let us apply this principle to the buyer of an option on the upside. His positive expectation is that of the buyer of a forward (§11), that is $\frac{\sqrt{\varphi(t)}}{2\sqrt{\pi}}$; his negative expectation is the value of the simple option denoted by the letter a. These quantities being equal, we have:

$$a = \frac{\sqrt{\varphi(t)}}{2\sqrt{\pi}}.$$

Given the value a of the premium relative to a certain liquidation date t (provided by quotes (cotes)), the probabilities for that same period are readily known.

The mean spread (§16) is double the simple option. The probable spread equals the simple option multiplied by the factor 1.688.

28.— The buyer of a simple option profits whenever the price at liquidation is between a and $+\infty$; referring to the table in paragraph 15, we can see that:

The probability of success of the simple option buyer is independent of the liquidation date; it is equal to 0.345.

This result does not depend on any hypothesis about the nature of the instability function.

29. — Assuming uniformity, we have:

$$a = k\sqrt{t}.$$

The value of the simple option must be proportional to the square root of time.
This allows for the computation of the coefficient k.

We have seen earlier that a number of results are independent of the instability function. It would be possible to determine the function $\varphi(t)$ if we were able to study options for all periods t. However, in reality we only treat options relative to certain periods $t_1, t_2, t_3, \ldots$ and make additional hypotheses on the nature of $\varphi(t)$ in order to compute probabilities for intermediate periods.

This simplest hypothesis consists in assuming that $\varphi(t)$ is proportional to t, which amounts to assuming uniformity.

Thus, the results obtained in our study are of three types:

1) invariable results (e.g., the probability of success for a buyer of a simple option).

2) Those relative to a period t during which an option is exercised. In this case, probabilities are known without the imposition of additional hypotheses; the instability function $\varphi(t)$ is not known but $\varphi(t_1)$ is and appears in the expression of probabilities.

3) Those relative to a period during which no option is exercised and that assume the uniformity law.

30. "Stellage" or double option. — A speculator A forecasting a large movement in one direction or the other and concerned with minimizing his risk, acquires a "stellage" or double option consisting of an option on the upside and an option on the downside.

The geometric representation of this operation is as follows: the line $y = -x - 2a$ for negative x and the line $y = x - 2a$ for x positive.

It is easy to see that the acquisition of a "stellage" yields a profit in the price intervals $2a, +\infty$ and $-2a, -\infty$. Hence, in view of the probability table in paragraph 15:

The probability of success with the acquisition of a "stellage" equals 0.425.

31. Options in general. — In a forward transaction, buyers and sellers expose themselves to theoretically unlimited losses. In the options market, the buyer pays more for his certificate than in the case of the forwards market, but his loss on the downside is limited in advance to the a certain sum which is the amount of the premium.

The seller of the option has the advantage of selling for more but cannot gain more than the value of the premium.

One could equally treat options on the downside that limit the loss of the seller; in which case, the transactions is made at a price lower than that of a forward.

One does not treat these options on speculations on assets; we obtain an option on the downside of a slightly different type by selling a forward and simultaneously buying an option on the upside. [19]

32. — Assume for instance that the 3 percent bonds cost 75^{fr}.

Speculator A, forecasting a rise at the end of the month, and wanting to avoid the risk associated with the purchase of a forward, purchases for the end of the month transaction date at the cost of 76.20, an option at 0.50. (It is customary to say an option *at* instead of an option of. [20])

The purchase price is higher than that of a forward but A's potential loss is limited to 0.50 regardless of what the decline might be. It is as if A has purchased a forward at 76.20 knowing that the price would not go lower than $76.20 - 0.50 = 75.70$.

If in reality the liquidation date price is less than 75.70, A does not loose more that 0.50 as if the decline did not go below 75.50.

The spread of the option is the difference $76.20 - 75 = 1.20$ between the price of the option and that of the forward.

The base of the option is the price $76.20 - 0.50 = 75.70$.

At the liquidation date, the option is said to be exercised if the price is greater than the base of the option, that is 75.70. The option is otherwise abandoned.

Since the actual price is assumed to be zero, the price of an option equals its spread.

33. — Generally there are three quantities to be considered for an option: the importance h of the option, that is the maximal sum the buyer is

19 This type of option is known as an option at a discount.

20 Bachelier uses the commonly used "une prime *dont*" which is translated into "an option at".

willing to risk; the price l at which the option is negotiated called *the spread of the option*; finally the liquidation period.

It is customary to say that a speculator has purchased an option at h meaning that he purchased an option with importance h.

The purchase of an option is equivalent to the purchase of a forward at price l; the price of the option cannot go lower than $l - h = m$.

When in reality the price at liquidation date is less than m, the buyer's loss is constant and equals h.

In other words, in the price interval $-\infty, m$ the loss is constantly equal to h. Between m and $m+h$ the loss decreases from h to zero. Starting from price $m + h$, the buyer gains in proportion to the increase.

The price m is said to be the base of the option; it is positive if $l > h$ and negative if $l < h$.

When at liquidation date the price is greater than $m = l - h$ the option is said to be exercised, otherwise it is said to be abandoned.

The purchase of an option is represented geometrically by the line $y = -h$ for $x < m$ and by the line $y = x - m - h$ for $x > m$.

The simple option which we studied is part of the general definition; we can say that it is an option whose importance equals the spread.

34. — We can also define the option by saying that the buyer pays a sum h to the seller in order to acquire the advantages of the buyer of a forward at the price m without incurring its risk, that is without loss.

The price m would be zero in the case of a simple option; m is therefore positive of negative depending on whether h is smaller or greater than a.

35. — We study options at any period, but only for liquidation dates that are fixed in advance at the middle and the end of each month, up to a maximum of ordinarily three months.

The price quoted at the liquidation date is called the *option response price.*[21]

A purchase of a forward can at any instant be cancelled by a sale with the realization of a loss or a gain. To the contrary, an option transaction necessarily runs up to liquidation date, that is to the response date.

