ISSN 0012-9593

quatrième série - tome 51 fascicule 5 septembre-octobre 2018

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Ling-Bing HE

Sharp bounds for Boltzmann and Landau collision operators

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 540 €. Hors Europe : 595 € ($ 863). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4$^{\text{e}}$ série, t. 51, 2018, p. 1253 à 1341

SHARP BOUNDS FOR BOLTZMANN AND LANDAU COLLISION OPERATORS

BY LING-BING HE

ABSTRACT. – The aim of the work is to provide a stable method to get sharp bounds for Boltzmann and Landau operators in weighted Sobolev spaces and in anisotropic spaces. The results and proofs have the following main features and innovations:

- All the sharp bounds are given for the original Boltzmann and Landau operators. The sharpness means the lower and upper bounds for the operators are consistent with the behavior of the linearized operators. Moreover, we make clear the difference between the bounds for the original operators and those for the linearized ones. It will be useful for the well-posedness of the original equations.
- According to the Bobylev's formula, we introduce two types of dyadic decompositions performed in both phase and frequency spaces to make full use of the interaction and the cancelation. It allows us to see clearly which part of the operator behaves like a Laplace type operator and which part is dominated by the anisotropic structure. It is the key point to get the sharp bounds in weighted Sobolev spaces and in anisotropic spaces.
- Based on the geometric structure of the elastic collision, we make a geometric decomposition to capture the anisotropic structure of the collision operator. More precisely, we make it explicit that the fractional Laplace-Beltrami operator really exists in the structure of the collision operator. It enables us to derive the sharp bounds in anisotropic spaces and then complete the entropy dissipation estimates.
- The structures mentioned above are so stable that we can apply them to the rescaled Boltzmann collision operator in the process of the grazing collisions limit. Then we get the sharp bounds for the Landau collision operator by passing to the limit. We remark that our analysis used here will shed light on the investigation of the asymptotics from Boltzmann equation to Landau equation.

RÉSUMÉ. – L'objectif de ce travail est de fournir une méthode robuste pour obtenir des estimations précises pour les opérateurs de Boltzmann et de Landau dans des espaces de Sobolev à poids et des espaces anisotropes. Les résultats et leur démonstration font ressortir les innovations suivantes :

- Toutes les estimations précises concernent les opérateurs originaux de Boltzmann et de Landau. Le mot 'précis' se réfère au fait que les estimations sont cohérentes avec le comportement des opérateurs linéarisés correspondants. Ceci est utile pour étudier le caractère bien posé des équations originales.

 doi:10.24033/asens.2375

– En accord avec la formule de Bobylev, on introduit deux types de décomposition dyadique, dans l'espace des phases et dans celui des fréquences, afin d'utiliser au maximum les annulations. Cela nous permet de voir clairement quelle partie de l'opérateur se comporte comme un opérateur de type Laplacien, et quelle partie est dominée par la structure anisotrope.
– En se basant sur la structure géométrique des collisions élastiques, on fait une décomposition géométrique pour capturer la structure anisotrope de l'opérateur de collision. Plus précisément, on explicite le fait que l'opérateur de Laplace-Beltrami apparaît bien dans l'opérateur de collision. Cela nous permet d'obtenir des estimations précises dans des espaces anisotropes et de finaliser les estimations sur la dissipation d'entropie.
– Les structures mentionnées ci-dessus sont si robustes qu'on peut les retrouver dans la limite des collisions rasantes. On obtient ainsi des estimations précises pour le noyau de collision de Landau en passant à la limite. On remarque que la présente analyse éclaire le passage à la limite de l'équation de Boltzmann vers celle de Landau.

1. Introduction

The aim of the present work is to provide a stable method to give a complete description of the behavior of the Boltzmann and Landau collision operators. We remark that it is related closely to the derivation of the Landau equation from the Boltzmann equation and also the asymptotics of the Boltzmann equation from short-range interactions to long-range interactions.

We first recall that the Boltzmann equation reads:

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \tag{1.1}$$

where $f(t, x, v) \geq 0$ is a distribution function of colliding particles which, at time $t \geq 0$ and position $x \in \mathbb{T}^3$, move with velocity $v \in \mathbb{R}^3$. We remark that the Boltzmann equation is one of the fundamental equations of mathematical physics and is a cornerstone of statistical physics.

The Boltzmann collision operator Q is a bilinear operator which acts only on the velocity variable v, that is,

$$Q(g, f)(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma)(g'_* f' - g_* f) d\sigma dv_*.$$

Here we use the standard shorthand $f = f(t, x, v)$, $g_* = g(t, x, v_*)$, $f' = f(t, x, v')$, $g'_* = g(t, x, v'_*)$ where (v, v_*) and (v', v'_*) are the velocities of particles before and after the collision. Here v' and v'_* are given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma\,, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma\,, \qquad \sigma \in \mathbb{S}^2\,. \tag{1.2}$$

The representation is consistent with the physical laws of the elastic collision:

$$\begin{aligned} v + v_* &= v' + v'_*, \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2. \end{aligned}$$

In the definition of Q, B is called the Boltzmann collision kernel. It is always assumed that $B \geq 0$ and that B depends only on $|v - v_*|$ and $\frac{v - v_*}{|v - v_*|} \cdot \sigma$. Usually, we introduce the angle variable θ through $\cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma$. Without loss of generality, we may assume that $B(v-v_*,\sigma)$ is supported in the set $0\le\theta\le\frac{\pi}{2}$, that is, $\frac{v-v_*}{|v-v_*|}\cdot\sigma\ge 0$. Otherwise, B can be replaced by its symmetrized form:

$$\bar{B}(v-v_*,\sigma) = [B(v-v_*,\sigma)+B(v-v_*,-\sigma)]\mathbf{1}_{\{\frac{v-v_*}{|v-v_*|}\cdot\sigma\ge 0\}}. \tag{1.3}$$

Here, $\mathbf{1}_A$ is the characteristic function of the set A. In this paper, we consider the collision kernel satisfying the following assumptions:

(A1). The kernel $B(v-v_*,\sigma)$ takes a product form

$$B(v-v_*,\sigma) = \Phi(|v-v_*|)b(\cos\theta), \tag{1.4}$$

where both Φ and b are nonnegative functions.

(A2). The angular function $b(t)$ satisfies for $\theta\in[0,\pi/2]$,

$$K\theta^{-1-2s}\le\sin\theta b(\cos\theta)\le K^{-1}\theta^{-1-2s},\quad \text{with } 0<s<1,\ K>0. \tag{1.5}$$

(A3). The kinetic factor Φ takes the form

$$\Phi(|v-v_*|) = |v-v_*|^\gamma, \tag{1.6}$$

where the parameter γ verifies that $\gamma+2s>-1$ and $\gamma\le 2$.

REMARK 1.1. – *For inverse repulsive potential, it holds that* $\gamma=\frac{p-5}{p-1}$ *and* $s=\frac{1}{p-1}$ *with* $p>2$. *It is easy to check that* $\gamma+4s=1$ *which makes sense of the assumption* $\gamma+2s>-1$. *Generally, the case* $\gamma>0$, *the case* $\gamma=0$, *and the case* $\gamma<0$ *correspond to so-called hard, Maxwellian, and soft potentials respectively.*

REMARK 1.2. – *If we replace the assumption* (1.5) *by*

$$K\theta^{-1-2s}\Big(1-\psi\big(\frac{\sin(\theta/2)}{\epsilon}\big)\Big)\le\sin\theta b(\cos\theta)\le K^{-1}\theta^{-1-2s}\Big(1-\psi\big(\frac{\sin(\theta/2)}{\epsilon}\big)\Big), \tag{1.7}$$

where ψ *is a non-negative and smooth function defined in* (1.33), *then the mathematical problem of the asymptotics of the Boltzmann equation from short-range interactions to long-range interactions can be formulated by the limit in which the parameter* ϵ *in* (1.7) *goes to zero. We remark that for fixed* ϵ, (1.7) *corresponds to the famous Grad's cut off assumption for the kernel* B.

The solutions of the Boltzmann Equation (1.1) enjoy the fundamental properties of the conservation of mass, momentum and the kinetic energy, that is, for all $t\ge 0$,

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} f(t,x,v)\phi(v)dvdx = \iint_{\mathbb{T}^3\times\mathbb{R}^3} f(0,x,v)\phi(v)dvdx,\quad \phi(v)=1,v,|v|^2.$$

Moreover, if the entropy $H(f)$ is defined by

$$H(f)(t)\stackrel{\text{def}}{=}\iint_{\mathbb{T}^3\times\mathbb{R}^3} f\ln f dvdx,$$

then the celebrated Boltzmann's H-theorem predicts that the entropy is decreasing over time, which formally is

$$\frac{d}{dt}H(f)(t) = \iint_{\mathbb{T}^3\times\mathbb{R}^3} Q(f,f)\ln f dvdx\le 0.$$

Before introducing the Landau equation, let us give the definition of the grazing collision limit. We introduce the rescaled Boltzmann's kernel B^ϵ which verifies the assumption.

(B1). The rescaled Boltzmann's kernel B^ϵ takes the simple product form

$$B^\epsilon(v - v_*, \sigma) = \Phi(|v - v_*|)b^\epsilon(\cos\theta),$$

where the kinetic factor Φ satisfies (1.6) and the angular function $b^\epsilon(t)$ satisfies for $\theta \in [0, \pi/2]$,

$$\sin\theta b^\epsilon(\cos\theta) = K'\epsilon^{2s-2}\psi\Big(\frac{\sin(\theta/2)}{\epsilon}\Big)\theta^{-1-2s}, \tag{1.8}$$

with $0 < s < 1$. Here K' is a positive constant and the function ψ is defined in (1.33).

The assumption (1.8) means that the deviation angles between relative velocities before and after collisions are restricted to be less than ϵ. Mathematically the grazing collision limit is defined by the process in which the parameter ϵ goes to zero. Thanks to the full Taylor expansion, by taking the limit $\epsilon \to 0$, the Boltzmann collision operator Q^ϵ with rescaled kernel B^ϵ will be reduced to a new operator, namely the Landau collision operator Q_L, defined by

$$Q_L(g, h) \stackrel{\text{def}}{=} \nabla_v \cdot \Big\{ \int_{\mathbb{R}^3} a(v - v_*)[g(v_*)\nabla_v h(v) - \nabla_v g(v_*)h(v)]dv_* \Big\}.$$

Here the nonnegative matrix a is given by

$$a_{ij}(v) = \Lambda\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right)|v|^{\gamma+2}, \qquad \gamma \in [-3, 1], \tag{1.9}$$

where Λ is a positive constant and can be calculated by

$$\Lambda = \frac{\pi}{8}K' \int_0^{\pi/2} \psi(\theta)\theta^{1-2s}d\theta.$$

Then the Landau equation can be written by

$$\partial_t f + v \cdot \nabla_x f = Q_L(f, f). \tag{1.10}$$

We remark that the equation was proposed by Landau in 1936 to model the behavior of a dilute plasma interacting through binary collisions. We also mention that the Landau equation possesses all the properties known for the Boltzmann equation, namely the conservation of mass, momentum and energy and the H-theorem.

1.1. Motivation and short review of the problem

The justification of the derivation of the Landau equation in the sense that the solution to the Boltzmann equation with rescaled kernel (see assumption (B1)) will converge to the solution to the corresponding Landau equation in the grazing collision limit had been proved by several authors. We refer readers to [18] and the references therein to check details. However, the physical problem of the justification is formulated as a higher-order correction to the limit. In other words, we should establish some kind of the asymptotic formula to the solutions in the limit process. Suppose that f^ϵ_B and f_L represent the solutions to Boltzmann and Landau equations. In [10], for the homogeneous case, that is, the solution does not depend on the position variable x, the author showed that the following asymptotic formula

$$f^\epsilon_B = f_L + O(\epsilon) \tag{1.11}$$

holds globally or locally in Sobolev spaces for almost all physical potentials except for Coulomb potential. It shows that the Landau equation is a good approximation to the Boltzmann equation when the parameter ϵ is small enough. It gives the validity of the Landau equation. However, it is very difficult to extend the similar result to the inhomogeneous case even in the close-to-equilibrium setting. The main obstruction is the lack of a complete description of the asymptotic behavior of collision operator in the limit process since the behavior of the operator is very sensitive to the parameter ϵ. The same problem happens when we study the asymptotics of the Boltzmann equation from short-range interactions to long-range interactions under the assumption (1.7). We remark that the investigation of the second asymptotics is related closely to the construction of approximate solutions to the equation with long-range interactions and also to the jump phenomenon of the spectral gap of the linearized collision operator (see (1.16) for the definition) for soft potentials ($-2s \le \gamma < 0$) when ϵ goes to zero.

Motivated by these two asymptotic problems, in the present work, we try to find out some stable structure inside the Boltzmann collision operator to obtain the sharp bounds and then extend them to the Landau operator via the grazing collision limit. Before going further, let us give a short review on the estimates of the Boltzmann collision operator. For simplicity, we only address the estimates for the Maxwellian molecular case, that is, $\gamma = 0$.

In what follows, we assume that all the functions depend only on the v variable recalling that the collision operator Q acts only on the v variable. We will use the duality method to get the lower and upper bounds of the operator. The inner product of f and g over $\mathbb{R}^3_v$, namely $\langle f, g\rangle_v$, is defined by

$$\langle f, g\rangle_v \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} f(v)g(v)dv.$$

Then by change of variables(see [1]), we have

$$\langle Q(g, f), f\rangle_v = \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} B(|v-v_*|,\sigma)g_* f(f'-f)d\sigma dv_* dv.$$

It is easy to check that

$$\langle Q(g, f), f\rangle_v = -\underbrace{\frac{1}{2}\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} B(|v-v_*|,\sigma)g_*(f'-f)^2 d\sigma dv_* dv}_{\mathcal{E}_g(f)}$$

$$+\frac{1}{2}\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} B(|v-v_*|,\sigma)g_*(f'^2-f^2)d\sigma dv_* dv. \tag{1.12}$$

In [1], the authors gave the first coercivity estimate of $\mathcal{E}_g(f)$ which can be stated as

$$\mathcal{E}_g(f) \ge C_g\|f\|^2_{H^s} - \|g\|_{L^1}\|f\|^2_{L^2},$$

where C_g is a constant depending on the lower bound of $\|g\|_{L^1}$ and the upper bounds of $\|g\|_{L^1_2}$ and $\|g\|_{L\log L}$ (see the definitions in Section 1.2). It gives a positive answer to the conjecture that the Boltzmann collision operator behaves like a fractional Laplace operator, that is,

$$-Q(g,\cdot) \sim C_g(-\triangle_v)^s + LOT. \tag{1.13}$$

This conjecture was further confirmed by the upper bound for the collision operator. Mathematically, it reads that if $a, b \in \mathbb{R}$ with $a + b = 2s$, then

$$|\langle Q(g,h), f\rangle_v| \lesssim \|g\|_{L^1_{2s}} \|h\|_{H^a_s} \|f\|_{H^b_s}, \tag{1.14}$$

which was proved in [3]. The weighted Sobolev space H^m_l is defined in Section 1.2. This upper bound is sharp in the sense that we have the freedom of choosing derivatives for functions h and f. For the general potentials, we refer readers to [3, 7, 11] on the lower and upper bounds in weighted Sobolev spaces.

Combining the lower and upper bounds, one may find

$$C_g\|f\|^2_{H^s} - \|g\|_{L^1}\|f\|^2_{L^2} \le \langle -Q(g,f), f\rangle_v \lesssim \|g\|_{L^1_{2s}}\|f\|^2_{H^s_s}. \tag{1.15}$$

We remark that additional condition for f is imposed in the upper bound. The reason lies in the fact that some anisotropic structure is hidden in the operator which can not be observed in weighted Sobolev spaces. Indeed, in [21], the authors show that the linearized collision operator $\mathcal{L}_B$, which is defined by

$$\mathcal{L}_B f \stackrel{\text{def}}{=} -\mu^{-\frac{1}{2}}\big(Q(\mu, \mu^{\frac{1}{2}} f) + Q(\mu^{\frac{1}{2}} f, \mu)\big), \quad \mu = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}, \tag{1.16}$$

is a self-adjoint operator and has explicit eigenvalues and eigenfunctions. In particular, the eigenfunction $E(v)$ takes the form

$$E(v) = f(|v|^2)Y(\sigma),$$

where $v = |v|\sigma$, f is a radial function and Y is a real spherical harmonic. In [19], Villani proved that

$$Q_L(f,f) = 3\nabla\cdot(\nabla f + vf) - \big(P_{ij}(f)\partial_{ij} f + \nabla\cdot(vf)\big) + \triangle_{\mathbb{S}^2} f,$$

where $P_{ij}(f) = \int_{\mathbb{R}^3} f v_i v_j dv$ and $(-\triangle_{\mathbb{S}^2})$ is the Laplace-Beltrami operator on the unit sphere. The special form of the eigenfunction of $\mathcal{L}_B$ and the Laplace-Beltrami operator in the expression of Q_L indicate that there should be some anisotropic structure inside the operator.

Mathematically the first attempts to capture the anisotropic structure of the operator were due to [5] and [8] (see also [4] and [9]). In fact, to describe the behavior of the operator, they introduce two types of anisotropic norms which are defined by

$$\|f\|^2 \stackrel{\text{def}}{=} \|f\|^2_{L^2_s} + \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\cos\theta)\mu_*(f'-f)^2 d\sigma dv_* dv \tag{1.17}$$

and

$$\|f\|^2_{N^s} \stackrel{\text{def}}{=} \|f\|^2_{L^2_s} + \underbrace{\iint_{v,v'\in\mathbb{R}^3} \langle v\rangle^{s+1/2}\langle v'\rangle^{s+1/2}\frac{|f-f'|^2}{d(v,v')^2} 1_{d(v,v')\le 1} dv dv'}_{\mathcal{A}(f)}, \tag{1.18}$$

where $d(v,v') = \sqrt{|v-v'|^2 + \frac{1}{4}(|v|^2 - |v'|^2)^2}$ and $\langle v\rangle = \sqrt{1+|v|^2}$. We remark that the second term in the righthand side of the Definition (1.17) is exactly the term $\mathcal{E}_\mu(f)$ defined in (1.12). Then the estimates can be stated as

$$\|f\|^2 - \|f\|^2_{L^2} \lesssim \langle \mathcal{L}_B f, f\rangle_v \lesssim |f|^2. \tag{1.19}$$

Moreover, the upper bound can be generalized to the original collision operator Q:

$$|\langle Q(g,h), f\rangle_v| \lesssim \|g\|_{L^2}|h||f|.$$

We emphasize that these two anisotropic norms are crucial to prove global small solutions in the perturbation framework. Since they are given in an implicit way, it does not help to understand the anisotropic property of the operator. In the grazing collision limit, for $\mathcal{E}_\mu(f)$, on the one hand, it is stable since it is given in an implicit way. On the other hand, we have no idea on the limit of this quantity. The similar problem occurs for $\mathscr{A}(f)$.