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21 In his thesis, Bachelier clearly defines *la réponse des primes* as taking place the day before the liquidation date, that is the day before the last day of the month, translated in Davis and Etheridge (2006) as the *call date of options*. An option is exercised or abandoned depending on whether the price at call date is above or below the exercise price of the option Thus, an option can be exercised at the call date, thus becoming a forward that liquidates the next day. The option clearing price then coincides with its price at call date.

For options, the importance h is fixed; only the spread varies. For example, ordinarily only options at 1 franc and options at $0^{fr}.50$ are considered for 3 percent bonds.

36. — The spread between the price of an option and that of a foward depends on a large number of factors and varies constantly.

At the same instant and for the same liquidation date, the spread is as larger as the option is smaller; for instance an option at 1^{fr} has obviously a price lesser than an option at $0^{fr}.50$.

The spread of an option decrease more or less regularly between the quotation date and the day before the response date, instant when this spread is very small.

However, depending on circumstances, it can stretch very irregularly to be greater few days prior to the response day than what it earlier was.

The tension of the spread of options is the instability index. It indicates that the market forecasts large fluctuations. This fact will be made precise in a mathematical way.

37. — It is obvious that for a same liquidation date, the spread of an option is larger as the importance of the option gets smaller.

A first question is in order (I solved this question well before I got interested in the study of probabilities): under the only assumption that the spread of an option decreases as its importance increases, does there exist speculative transactions yielding a gain for each possible price?

By a suitable choice of the ratios of the spread of three options, one can imagine infinitely many transactions yielding a gain at all prices.

The spreads required by those transactions do not go against common sense, and if the market never attains these spreads nor gets even close to them, it is precisely because it obeys the probability law without realizing it (*Théorie de la speculation*, p. 13). [22]

38. Law of the spread of options. — In order to find a relationship between the importance h of an option and its spread $m + h$, we apply to the buyer the mathematical expectation principle:

The mathematical expectation of every speculation vanishes.

Let ϖ be the probability of price x at expiration period t. We shall evaluate the expectation:

1) for prices included between $-\infty$ and m;

22 [Bachelier 1900].

2) for prices included between m and $m+h$;
3) for prices included between $m+h$ and $+\infty$.

1) For prices between $-\infty$ and m the buyer incurs a loss h. His mathematical expectation for a price in the interval above is $-\varpi h$; therefore for the entire interval it adds to

$$-h\int_{-\infty}^{m}\varpi dx.$$

2) For a price x between m and $m+h$, the buyer's loss is $m+h-x$. The corresponding mathematical expectation is $-\varpi(m+h-x)$; for the whole interval it amounts to

$$-\int_{m}^{m+h}\varpi(m+h-x)dx.$$

3) For a price x between $m+h$ and ∞, the buyer's profit is $x-m-h$. The corresponding mathematical expectation is $\varpi(x-m-h)$; for the whole interval it amounts to

$$\int_{m+h}^{\infty}\varpi(x-m-h)dx.$$

The total expectation principle yields the equation:

$$\int_{m+h}^{\infty}\varpi(x-m-h)dx-\int_{m}^{m+h}\varpi(m+h-x)dx-h\int_{-\infty}^{m}\varpi dx=0,$$

which becomes after reduction

$$h+m\int_{m}^{\infty}\varpi dx=\int_{m}^{\infty}\varpi x dx.$$

This definite integral equation establishes a relationship between the spread and the importance of an option.

39. — If in the equation above ϖ is replaced by its value

$$\varpi=\frac{1}{2\pi a}e^{-\frac{x^2}{4\pi a^2}},$$

and if the integral is expanded as a series, we obtain [23]

$$h-a+\frac{m}{2}-\frac{m^2}{4\pi a}+\frac{m^4}{96\pi^2 a^4}-\frac{m^6}{1920\pi^3 a^5}+\cdots=0.$$

This formula expresses the law of the spread of options. It provides a relationship between the importance h of an option, its spread $m+h$, and

23 The correct denominator in the fifth term is $96\pi^2 a^3$; a typographical error in Bachelier's paper.

the importance a of a simple option relative to the same period. It allows to compute any one of these quantities when the other ones are known.

If for instance m and $m+h$ are known, the above series yields the value of a.

40. — In practice, the importance of an option is always included between $\frac{a}{3}$ and $3a$. It follows that we can in the preceding formula eliminate the terms in m^6 and m^4 in order to obtain

$$a = \frac{\pi(2h+m) + \sqrt{\pi^2(2h+m)^2 - 4\pi m^2}}{4\pi}.$$

The same approximation leads to the expression of m in terms of a and h:

$$m = \pi a - \sqrt{\pi^2 a^2 - 4\pi a(a-h)}.$$

This formula is very appropriate for numerical computations although it presents two shortcomings: it is not easy to formulate in lay terms and it does not give a sufficiently clear idea about the variations of the spread and the importance of options. The formulas that we will obtain, although inferior to the preceding one from the numerical computability point of view, are very simple and expressive.

41. — We can introduce the value $l = m+h$ of the spread of the option in the complete formula of paragraph 39. The formula then becomes

$$\frac{h+l-a}{2} - \frac{(l-h)^2}{4\pi a} + \frac{(l-h)^4}{96\pi^2 a^3} - \frac{(l-h)^6}{1920\pi^3 a^5} + \cdots = 0.$$

The formula does not change if l and h are interchanged. Hence, if l is the spread of an option at h, the spread of an option at l is h.

This *reciprocity theorem* can be proved without analytical formulas; it suffices to suppose that a speculator buys an option at h with spread l and sells a forward simultaneously. It is not difficult to see that the resultant of this double transaction is an option on the downside with importance l and spread h.

The two transactions being equitable, so is their resultant; this proves the reciprocity theorem.

In the complete formula of paragraph 39,

$$h = a - \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + \frac{m^6}{1920\pi^3 a^5} - \cdots \tag{1}$$

and replacing h by l, we have

$$l = a + \frac{m}{2} + \frac{m^2}{4\pi a} - \frac{m^4}{96\pi^2 a^3} + \frac{m^6}{1920\pi^3 a^5} - \cdots \tag{2}$$

42. — As the spread of an option increases when its importance decreases, it is interesting to study the variation of the product of these two quantities.