Very recently, two groups gave explicit description of the anisotropic behavior of the linearized operator. Both of them started with the same point, that is, the well understanding of the behavior of the linearized Landau operator $\mathcal{L}_L$. In fact, they proved that

$$\mathcal{L}_L \sim (-\triangle + |v|^2/4) + (-\triangle_{\mathbb{S}^2}) \sim (-\triangle + |v|^2/4) + |D_v \times v|^2,$$

recalling that $-\triangle_{\mathbb{S}^2} = \sum_{1\le i<j\le 3} (v_i\partial_j - v_j\partial_i)^2$. In [2], the authors show that

$$\langle \mathcal{L}_B f, f\rangle_v + \|f\|^2_{L^2} \sim \|f\|^2_{L^2_s} + \|f\|^2_{H^s} + \||D_v \times v|^s f\|^2_{L^2}, \tag{1.20}$$

where $|D_v \times v|^s$ is a pseudo-differential operator with the symbol $|\xi \times v|^s$. Thanks to [21], by comparing the eigenvalues between Boltzmann and Landau collision operators, in [15], the authors show that

$$\mathcal{L}_B \sim \mathcal{L}_L^s \tag{1.21}$$

and

$$\langle \mathcal{L}_B f, f\rangle_v + \|f\|^2_{L^2} \sim \|f\|^2_{L^2_s} + \|f\|^2_{H^s} + \|(-\triangle_{\mathbb{S}^2})^{s/2} f\|^2_{L^2}. \tag{1.22}$$

We remark that the strategy to obtain (1.20) and (1.22) depends heavily on the linearized structure, for instance, the symmetric property of the operator and the fine properties of the Maxwellian state μ. Therefore it cannot be generalized to the original collision operator and to the rescaled operator under the assumption (1.7) or (1.8).

The short review can be summarized as follows:

1. The previous results on the description of the behavior of the operator are given in an implicit way or in an unstable way. It means that the anisotropic structure is still mysterious and not captured well.
2. The upper bound of the collision operator is far away from the sharpness. For instance, recalling (1.21), the typical upper bound for the operator should be in the form

$$|\langle Q(g,h), f\rangle_v| \lesssim C(g)\|(\mathcal{L}_L^{a/2} + 1)h\|_{L^2}\|(\mathcal{L}_L^{b/2} + 1)f\|_{L^2}, \tag{1.23}$$

   where $a, b \in [0, 2s]$ with $a + b = 2s$.
3. For the linearized collision operator $\mathcal{L}_B^\epsilon$ with rescaled kernel B^ϵ under the assumption (1.7) or (1.8), we have no available results on the complete description of the operator. We also have no idea on the sharp bounds of the original collision operator Q^ϵ in the process of the limit.

We end this subsection by the remark that points (1) and (2) are closely related to the Cauchy problem for the original Equation (1.1). And the point (3) is related to the investigation of two types of asymptotics mentioned before.

1.2. Notations and main results

Before stating our main results, we first introduce the function spaces which will be used throughout the paper.

1. For any integer $N \geq 0$, we define the Sobolev space H^N by
$$H^N \stackrel{\text{def}}{=} \Big\{ f(v) | \sum_{|\alpha| \leq N} \|\partial_v^\alpha f\|_{L^2} < +\infty \Big\},$$
where the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\partial_v^\alpha = \partial_{v_1}^{\alpha_1} \partial_{v_1}^{\alpha_2} \partial_{v_3}^{\alpha_2}$.
2. For real numbers m, l, we define the weighted Sobolev space H_l^m by
$$H_l^m \stackrel{\text{def}}{=} \Big\{ f(v) | \|f\|_{H_l^m} = \|\langle D\rangle^m \langle \cdot \rangle^l f\|_{L^2} < +\infty \Big\},$$
where $\langle v\rangle \stackrel{\text{def}}{=} (1+|v|^2)^{\frac{1}{2}}$. $a(D)$ is a pseudo-differential operator with the symbol $a(\xi)$ and is defined as
$$\big(a(D)f\big)(x) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(x-y)\xi} a(\xi) f(y) dy d\xi.$$
3. The general weighted Sobolev space $W_l^{N,p}$ with $p \in [1, \infty)$ is defined as
$$W_l^{N,p} \stackrel{\text{def}}{=} \Big\{ f(v) | \|f\|_{W_l^{N,p}} = \sum_{|\alpha| \leq N} \Big(\int_{\mathbb{R}^3} |\partial_v^\alpha f(v)|^p \langle v\rangle^{lp} dv \Big)^{1/p} < \infty \Big\}.$$
In particular, if $N = 0$, we introduce the weighted L_l^p space defined as
$$L_l^p \stackrel{\text{def}}{=} \Big\{ f(v) | \|f\|_{L_l^p} = \Big(\int_{\mathbb{R}^3} |f(v)|^p \langle v\rangle^{lp} dv \Big)^{\frac{1}{p}} < \infty \Big\}.$$
4. The $L \log L$ space is defined as
$$L \log L \stackrel{\text{def}}{=} \Big\{ f(v) | \|f\|_{L \log L} = \int_{\mathbb{R}^3} |f| \log(1+|f|) dv < \infty \Big\}.$$

Next we list some notations which will be used in the paper. We write $a \lesssim b$ to indicate that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$. We use the notation $a \sim b$ whenever $a \lesssim b$ and $b \lesssim a$. The notation a^+ means the maximum value of a and 0. The weight function W_l is defined by $W_l(v) \stackrel{\text{def}}{=} \langle v\rangle^l$ where $\langle v\rangle = \sqrt{1+|v|^2}$. Suppose A and B are two operators. Then the commutator $[A, B]$ between A and B is defined as follows:
$$[A, B] \stackrel{\text{def}}{=} AB - BA.$$
We denote $C(\lambda_1, \lambda_2, \dots, \lambda_n)$ by a constant depending on parameters $\lambda_1, \lambda_2, \dots, \lambda_n$.

Our first result is on the sharp upper bounds of the Boltzmann collision operator in weighted Sobolev spaces.

THEOREM 1.1. – *Let $w_1, w_2 \in \mathbb{R}, a, b \in [0, 2s]$ with $w_1 + w_2 = \gamma + 2s$ and $a + b = 2s$. Then for smooth functions g, h and f, we have*

1. *if $\gamma + 2s > 0$,*

$$|\langle Q(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{\gamma+2s+(-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2}) \|h\|_{H^a_{w_1}} \|f\|_{H^b_{w_2}}, \tag{1.24}$$

2. *if* $\gamma + 2s = 0$,

$$|\langle Q(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{w_3}} + \|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}, \tag{1.25}$$

where $w_3 = \max\{\delta, (-w_1)^+ + (-w_2)^+\}$ *with* $\delta > 0$ *which is sufficiently small,*

3. *if* $-1 < \gamma + 2s < 0$,

$$|\langle Q(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{w_4}} + \|g\|_{L^2_{-(\gamma+2s)}})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}, \tag{1.26}$$

where $w_4 = \max\{-(\gamma + 2s), \gamma + 2s + (-w_1)^+ + (-w_2)^+\}$.

REMARK 1.3. – *The estimates* (1.24–1.26) *are sharp in weighted Sobolev spaces in the sense that we have the freedom of choosing derivatives and weights for functions h and f. They will play a crucial role in solvability of the Cauchy problem for the Equation* (1.1) *in weighted Sobolev spaces. In particular, we will use them frequently to balance the energy estimates to close the argument. We will explain it in detail in our forthcoming paper* [12].

REMARK 1.4. – *We refer readers to the very recent work* [14] *and* [17] *on the similar results by using the Bobylev's formula (see* (2.2)*) or using the Random transform. We mention that their results only have the freedom of choosing derivatives for functions h and f.*

REMARK 1.5. – *The estimates are not sharp with respect to the function g. For instance, for* $\gamma + 2s > 0$ *and* $\gamma > -\frac{3}{2}$, *we can drop* $\|g\|_{L^2}$ *in the estimate. For* $\gamma > -\frac{5}{2}$, *we can replace* $\|g\|_{L^2_l}$ *by* $\|g\|_{L^{\frac{3}{2}}_l}$ *in the estimates. For the maxwellian molecular(*$\gamma = 0$*), the estimate can be rewritten as*

$$|\langle Q(g,h), f\rangle_v| \le C(a,b)\|g\|_{L^1_{2s+(-w_1)^+ + (-w_2)^+}}\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}, \tag{1.27}$$

where $a + b = 2s$ *and* $w_1 + w_2 = 2s$. *Here we remove the restriction* $a, b \in [0, 2s]$.

Next we will state our new coercivity estimates for the Boltzmann collision operator:

THEOREM 1.2. – *Suppose g is a non-negative and smooth function verifying that*

$$\|g\|_{L^1} > \delta \quad \textit{and} \quad \|g\|_{L^1_2} + \|g\|_{L\log L} < \lambda, \tag{1.28}$$

and let $\mathbf{A} = 0, 1$. *Then for sufficiently small* $\eta > 0$, *there exist constants* $\mathbf{C}_1(\delta,\lambda,\eta^{-1})$, $\mathbf{C}_2(\lambda,\delta)$, $\mathbf{C}_3(\delta,\lambda,\eta^{-1})$, $\mathbf{C}_4(\lambda,\delta)$ *and* $\mathbf{C}_5(\lambda,\delta)$ *such that*

1. *if* $\gamma + 2s \ge 0$,

$$\begin{aligned}\langle -Q(g,f), f\rangle_v \gtrsim \mathbf{A}\Big[&\mathbf{C}_1(\delta,\lambda,\eta^{-1})\big(\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|^2_{L^2_{\gamma/2}} + \|f\|^2_{H^s_{\gamma/2}}\big)\\ &- \eta\mathbf{C}_2(\delta,\lambda)\|f\|^2_{L^2_{\gamma/2+s}} - \mathbf{C}_3(\delta,\lambda,\eta^{-1})\|f\|^2_{L^2_{\gamma/2}}\Big]\\ &+ \mathbf{C}_4(\lambda,\delta)\|f\|^2_{H^s_{\gamma/2}} - \mathbf{C}_5(\lambda,\delta)\|f\|^2_{L^2_{\gamma/2}},\end{aligned}$$

2. *if* $-1-2s<\gamma<-2s$,

$$\langle -Q(g,f),f\rangle_v \gtrsim \mathbf{A}\Big[\mathbf{C}_1(\delta,\lambda,\eta^{-1})\big(\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|^2_{L^2_{\gamma/2}}+\|f\|^2_{H^s_{\gamma/2}}\big)-\eta\mathbf{C}_2(\delta,\lambda)\|f\|^2_{L^2_{\gamma/2+s}}$$
$$-\mathbf{C}_3(\delta,\lambda,\eta^{-1})(1+\|g\|^{\frac{(\gamma+2s+3)p}{(\gamma+2s+3)p-3}}_{L^p_{|\gamma|}})\|f\|^2_{L^2_{\gamma/2}}\Big]$$
$$+\mathbf{C}_4(\lambda,\delta)\|f\|^2_{H^s_{\gamma/2}}-\mathbf{C}_5(\lambda,\delta)(1+\|g\|^{\frac{(\gamma+2s+3)p}{(\gamma+2s+3)p-3}}_{L^p_{|\gamma|}})\|f\|^2_{L^2_{\gamma/2}},$$

with $p>\frac{3}{\gamma+2s+3}$.

REMARK 1.6. – *Here* $(-\triangle_{\mathbb{S}^2})^{s/2}$ *is the fractional Laplace-Beltrami operator. One may check the definition of the operator in* (5.18) *and* (5.21).

REMARK 1.7. – *Compared to the lower bound of the functional* $\langle \mathcal{L}_B f,f\rangle$ *(see* (1.22)*), we cannot get the control of* $\|f\|^2_{L^2_{\gamma/2+s}}$ *from the below of the functional* $\langle -Q(g,f),f\rangle_v$. *In fact, it is false to get*

$$\langle -Q(g,f),f\rangle_v \gtrsim \|f\|^2_{L^2_{\gamma/2+s}}-LOT. \tag{1.29}$$

Suppose it is true, then combining with upper bound(see Remark 1.12), we derive that

$$\|f\|^2_{L^2_{\gamma/2+s}}\lesssim \big(\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|^2_{L^2_{\gamma/2}}+\|f\|^2_{H^s_{\gamma/2}}\big).$$

It is obvious that the radial function does not verify such kind of the estimate. Then we get the contradiction. It shows on the one hand the lower bounds are sharp in anisotropic spaces. On the other hand, the behavior of the original operator is different from that of the linearized operator $\mathcal{L}_B$.

REMARK 1.8. – *We need an additional assumption* $f\in L^2_{\gamma/2+s}$ *to obtain the fractional Laplace-Beltrami derivative,* $\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|_{L^2_{\gamma/2}}$, *in the coercivity estimates. We comment that it is bad news to the Cauchy problem for the Equation* (1.1). *It means that the framework used in the close-to-equilibrium setting cannot be applied to the original equation since it will bring the trouble to close the energy estimates, in particular, in the case of* $\gamma+2s>0$. *We will explain it in detail in our forthcoming paper* [12].

As a direct consequence, now we can complete the entropy dissipation estimate as follows:

THEOREM 1.3. – *Suppose* f *is a non-negative function verifying the condition* (1.28). *Then it holds*

$$D_B(f)+\|f\|_{L^1_w}\gtrsim C(\lambda,\delta)\bigg(\|\sqrt{f}\|^2_{H^s_{\gamma/2}}+\|(-\triangle_{\mathbb{S}^2})^{s/2}\sqrt{f}\|^2_{L^2_{\gamma/2}}\bigg),$$

where $w=\max\{\gamma+2s,2\}$ *and* $D_B(f)=-\int_{\mathbb{R}^3}Q(f,f)\ln f dv$.

REMARK 1.9. – *Suppose that* f *is a solution to the spatially homogeneous Boltzmann equation with initial data* $f_0\in L^1_2\cap L\log L$. *Then we obtain that for* $\gamma+2s\le 2$, *there exists a constant* $C(f_0)$ *such that*

$$D_B(f)+\|f_0\|_{L^1_2}\gtrsim C(f_0)\bigg(\|\sqrt{f}\|^2_{H^s_{\gamma/2}}+\|(-\triangle_{\mathbb{S}^2})^{s/2}\sqrt{f}\|^2_{L^2_{\gamma/2}}\bigg).$$

REMARK 1.10. – *Compared with the entropy production estimates in* [9]*, our results do not need additional regularity assumption on f for soft potentials.*

Finally let us give the sharp bounds of the Boltzmann collision operator in anisotropic spaces:

THEOREM 1.4. – *Let $a, b \in [0, 2s], a_1, b_1, w_1, w_2 \in \mathbb{R}$ with $a + b = 2s$, $a_1 + b_1 = s$ and $w_1 + w_2 = \gamma + s$. Then for smooth functions g, h and f, it holds that*

1. *if $\gamma > 0$*

$$|\langle Q(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{\gamma+2s}} + \|g\|_{L^1_{\gamma+s+(-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2})$$
$$\times \left((\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}}) + \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{b_1}_{w_2}} \right),$$

2. *if $\gamma = 0$, for any $\delta > 0$,*

$$|\langle Q(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{2s+\delta}} + \|g\|_{L^1_{s+(-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2})$$
$$\times \left((\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f)\|_{L^2} + \|f\|_{H^b}) + \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{b_1}_{w_2}} \right),$$

3. *if $\gamma < 0$,*

$$|\langle Q(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{-\gamma+2s}} + \|g\|_{L^1_{\gamma+s+(-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2_{-\gamma}})$$
$$\times \left((\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}}) + \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{b_1}_{w_2}} \right).$$

REMARK 1.11. – *Thanks to the interpolation inequality*

$$\|h\|^2_{H^{a_1}_{w_1}} \le \|h\|_{H^{2a_1}_{\gamma/2}} \|h\|_{L^2_{2w_1-\gamma/2}},$$

take $a_1 = a/2, b_1 = b/2$ and $w_1 = \gamma/2 + a/2, w_2 = \gamma/2 + b/2$, then we have

$$\begin{aligned}|\langle Q(g,h), f\rangle_v| &\lesssim C(g)(\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}} + \|h\|_{L^2_{\gamma/2+a}}) \\ &\quad \times (\|(-\triangle_{\mathbb{S}^2})^{b/2} f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}} + \|f\|_{L^2_{\gamma/2+b}}) \\ &\sim C(g)\|(\mathscr{L}_L^{a/2} + 1)h\|_{L^2} \|(\mathscr{L}_L^{b/2} + 1)f\|_{L^2},\end{aligned}$$

where in the last equivalence we use the facts (1.21) *and* (1.22)*. In other words,* (1.23) *is proved.*

REMARK 1.12. – *By taking $a = b = s$, $a_1 = s, a_2 = 0$ $w_1 = \gamma/2, w_2 = \gamma/2 + s$, we deduce that for any $\eta > 0$,*

$$\begin{aligned}|\langle Q(g,h), f\rangle_v| &\lesssim C(g)(\|(-\triangle_{\mathbb{S}^2})^{s/2} h\|_{L^2_{\gamma/2}} + \eta^{-1}\|h\|_{H^s_{\gamma/2}}) \\ &\quad \times (\|(-\triangle_{\mathbb{S}^2})^{s/2} f)\|_{L^2_{\gamma/2}} + \|f\|_{H^s_{\gamma/2}} + \eta\|f\|_{L^2_{\gamma/2+s}}).\end{aligned}$$

Thanks to the symmetric property for functions h and f in the estimates, we also have

$$\begin{aligned}|\langle Q(g,h), f\rangle_v| &\lesssim C(g)(\|(-\triangle_{\mathbb{S}^2})^{s/2} h\|_{L^2_{\gamma/2}} + \|h\|_{H^s_{\gamma/2}} + \eta\|h\|_{L^2_{\gamma/2+s}}) \\ &\quad \times (\|(-\triangle_{\mathbb{S}^2})^{s/2} f)\|_{L^2_{\gamma/2}} + \eta^{-1}\|f\|_{H^s_{\gamma/2}}).\end{aligned}$$

We remark that both estimates improve the previous upper bounds in the two senses: the first one is that we only need to assume that one of h and f is in the space $L^2_{\gamma/2+s}$*; the second one is the free choice of the constant* η *in the estimates which enables us to prove that* (1.29) *is false.*

Thanks to the grazing collisions limit, we can extend the above estimates to the Landau collision operator.