Formulas (1) and (2) can be written

$$h = a - \frac{m}{2} + f(m) \text{ and } l = a + \frac{m}{2} + f(m).$$

The product hl has value

$$a^2 - \frac{m^2}{4} + 2af(m) + [f(m)]^2;$$

its derivative with respect to m

$$-\frac{m}{2} + 2af'(m) + 2f(m)f'(m)$$

vanishes for $m = 0$ since for this value f' also vanishes. Hence $h = l = a$.

The product of an option with its spread is therefore maximum when both terms of this product are equal; this is the case for simple options.

Multiplying the series (1) and (2) yields

$$hl = a^2 - \frac{(\pi - 2)m^2}{4\pi} + \frac{4m^4}{96\pi^2 a^2} - \cdots \tag{3}$$

As a first approximation we could thus set

$$hl = a^2;$$

that is, *the product of an option by its spread is constant.*

For given a and h, the value obtained for l from the above formula is by excess, whereas for given h and l, the value of a is by default.

If we postulate the preceding law together with uniformity, $a^2 = k^2 t$, and if we consider options with equal importance h, their spread l is proportional to time.

43. — The preceding law is not a satisfactory approximation; however, we obtain a very good approximation by proceeding as follows:

By adding the series (1) and (2) we obtain

$$h + l = 2a + \frac{2m^2}{4\pi a} - \frac{2m^4}{96\pi^2 a^3} + \frac{2m^6}{1920\pi^3 a^5} - \cdots$$

Multiplying this series by (3) gives

$$hl(h+l) = 2a^3 - \frac{2(\pi - 3)am^2}{4\pi} - \frac{(12\pi - 30)m^4}{96\pi^2 a} + \cdots$$

or approximately

$$hl(h+l) = 2a^3 - 0.0225am^2 - 0.0081\frac{m^4}{a} + \cdots$$

We can set as a second approximation

$$hl(h+l) = 2a^3.$$

The error is very small for small m, as in the ordinary case. For example, if $m = a$, the error on the second term is less than $\frac{2}{100}$.

In lay terms, the second approximation law can be stated in a simple way:

We multiply the option by its spread. We add the option and its spread. The product of these two quantities is the same for all options relative to the same period.

44. — In order to give an idea about the approximation obtained by the preceding formulas, let us consider the less favorable situations encountered in practice, assuming that we wish to determine the spread of a very small option $h = \frac{a}{3}$.

The first approximation formula in §42 gives the excessively large value $l = 3a$.

The second approximation formula in §43 yields the value $2.28a$.

The approximative formula in §40 gives $2.25a$.

The exact value is $l = 2.23a$.

45. — The fundamental formula providing the probability of the price x is

$$\frac{e^{-\frac{x^2}{4\pi a^2}}}{2\pi a}$$

and the formulas deduced from this one contain the quantity a which is precisely the value of the simple option.

We can therefore compute probabilities if we are studying a simple option (or a stellage).

We can also do this when studying an arbitrary option by using the formula in §43 from which the value a is derived.

The knowledge of quantity a allows for the determination of all analogous quantities; for example, the probable spread is $1.688a$ and the mean spread is $2a$.

46. — Knowing the spread l_1 of an option at h_1 for a certain period t_1, we can deduce the spread l of an option at h for the period t.

One has to obviously admit the uniformity expressed by the formula $a = k\sqrt{t}$.

According to the result in §43, we have

$$hl(h+l) = \frac{\sqrt{t^3}}{\sqrt{t_1^3}} h_1 l_1 (h_1 + l_1),$$

from which we deduce l.

47. — In order to apply the law of the spread of options, one has to remark that the spread of an option is the difference between its price and the true price relative to the liquidation date.

The true price is equal to the quoted price corrected by the effect of the "coupons and reports". The difference between the two prices is generally negligible, but it is sometimes substantial. In such cases we not taking then into account might be misleading.

The process by which we determine the true price relative to a certain period is described in my treatise on the *Théorie de la Spéculation*.

48. Options or call-of-mores. — We treat, in certain markets, of operations somehow intermediate between futures and options known as options or call-of-mores.

Suppose that the price of a given commodity is 60^{fr}. Instead of purchasing a unit at the price of 60^{fr} for a given period, we could purchase a call-of-more of order 2 for the same period at 62^{fr} for example. This is to say that for every difference below 62^{fr} we only loose on one unit, whereas for every higher difference, we gain on two units.

We could have purchased a call-of-more of order 3 at 63^{fr} for example, that is for every difference below 63^{fr} we only loose on one unit, whereas for every higher difference, we gain on three units. We can imagine call-of-more of order 4 and more generally multiple order call-of-mores.

We also treat call-of-mores on the downside necessarily with spread equal to that of upside call-of-mores of the same order of multiplicity.

49. — In order to obtain the value of the spread of a call-of-more, we make use of the mathematical expectation principle, scientific expression of the law of offer and demand.

Being completely free, the purchase or sale of call-of-mores cannot be a priori advantageous nor disadvantageous; *the mathematical expectation of the buyer of a call-of-more is zero*

Let us apply this principle to the purchase of a call-of-more of order n treated at spread r.

The call-of-more of order n can be considered as the composition of two operations:

1) the fixed futures purchase of a unit at price r;

2) a fixed futures purchase of $(n-1)$ units at price r, this purchase only to be considered in the interval r, ∞.

The first operation has a mathematical expectation of $-r$; the second has expectation

$$(n-1)\int_r^{\infty} \varpi(x-r)\,dx.$$

We must therefore have

$$r = (n-1)\int_r^{\infty} \varpi(x-r)\,dx,$$

or, after replacing ϖ by its value

$$\varpi = \frac{e^{-\frac{x^2}{4\pi a^2}}}{2\pi a}$$

and a series expansion

$$2\pi - \pi\frac{n+1}{n-1}\left(\frac{r}{a}\right) + \frac{1}{2}\left(\frac{r}{a}\right)^2 - \frac{1}{48\pi}\left(\frac{r}{a}\right)^4 + \cdots = 0.$$

50. — By only retaining the first three terms, we obtain

$$r = a\left[\frac{n+1}{n-1}\pi - \sqrt{(\frac{n+1}{n-1}\pi)^2 - 4\pi}\right].$$

If $n = 2, r = 0.68a$.

The spread of the double call-of-more must be approximately $\frac{2}{3}$ *the value of the simple option.*

If $n = 3, r = 1.096a$.

The spread of the triple call-of-more must be higher than the value of the simple option by approximately $\frac{1}{10}$ *of this value.*

51. — The preceding formulas show that the spreads of options are proportional to the quantity a; it follows that the probability of success of these operations is independent of the liquidation period.

The probability of success of the double call-of-more is 0.394; *the operation yields a gain four times out of ten.*

The probability of triple call-of-more is 0.33; *the operation succeeds once every three times.*

52. — When selling a futures and simultaneously buying a double call-of-more we obtain an option with importance $r = 0.68a$ and with spread $2r$.

The probability of success for the operation is 0.30.

By analogy with transactions in options, we could call "*call-of-more straddles of order n*" the operation resulting in two call-of-mores of order n one at a premium, the other at a discount.

The call-of-more straddle or order two is a strange operation; between prices $\pm r$, the potential loss is constant and equal to $2r$. The potential loss then decreases progressively to the prices $\pm 3r$ where it vanishes. There is profit outside of the interval $\pm 3r$. The probability of success is 0.42.

If the geometric representation of this operation is compared to that of a call-of-more straddle, one can observe that the spread of a call-of-more of order 2 must be greater than $\frac{2}{3}$ of the simple option.

In France, call-of-mores are not treated for securities; an operation analogous to a double call-of-more with a premium is obtained by simultaneously purchasing futures and options. Similarly, an operation analogous to a double call-of-more with discount is obtained by purchasing an option and simultaneously selling a double quantities of futures.

53. Complex operations. — As we treat futures together with up to three options for the same period, we could undertake triple and even quadruple transactions at the same time.

Triple transactions are beyond what are considered classical operations; their study is very interesting but too long to be discussed here. We shall limit ourselves to double transactions.

These can be divided into two groups depending on whether they contain futures or not.

Transactions containing futures consist of a futures purchase and a sale of options or conversely.

Option against option transactions consist of a sale of a large option and the purchase of a smaller one or conversely.

The ratio of sales and purchase can vary to infinity. For the sake of simplicity, we shall only study two very simple proportions:

1) the second operation involves the same number as the first;
2) it involves double that number.

54. Purchase of futures against option sale. — This is about an option at h with spread l.

For prices lesser than the base of the option $(m = l - h)$, a profit h is produced and added to the profit x (positive or negative) of the futures purchase.

The profit at price x is therefore $x + h$; at the base of the option it is $m + h = l$.

Above this price, the purchase and sale cancel each other and the gain is always $m + h = l$.

Gain is limited while risk is unlimited.

Geometric representation: the line $y = x + h$ for $x < m = (l - h)$ and the line $y = l$ for all other values of x.

The operation yields a loss when the drop exceeds the price h, the probability of this event being $\mathcal{P}_h$. For example if $l = 0.31h$, the probability of a loss is 0.25.

The speculator initiates such an operation when he foresees a market stagnation with a slight upside tendency.

55. Futures sale against option purchase. — This is the counterpart to the preceding operation.

Profit is unlimited on the downside, whereas loss on the upside is limited to the spread of the option.

This operation replaces downside options which are not treated on securities.

But here the spread of the option is h and its importance l. For this type of option on the downside, to the contrary of upside options, the spread is fixed and the importance varies.

The probability of success is $\mathcal{P}_h$.

This operation is undertaken in the hope of a market decrease.

56. Futures purchase against double option sale. — This operation is analogous to the sale of a "stellage" with a limited gain and unlimited loss on either the up or down sides.

Gain occurs in the interval $-2h, +2l$; it is maximal at the base of the options $x = m = l - h$. Its value is therefore

$$m + 2h = l + h.$$

The operation is represented by the line $y = x + 2h$ when x is smaller than m and by the line $y = -x + 2l$ when x is greater than m.

The probability of loss is $\mathcal{P}_{2h} + \mathcal{P}_{2l}$.

The speculator engages in this operation whenever he believes in a market stagnation.

57. Futures sale against a double "faculté". — This operation is analogous to a stellage; it is performed by a speculator who believes in a wide fluctuation on either the up or down sides. It is the inverse of the previous one with unlimited risk and limited gain.

58. Purchase of a large option against sale of a small one. — For instance, on a 3 percent bond, purchase of an option at 1^{fr} with spread L against the sale of an option at 0.50^{fr} with spread l.

This is an operation with gains on the upside and limited risks.

Below the basis of the larger option $(M = L - 1)$ both options are abandoned with a loss of 0.50^{fr}.

Above the price M, the loss decreases to vanish at price $M + 0.50^{\text{fr}} = L - 0.50^{\text{fr}}$.

For higher prices there is gain up to the basis $m = l - 0.50^{\text{fr}}$ of the smaller option where the gain is $l - L$.

On the upside, both options being exercised, the gain is always $l - L$.

Geometric representation: the line $y = -0.50$ for the values of x less than $M = L - 1$. The line $y = x + 0.50 - L$ for the values of x between M and $m = l - 0.50$. The line $y = l - L$ for the values of x greater than m.

The operation yields a gain if the price is greater than $L - 0.50$, the probability of this event being $\mathcal{P}_{L-0.50}$ if $L - 0.50$ is positive and $1 - \mathcal{P}_{0.50-L}$ in the opposite case.

If for example, an option at 1^{fr} has a spread of 0.70, and an option at 0.50 has normal spread of 1.30^{fr}, then $a = 0.98$.