THEOREM 1.5. – *Let* $w_1, w_2, w_3, w_4, a_1, b_1 \in \mathbb{R}, a, b \in [0,2]$ *with* $w_1 + w_2 = \gamma + 2$, $w_3 + w_4 = \gamma + 1$, $a + b = 2$ *and* $a_1 + b_1 = 1$. *Then for smooth functions* g, h *and* f, *it holds*

1. *if* $\gamma + 2 > 0$,

$$|\langle Q_L(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{\gamma+2+(-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}, \tag{1.30}$$

2. *if* $\gamma + 2 = 0$,

$$|\langle Q_L(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{w_5}} + \|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}, \tag{1.31}$$

where $w_5 = \max\{\delta, (-w_1)^+ + (-w_2)^+\}$ *with* $\delta > 0$ *which is sufficiently small,*

3. *if* $\gamma + 2 < 0$,

$$|\langle Q_L(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{w_6}} + \|g\|_{L^2_{-(\gamma+2s)}})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}, \tag{1.32}$$

where $w_6 = \max\{-(\gamma+2), \gamma + 2 + (-w_1)^+ + (-w_2)^+\}$,

4. *if* $\gamma > 0$

$$|\langle Q_L(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{\gamma+2}} + \|g\|_{L^1_{\gamma+1+(-w_3)^+ + (-w_4)^+}} + \|g\|_{L^2})$$
$$\times\Big((\|(-\triangle_{\mathbb{S}^2})^{a/2}h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}})(\|(-\triangle_{\mathbb{S}^2})^{b/2}f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}}) + \|h\|_{H^{a_1}_{w_3}}\|f\|_{H^{b_1}_{w_4}}\Big),$$

5. *if* $\gamma = 0$,

$$|\langle Q_L(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{2+\delta}} + \|g\|_{L^1_{\gamma+1+(-w_3)^+ + (-w_4)^+}} + \|g\|_{L^2})$$
$$\times\Big((\|(-\triangle_{\mathbb{S}^2})^{a/2}h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2}f)\|_{L^2} + \|f\|_{H^b}) + \|h\|_{H^{a_1}_{w_3}}\|f\|_{H^{b_1}_{w_4}}\Big),$$

6. *if* $\gamma < 0$,

$$|\langle Q_L(g,h), f\rangle_v| \lesssim (\|g\|_{L^1_{-\gamma+2}} + \|g\|_{L^1_{\gamma+1+(-w_3)^+ + (-w_4)^+}} + \|g\|_{L^2_{-\gamma}})$$
$$\times\Big((\|(-\triangle_{\mathbb{S}^2})^{a/2}h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}})(\|(-\triangle_{\mathbb{S}^2})^{b/2}f)\|_{L^2_{\gamma/2}} + \|h\|_{H^{a_1}_{w_3}}\|f\|_{H^{b_1}_{w_4}}\Big).$$

THEOREM 1.6. – *Suppose* g *is a non-negative and smooth function verifying* (1.28) *and let* $\mathbf{A} = 0, 1$. *Then for sufficiently small* $\eta > 0$, *there exist constants* $\mathbf{C}_1(\delta, \lambda, \eta^{-1})$, $\mathbf{C}_2(\lambda, \delta)$, $\mathbf{C}_3(\delta, \lambda, \eta^{-1})$, $\mathbf{C}_4(\lambda, \delta)$ *and* $\mathbf{C}_5(\lambda, \delta)$ *such that*

1. *if* $\gamma + 2 > 0$,

$$\begin{aligned}\langle -Q_L(g,f), f\rangle_v \gtrsim \mathbf{A}\Big[&\mathbf{C}_1(\delta,\lambda,\eta^{-1})\big(\|(-\triangle_{\mathbb{S}^2})^{1/2} f\|^2_{L^2_{\gamma/2}} + \|f\|^2_{H^1_{\gamma/2}}\big)\\ &- \eta\mathbf{C}_2(\delta,\lambda)\|f\|^2_{L^2_{\gamma/2+1}} - \mathbf{C}_3(\delta,\lambda,\eta^{-1})\|f\|^2_{L^2_{\gamma/2}}\Big]\\ &+ \mathbf{C}_4(\lambda,\delta)\|f\|^2_{H^1_{\gamma/2}} - \mathbf{C}_5(\delta,\lambda)\|f\|^2_{L^2_{\gamma/2}},\end{aligned}$$

2. *if* $\gamma = -2$,

$$\begin{aligned}|\langle Q_L(g,f), f\rangle_v| \gtrsim \mathbf{A}\Big[&\mathbf{C}_1(\delta,\lambda,\eta^{-1})\big(\|(-\triangle_{\mathbb{S}^2})^{1/2} f\|^2_{L^2_{\gamma/2}} + \|f\|^2_{H^1_{\gamma/2}}\big) - \eta\mathbf{C}_2(\delta,\lambda)\|f\|^2_{L^2_{\gamma/2+1}}\\ &- \exp\big\{\big(\mathbf{C}_3(\delta,\lambda,\eta^{-1})(1+w^{-1}\|g\|_{L^1_{w+2}})\big)^{\frac{2+w}{w}}\big\}\|f\|^2_{L^2_{\gamma/2}}\Big]\\ &+ \mathbf{C}_4(\lambda,\delta)\|f\|^2_{H^1_{\gamma/2}} - \exp\big\{\big(\mathbf{C}_5(\delta,\lambda)(1+w^{-1}\|g\|_{L^1_{w+2}})\big)^{\frac{2+w}{w}}\big\}\|f\|^2_{L^2_{\gamma/2}},\end{aligned}$$

where $w > 0$,

3. *if* $-3 < \gamma < -2$,

$$\begin{aligned}|\langle Q_L(g,f), f\rangle_v| \gtrsim \mathbf{A}\Big[&\mathbf{C}_1(\delta,\lambda,\eta^{-1})\big(\|(-\triangle_{\mathbb{S}^2})^{1/2} f\|^2_{L^2_{\gamma/2}} + \|f\|^2_{H^1_{\gamma/2}}\big) - \eta\mathbf{C}_2(\delta,\lambda)\|f\|^2_{L^2_{\gamma/2+1}}\\ &- \mathbf{C}_3(\delta,\lambda,\eta^{-1})(1+\|g\|^{\frac{(\gamma+5)p}{(\gamma+5)p-3}}_{L^p_{|\gamma|}})\|f\|^2_{L^2_{\gamma/2}}\Big]\\ &+ \mathbf{C}_4(\lambda,\delta)\|f\|^2_{H^1_{\gamma/2}} - \mathbf{C}_5(\delta,\lambda)(1+\|g\|^{\frac{(\gamma+5)p}{(\gamma+5)p-3}}_{L^p_{|\gamma|}})\|f\|^2_{L^2_{\gamma/2}},\end{aligned}$$

with $p > \frac{3}{\gamma+5}$.

THEOREM 1.7. – *Suppose f is a non-negative and smooth function verifying* (1.28). *Then we have*

$$D_L(f) + \|f\|_{L^1_w} \gtrsim C(\lambda,\delta)\Big(\|\sqrt{f}\|^2_{H^1_{\gamma/2}} + \|(-\triangle_{\mathbb{S}^2})^{1/2}\sqrt{f}\|^2_{L^2_{\gamma/2}}\Big),$$

where $w = \max\{\gamma+2, 2\}$ *and* $D_L(f) = -\int_{\mathbb{R}^3} Q_L(f,f)\ln f dv$.

1.3. The new strategy: dyadic and geometric decompositions

In this subsection, we will illustrate how to catch the structure of the Boltzmann collision operator to get the sharp bounds. Roughly speaking, the new strategy relies on the two types of the dyadic decomposition performed in both phase and frequency spaces and also the geometric structure of the elastic collision.

1.3.1. *Dyadic decompositions in phase and frequency spaces.* – We first introduce two types of the dyadic decomposition. Let $B_{\frac{4}{3}} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3 \mid |\xi| \le \frac{4}{3}\}$ and $C \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^3 \mid \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. Then one may introduce two radial functions $\psi \in C_0^\infty(B_{\frac{4}{3}})$ and $\varphi \in C_0^\infty(C)$ which satisfy

$$0 \le \psi, \varphi \le 0, \quad \text{and} \quad \psi(\xi) + \sum_{j\ge 0}\varphi(2^{-j}\xi) = 1, \;\; \xi \in \mathbb{R}^3. \tag{1.33}$$

The first decomposition is performed in the phase space. We introduce the dyadic operator $\mathscr{P}_j$ defined as

$$\mathscr{P}_{-1} f(x) = \psi(x) f(x), \quad \mathscr{P}_j f(x) = \varphi(2^{-j}x) f(x), \ (j \geq 0).$$

We also introduce operators $\tilde{\mathscr{P}}_j$ and $\mathscr{U}_j$ which are related to $\mathscr{P}_j$:

$$\tilde{\mathscr{P}}_j f(x) = \sum_{|k-j|\leq N_0} \mathscr{P}_k f(x) \quad \text{and} \quad \mathscr{U}_j f(x) = \sum_{k\leq j} \mathscr{P}_k f(x).$$

Here N_0 is an integer which will be chosen in the later. Then for any $f \in L^2(\mathbb{R}^3)$, it holds

$$f = \mathscr{P}_{-1} f + \sum_{j\geq 0} \mathscr{P}_j f.$$

We set

$$\Phi_k^\gamma(v) \stackrel{\text{def}}{=} \begin{cases} |v|^\gamma \varphi(2^{-k}|v|), & \text{if} \quad k \geq 0; \\ |v|^\gamma \psi(|v|), & \text{if} \quad k = -1. \end{cases} \tag{1.34}$$

Then we derive that

$$\langle Q(g,h), f\rangle_v = \sum_{k=-1}^{\infty} \langle Q_k(g,h), f\rangle_v = \sum_{k=-1}^{\infty} \sum_{j=-1}^{\infty} \langle Q_k(\mathscr{P}_j g, h), f\rangle_v,$$

where

$$Q_k(g,h) = \iint_{\sigma\in\mathbb{S}^2, v_*\in\mathbb{R}^3} \Phi_k^\gamma(|v-v_*|) b(\cos\theta)(g'_* h' - g_* h) d\sigma dv_*.$$

The second decomposition is performed in the frequency space. In fact, it is the standard Littlewood-Paley theory. We denote $\tilde{m} \stackrel{\text{def}}{=} \mathscr{F}^{-1}\psi$ and $\phi \stackrel{\text{def}}{=} \mathscr{F}^{-1}\varphi$, where they are the inverse Fourier Transform of φ and ψ. If we set $\phi_j(x) \stackrel{\text{def}}{=} 2^{3j}\phi(2^j x)$, then the dyadic operators $\mathfrak{F}_j$ can be defined as follows

$$\mathfrak{F}_{-1} f(x) = \int_{\mathbf{R}^3} \tilde{m}(x-y) f(y) dy, \quad \mathfrak{F}_j f(x) = \int_{\mathbf{R}^3} \phi_j(x-y) f(y) dy, \ (j \geq 0).$$

We also introduce operators $\tilde{\mathfrak{F}}_j$ and $\mathscr{S}_j$ which are related to $\mathfrak{F}_j$:

$$\tilde{\mathfrak{F}}_j f(x) = \sum_{|k-j|\leq 3N_0} \mathfrak{F}_k f(x) \ \text{ and } \ \mathscr{S}_j f(x) = \sum_{k\leq j} \mathfrak{F}_k f.$$

Then for any $f \in \mathbf{S}'(\mathbb{R}^3)$, it holds

$$f = \mathfrak{F}_{-1} f + \sum_{j\geq 0} \mathfrak{F}_j f.$$

It yields that

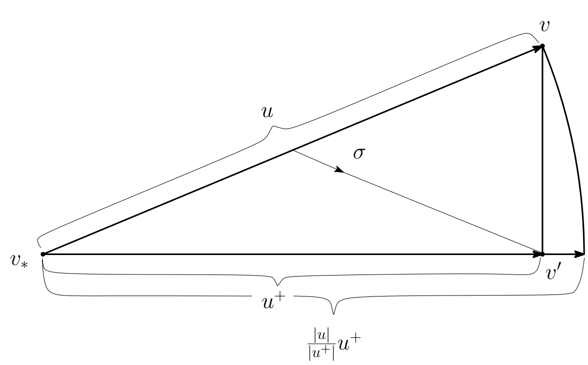
$$\langle Q_k(g,h), f\rangle_v = \sum_{p=-1}^{\infty} \sum_{l=-1}^{\infty} \langle Q_k(\mathfrak{F}_p g, \mathfrak{F}_l h), f\rangle_v.$$

Let us give some remarks on these two decompositions:

1. The main purpose of the introduction of the dyadic decomposition in the frequency space is to make full use of the interaction and the cancelation between the different parts of frequency of functions h and f. It will enable us to have the freedom of choosing derivatives for functions h and f. However, it is not enough to get the sharp bounds considering the fact that additional weight is paid in the upper bound compared to that in the lower bound. It is obviously due to the anisotropic structure of the operator.
2. To clarify where the additional weight comes from, we introduce the dyadic decomposition in the phase space. By careful analysis, we can distinguish which part of the operator is the worst term that brings the additional weight to h and f. In fact, the worst situation happens in the case where functions h and f are localized in the same region both in phase and frequency spaces and at the same time the relative velocity $|v - v_*|$ is far away from the zero. In other words, in such a situation the collision operator is dominated by the anisotropic structure. It is the key point to obtain the estimate (1.23) in anisotropic spaces.
3. These two dyadic decompositions are consistent with the new profiles of the weighted Sobolev spaces(see Theorem 5.1 in Section 5).
4. The decompositions are very stable in the grazing collision limit. In fact, we can apply them to the rescaled Boltzmann operator to get the upper bounds. By taking the grazing collision limit, these upper bounds turn to be the sharp upper bounds of the Landau operator in weighted Sobolev spaces. The reader may check details in Section 4.

1.3.2. *Key observation: geometric decomposition.* – The key observation which enables us to get the new sharp bounds for the original collision operator is due to the geometric structure of the elastic collision. To explain it clearly, in what follows, we only focus on the Maxwellian molecular case($\gamma = 0$).

We revisit the quantity $\mathcal{E}_g(f)$. In particular, we look for a new decomposition for the term $f' - f$ contained in $\mathcal{E}_g(f)$. Our main observation can be depicted schematically as follows:



We first note that $v' = v_* + u^+$ and $v = v_* + u$, where $u = v - v_*$ and $u^+ = \frac{u+|u|\sigma}{2}$. Now assuming $u = r\tau$ with $r = |u|$ and $\tau \in \mathbb{S}^2$, we obtain that

$$v = v_* + r\tau, v' = v_* + r\frac{\tau + \sigma}{2}.$$

Let $\varsigma=\frac{\tau+\sigma}{|\tau+\sigma|}\in\mathbb{S}^2$. Then we have the geometric decomposition:

$$\begin{aligned} f(v')-f(v) &= \big(f(v_*+\frac{|\tau+\sigma|}{2}r\varsigma)-f(v_*+r\varsigma)\big)+\big(f(v_*+r\varsigma)-f(v_*+r\tau)\big)\\ &= \big(f(v_*+u^+)-f(v_*+|u|\frac{u^+}{|u^+|})\big)+\big((T_{v_*}f)(r\varsigma)-(T_{v_*}f)(r\tau)\big), \end{aligned} \tag{1.35}$$

where $T_v f\overset{\text{def}}{=}f(v+\cdot)$. Applying it to the functional $\langle Q(g,h),f\rangle$, we have

$$\begin{aligned} \langle Q(g,h),f\rangle =& \underbrace{\iint_{\sigma\in\mathbb{S}^2,u,v_*\in\mathbb{R}^3} b(\cos\theta)g_*(T_{v_*}h)(u)\big(f(v_*+u^+)-f(v_*+|u|\frac{u^+}{|u^+|})\big)d\sigma dv_* du}_{=\langle \mathcal{Q}_g h,f\rangle}\\ &+\underbrace{\iint_{\sigma\in\mathbb{S}^2,u,v_*\in\mathbb{R}^3} b(\cos\theta)g_*(T_{v_*}h)(u)\big((T_{v_*}f)(r\varsigma)-(T_{v_*}f)(r\tau)\big)d\sigma dv_* du}_{=\langle \mathcal{G}_g h,f\rangle_v}. \end{aligned}$$

Notice that $|u^+-|u|\frac{u^+}{|u^+|}|\sim\theta^2|u|$. Then by technical argument(see the proof of Theorem 1.2), the operator $\mathcal{Q}_g$ behaves like

$$\mathcal{Q}_g\sim C_g\langle v\rangle^s(-\triangle_v)^{s/2}.$$

Recalling the rough behavior of $Q(g,\cdot)$ (see (1.13)), we may regard $\mathcal{Q}_g$ as the lower order term.

Now we concentrate on the functional $\langle\mathcal{G}_g h,f\rangle_v$. By (1.3), it is easy to check

$$\begin{aligned} &\langle \mathcal{G}_g h,f\rangle_v\\ &=\iint_{r>0,\sigma,\tau\in\mathbb{S}^2,v_*\in\mathbb{R}^3} b(\sigma\cdot\tau)1_{\sigma\cdot\tau\ge0}g_*(T_{v_*}h)(r\tau)\big((T_{v_*}f)(r\varsigma)-(T_{v_*}f)(r\tau)\big)r^2 d\sigma dv_* d\tau dr. \end{aligned}$$

For fixed v_*, τ and r, if τ is chosen to be the polar direction, one has

$$d\sigma=\sin\theta d\theta d\,\mathbb{S}^1,\, d\varsigma=\sin\phi d\phi d\,\mathbb{S}^1,$$

where $\theta=2\phi$. We deduce that

$$d\sigma=4\cos\phi d\varsigma.$$

Then by change of the variable from σ to ς, we derive that

$$\begin{aligned} \langle \mathcal{G}_g h,f\rangle_v =& \iint_{r>0,\varsigma,\tau\in\mathbb{S}^2,v_*\in\mathbb{R}^3} b\big(2(\varsigma\cdot\tau)^2-1\big)1_{\varsigma\cdot\tau\ge\sqrt{2}/2}4(\varsigma\cdot\tau)g_*(T_{v_*}h)(r\tau)\\ &\times\big((T_{v_*}f)(r\varsigma)-(T_{v_*}f)(r\tau)\big)r^2 d\varsigma d\tau dr dv_*. \end{aligned}\tag{1.36}$$

By the symmetric property of τ and ς, we get

$$\begin{aligned} \langle \mathcal{G}_g h,f\rangle_v =& \frac12\iint_{r>0,\varsigma,\tau\in\mathbb{S}^2,v_*\in\mathbb{R}^3} b\big(2(\varsigma\cdot\tau)^2-1\big)1_{\varsigma\cdot\tau\ge\sqrt{2}/2}4(\varsigma\cdot\tau)g_*\big((T_{v_*}h)(r\tau)-(T_{v_*}h)(r\varsigma)\big)\\ &\times\big((T_{v_*}f)(r\varsigma)-(T_{v_*}f)(r\tau)\big)r^2 d\varsigma d\tau dr dv_*. \end{aligned}$$

Thus to give the estimate of $\mathcal{G}_g$, it suffices to consider the functional

$$\mathfrak{G}(h,f)\overset{\text{def}}{=}\int_{\varsigma,\tau\in\mathbb{S}^2}\big(h(\tau)-h(\varsigma)\big)\big(f(\varsigma)-f(\tau)\big)H(\varsigma\cdot\tau)d\varsigma d\tau,$$

where $H(\varsigma\cdot\tau) = b\big(2(\varsigma\cdot\tau)^2-1\big)4(\varsigma\cdot\tau)1_{\varsigma\cdot\tau\geq\sqrt{2}/2}$. Recall that the assumption (1.5) is equivalent to $b(\sigma\cdot\tau)\sim|\sigma-\tau|^{-2-2s}$, then by the fact $|\sigma-\tau|\sim|\varsigma-\tau|$, we get

$$H(\varsigma\cdot\tau)\sim|\varsigma-\tau|^{-2-2s}1_{|\varsigma-\tau|^2\leq 2-\sqrt{2}}.$$

Let us first consider the lower bound of the operator. Thanks to Lemma 5.5, we get

$$\begin{aligned}(1.37)\qquad -\mathfrak{G}(f,f)+\|f\|^2_{L^2(\mathbb{S}^2)} &\sim \int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}d\sigma d\tau+\|f\|^2_{L^2(\mathbb{S}^2)}\\ &\sim \|(-\triangle_{\mathbb{S}^2})^{s/2}f\|_{L^2(\mathbb{S}^2)}+\|f\|^2_{L^2(\mathbb{S}^2)}.\end{aligned}$$

Then we have

$$\langle-\mathcal{G}_g f,f\rangle_v\gtrsim\int_{\mathbb{R}^3}g_*\|(-\triangle_{\mathbb{S}^2})^{s/2}T_{v_*}f\|^2_{L^2}dv_*-LOT,$$

which almost yields

$$\langle-Q(g,f),f\rangle_v\gtrsim C_g\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|^2_{L^2}-LOT.$$

Now what remains is to prove

$$(1.38)\qquad \|(-\triangle_{\mathbb{S}^2})^{s/2}T_{v_*}f\|_{L^2}\lesssim\langle v_*\rangle^s(\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|_{L^2}+\|f\|_{H^s}).$$

It is easy to check that it holds for $s=0$ and $s=1$. Unfortunately because $-\triangle_{\mathbb{S}^2}$ does not commutate with ∇, the standard real interpolation method cannot be applied directly. To solve the problem, we develop a new interpolation theory(which is of independent interest) to overcome the difficulty. We refer readers to check the theory in Section 5. Then by combining the coercivity estimate in Sobolev spaces (see (1.15)), roughly speaking, we derive that

$$\langle-Q(g,f),f\rangle_v\gtrsim C_g(\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|^2_{L^2}+\|f\|^2_{H^s})-LOT.$$

We remark that the estimate is sharp since it is consistent with the behavior of the linearized operator (1.19).