There is gain if the price is higher than 0.20 or $0.204a$. The probability that profit occurs is $\mathcal{P}_{0.204a} = 0.46$.

59. Sale of an option with a large premium against the purchase of an option with a small premium. — Operation on the downside with limited risks and profits opposite to the previous one.

60. Purchase of an option with a large premium against the sale of an option with a small premium in double quantity. — For example, on a 3 percent bond, purchase of an option at 1^{fr} with spread 0.70 against the sale of an option at 0.50 with spread 1.30 in double quantity.

Below the price -0.30 both options are abandoned and match; the operation is neutral.

From the price -0.30 the large option is exercised and the operation yields a gain equal to $x + 0.30$ which is maximum at the basis 0.80 of the small option, that is equal to 1.10.

The gain then decreases; it vanishes at the price 1.90 and at price x the loss is $x - 1.90$.

Limited profit, unlimited loss. Geometric representation: the $x-$axis for values up to the the basis M of the large option. For the values of x between M and m, the line $y = -x + 2m - M$. For the values of x greater than m, the line $y = -x + 2m - M$.

Loss occurs when the price exceeds $2m-M$, the probability of this event being $\mathcal{P}_{2m-M}$. For the example cited above, this probability is 0.22.

We undertake this operation when we believe a slight movement on the upside but we fear the possibility of a downside movement.

61. Sale of a large option against the purchase of a small option in double quantity. — Inverse operation to the previous one. Zero risk on an appreciable movement downside, limited on the upside; unlimited gain on very strong upside movement.

CHAPTER III

THE SECOND PROBLEM OF SPECULATION THEORY

62. — In the first chapter we have computed the probabilities relative to a given period t.

Save for the continuity of the variables t and x and of the fact that the formulas are exact and non-asymptotic, our computations remain till now in what can be considered as a classical framework.

But we can conceive being confronted by a multitude of other very interesting and much more difficult problems.

It is not enough to study prices relative to a period t, we must also study *the movements of prices in a time interval* t determined or not.

This study is the object of this chapter; it is based on the theory of images which I used right at the start of my investigations and which made possible and almost elementary the resolution of seemingly intractable problems.

The proofs will not be given here, they are presented in my *Traité*.[24]

63. — As we did earlier, we will denote by $\mathcal{P}$ the probability that a given price will be attained or exceeded at time t. $\mathcal{P}$ is the probability of the first kind. P will denote the probability that the price considered will be attained or exceeded during a time interval t, that is before period t. P is the probability of the second kind.

The probability that a price will be exceeded at the moment t *is one-half of the probability that the same price will be exceeded during the interval of time* t.

Indeed, the price cannot be exceeded at the moment t without having been attained previously. The probability $\mathcal{P}$ is thus equal to the probability P multiplied by the the probability that, given that the price has been quoted prior to the moment t, it will exceed at moment t, that is multiplied by $\frac{1}{2}$.

Therefore

$$\mathcal{P} = \frac{P}{2}.$$

24 [Bachelier 1912].

The probability that the price c will be exceeded during an interval of time t is expressed as

$$P = 2\mathcal{P} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{\sqrt{\varphi(t)}}} e^{-\lambda^2} d\lambda$$

or

$$P = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{2\sqrt{\pi}a}} e^{-\lambda^2} d\lambda.$$

64. Some applications. — The tables of the function Θ allow for the computation of the function

$$P = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{2\sqrt{\pi}a}} e^{-\lambda^2} d\lambda = 1 - \Theta(\frac{c}{2\sqrt{\pi}a});$$

but it is generally much simpler to resort to the probability table in paragraph 15 by doubling its entries since $P = 2\mathcal{P}$.

65. — Let us look, for example, at the probability that the spread of the simple option a will be attained before the expiration of this option.

The probability $\mathcal{P}$ relative to price a is (§15) 0.345; the probability at the same price is

$$2 \times 0.345 = 0.69.$$

It is important to note that this number expresses the probability that the spread a will be attained in a given direction, say on the upside. The search for the probability that the interval $\pm a$ will be exceeded before the moment t in a given direction, constitutes a much more complicated problem which will be resolved later (§77).

66. — Let us look again for the probability that the spread $2a$ will be attained (in one direction) before the moment t.

The probability $\mathcal{P}$ relative to the price $2a$ being 0.213, the probability P relative to the same price is double this quantity, namely 0.426.

The probability for the spread of a call-of-more of order 2 to be attained is 0.78. For a call-of-more of order 3, the probability would be 0.66.

67. Elementary probability. — We have denoted by ϖ the probability that a price will be quoted at a moment t; this is the elementary probability of the first kind. We will denote by Π the probability that this price will be quoted for the first time at the moment t, that is the elementary probability of the second kind. Π is therefore the probability that the price will be quoted at the moment t given that it has not been quoted previously.

Evidently, we have

$$\int_0^t \Pi dt = P \text{ or } \Pi = \frac{\partial P}{\partial t},$$

subsequently

$$\Pi = \frac{c\varphi'(t)e^{-\frac{c^2}{\varphi(t)}}}{\sqrt{\pi}\varphi(t)\sqrt{\varphi(t)}}dt.$$

Such is the expression of the probability that the price c *will be attained for the first time at the moment* t*.*

$\varphi'(t)$ is the derivative of the instability function $\varphi(t)$ which, in all generality, is a given of the problem.

If we assume uniformity, we have

$$\Pi = \frac{ce^{-\frac{c^2}{4\pi k^2 t}}}{2\pi kt\sqrt{t}}dt.$$

68. Probability distributions. — The knowledge of the probability that the price c will be attained at a given moment t does not completely solve the problem we have proposed; it remains tho study the case where the price c is not attained before time t.

For example, a speculator having purchased at price zero proposes to sell at price c. It is useful not only to know the probabilities that he could effect the resale at each period prior to time t, but also to know the probability of earning an amount x at time t if the resale did not take place, due to the price c not being attained.

The probability of price x at time t, if the price c has not been attained prior to t is

$$\frac{1}{2\pi a}[e^{-\frac{x^2}{4\pi a^2}} - e^{-\frac{(2c-x)^2}{4\pi a^2}}].$$

69. — To obtain the price for which the probability is the greatest, in case where the price c has not been attained, it suffices to set to zero the derivative of the expression above, yielding

$$\frac{x}{2c-x} + e^{-\frac{c(c-x)}{\pi a^2}} = 0.$$

If one supposes that $c = a$, one obtains $x_m = -1.5a$.

If one supposes that $c = 2a$, one obtains $x_m = -0.4a$.

Finally, one would obtain $x_m = -c$ if c were equal to $1.33a$.

70. — The probability for the price to be, at time t, between 0 and c, given that the price c has not been attained earlier is

$$\frac{1}{2} - 2\mathcal{P}_c + \mathcal{P}_{2c}.$$

c is expressed using a is a unit, $\mathcal{P}$ is computed according to the Table in paragraph 15.

71. — If $c = a$, the probability is 0.024. Hence, the buyer proposing to sell with profit a, if price a is attained prior to time t, has 0.69 probability of resale, 0.024 probability of a gain between 0 and a at time t, and 0.286 probability of loss.

If $c = 2a$, the same probabilities are respectively 0.425, 0.127, and 0.448.

72. Most probable instant. — If no time limit is fixed, at what instant is price c most probably attained?

One supposes uniformity. By setting to zero the derivative in t of the elementary probability (§67), one obtains

$$t = \frac{c^2}{6\pi k^2}.$$

If for example, c has the value $2a = 2k\sqrt{t}$ of a double option or the average spread relative to a settlement date t_1, one has $t = \frac{2t_1}{3\pi}$.

73. Probable instant. — If no time limit is fixed, what is the instant for which there is one chance in two for the price c to be exceeded prior to that instant?

One assumes uniformity. From the equality $P = \frac{1}{2}$, one deduces

$$t = \frac{c^2}{2.89k^2}.$$

The probable instant varies, in the same manner as the most probable instant, proportionally to the square of the spread c; it is about six times greater than the most probable instant.

74. Probability of a sequence of movements. — One asks, for example, the probability for the sequence of movements defined in this way to happen.

The price will exceed on the upside the value $+c$, then the value $-b$ on the downside, and exceed the value $+c$ on the upside again to be finally arbitrary at instant t.

This would be, for example, the case of a speculator buying in order to resell with profit c, waiting for the price to decrease below the value $-b$, in order to purchase again with the idea of reselling with a new profit $b+v$.

It would matter to know the probability for these three operations to be carried out before instant t, thus producing the profit $c+b+v$.

The desired probability is

$$P_{2c+2b+v} = 2\mathcal{P}_{2c+2b+v} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{2c+2b+v}{2\sqrt{\pi}a}} e^{-\lambda^2} d\lambda.$$

75. Largest spread. — What is the probability for the price c to be the greatest price quoted in the time interval t?

The probability looked for is the difference between the probabilities that the prices c and $c+dc$ be attained or exceeded, that is

$$\frac{\partial P}{\partial c} = \frac{e^{-\frac{c^2}{4\pi a^2}}}{\pi a} = 2\varpi.$$

The probability for a price to be the greatest price quoted in the time interval t is double the probability for that same price to be quoted at instant t.

76. — One could imagine a new type of option: against payment of a certain option, one would pocket the difference between the actual price and the highest price achieved in the time interval t. The value of this option

$$\int_0^\infty 2\varpi x dx = 2a$$

should be double the value of the simple option relative to the same period.

An option of same value would be relative to the difference between the actual price and the lowest price quoted in the time interval t.

In buying these two options, that is in paying the amount $4a$, one would earn the largest spread in both directions, that is the difference between the highest and the lowest prices quoted in the time interval t.

77. Probability of the third type. — *What is the probability for the price c (assumed to be positive in order to fix the ideas) to be attained or exceeded in the time interval t, if the downside variations never reached a given price $-b$.*

I believe it is useful to present the statement in a more explicit form by applying it to an example:

I purchase a (fixed) bond with the intent of reselling within a month if, during that time interval, it exceeds by 1^{fr} its actual price. Wanting to limit my risk to 2^{fr}, I commit to resell my bond if, in the course of the month, a 2^{fr} decrease below the actual price occurs.

One asks: the probability that, during the month, I could resell with a 1^{fr} profit; the probability that I would have resold with a loss of 2^{fr}, and the probability that, at the end of the month, I could not perform any resale.

Let us designate by $P_{c,\infty}$ the probability already computed (nb.. 63)

$$P_{c,\infty} = 2\mathcal{P}_{c,\infty} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{2\sqrt{\pi}a}} e^{-\lambda^2} d\lambda$$
$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{c}{\sqrt{\varphi(t)}}} e^{-\lambda^2} d\lambda = 1 - \Theta(\frac{c}{2\sqrt{\pi}a}),$$

for the price c to be exceeded assuming b infinite.

The probability asked for is expressed by the series

$$P_{c,b} = P_{c,\infty} - P_{c+2b,\infty} + P_{3c+2b,\infty} - P_{3c+4b,\infty} + P_{5c+4b,\infty} - \cdots,$$

or by the following

$$P_{c,b} = 2\mathcal{P}_{c,\infty} - 2\mathcal{P}_{c+2b,\infty} + 2\mathcal{P}_{3c+2b,\infty} - 2\mathcal{P}_{3c+4b,\infty} + 2\mathcal{P}_{5c+4b,\infty} - \cdots.$$

The quantities $P_{c,b}$ and $P_{b,c}$ are the probabilities of the third type.