Next we turn to the upper bounds. We will show that our new observation can be used to prove (1.23). Indeed, by Cauchy-Schwarz inequality, we have

$$(1.39)\quad |\mathfrak{G}(h,f)|\lesssim\Big(\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|h(\sigma)-h(\tau)|^2}{|\sigma-\tau|^{2+2s}}d\sigma d\tau\Big)^{\frac12}\Big(\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}d\sigma d\tau\Big)^{\frac12}.$$

Thanks to the Addition Theorem(see the statement in Section 5), we deduce that for constants $a,b\in\mathbb{R}$ with $a+b=2s$,

$$|\mathfrak{G}(h,f)|\lesssim\|(1-\triangle_{\mathbb{S}^2})^{a/2}h\|_{L^2}\|(1-\triangle_{\mathbb{S}^2})^{b/2}f\|_{L^2},$$

which implies

$$\begin{aligned}&\langle-\mathcal{G}_g h,f\rangle_v\\ &\lesssim\Big(\int_{\mathbb{R}^3}g_*\|(-\triangle_{\mathbb{S}^2})^{a/2}T_{v_*}h\|^2_{L^2}dv_*\Big)^{\frac12}\Big(\int_{\mathbb{R}^3}g_*\|(-\triangle_{\mathbb{S}^2})^{b/2}T_{v_*}f\|^2_{L^2}dv_*\Big)^{\frac12}+LOT.\end{aligned}$$

Together with the sharp upper bounds in weighted Sobolev spaces and (1.38), we finally arrive at (1.23).

Some remarks are in order:

1. The geometric decomposition plays an essential role to catch the anisotropic structure of the operator in the lower and upper bounds. Since it does not use the symmetry and regularity of the function g, it is more robust than the previous work.
2. The geometric decomposition is stable in the process of the grazing collision limit. Actually we can give an explicit description of the asymptotic behavior of the anisotropic structure in the limit (see Lemma 4.1). Roughly speaking, in the process of the limit, the behavior of collision operator depends on the parameter ϵ. If the eigenvalue of $-\triangle_{\mathbb{S}^2}$ is less than ϵ^{-2}, then the operator behaves like $-\triangle_{\mathbb{S}^2}$. And if the eigenvalue of $-\triangle_{\mathbb{S}^2}$ is bigger than ϵ^{-2}, then the operator behaves like $\epsilon^{2s-2}(-\triangle_{\mathbb{S}^2})^s$. It reflects the strong connection between Boltzmann and Landau collision operators.
3. We remark that the geometric decomposition can also be applied to the operator in frequency space (recalling (2.2)) to catch the anisotropic structure. It will give an alternative proof to the lower and upper bounds of the operator in anisotropic spaces. But it only works well for the Maxwellian case ($\gamma = 0$) because of the simplicity of (2.2) in this case.

1.4. Organization of the paper

In Section 2, based on two types of the dyadic decomposition performed both in phase and frequency spaces, we give a complete proof to the sharp bounds of the collision operator in weighted Sobolev spaces.

Based on the geometric decomposition, we give a proof to the sharp bounds of the collision operator in anisotropic spaces in Section 3.

In Section 4, we show that the strategy is so stable that we can extend all the estimates to the Landau collision operator by taking the grazing collision limit. We also show the asymptotic behavior of the anisotropic structure of collision operator in the process of the grazing collision limit.

In Section 5, we list some important lemmas which are of independent interest to the proof of the main theorems. We first give some auxiliary lemmas on the new profiles of the weighted Sobolev Spaces, the new version of the interpolation theory and the basic properties of the real spherical harmonics. Then in the next we give the L^2 profile of the fractional Laplace-Beltrami operator. Finally we give the proof to (1.38).

At the end of the paper, we give the conclusions and perspectives.

2. Upper bound for the Boltzmann collision operator in weighted Sobolev Spaces

In this section, we will make full use of dyadic decompositions which are performed in both phase and frequency spaces to give the precise estimates of the collision operator in weighted Sobolev spaces. Of course these estimates are not optimal. But they are still interesting. In fact, they have two advantages. The first one is that we have the freedom of choosing derivatives and weights for the functions compared to the previous work. The second one is that the two decompositions used in the proof enable us to find out where the additional weight comes from in the upper bound. It will be crucial to improve the upper bound of the operator in anisotropic spaces.

We first show how to reduce the estimate of the functional $\langle Q(g,h), f\rangle_v$ with the aid of the two types of the decomposition. For Q_k, relative velocity $v-v_*$ is localized in the ring $\{\frac{3}{4}2^k \le |v-v_*| \le \frac{8}{3}2^k\}$. Suppose that g is localized in the ring $\{\frac{3}{4}2^j \le |v_*| \le \frac{8}{3}2^j\}$. Then thanks to the fact $\frac{\sqrt{2}}{2}|v-v_*| \le |v'-v_*| \le |v-v_*|$, we have

- If $j \le k-N_0$, then $|v|, |v'| \in [(\frac{3}{4}-\frac{8}{3}2^{-N_0})2^k, \frac{8}{3}(1+2^{-N_0})2^k]$;
- If $j \ge k+N_0$, then $|v| \in [(\frac{3}{4}-\frac{8}{3}2^{-N_0})2^j, \frac{8}{3}(1+2^{-N_0})2^j]$ and $|v'| \in [(\frac{\sqrt{2}}{2}\frac{3}{4}-\frac{8}{3}2^{-N_0})2^j, \frac{8}{3}(1+2^{-N_0})2^j]$;
- If $|j-k| < N_0$, then $|v| \le 22^{k+N_0}, |v'| \le 22^{k+N_0}$.

Then, together with the fact that

$$\langle Q_k(g,h), f\rangle_v = \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \Phi_k^\gamma(|v-v_*|) b(\cos\theta) g_* h (f'-f) d\sigma dv_* dv,$$

we deduce that there exists an integer $N_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\langle Q(g,h), f\rangle_v &= \sum_{k\ge -1}\sum_{j\ge -1} \langle Q_k(\mathcal{P}_j g, h), f\rangle_v \\
&= \sum_{j\le k-N_0} \langle Q_k(\mathcal{P}_j g, \tilde{\mathcal{P}}_k h), \tilde{\mathcal{P}}_k f\rangle_v + \sum_{j\ge k+N_0} \langle Q_k(\mathcal{P}_j g, \tilde{\mathcal{P}}_j h), \tilde{\mathcal{P}}_j f\rangle_v \\
&\quad + \sum_{|j-k|\le N_0} \langle Q_k(\mathcal{P}_j g, \mathcal{U}_{k+N_0} h), \mathcal{U}_{k+N_0} f\rangle_v \qquad (2.1)\\
&= \sum_{k\ge N_0-1} \langle Q_k(\mathcal{U}_{k-N_0} g, \tilde{\mathcal{P}}_k h), \tilde{\mathcal{P}}_k f\rangle_v + \sum_{j\ge k+N_0} \langle Q_k(\mathcal{P}_j g, \tilde{\mathcal{P}}_j h), \tilde{\mathcal{P}}_j f\rangle_v \\
&\quad + \sum_{|j-k|\le N_0} \langle Q_k(\mathcal{P}_j g, \mathcal{U}_{k+N_0} h), \mathcal{U}_{k+N_0} f\rangle_v.
\end{aligned}$$

We recall that the Bobylev's formula of the operator can be stated as

$$\begin{aligned}
(2.2)\quad &\langle \mathcal{F}(Q_k(g,h)), \mathcal{F} f\rangle \\
&= \iint_{\sigma\in\mathbb{S}^2, \eta_*, \xi\in\mathbb{R}^3} b(\frac{\xi}{|\xi|}\cdot\sigma)\big[\mathcal{F}(\Phi_k^\gamma)(\eta_*-\xi^-) - \mathcal{F}(\Phi_k^\gamma)(\eta_*)\big] \\
&\qquad\qquad \times (\mathcal{F} g)(\eta_*)(\mathcal{F} h)(\xi-\eta_*)\overline{(\mathcal{F} f)}(\xi) d\sigma d\eta_* d\xi,
\end{aligned}$$

where $\mathcal{F} f$ denotes the Fourier transform of f and $\xi^- = \frac{\xi-|\xi|\sigma}{2}$. Suppose that functions g and h are localized in rings $\{\frac{3}{4}2^p \le |\xi| \le \frac{8}{3}2^p\}$ and $\{\frac{3}{4}2^l \le |\xi| \le \frac{8}{3}2^l\}$ respectively in the frequency space. Due to (2.2), we have $\frac{3}{4}2^p \le |\eta_*| \le \frac{8}{3}2^p$ and $\frac{3}{4}2^l \le |\xi-\eta_*| \le \frac{8}{3}2^l$. Then

- If $l \le p-N_0$, then $|\xi| \in [(\frac{3}{4}-\frac{8}{3}2^{-N_0})2^p, \frac{8}{3}(1+2^{-N_0})2^p]$. Set $\xi^+ \stackrel{\text{def}}{=} \xi-\xi^-$, then it verifies $|\xi^+| \in [\frac{\sqrt{2}}{2}(\frac{3}{4}-\frac{8}{3}2^{-N_0})2^p, \frac{8}{3}(1+2^{-N_0})2^p]$. Notice that $|\eta_*-\xi^-| = |(\eta_*-\xi)+\xi^+|$, then one has $|\eta_*-\xi^-| \in [\frac{\sqrt{2}}{2}(\frac{3}{4}-(1+\sqrt{2})\frac{8}{3}2^{-N_0})2^p, \frac{8}{3}(1+22^{-N_0})2^p]$.
- If $l \ge p+N_0$, then $|\xi| \in [(\frac{3}{4}-\frac{8}{3}2^{-N_0})2^l, \frac{8}{3}(1+2^{-N_0})2^l]$ and $|\eta_*-\xi^-|, |\eta_*| \le (1+2\frac{8}{3}2^{-N_0})2^l$.
- If $|l-p| < N_0$, then $|\xi| \le 2\frac{8}{3}2^{p+N_0}$. Now let $|\xi| \in [\frac{3}{4}2^m, \frac{8}{3}2^m]$. In the case of $|m-p| \le 2N_0$, one has $|\xi| \in [2^{-2N_0}\frac{3}{4}2^p, 2^{2N_0}\frac{8}{3}2^p]$. While in the case of $m < p-2N_0$, one has $|\eta_*|, |\eta_*-\xi^-| \in [(2^{-N_0}\frac{3}{4}-\frac{8}{3}2^{-2N_0})2^p, (\frac{8}{3}2^{2N_0}+\frac{8}{3}2^{-2N_0})2^p]$.

Then we have

$$\begin{aligned}\langle Q_k(g,h), f\rangle_v &= \sum_{p,l\geq -1} \langle Q_k(\mathfrak{F}_p g, \mathfrak{F}_l h), f\rangle_v \\ &= \sum_{l\leq p-N_0} \mathfrak{W}^1_{k,p,l} + \sum_{l\geq p+N_0} \mathfrak{W}^2_{k,p,l} \\ &\quad + \sum_{|l-p|<N_0} \Big(\sum_{|m-p|\leq 2N_0} \mathfrak{W}^3_{k,p,l,m} + \sum_{m<p-2N_0} \mathfrak{W}^4_{k,p,l,m}\Big),\end{aligned}$$

where

$$\mathfrak{W}^1_{k,p,l} \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \big(\tilde{\mathfrak{F}}_p \Phi^\gamma_k\big)(|v-v_*|) b(\cos\theta) (\mathfrak{F}_p g)_* (\mathfrak{F}_l h)\big[(\tilde{\mathfrak{F}}_p f)' - \tilde{\mathfrak{F}}_p f\big] d\sigma dv_* dv,$$

$$\mathfrak{W}^2_{k,p,l} \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \Phi^\gamma_k(|v-v_*|) b(\cos\theta) (\mathfrak{F}_p g)_* (\mathfrak{F}_l h)\big[(\tilde{\mathfrak{F}}_l f)' - \tilde{\mathfrak{F}}_l f\big] d\sigma dv_* dv,$$

$$\mathfrak{W}^3_{k,p,l,m} \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \Phi^\gamma_k(|v-v_*|) b(\cos\theta) (\mathfrak{F}_p g)_* (\mathfrak{F}_l h)\big[(\mathfrak{F}_m f)' - \mathfrak{F}_m f\big] d\sigma dv_* dv,$$

$$\mathfrak{W}^4_{k,p,l,m} \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \big(\tilde{\mathfrak{F}}_p \Phi^\gamma_k\big)(|v-v_*|) b(\cos\theta) (\mathfrak{F}_p g)_* (\mathfrak{F}_l h)\big[(\mathfrak{F}_m f)' - \mathfrak{F}_m f\big] d\sigma dv_* dv.$$

We remark that we use the fact that the Fourier transform maps a radial function into a radial function.

By simple calculation, we arrive at

$$\langle Q_k(g,h), f\rangle_v = \sum_{l\leq p-N_0} \mathfrak{W}^1_{k,p,l} + \sum_{l\geq -1} \mathfrak{W}^2_{k,l} + \sum_{p\geq -1} \mathfrak{W}^3_{k,p} + \sum_{m<p-N_0} \mathfrak{W}^4_{k,p,m}, \tag{2.3}$$

where

$$\mathfrak{W}^2_{k,l} \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \Phi^\gamma_k(|v-v_*|) b(\cos\theta) (\mathcal{S}_{l-N_0} g)_* (\mathfrak{F}_l h)\big[(\tilde{\mathfrak{F}}_l f)' - \tilde{\mathfrak{F}}_l f\big] d\sigma dv_* dv,$$

$$\mathfrak{W}^3_{k,p} \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \Phi^\gamma_k(|v-v_*|) b(\cos\theta) (\mathfrak{F}_p g)_* (\tilde{\mathfrak{F}}_p h)\big[(\tilde{\mathfrak{F}}_p f)' - \tilde{\mathfrak{F}}_p f\big] d\sigma dv_* dv,$$

$$\mathfrak{W}^4_{k,p,m} \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \big(\tilde{\mathfrak{F}}_p \Phi^\gamma_k\big)(|v-v_*|) b(\cos\theta) (\mathfrak{F}_p g)_* (\tilde{\mathfrak{F}}_p h)\big[(\mathfrak{F}_m f)' - \mathfrak{F}_m f\big] d\sigma dv_* dv.$$

Now the estimate of the functional $\langle Q(g,h), f\rangle_v$ is reduced to the estimates of the terms in the righthand sides of (2.1) and (2.3).

We begin with two useful propositions.

PROPOSITION 2.1. – *Suppose* $\varpi \in (0,1]$. *Then for* $|y| \neq 0$, *one has*

$$\big||x|^{2\varpi} - |y|^{2\varpi}\big| \lesssim \begin{cases} |x-y||y|^{2\varpi-1}, & \text{if} \quad 0<\varpi\leq \frac{1}{2}; \\ |x-y||y|^{2\varpi-1} + |x-y|^{2\varpi}, & \text{if} \quad \frac{1}{2}<\varpi\leq 1. \end{cases}$$

Proof. – We first treat the case $2\varpi \leq 1$. Suppose that $|x| \leq b|y|$ with $0<b<1$. Then one has

$$(1-b)|y| \leq |x-y| \leq (1+b)|y|,$$

which implies

$$\big||x|^{2\varpi}-|y|^{2\varpi}\big|\lesssim |y|^{2\varpi}\lesssim |x-y||y|^{2\varpi-1}.$$

Next we handle the case $|x|>b|y|$. If $\theta\in[0,1]$, we have

$$(1-\theta)|x|+\theta|y|\geq[(1-\theta)b+\theta]|y|.$$

From which together with the fact

$$\big||x|^{2\varpi}-|y|^{2\varpi}\big|\lesssim |x-y|\int_0^1[(1-\theta)|x|+\theta|y|]^{2\varpi-1}d\theta, \tag{2.4}$$

we get the desired result.

When $2\varpi>1$, the proposition is easily followed from (2.4) and the fact

$$(1-\theta)|x|+\theta|y|\leq|y|+|x-y|. \qquad \square$$

PROPOSITION 2.2. – *Suppose* $\varpi\in(0,1]$ *and* $N\in\mathbb{N}$. *Recall that* Φ_k^γ *is defined in* (1.34).

1. *Set* $A_k^\varpi(v)\overset{\text{def}}{=}(\tilde{\mathfrak{F}}_p\Phi_k^\gamma)(v)|v|^{2\varpi}$. *Then if* $k\geq0$, *we have*

$$\|A_k^\varpi\|_{L^\infty}\lesssim 2^{k(\gamma+\frac32-N)}2^{-pN}\|\Phi_0^\gamma\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}.$$

If $\gamma=0$ *and* $k=-1$,

$$\|A_{-1}^\varpi\|_{L^\infty}\lesssim 2^{-pN}\|\psi\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}.$$

2. *Set* $B_{-1}^\varpi(v)\overset{\text{def}}{=}(\tilde{\mathfrak{F}}_p\Phi_{-1}^\gamma)(v)|v|^{2\varpi}-(\tilde{\mathfrak{F}}_p\Phi_{-1}^{\gamma+2\varpi})(v)$. *Then if* $\gamma+2\varpi>0$, *there exists a constant* $\eta_1=\min\{\gamma+2\varpi,1\}$ *such that*

$$|B_{-1}^\varpi|\leq|B_1|+|B_2|,$$

where

$$\|B_1\|_{L^2}\leq 2^{-(\eta_1+\frac32)p}\quad\text{and}\quad\|B_2\|_{L^\infty}\lesssim 2^{-\eta_1 p}.$$

If $\gamma+2\varpi>-1$, *then there exists a constant* η_2 *such that*

$$\|B_{-1}^\varpi\|_{L^2}\lesssim 2^{-(\eta_2+\frac12)p},$$

where

$$\eta_2=\begin{cases}\dfrac12, & \text{if}\quad \gamma+2\varpi>-\dfrac12;\\[2mm] \dfrac12-(\log_2 p)/p, & \text{if}\quad \gamma+2\varpi=-\dfrac12;\\[2mm] \gamma+2\varpi+1, & \text{if}\quad -1<\gamma+2\varpi<-\dfrac12.\end{cases}$$

Proof. – (i). For $k\geq0$, we recall that $\Phi_k^\gamma(v)=|v|^\gamma\varphi(2^{-k}v)$. Then by direct calculation, we have

$$\mathscr{F}(\Phi_k^\gamma)(\xi)=2^{(\gamma+3)k}\mathscr{F}(\Phi_0^\gamma)(2^k\xi), \tag{2.5}$$

which yields

$$\begin{aligned}\|\tilde{\mathfrak{F}}_p\Phi_k^\gamma\|_{L_2^\infty}&\lesssim\|(-\triangle)\mathscr{F}\big(\tilde{\mathfrak{F}}_p\Phi_k^\gamma\big)\|_{L^1}+\|\mathscr{F}(\tilde{\mathfrak{F}}_p\Phi_k^\gamma)\|_{L^1}\lesssim\|\mathscr{F}(\tilde{\mathfrak{F}}_p\Phi_k^\gamma)\|_{H_2^2}\\&\lesssim 2^{k(\gamma+\frac32-N)}2^{-pN}\|\Phi_0^\gamma\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}.\end{aligned}$$

From this we obtain that

$$\|A_k^{\varpi}\|_{L^\infty} \lesssim 2^{k(\gamma+\frac{3}{2}-N)} 2^{-pN} \|\Phi_0^{\gamma}\|_{H^{N+2}} \|\varphi\|_{W_N^{2,\infty}}.$$

If $\gamma = 0$ and $k = -1$, by the definition, we have

$$\|A_{-1}^{\varpi}\|_{L^\infty} \lesssim \|(-\triangle)\mathscr{F}(\tilde{\mathfrak{F}}_p\psi)\|_{L^1} + \|\mathscr{F}(\mathfrak{F}_p\psi)\|_{L^1} \lesssim 2^{-pN}\|\psi\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}.$$

(ii). Let $\tilde{\phi}_p = \sum_{|m-p|\le N_0} \phi_m$. Then by the definition of $\tilde{\mathfrak{F}}_p$, we have

$$|B_{-1}^{\varpi}(v)| = \left|\int_{\mathbb{R}^3} \tilde{\phi}_p(v-y)\Phi_{-1}^{\gamma}(y)(|v|^{2\varpi} - |y|^{2\varpi})dy\right|.$$

Thanks to Proposition 2.1, it can be reduced to bound the terms B_{-1}^1 and B_{-1}^2 which are defined by

$$B_{-1}^1 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\tilde{\phi}_p(v-y)|\Phi_{-1}^{\gamma+2\varpi-1}(y)|v-y|dy,$$
$$\text{and} \quad B_{-1}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\tilde{\phi}_p(v-y)|\Phi_{-1}^{\gamma}(y)|v-y|^{2\varpi}dy.$$

We remind the reader that the term B_{-1}^2 is only needed to be considered in the case of $2\varpi > 1$.

We begin with the estimate of B_{-1}^1. Observe that

$$\begin{aligned} B_{-1}^1 &= \int_{\mathbb{R}^3} |\tilde{\phi}_p(v-y)|\Phi_{-1}^{\gamma+2\varpi-1}(y)1_{|y|\le 2^{-p}}|v-y|dy \\ &\quad + \int_{\mathbb{R}^3} |\tilde{\phi}_p(v-y)|\Phi_{-1}^{\gamma+2\varpi-1}(y)1_{|y|\ge 2^{-p}}|v-y|dy \\ &\stackrel{\text{def}}{=} B_{-1}^{1,1} + B_{-1}^{1,2}. \end{aligned}$$

It is easy to check that

$$\|B_{-1}^{1,1}\|_{L^2} \lesssim \left(\int_{\mathbb{R}^3} |\tilde{\phi}_p|^2|y|^2dy\right)^{1/2} 2^{-(\gamma+2\varpi-1+3)p} \lesssim 2^{-(\gamma+2\varpi+\frac{3}{2})p}.$$

We turn to the estimate of $B_{-1}^{1,2}$. To make full use of the structure, we separate the estimate into several cases.

- Case 1: $\gamma + 2\varpi > 0$. Then it holds

  $$\|B_{-1}^{1,2}\|_{L^\infty} \lesssim 2^{-\eta p},$$

  where $\eta = \min\{1, \gamma+2\varpi\}$. Nevertheless, by Young inequality, we also have

  $$\|B_{-1}^{1,2}\|_{L^2} \lesssim 2^{-p}.$$

- Case 2: $-\frac{1}{2} \le \gamma + 2\varpi \le 0$. Then by Young inequality, one has that if $\gamma + 2s > -\frac{1}{2}$,

  $$\|B_{-1}^{1,2}\|_{L^2} \lesssim 2^{-p}.$$

  However, if $\gamma + 2s = -\frac{1}{2}$, we have

  $$\|B_{-1}^{1,2}\|_{L^2} \lesssim 2^{-p}p \lesssim 2^{-p(1-(\log_2 p)/p)}.$$

- Case 3: $-1 < \gamma + 2\varpi < -\frac{1}{2}$. We have

$$\|B_{-1}^{1,2}\|_{L^2} \lesssim 2^{-p}\Big(\int_{2^{-p}\le|y|\le 2} |y|^{2(\gamma+2\varpi-1)}dy\Big)^{\frac{1}{2}} \lesssim 2^{-(\gamma+2\varpi+\frac{3}{2})p}.$$

The similar argument can be applied to B_{-1}^{2} with $2\varpi > 1$. Notice that

$$\begin{aligned} B_{-1}^{2} &= \int_{\mathbb{R}^3} |\tilde{\phi}_p(v-y)|\Phi_{-1}^{\gamma}(y)1_{|y|\le 2^{-p}}|v-y|^{2\varpi}dy \\ &+ \int_{\mathbb{R}^3} |\tilde{\phi}_p(v-y)|\Phi_{-1}^{\gamma}(y)1_{|y|\ge 2^{-p}}|v-y|^{2\varpi}dy \\ &\stackrel{\text{def}}{=} B_{-1}^{2,1} + B_{-1}^{2,2}. \end{aligned}$$

It is easy to check that

$$\|B_{-1}^{2,1}\|_{L^2} \lesssim \Big(\int |\tilde{\phi}_p|^2|y|^{4\varpi}dy\Big)^{1/2} 2^{-(\gamma+3)p} \lesssim 2^{-(\gamma+2\varpi+\frac{3}{2})p}.$$

Similarly, the estimate of $B_{-1}^{2,2}$ falls in several cases.

- Case 1: $\gamma \ge 0$. Then it holds

$$\|B_{-1}^{2,2}\|_{L^\infty} \lesssim \int_{\mathbb{R}^3} |\tilde{\phi}_p||y|^{2\varpi}dy \lesssim 2^{-2\varpi p}.$$

- Case 2: $\gamma + 2\varpi > 0$ and $\gamma < 0$. Using the fact that $|\Phi_{-1}^{\gamma}(y)1_{|y|\ge 2^{-p}}| \le 2^{-\gamma p}$, then one has

$$\|B_{-1}^{2,2}\|_{L^\infty} \lesssim 2^{-\gamma p}\int_{\mathbb{R}^3} |\tilde{\phi}_p||y|^{2\varpi}dy \lesssim 2^{-(\gamma+2\varpi)p}.$$

- Case 3: $\gamma \ge -\frac{3}{2}$. By Young inequality, we have

$$\|B_{-1}^{2,2}\|_{L^2} \lesssim \Big(\int_{\mathbb{R}^3} |\Phi_{-1}^{\gamma}|^2 1_{|y|\ge 2^{-p}}dy\Big)^{1/2}\big(\int_{\mathbb{R}^3} |\tilde{\phi}_p||y|^{2\varpi}dy\big) \lesssim 2^{-(\eta+\frac{1}{2})p},$$

where $\eta = 2\varpi - \frac{1}{2}$ if $\gamma > -\frac{3}{2}$ and $\eta = 2\varpi - \frac{1}{2} - (\log_2 p)/p$ if $\gamma = -\frac{3}{2}$.

- Case 4: $\gamma < -\frac{3}{2}$. Again by Young inequality, we have

$$\|B_{-1}^{2,2}\|_{L^2} \lesssim \Big(\int_{\mathbb{R}^3} |\Phi_{-1}^{\gamma}|^2 1_{|y|\ge 2^{-p}}dy\Big)^{1/2}\big(\int_{\mathbb{R}^3} |\tilde{\phi}_p||y|^{2\varpi}dy\big) \lesssim 2^{-(\gamma+2\varpi+\frac{3}{2})p}.$$

Notice that there are only two types of the estimates, L^2 and L^∞ estimates, in the proof; then the proposition is easily obtained by patching together all the estimates. □

2.1. Estimates of $\mathfrak{W}_{k,p,l}^1$ and $\mathfrak{W}_{k,p,m}^4$ defined in (2.3)

We first give the estimate to $\mathfrak{W}_{k,p,l}^1$.

Lemma 2.1. – *Suppose $N \in \mathbb{N}$. For $k \ge 0$, it holds*

$$\begin{aligned} |\mathfrak{W}_{k,p,l}^1| &\lesssim 2^{k(\gamma+\frac{5}{2}-N)}(2^{-p(N-2s)}2^{2s(l-p)} + 2^{-(N-\frac{5}{2})p}2^{\frac{3}{2}(l-p)}) \\ &\quad\times \|\Phi_0^{\gamma}\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}. \end{aligned}$$

If $k = -1$, we have

1. *if* $\gamma = 0$,

$$|\mathfrak{W}^1_{-1,p,l}| \lesssim \big(2^{-p(N-2s)}2^{2s(l-p)} + 2^{-(N-1)p}2^{\frac{3}{2}l}\big) \times \|\psi\|_{H^{N+2}}\|\varphi\|_{W^{2,\infty}_N}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},$$

2. *if* $\gamma + 2s > 0$ *and* $\gamma > -\frac{3}{2}$,

$$|\mathfrak{W}^1_{-1,p,l}| \lesssim 2^{\tilde{\eta}(l-p)}2^{(2s-\eta_1)l}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},$$

3. *if* $\gamma + 2s > -1$ *and* $\gamma > -\frac{5}{2}$,

$$|\mathfrak{W}^1_{-1,p,l}| \lesssim 2^{\tilde{\eta}(l-p)}2^{(2s-\eta_2)l}\|\mathfrak{F}_p g\|_{L^{\frac{3}{2}}}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},$$

4. *if* $\gamma + 2s > -1$,

$$\begin{aligned}|\mathfrak{W}^1_{-1,p,l}| &\lesssim (2^{2sl}2^{-(\frac{1}{2}+\eta_2)p} + 2^{\frac{3}{2}l}2^{-(\gamma+3)p})\|\mathfrak{F}_p g\|_{L^2}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}\\ &\lesssim 2^{\tilde{\eta}(l-p)}2^{(2s-\frac{1}{2}-\eta_2)l}\|\mathfrak{F}_p g\|_{L^2}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},\end{aligned}$$

where $\tilde{\eta}$ *is a positive constant which depends only on* γ *and* s *and varies for different cases. We remark that constants* η_1 *and* η_2 *are stated in Proposition 2.2 and functions* ψ *and* φ *are defined in* (1.33).

Proof. – We recall that l and p verify the condition $l < p - N_0$. To make full use of the cancelation and to handle the singularity caused by the angular function, we make the following decomposition:

$$\mathfrak{W}^1_{k,p,l} = \mathfrak{D}^1_k + \mathfrak{D}^2_k,$$

where

$$\mathfrak{D}^1_k \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \big(\tilde{\mathfrak{F}}_p\Phi^\gamma_k\big)(|v-v_*|)b(\cos\theta)(\mathfrak{F}_p g)_*\big[(\mathfrak{F}_l h) - (\mathfrak{F}_l h)'\big](\tilde{\mathfrak{F}}_p f)' d\sigma dv_* dv,$$

$$\mathfrak{D}^2_k \stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \big(\tilde{\mathfrak{F}}_p\Phi^\gamma_k\big)(|v-v_*|)b(\cos\theta)(\mathfrak{F}_p g)_*\big[(\mathfrak{F}_l h)'(\tilde{\mathfrak{F}}_p f)' - (\mathfrak{F}_l h)\tilde{\mathfrak{F}}_p f\big] d\sigma dv_* dv.$$

The proof falls in several steps.

Step 1: Estimate of $\mathfrak{D}^1_k$. – Observe the facts

$$\begin{aligned}(2.6)\quad &(\mathfrak{F}_l h)(v) - (\mathfrak{F}_l h)(v')\\ &= (v-v')\cdot(\nabla\mathfrak{F}_l h)(v') + \frac{1}{2}\int_0^1 (1-\kappa)(v-v')\otimes(v-v') : (\nabla^2\mathfrak{F}_l h)(\kappa(v))d\kappa,\end{aligned}$$

where $\kappa(v) = v' + \kappa(v-v')$, and

$$\begin{aligned}(2.7)\quad &\iint_{\sigma\in\mathbb{S}^2, v\in\mathbb{R}^3} \Gamma(|v-v_*|)b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)w(|v'-v|)(v-v')\rho(v')d\sigma dv\\ &= 4\iint_{\sigma\in\mathbb{S}^2, v\in\mathbb{R}^3} \Gamma(|T_\sigma(v')-v_*|)b(\frac{T_\sigma(v')-v_*}{|T_\sigma(v')-v_*|}\cdot\sigma)\\ &\qquad\times w(|v'-T_\sigma(v')|)\frac{T_\sigma(v')-v'}{\big(\frac{v'-v_*}{|v'-v_*|}\cdot\sigma\big)^2}\rho(v')d\sigma dv' = 0,\end{aligned}$$

where w and ρ are smooth functions and T_σ represents the transform such that $T_\sigma(v') = v$. We refer readers to [1] or [6] to check the change of the variable from v to v'. Then in order to get the optimal estimate, we follow the idea in [7] to introduce the function ψ (defined in (1.33)) to decompose $\mathfrak{D}_k^1$ into angular cutoff part and angular non cutoff part, that is,

$$\mathfrak{D}_k^1 = \mathfrak{D}_k^{1,1} + \mathfrak{D}_k^{1,2},$$

where

$$\begin{aligned}\mathfrak{D}_k^{1,1} &\stackrel{\text{def}}{=} \frac{1}{2}\int_0^1 \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} A_k^s(|v-v_*|)b(\cos\theta)\psi(2^l(v'-v))(\mathfrak{F}_p g)_* \\ &\qquad \times|v-v_*|^{-2s}\big((v-v')\otimes(v-v'):(\nabla^2\mathfrak{F}_l h)(\kappa(v))\big)(\tilde{\mathfrak{F}}_p f)' d\sigma dv_* dv d\kappa, \\ \mathfrak{D}_k^{1,2} &\stackrel{\text{def}}{=} \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} A_k^s(|v-v_*|)b(\cos\theta)(1-\psi(2^l(v'-v)))(\mathfrak{F}_p g)_* \\ &\qquad \times|v-v_*|^{-2s}\big[(\mathfrak{F}_l h)-(\mathfrak{F}_l h)'\big](\tilde{\mathfrak{F}}_p f)' d\sigma dv_* dv,\end{aligned}$$

where (2.6) and (2.7) are used and A_k^s is defined in Proposition 2.2.

Step 1.1: Estimate of $\mathfrak{D}_k^{1,1}$. – We divide the estimate into three cases.

CASE 1: $k \geq 0$. — By Cauchy-Schwarz inequality, one has

$$\begin{aligned}|\mathfrak{D}_k^{1,1}| &\lesssim \|A_k^s\|_{L^\infty}\bigg(\int_0^1\iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*|b(\cos\theta)\sin^2(\theta/2)|\psi(2^l(v'-v))| \\ &\quad \times|v-v_*|^{2-2s}|(\tilde{\mathfrak{F}}_p f)'|^2 d\sigma dv_* dv d\kappa\bigg)^{\frac12}\bigg(\int_0^1\iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3}|(\mathfrak{F}_p g)_*|b(\cos\theta) \\ &\quad \times\sin^2(\theta/2)|\psi(2^l(v'-v))||v-v_*|^{2-2s}\big|(\nabla^2\mathfrak{F}_l h)(\kappa(v))\big|^2 d\sigma dv_* dv d\kappa\bigg)^{\frac12},\end{aligned}$$

where we use the fact $|v-v'| = |v-v_*|\sin(\theta/2)$.