78. — The probability for the price $-b$ to be attained in the time interval t, the upward variations never having attained the price c, will be obtained by replacing b by c and c by b in the preceding formula. The probability that, up to instant t, the price did not exit the interval $-b, +c$, is

$$1 - P_{b,c} - P_{c,b}.$$

79. Elementary probability. — Differentiating with respect to t the formula

$$P_{b,c} = P_{b,\infty} - P_{b+2c,\infty} + P_{3b+2c,\infty} - P_{3b+4c,\infty} + \cdots,$$

one obtains the probability for the price $-b$ to be attained for the first time at instant t, the upward variations having not previously attained the price c,

$$\Pi_{b,c} = \Pi_{b,\infty} - \Pi_{b+2c,\infty} + \Pi_{3b+2c,\infty} - \Pi_{3b+4c,\infty} + \cdots.$$

The elementary probabilities Π are computed by the formula (§67)

$$\Pi_{b,\infty} = \frac{b\varphi'(t)e^{-\frac{b^2}{\varphi(t)}}}{\sqrt{\pi}\varphi(t)\sqrt{\varphi(t)}}dt.$$

The probabilities of third type are thus expressed by series of probabilities of the second type.

80. — If no time limit is fixed, that is $t = \infty$, one has

$$P_{c,b} = \frac{b}{c+b}, \qquad R_{b.c} = \frac{c}{c+b}.$$

81. Applications. — The preceding formulas are susceptible of a wide number of interesting applications:

1) If one assumes $b = c = a = k\sqrt{t}$, $P_{a,a}$ is equal to 0.498. The probability that the price does not exit the interval is very small: $1 - 2 \times 0.498 = 0.004$.

2) When $b = c = 2a$, $P_{2a,2a} = 0.410$; the probability that the price remains within the interval $\pm 2a$ is $1 - 2 \times 0.41 = 0.18$.

If one purchases a double option with the preconceived idea of resale of a forward contract if the spread $2a$ is attained on the upside, or to repurchase a forward if the spread $2a$ is attained on the downside, the probability that one of the two transactions can occur is 0.82. Let us remark that $P_{2a,2a} = 0.41$ whereas $P_{2a,\infty} = 0.425$. When the spread on the upside or downside is greater than $2a$, the probability for the price to be attained in one direction is almost the same as if the variations in the other directions could have been arbitrary.

3) Let us suppose that $c = a$ and that $b = 2a$; the probability $P_{c,b}$ for the price c to be attained is 0.652, and the probability for the price $b = -2a$ to be attained is $P_{b,c} = 0.325$. The probability for the price to remain in the interval under consideration is 0.023.

If we have assumed a priori this probability to be negligible, we would have obtained through the formulas of paragraph 80 the very much close values

$$P_{c,b} = 0.666 \text{ and } P_{b,c} = 0.333.$$

82. Maximum spread. — *What is the probability that, in the time interval t, the largest spread in one sense or the other has a given value c?*

The probability asked for is

$$\frac{4dc}{\sqrt{\pi}\sqrt{\varphi(t)}}\left[e^{-\frac{c^2}{\varphi(t)}} - 3e^{-\frac{(3c)^2}{\varphi(t)}} + 5e^{-\frac{(5c)^2}{\varphi(t)}} - 7e^{-\frac{(7c)^2}{\varphi(t)}} + \cdots\right].$$

83. Second probability curve. — The curve that represents the variation of the probability of the maximum spread is tangent to the x-axis at the origin and at infinity, it exhibits two inflection points; the ordinate is maximum when

$$c = 0.642\ldots\ \sqrt{\varphi(t)}.$$

The most probable of the maximum spread is $0.642\ldots\ \sqrt{\varphi(t)}$.

The maximum probability that corresponds to the maximum ordinate of the curve decreases as $\frac{1}{\sqrt{\varphi(t)}}$.

The abscissa of the inflection points increase proportional to $\sqrt{\varphi(t)}$.

84. Second mean spread. — We shall call *second mean spread* the mean value of the largest spread existing between the current price and all prices quoted in the time interval t.

The second mean spread has value

$$\frac{\sqrt{\pi}}{2}\sqrt{\varphi(t)|} = \pi a;$$

it is proportional to the square root of the instability function and equal to the first mean spread multiplied by $\frac{\pi}{2}$.

85.— One could imagine new types of options: against abandoning a certain option, one would receive the largest spread between the current price and those quoted in the time interval t, whether that spread is positive or negative.

The value of the option, according to the result of the preceding paragraph, should be πa.

We have seen (§76) that the promise of receiving the difference between the highest and the lowest prices quoted during the period t, that is the promise of receiving the sum of the two largest spreads in both directions, had value $4a$. The promise of receiving the largest of the two spreads being worth πa, the promise to receive the smallest would be worth $(4 - \pi)a$, a little less than the simple option.

86. Second probable spread. — We call *second probable spread* the interval $\pm\gamma$ such that, during period t, the price has as much chance to remain within that interval than chances to exceed it.

One must have $P_{\gamma,\gamma} = \frac{1}{4}$, thus

$$\gamma = 2.9a = 0.8062\ldots \sqrt{\varphi(t)}.$$

The second probable spread is proportional to the square root of the instability function, it is equal to the first probable spread multiplied by 1.7...

One understands the difference that exists between the two probable spreads: the first has equal chances to be or not to be exceeded at period t, whereas the second has equal probability to be or not to be exceeded before period t.

87. Second isoprobable spreads. — Consider a maximum spread c such that the probability for that spread not to be exceeded is equal to a given number u. One must have

$$\int_0^c \frac{4}{\sqrt{\pi}\sqrt{\varphi(t)}}[e^{-\frac{c^2}{\varphi(t)}} - 3e^{-\frac{(3c)^2}{\varphi(t)}} + 5e^{-\frac{(5c)^2}{\varphi(t)}} - 7e^{-\frac{(7c)^2}{\varphi(t)}} + \cdots]dc = u.$$

By setting $\frac{c^3}{\varphi(t)} = \lambda^2$, this inequality becomes

$$\int_0^{\frac{c}{\sqrt{\varphi(t)}}} \frac{4}{\sqrt{\pi}}(e^{-\lambda^2} - 3e^{-9\lambda^2} + 5e^{-25\lambda^2} - 7e^{-49\lambda^2} + \cdots]d\lambda = u,$$

u being constant, so is $\frac{c}{\sqrt{\varphi(t)}}$ and c is proportional to $\sqrt{\varphi(t)}$. Therefore:

The second isoprobable spreads are proportional to the square root of the instability function.