Then we follow the change of variables: $(v_*, v)\to(v_*, u_1 = v')$ and $(v_*, v)\to\big(v_*, u_2 = \kappa(v)\big)$. Thanks to the fact that

$$|\frac{\partial u_2}{\partial v}| = (1-\frac{\kappa}{2})^2\Big\{(1-\frac{\kappa}{2}) + \frac{\kappa}{2}\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big\}, \tag{2.8}$$

we derive that

$$\begin{aligned}|\mathfrak{D}_k^{1,1}| &\lesssim \|A_k^s\|_{L^\infty}\bigg(\int_0^1\iint_{\sigma\in\mathbb{S}^2, v_*, u_1\in\mathbb{R}^3}|(\mathfrak{F}_p g)_*|b(\cos\theta)\sin^2(\theta/2)|\psi(2^l|v-v_*|\sin(\theta/2))| \\ &\quad \times|u_1-v_*|^{2-2s}|(\tilde{\mathfrak{F}}_p f)(u_1)|^2 d\sigma dv_* du_1 d\kappa\bigg)^{\frac12}\bigg(\int_0^1\iint_{\sigma\in\mathbb{S}^2, v_*, u_2\in\mathbb{R}^3}|(\mathfrak{F}_p g)_*|b(\cos\theta) \\ &\quad \times\sin^2(\theta/2)|\psi(2^l|v-v_*|\sin(\theta/2))||u_2-v_*|^{2-2s}\big|(\nabla^2\mathfrak{F}_l h)(u_2)\big|^2 d\sigma dv_* du_2 d\kappa\bigg)^{\frac12},\end{aligned}$$

where we use the fact $|v-v_*| \sim |u_1-v_*| \sim |u_2-v_*|$. It is not difficult to check

$$
\begin{aligned}
&|v-v_*|^{2-2s}\int_{\sigma\in\mathbb{S}^2} b(\cos\theta)\sin^2(\theta/2)\psi(2^l|v-v_*|\sin(\theta/2))d\sigma \\
&\lesssim |u_2-v_*|^{2-2s}\int_0^{\frac{8}{3}2^{-l}|u_2-v_*|^{-1}} b(\cos\theta)\theta^2\sin\tilde{\theta}d\tilde{\theta} \\
&\lesssim 2^{-(2-2s)l},
\end{aligned}
\tag{2.9}
$$

where $\tilde{\theta}$ verifies $\cos\tilde{\theta} = \frac{u_2-v_*}{|u_2-v_*|}\cdot\sigma$ and $\theta/2 \le \tilde{\theta} \le \theta$. By Bernstein inequalities (5.1)-(5.3), it is easy to derive that

$$
\begin{aligned}
|\mathfrak{D}_k^{1,1}| &\lesssim 2^{2sl}\|A_k\|_{L^\infty}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2} \\
&\lesssim 2^{k(\gamma+\frac{3}{2}-N)}2^{-p(N-2s)}2^{2s(l-p)}\|\Phi_0^\gamma\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}.
\end{aligned}
$$

CASE 2: $k=-1$ AND $\gamma=0$. — Thanks to Proposition 2.2, we easily get

$$
|\mathfrak{D}_{-1}^{1,1}| \lesssim 2^{-p(N-2s)}2^{2s(l-p)}\|\psi\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}.
$$

CASE 3: $k=-1$ AND $\gamma\neq 0$. — For the general case, following the decomposition in Proposition 2.2:

$$
A_{-1}^s = B_{-1}^s + \tilde{\mathfrak{F}}_p\Phi_{-1}^{\gamma+2s},
$$

one has the corresponding decomposition:

$$
\mathfrak{D}_{-1}^{1,1} = \mathfrak{D}_{-1,1}^{1,1} + \mathfrak{D}_{-1,2}^{1,1}.
$$

We first have

$$
\begin{aligned}
|\mathfrak{D}_{-1,2}^{1,1}| \lesssim &\Big(\int_0^1\iint_{\sigma\in\mathbb{S}^2,v_*,v\in\mathbb{R}^3}\big|\big(\tilde{\mathfrak{F}}_p\Phi_{-1}^{\gamma+2s}\big)(|v-v_*|)\big|^2 b(\cos\theta)\sin^2(\theta/2)|\psi(2^l(v'-v))| \\
&\times|v-v_*|^{2-2s}|(\tilde{\mathfrak{F}}_p f)'|^2 d\sigma dv_* dv d\kappa\Big)^{\frac{1}{2}}\Big(\int_0^1\iint_{\sigma\in\mathbb{S}^2,v_*,v\in\mathbb{R}^3}|(\mathfrak{F}_p g)_*|^2 b(\cos\theta) \\
&\times\sin^2(\theta/2)|\psi(2^l(v'-v))||v-v_*|^{2-2s}\big|(\nabla^2\mathfrak{F}_l h)(\kappa(v))\big|^2 d\sigma dv_* dv d\kappa\Big)^{\frac{1}{2}}.
\end{aligned}
$$

By change of variables from (v_*,v) to $(u_1=v-v_*, u_2=v')$ and from (v_*,v) to $(v_*,u_3=\kappa(v))$, we get

$$
\begin{aligned}
|\mathfrak{D}_{-1,2}^{1,1}| \lesssim &\Big(\int_0^1\iint_{\sigma\in\mathbb{S}^2,u_1,u_2\in\mathbb{R}^3}\big|\big(\tilde{\mathfrak{F}}_p\Phi_{-1}^{\gamma+2s}\big)(|u_1|)\big|^2 b(\cos\theta)\sin^2(\theta/2)|\psi(2^l|u_1|\sin(\theta/2))| \\
&\times|u_1|^{2-2s}|(\tilde{\mathfrak{F}}_p f)(u_2)|^2 d\sigma du_1 du_2 d\kappa\Big)^{\frac{1}{2}}\Big(\int_0^1\iint_{\sigma\in\mathbb{S}^2,v_*,u_3\in\mathbb{R}^3}|(\mathfrak{F}_p g)_*|^2 b(\cos\theta) \\
&\times\sin^2(\theta/2)|\psi(2^l|v-v_*|\sin(\theta/2))||u_3-v_*|^{2-2s}\big|(\nabla^2\mathfrak{F}_l h)(u_3)\big|^2 d\sigma dv_* du_3 d\kappa\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Then we have

$$
|\mathfrak{D}_{-1,2}^{1,1}| \lesssim 2^{2sl}\|\tilde{\mathfrak{F}}_p\Phi_{-1}^{\gamma+2s}\|_{L^2}\|\mathfrak{F}_p g\|_{L^2}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}.
$$

Due to the fact (see [5]) that for $\gamma > -3$ and the multi-index α with $|\alpha| = k \in \mathbb{N}$,

$$|\big(\partial_\xi^\alpha \mathscr{F}(\Phi_{-1}^\gamma)\big)(\xi)| \lesssim \langle\xi\rangle^{-3-\gamma-k}, \tag{2.10}$$

we have

$$\begin{aligned}\|\tilde{\mathfrak{F}}_p \Phi_{-1}^{\gamma+2s}\|_{L^2}^2 &\lesssim \int_{\mathbb{R}^3} |\varphi_p(\xi)|^2 |\xi|^{-2(\gamma+2s)-6} d\xi \\ &\lesssim 2^{-2(\gamma+2s+\frac{3}{2})p}.\end{aligned}$$

We deduce that if $\gamma + 2s > -1$,

$$|\mathfrak{D}_{-1,2}^{1,1}| \lesssim 2^{2sl} 2^{-(\gamma+2s+\frac{3}{2})p} \|\mathfrak{F}_p g\|_{L^2} \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2}.$$

Due to the Bernstein inequalities (5.1)-(5.3), if $\gamma + 2s > 0$, then we have

$$|\mathfrak{D}_{-1,2}^{1,1}| \lesssim 2^{2sl} 2^{-(\gamma+2s)p} \|\mathfrak{F}_p g\|_{L^1} \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2}.$$

Next we turn to the estimate of $\mathfrak{D}_{-1,1}^{1,1}$. Notice that B_{-1}^s can be separated into two parts B_1 and B_2 which can be controlled in L^2 and L^∞ spaces thanks to Proposition 2.2. Thus we may copy the argument for $\mathfrak{D}_{-1,2}^{1,1}$ to $\mathfrak{D}_{-1,1}^{1,1}$ when B_1 is bounded in L^2 space and apply the argument for $\mathfrak{D}_k^{1,1}$ to $\mathfrak{D}_{-1,1}^{1,1}$ when B_2 is controlled in L^∞ space. Finally we obtain that in the case of $\gamma + 2s > 0$,

$$\begin{aligned}|\mathfrak{D}_{-1,1}^{1,1}| &\lesssim 2^{2sl} (\|B_1\|_{L^2} \|\mathfrak{F}_p g\|_{L^2} + \|B_2\|_{L^\infty} \|\mathfrak{F}_p g\|_{L^1}) \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2} \\ &\lesssim 2^{2sl} 2^{-\eta_1 p} \|\mathfrak{F}_p g\|_{L^1} \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2}.\end{aligned}$$

While in the case of $\gamma + 2s > -1$,

$$|\mathfrak{D}_{-1,1}^{1,1}| \lesssim 2^{2sl} 2^{-(\frac{1}{2}+\eta_2)p} \|\mathfrak{F}_p g\|_{L^2} \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2}.$$

Step 1.2: Estimate of $\mathfrak{D}_k^{1,2}$. – We separate the estimate into several cases.

Case 1: $k \geq 0$. — By Cauchy-Schwarz inequality, one has

$$\begin{aligned}|\mathfrak{D}_k^{1,2}| \lesssim \|A_k^s\|_{L^\infty} \bigg(&\iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*| b(\cos\theta) |1 - \psi(2^l(v'-v))| \\ &\times |v - v_*|^{-2s} |(\tilde{\mathfrak{F}}_p f)'|^2 d\sigma dv_* dv \bigg)^{\frac{1}{2}} \bigg(\iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*| b(\cos\theta) \\ &\times |1 - \psi(2^l(v'-v))| |v - v_*|^{-2s} \big|(\mathfrak{F}_l h)(v)\big|^2 d\sigma dv_* dv \bigg)^{\frac{1}{2}} \\ &+ \|A_k^s\|_{L^\infty} \bigg(\iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*| b(\cos\theta) |1 - \psi(2^l(v'-v))| \\ &\times |v - v_*|^{-2s} |(\tilde{\mathfrak{F}}_p f)'|^2 d\sigma dv_* dv \bigg)^{\frac{1}{2}} \bigg(\int_0^1 \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*| b(\cos\theta) \\ &\times |1 - \psi(2^l(v'-v))| |v - v_*|^{-2s} \big|(\mathfrak{F}_l h)(v')\big|^2 d\sigma dv_* dv \bigg)^{\frac{1}{2}}.\end{aligned}$$

Following the change of variables: $(v_*, v) \to (v_*, u_2 = v')$ and the fact (2.8), we get

$$
\begin{aligned}
|\mathfrak{D}_k^{1,2}| \lesssim\ & \|A_k^s\|_{L^\infty}\Big(\iint_{\sigma\in\mathbb{S}^2, v_*, u_2\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*| b(\cos\theta)|1-\psi(2^l|v-v_*|\sin(\theta/2))| \\
& \times |v-v_*|^{-2s}|(\tilde{\mathfrak{F}}_p f)(u_2)|^2 d\sigma dv_* du_2\Big)^{\frac{1}{2}}\Big(\iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*| b(\cos\theta) \\
& \times |1-\psi(2^l|v-v_*|\sin(\theta/2))||v-v_*|^{-2s}\big|(\mathfrak{F}_l h)(v)\big|^2 d\sigma dv_* dv\Big)^{\frac{1}{2}} \\
& + \|A_k^s\|_{L^\infty}\Big(\iint_{\sigma\in\mathbb{S}^2, v_*, u_2\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*| b(\cos\theta)|1-\psi(2^l|v-v_*|\sin(\theta/2))| \\
& \times |v-v_*|^{-2s}|(\tilde{\mathfrak{F}}_p f)(u_2)|^2 d\sigma dv_* du_2\Big)^{\frac{1}{2}}\Big(\iint_{\sigma\in\mathbb{S}^2, v_*, u_2\in\mathbb{R}^3} |(\mathfrak{F}_p g)_*| b(\cos\theta) \\
& \times |1-\psi(2^l|v-v_*|\sin(\theta/2))||u_2-v_*|^{-2s}\big|(\mathfrak{F}_l h)(u_2)\big|^2 d\sigma dv_* du_2\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Notice

$$
\begin{aligned}
|v-v_*|^{-2s} & \int_{\mathbb{S}^2} b(\cos\theta)(1-\psi(2^l|v-v_*|\sin(\theta/2)))d\sigma \\
& \lesssim \begin{cases} |v-v_*|^{-2s}\displaystyle\int_{\frac{4}{3}2^{-l}|v-v_*|^{-1}}^{\pi/2} b(\cos\theta)\sin\theta d\theta \\ |u_2-v_*|^{-2s}\displaystyle\int_{\frac{4}{3}2^{-l}|u_2-v_*|^{-1}}^{\pi/2} b(\cos\theta)\sin\tilde{\theta} d\tilde{\theta} \end{cases} \\
& \lesssim 2^{2sl},
\end{aligned}
\tag{2.11}
$$

where $\tilde{\theta}$ verifies $\cos\tilde{\theta} = \frac{u_2-v_*}{|u_2-v_*|}\cdot\sigma$ and $\theta/2 \le \tilde{\theta} \le \theta$. Finally we get the estimate to $\mathfrak{D}_k^{1,2}$, that is, for any $N \in \mathbb{N}$,

$$
|\mathfrak{D}_k^{1,2}| \lesssim 2^{(\gamma+\frac{3}{2}-N)k} 2^{2s(l-p)} 2^{(2s-N)p} \|\Phi_0^\gamma\|_{H^{N+2}} \|\varphi\|_{W_N^{2,\infty}} \|\mathfrak{F}_p g\|_{L^1} \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2}.
$$

Case 2: $k = -1$. Following the similar argument used in the previous step, we conclude that in the case of $\gamma = 0$,

$$
|\mathfrak{D}_{-1}^{1,2}| \lesssim 2^{2s(l-p)} 2^{(2s-N)p} \|\psi\|_{H^{N+2}} \|\varphi\|_{W_N^{2,\infty}} \|\mathfrak{F}_p g\|_{L^1} \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2},
$$

in the case of $\gamma + 2s > 0$,

$$
\begin{aligned}
|\mathfrak{D}_{-1}^{1,2}| &\lesssim 2^{2sl}(\|B_1\|_{L^2}\|\mathfrak{F}_p g\|_{L^2} + \|B_2\|_{L^\infty}\|\mathfrak{F}_p g\|_{L^1})\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2} \\
&\lesssim 2^{2sl} 2^{-\eta_1 p}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},
\end{aligned}
$$

and in the case of $\gamma + 2s > -1$,

$$
|\mathfrak{D}_{-1}^{1,2}| \lesssim 2^{2sl} 2^{-(\frac{1}{2}+\eta_2)p} \|\mathfrak{F}_p g\|_{L^2} \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2}.
$$

Step 2: Estimate of $\mathfrak{D}_k^2$. – Thanks to the Cancelation Lemma in [1], we obtain that

$$\mathfrak{D}_k^2 = |\mathbb{S}^1| \iint_{\theta\in[0,\pi/2],v_*,v\in\mathbb{R}^3} \Big[(\tilde{\mathfrak{F}}_p \Phi_k^\gamma)\Big(\frac{|v-v_*|}{\cos\frac{\theta}{2}}\Big)\frac{1}{\cos^3\frac{\theta}{2}} - (\tilde{\mathfrak{F}}_p\Phi_k^\gamma)(|v-v_*|)\Big] b(\cos\theta)\sin\theta$$
$$\times (\mathfrak{F}_p g)_*(\mathfrak{F}_l h)(\tilde{\mathfrak{F}}_p f) d\theta dv_* dv.$$

Notice that

$$C_k(\theta,\xi) \stackrel{\text{def}}{=} \frac{1}{\cos^3\frac{\theta}{2}} \mathscr{F}\Big((\tilde{\mathfrak{F}}_p\Phi_k^\gamma)(\frac{\cdot}{\cos\frac{\theta}{2}})\Big)(\xi) - \mathscr{F}\Big((\tilde{\mathfrak{F}}_p\Phi_k^\gamma)(\cdot)\Big)(\xi)$$
$$= \varphi_p(\xi)2^{(\gamma+3)k}\Big(\mathscr{F}(\Phi_0^\gamma)(\cos\frac{\theta}{2}2^k\xi) - (\mathscr{F}(\Phi_0^\gamma)(2^k\xi)\Big),$$

where we use (2.5). We split the estimate into several cases.

CASE 1: $k\ge 0$ OR $k=-1$ WITH $\gamma=0$. — Thanks to the mean value theorem, we have

$$\|C_k(\theta,\cdot)\|_{L^2} \lesssim \theta^2 2^{(\gamma+\frac{5}{2}-N)k}2^{(1-N)p}\|\Phi_0^\gamma\|_{H^N}\|\varphi\|_{W_N^{2,\infty}}$$

and

$$\|C_{-1}(\theta,\cdot)\|_{L^2} \lesssim \theta^2 2^{(\gamma+\frac{5}{2}-N)k}2^{(1-N)p}\|\psi\|_{H^N}\|\varphi\|_{W_N^{2,\infty}}.$$

Then by Bernstein inequalities (5.1)-(5.3), for any $N\in\mathbb{N}$, we get

$$|\mathfrak{D}_k^2| \lesssim \int_{\mathbb{S}^2} b(\cos\theta)\|C_k(\theta,\cdot)\|_{L^2}\|\mathscr{F}(\mathfrak{F}_p g)\|_{L^\infty}\|\mathscr{F}(\mathfrak{F}_l h\tilde{\mathfrak{F}}_p f)\|_{L^2} d\sigma$$
$$\lesssim 2^{(\gamma+\frac{5}{2}-N)k}2^{(1-N)p}\|\varphi\|_{W_N^{2,\infty}}\|\Phi_0^\gamma\|_{H^N}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^\infty}\|\tilde{\mathfrak{F}}_p f\|_{L^2}$$
$$\lesssim 2^{(\gamma+\frac{5}{2}-N)k}2^{(1-N)p}2^{\frac{3}{2}l}\|\varphi\|_{W_N^{2,\infty}}\|\Phi_0^\gamma\|_{H^N}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}$$

and

$$|\mathfrak{D}_{-1}^2| \lesssim 2^{(1-N)p}2^{\frac{3}{2}l}\|\varphi\|_{W_N^{2,\infty}}\|\psi\|_{H^N}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}.$$

CASE 2: $k=-1$ WITH GENERAL POTENTIALS. — Due to the fact (2.10), it is easy to check that

$$|C_{-1}(\theta,\xi)| \lesssim \theta^2|\xi||\varphi_p(\xi)|\|\nabla\mathscr{F}(\Phi_{-1}^\gamma)\|_{L^\infty}$$
$$\lesssim 2^{-(\gamma+3)p}\theta^2.$$

Thanks to Plancherel theorem and Bernstein inequalities (5.1)-(5.3), one has

$$|\mathfrak{D}_{-1}^2| \lesssim 2^{-(\gamma+3)p}\|\mathfrak{F}_p g\|_{L^2}\|(\mathfrak{F}_l h)(\tilde{\mathfrak{F}}_p f)\|_{L^2}$$
$$\lesssim 2^{-(\gamma+3)p}2^{\frac{3}{2}l}\|\mathfrak{F}_p g\|_{L^2}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}.$$

Thus we have

1. if $\gamma+2s>0$ and $\gamma>-\frac{3}{2}$,
$$|\mathfrak{D}_{-1}^2| \lesssim 2^{(l-p)(\gamma+\frac{3}{2})}2^{(2s-(\gamma+2s))l}\|\mathfrak{F}_p g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},$$
2. if $-1<\gamma+2s\le 0$ and $\gamma>-\frac{5}{2}$,
$$|\mathfrak{D}_{-1}^2| \lesssim 2^{(l-p)(\gamma+\frac{5}{2})}2^{-(\gamma+2s+1)l}2^{2sl}\|\mathfrak{F}_p g\|_{L^{\frac{3}{2}}}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},$$

3. if $-1 < \gamma + 2s$,

$$|\mathfrak{D}^2_{-1}| \lesssim 2^{(l-p)(\gamma+3)} 2^{-(\gamma+2s+1)l} 2^{(2s-\frac{1}{2})l} \|\mathfrak{F}_p g\|_{L^2} \|\mathfrak{F}_l h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2}.$$

Now combining all the estimates in *Step 1* and *Step 2* and using Bernstein inequalities (5.1)-(5.3), we finally get the desired results in the lemma. □

Next we have

LEMMA 2.2. – *Suppose $N \in \mathbb{N}$. For $k \geq 0$, it holds*

$$|\mathfrak{W}^4_{k,p,m}| \lesssim 2^{2s(m-p)} 2^{(\gamma+\frac{3}{2}-N)k} 2^{-p(N-\frac{5}{2})} \|\Phi^\gamma_0\|_{H^{N+2}} \|\varphi\|_{W^{2,\infty}_N} \|\mathfrak{F}_p g\|_{L^1} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2}.$$

For $k = -1$, we have

1. *if $\gamma = 0$,*

$$|\mathfrak{W}^4_{-1,p,m}| \lesssim 2^{2s(m-p)} 2^{-p(N-\frac{5}{2})} \|\psi\|_{H^{N+2}} \|\varphi\|_{W^{2,\infty}_N} \|\mathfrak{F}_p g\|_{L^1} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2},$$

2. *if $\gamma + 2s > 0$,*

$$|\mathfrak{W}^4_{-1,p,m}| \lesssim 2^{\tilde{\eta}(m-p)} 2^{(2s-\eta_1)m} \|\mathfrak{F}_p g\|_{L^1} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2},$$

3. *if $\gamma + 2s > -1$,*

$$\begin{aligned} |\mathfrak{W}^4_{-1,p,m}| &\lesssim 2^{2sm} 2^{-(\frac{1}{2}+\eta_2)p} \|\mathfrak{F}_p g\|_{L^2} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2} \\ &\lesssim 2^{\eta(m-p)} 2^{-\frac{1}{2}p} 2^{(2s-\eta_2)m} \|\mathfrak{F}_p g\|_{L^2} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2}, \end{aligned}$$

where $\tilde{\eta}$ is a positive constant which depends only on γ and s. We remark that constants η_1 and η_2 are stated in Proposition 2.2 and functions ψ and φ are defined in (1.33).

Proof. – Noticing the fact that $m < p - N_0$, we following the similar decomposition used in Lemma 2.1 to get

$$\begin{aligned} \mathfrak{W}^4_{k,p,m} &= \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \big(\tilde{\mathfrak{F}}_p \Phi^\gamma_k\big)(|v-v_*|) b(\cos\theta) \psi(2^m(v'-v)) (\mathfrak{F}_p g)_* (\tilde{\mathfrak{F}}_p h) \\ &\quad \times \big[(\mathfrak{F}_m f)' - \mathfrak{F}_m f\big] d\sigma dv_* dv + \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \big(\tilde{\mathfrak{F}}_p \Phi^\gamma_k\big)(|v-v_*|) b(\cos\theta) \\ &\quad \times \big(1 - \psi(2^m(v'-v))\big) (\mathfrak{F}_p g)_* (\tilde{\mathfrak{F}}_p h) \big[(\mathfrak{F}_m f)' - \mathfrak{F}_m f\big] d\sigma dv_* dv \\ &\stackrel{\text{def}}{=} \mathfrak{E}^1_k + \mathfrak{E}^2_k. \end{aligned}$$

Observe the fact that

$$\begin{aligned} (2.12)\quad & (\mathfrak{F}_m f)(v') - (\mathfrak{F}_m f)(v) \\ &= (v'-v)\cdot(\nabla\mathfrak{F}_m f)(v) + \frac{1}{2}\int_0^1 (1-\kappa)(v'-v)\otimes(v'-v) : (\nabla^2 \mathfrak{F}_m f)(\kappa(v)) d\kappa, \end{aligned}$$

where $\kappa(v) = v + \kappa(v'-v)$, then we have the further decomposition:

$$\mathfrak{E}^1_k = \mathfrak{E}^{1,1}_k + \mathfrak{E}^{1,2}_k,$$

where

$$\mathfrak{E}_k^{1,1} = \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \big(\tilde{\mathfrak{F}}_p \Phi_k^\gamma\big)(|v-v_*|) b(\cos\theta)\psi(2^m(v'-v))(\mathfrak{F}_p g)_*(\tilde{\mathfrak{F}}_p h) \times (v'-v)\cdot(\nabla \mathfrak{F}_m f)(v) d\sigma dv_* dv,$$

$$\mathfrak{E}_k^{1,2} = \int_0^1 \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} A_k^s(|v-v_*|) b(\cos\theta)\psi(2^m(v'-v))(\mathfrak{F}_p g)_*(\tilde{\mathfrak{F}}_p h) \times |v-v_*|^{-2s}\big[(v'-v)\otimes(v'-v) : (\nabla^2 \mathfrak{F}_m f)(\kappa(v))\big] d\sigma dv_* dv d\kappa,$$

where A_k^s is defined in Proposition 2.2.

It is not difficult to check that the main structures of $\mathfrak{E}_k^{1,2}$ and $\mathfrak{E}_k^2$ are almost the same as those of $\mathfrak{D}_k^{1,1}$ and $\mathfrak{D}_k^{1,2}$. We conclude that for any $N \in \mathbb{N}$,

$$|\mathfrak{E}_k^{1,2}| + |\mathfrak{E}_k^2| \lesssim \begin{cases} 2^{(\gamma+\frac{3}{2}-N)k} 2^{(2s-N)p} 2^{2s(m-p)} \|\Phi_0^\gamma\|_{H^{N+2}} \|\varphi\|_{W_N^{2,\infty}} \|\mathfrak{F}_p g\|_{L^1} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2}, \text{if } k \geq 0, \\ 2^{2s(m-p)} 2^{(2s-N)p} \|\psi\|_{H^{N+2}} \|\varphi\|_{W_N^{2,\infty}} \|\mathfrak{F}_p g\|_{L^1} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2}, \text{if } k=-1 \text{ and } \gamma = 0, \\ 2^{2sm} 2^{-\eta_1 p} \|\mathfrak{F}_p g\|_{L^1} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2}, \text{ if } k=-1 \text{ and } \gamma+2s>0, \\ 2^{2sm} 2^{-(\frac{1}{2}+\eta_2)p} \|\mathfrak{F}_p g\|_{L^2} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\mathfrak{F}_m f\|_{L^2}, \text{ if } k=-1 \text{ and } \gamma+2s>-1. \end{cases}$$

Now we only need to give the bound to $\mathfrak{E}_k^{1,1}$. Thanks to the fact that

$$\begin{aligned} &\int_{\mathbb{S}^2} b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)(v-v')\psi(2^m|v-v'|)d\sigma \\ &= \int_{\mathbb{S}^2} b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\frac{v-v'}{|v-v'|}\cdot\frac{v-v_*}{|v-v_*|}|v-v'|\psi(|2^m|v-v'|)\frac{v-v_*}{|v-v_*|}d\sigma \\ &= \int_{\mathbb{S}^2} b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\frac{1-\langle\frac{v-v_*}{|v-v_*|},\sigma\rangle}{2}\psi(2^m|v-v'|)d\sigma(v-v_*), \end{aligned}$$

one has

(2.13)

$$\begin{aligned} &2^{(2-2s)m}|v-v_*|^{-2s}\left|\int_{\mathbb{S}^2} b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)(v-v')\psi(2^m|v-v'|)d\sigma\right| \\ &\lesssim 2^{(2-2s)m}|v-v_*|^{-2s}\int_{\sqrt{\frac{1-\frac{v-v_*}{|v-v_*|}\cdot\sigma}{2}}\lesssim 2^{-m}|v-v_*|^{-1}} b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\frac{1-\frac{v-v_*}{|v-v_*|}\cdot\sigma}{2}d\sigma|v-v_*| \\ &\lesssim 2^{(2-2s)m}|v-v_*|^{-2s}\int_0^{2^{-m}|v-v_*|^{-1}} \theta^{1-2s}d\theta|v-v_*| \\ &\lesssim |v-v_*|^{-1}. \end{aligned}$$

In other words, if we set

$$U(v) \stackrel{\text{def}}{=} 2^{(2-2s)m}|v|^{-2s}v\int_{\mathbb{S}^2} b\Big(\frac{v}{|v|}\cdot\sigma\Big)\frac{1-\langle\frac{v}{|v|},\sigma\rangle}{2}\psi\Big(2^{m-1}|v|\sqrt{1-\langle\frac{v}{|v|},\sigma\rangle}\sqrt{2}\Big)d\sigma,$$

then (2.13) yields $|U(v)| \lesssim |v|^{-1}$.

Due to this observation, we have

$$|\mathfrak{E}_k^{1,1}| = \left|2^{(2s-2)m} \iint_{v,v_*\in\mathbb{R}^3} A_k^s(|v-v_*|)(\mathfrak{F}_p g)_*(\tilde{\mathfrak{F}}_p h)(v)U(v-v_*)\cdot\nabla\mathfrak{F}_m f(v)dv_*dv\right|.$$

We divide the estimate into three cases.

CASE 1: $k \geq 0$. — By the definition, we have $|A_k^s U| \lesssim A_k^{s-\frac{1}{2}}$. If $s \geq 1/2$, then

$$\begin{aligned}|\mathfrak{E}_k^{1,1}| &\lesssim 2^{(2s-2)m}\|A_k^{s-\frac{1}{2}}\|_{L^\infty}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\nabla\mathfrak{F}_m f\|_{L^2}\\ &\lesssim 2^{2s(m-p)}2^{(\gamma+\frac{3}{2}-N)k}2^{-p(N-2s)}\|\Phi_0^\gamma\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\mathfrak{F}_m f\|_{L^2}.\end{aligned}$$

In the case of $s < 1/2$, one has

$$\begin{aligned}|\mathfrak{E}_k^{1,1}| &\lesssim 2^{(2s-2)m}\|A_k^{s-\frac{1}{2}}\|_{L^2}\|\mathfrak{F}_p g\|_{L^2}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\nabla\mathfrak{F}_m f\|_{L^2}\\ &\lesssim 2^{2s(m-p)}2^{(\gamma+\frac{3}{2}-N)k}2^{-p(N-2s-\frac{3}{2})}\|\Phi_0^\gamma\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\mathfrak{F}_m f\|_{L^2},\end{aligned}$$

where we use the Hardy inequality to get

$$\|A_k^{s-\frac{1}{2}}\|_{L^2} \leq \|\tilde{\mathfrak{F}}_p\Phi_k^\gamma\|_{H^{1-2s}} \lesssim 2^{(\gamma+\frac{3}{2}-N)k}2^{-pN}\|\Phi_0^\gamma\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}.$$

CASE 2: $k = -1$ AND $\gamma = 0$. — In this case, we only need to copy the argument in *Case* 1 to get

$$|\mathfrak{E}_{-1}^{1,1}| \lesssim 2^{2s(m-p)}2^{-p(N-\frac{5}{2})}\|\psi\|_{H^{N+2}}\|\varphi\|_{W_N^{2,\infty}}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\mathfrak{F}_m f\|_{L^2}.$$

CASE 3: $k = -1$ WITH GENERAL POTENTIALS. — Following the decomposition

$$A_{-1}^s = \tilde{\mathfrak{F}}_p\Phi_{-1}^{\gamma+2s} + B_{-1}^s,$$

we split $\mathfrak{E}_{-1}^{1,1}$ into two parts: $\mathfrak{E}_{-1}^{1,1,1}$ and $\mathfrak{E}_{-1}^{1,1,2}$ which are defined by

$$\begin{aligned}\mathfrak{E}_{-1}^{1,1,1} &= 2^{(2s-2)m}\iint_{v,v_*\in\mathbb{R}^3}\left(\tilde{\mathfrak{F}}_p\Phi_{-1}^{\gamma+2s}\right)(|v-v_*|)(\mathfrak{F}_p g)_*(\tilde{\mathfrak{F}}_p h)(v)U(v-v_*)\cdot\nabla\mathfrak{F}_m f(v)dv_*dv,\\ \mathfrak{E}_{-1}^{1,1,2} &= 2^{(2s-2)m}\iint_{v,v_*\in\mathbb{R}^3} B_{-1}^s(|v-v_*|)(\mathfrak{F}_p g)_*(\tilde{\mathfrak{F}}_p h)(v)U(v-v_*)\cdot\nabla\mathfrak{F}_m f(v)dv_*dv.\end{aligned}$$

Thanks to Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathfrak{E}_{-1}^{1,1,1}| &\leq 2^{(2s-2)m}\left(\iint_{v,v_*\in\mathbb{R}^3}|(\tilde{\mathfrak{F}}_p\Phi_k^{\gamma+2s})(|v-v_*|)|^2|(\mathfrak{F}_p g)_*||\tilde{\mathfrak{F}}_p h|dvdv_*\right)^{\frac{1}{2}}\\ &\quad\times\left(\iint_{v,v_*\in\mathbb{R}^3}|v-v_*|^{-2}|(\mathfrak{F}_p g)_*||\tilde{\mathfrak{F}}_p h||\nabla\mathfrak{F}_m f|^2dvdv_*\right)^{\frac{1}{2}}\\ &\lesssim 2^{(2s-2)m}\|\tilde{\mathfrak{F}}_p\Phi_k^{\gamma+2s}\|_{L^2}\|\mathfrak{F}_p g\|_{L^2}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\nabla\mathfrak{F}_m f\|_{L^6}\\ &\lesssim 2^{2sm}2^{-(\gamma+2s+\frac{3}{2})p}\|\mathfrak{F}_p g\|_{L^2}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\mathfrak{F}_m f\|_{L^2},\end{aligned}$$

where we use Young and Hardy-Littlewood-Sobolev inequalities and Lemma 5.1.

Finally we turn to the estimate of $\mathfrak{E}_{-1}^{1,1,2}$. If $\gamma + 2s > 0$, thanks to Proposition 2.2, Young and Hardy-Littlewood-Sobolev inequalities, we have for $\delta < \eta_1/3$ where η_1 is defined in Proposition 2.2,

$$\begin{aligned}
|\mathfrak{E}_{-1}^{1,1,2}| &\lesssim 2^{(2s-2)m}\|B_1\|_{L^2}\|\mathfrak{F}_p g\|_{L^2}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\nabla \mathfrak{F}_m f\|_{L^6}\\
&\quad + 2^{(2s-2)m}\|B_2\|_{L^\infty}\|\mathfrak{F}_p g\|_{L^{1+\delta}}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\nabla \mathfrak{F}_m f\|_{L^{\frac{6(1+\delta)}{1+7\delta}}}\\
&\lesssim 2^{2sm}2^{-\eta_1 p}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\mathfrak{F}_m f\|_{L^2}\\
&\quad + 2^{-(\eta_1 - \frac{3\delta}{1+\delta})p}2^{(2s-\frac{3\delta}{1+\delta})m}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\mathfrak{F}_m f\|_{L^2}\\
&\lesssim (2^{2sm}2^{-\eta_1 p} + 2^{(2s-\eta_1)m}2^{(\eta_1-\frac{3\delta}{1+\delta})(m-p)})\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\mathfrak{F}_m f\|_{L^2}.
\end{aligned}$$

While in the case of $\gamma + 2s > -1$, with the help of Proposition 2.2, following the similar argument applied to $\mathfrak{E}_{-1}^{1,1,1}$, we get

$$\begin{aligned}
|\mathfrak{E}_{-1}^{1,1,2}| &\lesssim 2^{(2s-2)m}\|B_{-1}^s\|_{L^2}\|\mathfrak{F}_p g\|_{L^2}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\nabla \mathfrak{F}_m f\|_{L^6}\\
&\lesssim 2^{2sm}2^{-(\frac{1}{2}+\eta_2)p}\|\mathfrak{F}_p g\|_{L^2}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\mathfrak{F}_m f\|_{L^2}.
\end{aligned}$$

Now patching together all the estimates, we are led to the desired results. □

2.2. Estimates of $\mathfrak{W}_{k,l}^2$ and $\mathfrak{W}_{k,p}^3$ defined in (2.3)

Since $\mathfrak{W}_{k,l}^2$ enjoys almost the same structure as that of $\mathfrak{W}_{k,p}^3$, it suffices to give the estimate of $\mathfrak{W}_{k,p}^3$.

LEMMA 2.3. – *If $k \geq 0$, we have*

$$\begin{aligned}
|\mathfrak{W}_{k,l}^2| &\lesssim 2^{(\gamma+2s)k}2^{2sl}\|\mathcal{S}_{l-N_0} g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_l f\|_{L^2},\\
|\mathfrak{W}_{k,p}^3| &\lesssim 2^{(\gamma+2s)k}2^{2sp}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}.
\end{aligned}$$

If $k = -1$, we have

$$|\mathfrak{W}_{-1,l}^2| \lesssim \begin{cases} 2^{2sl}\|\mathcal{S}_{l-N_0} g\|_{L^1}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_l f\|_{L^2}, & \text{if } \gamma + 2s > 0,\\ 2^{2sl}\|\mathcal{S}_{l-N_0} g\|_{L^2}\|\mathfrak{F}_l h\|_{L^2}\|\tilde{\mathfrak{F}}_l f\|_{L^2}, & \text{if } \gamma + 2s \leq 0,\end{cases}$$

$$|\mathfrak{W}_{-1,p}^3| \lesssim \begin{cases} 2^{2sp}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}, & \text{if } \gamma + 2s > 0,\\ 2^{2sp}\|\mathfrak{F}_p g\|_{L^2}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}, & \text{if } \gamma + 2s \leq 0.\end{cases}$$

Proof. – We introduce the function ψ defined in (1.33) to decompose $\mathfrak{W}_{k,p}^3$ into two parts:

$$\begin{aligned}
\mathfrak{W}_{k,p}^3 = &\iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \Phi_k^\gamma(|v-v_*|)b(\cos\theta)\psi(2^p(v'-v))(\mathfrak{F}_p g)_*(\tilde{\mathfrak{F}}_p h)\\
&\times\big[(\tilde{\mathfrak{F}}_p f)' - \tilde{\mathfrak{F}}_p f\big]d\sigma dv_* dv + \iint_{\sigma\in\mathbb{S}^2, v_*, v\in\mathbb{R}^3} \Phi_k^\gamma(|v-v_*|)b(\cos\theta)\\
&\times\big(1-\psi(2^p(v'-v))\big)(\mathfrak{F}_p g)_*(\tilde{\mathfrak{F}}_p h)\big[(\tilde{\mathfrak{F}}_p f)' - \tilde{\mathfrak{F}}_p f\big]d\sigma dv_* dv.
\end{aligned}$$

By using Taylor expansion (2.12), the facts (2.9), (2.13) and Hölder inequality, we deduce that for $k \geq 0$,

$$|\mathfrak{W}_{k,p}^3| \lesssim 2^{(\gamma+2s)k}2^{2sp}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},$$

and

$$|\mathfrak{W}^3_{-1,p}| \lesssim \begin{cases} 2^{2sp}\|\mathfrak{F}_p g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}, & \text{if } \gamma+2s>0,\\ 2^{2sp}\|\mathfrak{F}_p g\|_{L^2}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2}, & \text{if } \gamma+2s\le 0.\end{cases}$$

The similar results hold for $\mathfrak{W}^2_{k,l}$. This completes the proof of the lemma. □

2.3. Proof of Theorem 1.1

Now we are ready to give the proof to Theorem 1.1.