If one admits uniformity, the second spreads are proportional to the square root of time.

88. Most probable instant. — We have studied problems in which we have considered

a fixed time interval and variable spreads; we shall now suppose that the spreads are fixed and the duration of the operation variable.

The *most probable* instant at which the price will exist the interval $(c, -b)$, quoting the price c, is given by the formula

$$\frac{\partial^2 P_{c,b}}{\partial t^2} = 0.$$

The most probable instant at which the price will attain the limit $-b$ is obtained by solving the equation

$$\frac{\partial^2 P_{b,c}}{\partial t^2} = 0.$$

The most probable instant at which the price will exit the interval $(c, -b)$ is given by the equality

$$\frac{\partial^2}{\partial t^2}(P_{c.b} + P_{b,c}) = 0.$$

89. Probable instant. — The probable instant at which the price exits the interval $c, -b$ is obtained by the resolution of the equation

$$P_{c,b} + P_{b,c} = \frac{1}{2}.$$

Let first $b = c$; we must have, assuming uniformity,

$$t = \frac{c^2}{8.24k^2}.$$

If, for example, $c = a = k\sqrt{t_1}$, we shall have $t = \frac{t_1}{8.24}$; if $c = 2a = 2k\sqrt{t_1}$, we shall have $t = \frac{t_1}{2.06}$.

Let us assume now that $b = 2c$; the probable instant corresponds to

$$t = \frac{c^2}{4.6k^2} = \frac{bc}{9.2k^2}.$$

If, for example, $c = a = k\sqrt{t_1}, b = 2k\sqrt{t_1}$, we have

$$t = \frac{t_1}{4.6}.$$

90. Average time. — Since the price could exit the interval $+c, -b$ at all instants from zero to infinity, the average time corresponding to this interval is the sum of the products of the probabilities that the price exits the interval at time t by the time t itself.

The average time at which the price exits the interval $+c, -b$ is given by the formula

$$\frac{cb}{2\pi k^2}.$$

If, for example, $b = c$, the average time $\frac{c^2}{2\pi k^2}$ is greater than the probable time in the ratio of 4 to 3, approximately.

91. — Let us reconsider, by way of illustration the following example:

One purchases bonds with the intention of reselling with profit $a = k\sqrt{t}$ or with loss $2a$; one concludes the transaction if at time t the resale has not taken place. What are the principal results provided by the computation of probabilities for this transaction?

The probability of resale with profit a is 0.652.
The probability of resale with loss $2a$ is 0.325.
The probability that the resale has not taken place before time t is 0.023.
The most probable time of resale with profit a is $\frac{t}{6}$.
The most probable time of resale with loss $2a$ is $\frac{2}{3}t$.
The probable time of resale is $\frac{t}{4.6}$.
The average time is $\frac{t}{\pi}$.

92. Distribution of the probabilities.-. — It remains for us to solve the probabilities distribution problem, that is to look for the probability for the price to be x at time t, if prior variations have never exceed the interval $-b + c$.

This probability is given by the formula

$$\frac{dx}{\sqrt{\pi}\sqrt{\varphi(t)}}[e^{\frac{x^2}{\varphi(t)}} - e^{\frac{(2c-x)^2}{\varphi(t)}} + e^{\frac{(2c+2b-x)^2}{\varphi(t)}} - e^{\frac{(4c+2b-x)^2}{\varphi(t)}} + \cdots$$
$$- e^{\frac{(2b+x)^2}{\varphi(t)}} + e^{\frac{(2b+2c+x)^2}{\varphi(t)}} - e^{\frac{(4b+2c+x)^2}{\varphi(t)}} + \cdots].$$

93. — By solely adopting a mathematical point of view completely removed from any idea of rapprochement with reality, one could attempt to build a theory more general than the one presented here.

Such a theory would admit, for example, that the price variations depend on the absolute value of that price, would deny the mathematical expectation principle, scientific expression of the law of the supply and demand, would loose contact with reality and, mostly, would draw its interest from its difficulty. It would be the analogue, for the problem occupying us, of non-Euclidean geometries or relativity. The analysis developed in §§308 and 314 of my *Traité*[25] would allow the building of such a theory limited to the first problem.

The analysis developed in §607 based on altogether other hypotheses would also lead to absolutely mathematical laws having, however, no relation with the speculation problem.

I have named *connected probabilities* those that are related to a large number of trials or to continuous variations in time without there being independence between successive trials or time elements.

Laplace has resolved various difficult problems on these subjects, but without tying them with a unifying idea and without going as far as I did in my Traité and in the *Mémoires* which I have previously published.

In §367 of this *Traité*, a kind of connectdeness very different from the preceding ones is studied. I also refer, for these questions, to my work on the *Jeu, la chance et le hasard*, where the general ideas are exposed without formulas.

For a while, the term *chain forming probabilities*[26] is sometimes used to designate certain classes of connected probabilities; this term is not fortu-

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[25] By *Traité*, Bachelier refers to his *Calcul des Probabilités* published by Gauthier-Villars in 1912.

[26] Bachelier uses the terminology "*probabilités formant chaîne*".

nate, a chain giving the idea of an absolute rather than random link. The use of such an expression, even restricted to a limited class of questions, is therefore illogical; it self-imply some sort of contradiction.

94. — Let us place ourselves on the experimental view point. A first series of observations described in my work on the *Théorie de la spéculation*[27] has demonstrated the perfect agreement between calculated and observed data.

I have made the results of a second series of experiments known in my book on the *Jeu, la chance et le hasard* (p. 212).[28]

Both series show that the theory is in perfect harmony with reality, which, a priori, was indeed very likely.

This theory does not only have the merit of being the starting point of a sequence of studies providing a new activity to a science dormant since the admirable works of Laplace, nearly a century ago, but it also has the merit to be, as a whole, the expression of reality.

27 [Bachelier 1900].

28 [Bachelier 1914].