Proof. – Let $a_1, b_1 \in \mathbb{R}$ with $a_1 + b_1 = 2s$. Then by Lemma 2.1, Lemma 2.2 and Lemma 2.3, we conclude that for $k \ge 0$ or $\gamma = 0$ with $k \ge -1$,

$$\sum_{l\le p-N_0} |\mathfrak{W}^1_{k,p,l}| + \sum_{m<p-N_0} |\mathfrak{W}^4_{k,p,m}| \lesssim C(a_1,b_1)\|g\|_{L^1}\|h\|_{H^{a_1}}\|f\|_{H^{b_1}},$$

$$\sum_{l\ge -1} |\mathfrak{W}^2_{k,l}| + \sum_{p\ge -1} |\mathfrak{W}^3_{k,p}| \lesssim 2^{(\gamma+2s)k}\|g\|_{L^1}\|h\|_{H^{a_1}}\|f\|_{H^{b_1}}.$$

Then we are led to the fact that if $k \ge 0$ or $\gamma = 0$ with $k \ge -1$,

$$|\langle Q_k(g,h), f\rangle_v| \lesssim C(a_1,b_1)2^{(\gamma+2s)k}\|g\|_{L^1}\|h\|_{H^{a_1}}\|f\|_{H^{b_2}}. \tag{2.14}$$

Let $a, b \in [0, 2s]$ with $a + b = 2s$. If $k = -1$, then by Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have

$$\sum_{l\le p-N_0} |\mathfrak{W}^1_{k,p,l}| + \sum_{l\ge -1} |\mathfrak{W}^2_{k,l}| + \sum_{p\ge -1} |\mathfrak{W}^3_{k,p}| + \sum_{m<p-N_0} |\mathfrak{W}^4_{k,p,m}| \lesssim (\|g\|_{L^1} + \|g\|_{L^2})\|h\|_{H^a}\|f\|_{H^b},$$

which yields

$$|\langle Q_{-1}(g,h), f\rangle_v| \lesssim (\|g\|_{L^1} + \|g\|_{L^2})\|h\|_{H^a}\|f\|_{H^b}. \tag{2.15}$$

Now we are in a position to give the upper bound for the collision operator in weighted Sobolev space. Let $w_1, w_2 \in \mathbb{R}$ with $w_1 + w_2 = \gamma + 2s$. Recalling (2.1), we infer from (2.14) and (2.15),

$$\begin{aligned} |\langle Q(g,h), f\rangle_v| \lesssim & \sum_{k\ge N_0-1} 2^{(\gamma+2s)k}\|\mathcal{U}_{k-N_0} g\|_{L^1}\|\tilde{\mathcal{P}}_k h\|_{H^a}\|\tilde{\mathcal{P}}_k f\|_{H^b} \\ & + \Big(\sum_{j\ge k+N_0, k\ge 0} 2^{(\gamma+2s)k}\|\mathcal{P}_j g\|_{L^1}\|\tilde{\mathcal{P}}_j h\|_{H^a}\|\tilde{\mathcal{P}}_j f\|_{H^b} \\ & + \sum_{j\ge -1+N_0} (\|\mathcal{P}_j g\|_{L^1} + \|\mathcal{P}_j g\|_{L^2})\|\tilde{\mathcal{P}}_j h\|_{H^a}\|\tilde{\mathcal{P}}_j f\|_{H^b}\Big) \\ & + \Big(\sum_{k\ge 0, |j-k|\le N_0} 2^{(\gamma+2s)k}\|\mathcal{P}_j g\|_{L^1}\|\mathcal{U}_{k+N_0} h\|_{H^a}\|\mathcal{U}_{k+N_0} f\|_{H^b} \\ & + (\|g\|_{L^1} + \|g\|_{L^2})\|\mathcal{U}_{N_0} h\|_{H^a}\|\mathcal{U}_{N_0} f\|_{H^b}\Big) \\ \stackrel{\text{def}}{=} & \, \mathfrak{U}_1 + \mathfrak{U}_2 + \mathfrak{U}_3. \end{aligned}$$

For the term $\mathfrak{U}_1$, thanks to Theorem 5.1, one has

$$\mathfrak{U}_1 \lesssim \|g\|_{L^1}\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}.$$

For the term $\mathfrak{U}_2$, we separate the estimate into three cases. If $\gamma+2s>0$, we have

$$\begin{aligned}\mathfrak{U}_2 &\lesssim \sum_{j\geq -1} 2^{(\gamma+2s)j}(\|\mathcal{P}_j g\|_{L^1}+\|\mathcal{P}_j g\|_{L^2})\|\tilde{\mathcal{P}}_j h\|_{H^a}\|\tilde{\mathcal{P}}_j f\|_{H^b}\\ &\lesssim (\|g\|_{L^1}+\|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}.\end{aligned}$$

In the case of $\gamma+2s=0$, it holds for any $\delta>0$,

$$\begin{aligned}\mathfrak{U}_2 &\lesssim \sum_{j\geq -1}(\|g\|_{L^1_\delta}+\|g\|_{L^2})\|\tilde{\mathcal{P}}_j h\|_{H^a}\|\tilde{\mathcal{P}}_j f\|_{H^b}\\ &\lesssim (\|g\|_{L^1_\delta}+\|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}.\end{aligned}$$

While in the case of $\gamma+2s<0$, we have

$$\begin{aligned}\mathfrak{U}_2 &\lesssim \sum_{j\geq -1}(\|\mathcal{P}_j g\|_{L^1}+\|\mathcal{P}_j g\|_{L^2})\|\tilde{\mathcal{P}}_j h\|_{H^a}\|\tilde{\mathcal{P}}_j f\|_{H^b}\\ &\lesssim (\|g\|_{L^1_{-(\gamma+2s)}}+\|g\|_{L^2_{-(\gamma+2s)}})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}.\end{aligned}$$

We remark that in each case Theorem 5.1 is used in the last inequality.

Now we turn to the term $\mathfrak{U}_3$. We first claim that it holds

$$\|\mathcal{U}_{k+N_0}h\|_{H^a} \lesssim 2^{k(-w_1)^+}\|h\|_{H^a_{w_1}}. \tag{2.16}$$

By the definition of $\mathcal{U}$, we have

$$(\mathcal{U}_{k+N_0}h)(v) = \Big[\sum_{j\leq k+N_0}\varphi(2^{-j}v)+\psi(v)\Big]h(v) \stackrel{\text{def}}{=} \tilde{\psi}_{k+N_0}(v)h(v).$$

Thanks to Lemma 5.3 and the facts, if $w_1\geq 0$,

$$\partial_v^\alpha\big(\tilde{\psi}_{k+N_0}\langle v\rangle^{-w_1}\big) \lesssim \langle v\rangle^{-w_1-|\alpha|} \lesssim \langle v\rangle^{-|\alpha|},$$

and if $w_1<0$,

$$\partial_v^\alpha\big(2^{kw_1}\tilde{\psi}_{k+N_0}\langle v\rangle^{-w_1}\big) \lesssim \langle v\rangle^{-|\alpha|},$$

we deduce that

$$\begin{aligned}\|\mathcal{U}_{k+N_0}h\|_{H^a} &\lesssim \|\tilde{\psi}_{k+N_0}W_{-w_1}(W_{w_1}h)\|_{H^a}\\ &\lesssim 2^{k(-w_1)^+}\|h\|_{H^a_{w_1}},\end{aligned}$$

which completes the proof to the claim. Now we apply the claim to the estimate of $\mathfrak{U}_3$. It is easy to get

$$\mathfrak{U}_3 \lesssim (\|g\|_{L^1_{\gamma+2s+(-w_1)^++(-w_2)^+}}+\|g\|_{L^2})\|h\|_{H^a_{w_1}}\|f\|_{H^b_{w_2}}.$$

Now patching together all the estimates for $\mathfrak{U}_1$, $\mathfrak{U}_2$ and $\mathfrak{U}_3$, we obtain the desired results. For the special case $\gamma=0$, by the estimate (2.14) and the similar argument, we can easily get (1.27). $\square$

3. Lower and upper bounds for the Boltzmann collision operator in anisotropic spaces

In this section, we will give the proof to the sharp bounds for the collision operator in anisotropic spaces. The main idea is to use the geometric decomposition explained in the introduction and also the L^2 profile of the fractional Laplace-Beltrami operator in Section 5.

3.1. Proof of Theorem 1.2

We take the following decomposition:

$$\begin{aligned}\langle -Q(g,f), f\rangle_v &= -\int_{\mathbb{R}^6} dv_* dv \int_{\mathbb{S}^2} B(|v-v_*|,\sigma) g_* f(f'-f) d\sigma \\ &= \underbrace{-\frac{1}{2}\int_{\mathbb{R}^6} dv_* dv \int_{\mathbb{S}^2} B(|v-v_*|,\sigma) g_* (f'^2-f^2) d\sigma}_{\mathscr{L}} \\ &\quad + \underbrace{\frac{1}{2}\int_{\mathbb{R}^6} dv_* dv \int_{\mathbb{S}^2} B(|v-v_*|,\sigma) g_* (f'-f)^2 d\sigma}_{\mathcal{E}_g^{\gamma}(f)} .\end{aligned} \tag{3.1}$$

By change of variables, we have

$$\begin{aligned}|\mathscr{L}| &= |\mathbb{S}^1|\left|\int_{\mathbb{R}^6}\int_0^{\frac{\pi}{2}} \sin\theta\Big(\frac{1}{\cos^3\frac{\theta}{2}} B(\frac{|v-v_*|}{\cos\frac{\theta}{2}},\cos\theta) - B(|v-v_*|,\cos\theta)\Big) g_* f^2 d\theta dv_* dv\right| \\ &\lesssim \int_{\mathbb{R}^6} |v-v_*|^{\gamma} g_* f^2 dv_* dv \stackrel{\text{def}}{=} \mathscr{R}.\end{aligned}$$

We first show that $\mathscr{R}$ can be estimated by the following lemma:

Lemma 3.1. – *Let $\varpi \in (0,1]$. For smooth functions g and f, there exist a sufficiently small constant η and a universal constant $a \in (0,1)$ such that*

1. *if $\gamma \geq 0$,*
$$|\mathscr{R}| \lesssim \|g\|_{L^1_\gamma}\|f\|_{L^2}^2 + \|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}},$$
2. *if $-2\varpi < \gamma < 0$,*
$$|\mathscr{R}| \lesssim \eta^{\frac{\gamma}{\gamma+2\varpi}} \frac{\gamma+2\varpi}{2\varpi}\|f\|^2_{L^2_{\gamma/2}} + \eta\|f\|^2_{H^{\varpi}_{\gamma/2}},$$
3. *if $\gamma + 2\varpi = 0$,*
$$\begin{aligned}|\mathscr{R}| \lesssim \big[\|g\|_{L^1_{|\gamma|}} &+ \exp([\eta^{-1}(1-a)\|g\|_{L\log L} \\ &+ 2\eta^{-1}(1-a)^{-1}\|g\|_{L^1_{2\varpi/a}}]^{\frac{1}{1-a}})\big]\|f\|^2_{L^2_{\gamma/2}} + \eta\|f\|^2_{H^{\varpi}_{\gamma/2}},\end{aligned}$$
4. *if $-1 \leq \gamma + 2\varpi < 0$ and $p > 3/2$,*
$$|\mathscr{R}| \lesssim \eta^{-\frac{3}{(\gamma+2\varpi+3)p-3}}\|g\langle\cdot\rangle^{|\gamma|}\|_{L^p}^{\frac{(\gamma+2\varpi+3)p}{(\gamma+2\varpi+3)p-3}}\|f\|^2_{L^2_{\gamma/2}} + \eta\|f\|^2_{H^{\varpi}_{\gamma/2}}.$$

Proof. – It is easy to check that if $\gamma > 0$, we have

$$|\mathscr{R}| \lesssim \|g\|_{L^1_\gamma}\|f\|^2_{L^2} + \|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}}.$$

For $\gamma < 0$, we observe that

$$\begin{aligned}
|\mathscr{R}| &= \left|\iint_{v_*,v\in\mathbb{R}^3} |v-v_*|^\gamma \big(\langle v\rangle\langle v_*\rangle^{-1}\big)^{|\gamma|} g_*\langle v_*\rangle^{|\gamma|}(f\langle v\rangle^{\gamma/2})^2 dv_* dv\right| \\
&= \left|\iint_{v_*,v\in\mathbb{R}^3} |v-v_*|^\gamma \big(\langle v\rangle\langle v_*\rangle^{-1}\big)^{|\gamma|} G_* F^2 dv_* dv\right| \\
&\lesssim \iint_{v_*,v\in\mathbb{R}^3} \frac{\langle v-v_*\rangle^{|\gamma|}}{|v-v_*|^{|\gamma|}}|G_*|F^2 dv_* dv,
\end{aligned}$$

where $G = g\langle v\rangle^{|\gamma|}$, $F = f\langle v\rangle^{\gamma/2}$.

In the case of $-2\varpi < \gamma < 0$, by using Hardy inequality, we get

$$|\mathscr{R}| \lesssim \|g\|_{L^1_{|\gamma|}}\|f\|^2_{H^{|\gamma|/2}_{\gamma/2}}.$$

Thanks to the interpolation inequality and the condition $\gamma + 2\varpi > 0$, we derive that

$$|\mathscr{R}| \lesssim \eta^{\frac{\gamma}{\gamma+2\varpi}}\frac{\gamma+2\varpi}{2\varpi}\|g\|^{2\varpi/|\gamma|}_{L^1_{|\gamma|}}\|f\|^2_{L^2_{\gamma/2}} + \eta\|f\|^2_{H^\varpi_{\gamma/2}}.$$

For the case of $\gamma + 2\varpi = 0$, we have

$$\begin{aligned}
|\mathscr{R}| &\lesssim \|G\|_{L^1}\|F\|^2_{L^2} + M\|F\|^2_{L^2} + \iint_{v,v_*\in\mathbb{R}^3} |v-v_*|^{-2\varpi}1_{|v-v_*|\le 1}(G1_{|G|\ge M})_* F^2 dv_* dv \\
&\lesssim \|g\|_{L^1_{|\gamma|}}\|f\|^2_{L^2_{\gamma/2}} + M\|f\|^2_{L^2_{\gamma/2}} + \|G1_{|G|\ge M}\|_{L^1}\|F\|^2_{H^\varpi},
\end{aligned}$$

where the Hardy inequality is used in the last step. Choose $a \in (0,1)$, then we get

$$\begin{aligned}
\|G1_{|G|\ge M}\|_{L^1} &\lesssim (\log M)^{-(1-a)}\|G(\log G)^{1-a}\|_{L^1} \\
&\lesssim (\log M)^{-(1-a)}[\|g\|^{1-a}_{L\log L}\|g\|^a_{L^1_{2\varpi/a}} + (1-a)^{-1}|\gamma|\|g\|_{L^1_{2\varpi/a}}].
\end{aligned}$$

It yields

$$\begin{aligned}
|\mathscr{R}| &\lesssim (M + \|g\|_{L^1_{|\gamma|}})\|f\|^2_{L^2_{\gamma/2}} \\
&\quad + (\log M)^{-(1-a)}[\|g\|^{1-a}_{L\log L}\|g\|^a_{L^1_{2\varpi/a}} + (1-a)^{-1}|\gamma|\|g\|_{L^1_{2\varpi/a}}]\|f\|^2_{H^\varpi_{\gamma/2}}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
|\mathscr{R}| &\lesssim \big[\|g\|_{L^1_{|\gamma|}} + \exp([\eta^{-1}\|g\|^{1-a}_{L\log L}\|g\|^a_{L^1_{2\varpi/a}} \\
&\quad + \eta^{-1}(1-a)^{-1}|\gamma|\|g\|_{L^1_{2\varpi/a}}]^{\frac{1}{1-a}})\big]\|f\|^2_{L^2_{\gamma/2}} + \eta\|f\|^2_{H^\varpi_{\gamma/2}}.
\end{aligned}$$

In particular, if $\varpi \in (0,1)$, then for $a \in (\varpi, 1)$,

$$\begin{aligned}
|\mathscr{R}| &\lesssim \big[\|g\|_{L^1_2} + \exp([\eta^{-1}(1-a)\|g\|_{L\log L} \\
&\quad + \eta^{-1}(1-a)^{-1}|\gamma|\|g\|_{L^1_2})]^{\frac{1}{1-a}})\big]\|f\|^2_{L^2_{-\varpi}} + \eta\|f\|^2_{H^\varpi_{-\varpi}}.
\end{aligned}$$

If $\varpi = 1$, then for any $\delta > 0$,

$$|\mathscr{R}| \lesssim \big[\|g\|_{L^1_2} + \exp([\eta^{-1}\|g\|_{L\log L} + \eta^{-1}\delta^{-1}\|g\|_{L^1_{2+\delta}})]^{\frac{2+\delta}{\delta}})\big]\|f\|^2_{L^2_{-1}} + \eta\|f\|^2_{H^{\varpi}_{-1}}.$$

Finally we handle the case $-1 < \gamma + 2\varpi < 0$. By Hardy-Littlewood-Sobolev inequality, we have

$$|\mathscr{R}| \lesssim M\|f\|^2_{L^2_{\gamma/2}} + \|G1_{|G|\geq M}\|_{L^{\frac{3}{\gamma+2\varpi+3}}}\|F\|^2_{H^{\varpi}}.$$

Since $p > 3/2$, we have

$$\|G1_{|G|\geq M}\|_{L^{\frac{3}{\gamma+2\varpi+3}}} \lesssim M^{-\frac{(\gamma+2\varpi+3)p-3}{3}}\|G\|_{L^p}^{\frac{(\gamma+2\varpi+3)p}{3}},$$

Thus we get

$$|\mathscr{R}| \lesssim \eta^{-\frac{3}{(\gamma+2\varpi+3)p-3}}\|g\|_{L^p_{|\gamma|}}^{\frac{(\gamma+2\varpi+3)p}{(\gamma+2\varpi+3)p-3}}\|f\|^2_{L^2_{\gamma/2}} + \eta\|f\|^2_{H^{\varpi}_{\gamma/2}}.$$

We complete the proof of the lemma. □

From now on, we focus on the estimate of the elliptic part $\mathcal{E}^{\gamma}_g$. We begin with two useful lemmas to deal with the simple case $\gamma = 0$ and then extend the results to the general cases.

LEMMA 3.2. – *Suppose g is a non-negative and smooth function. Then for any $\eta > 0$,*

$$\mathcal{E}^0_g(f) \gtrsim \mathcal{C}_4(g)\big(\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|^2_{L^2} + \|f\|^2_{H^s}\big) - \eta\|g\|_{L^1_s}\|f\|^2_{L^2_s} - (\|g\|_{L^1_s}\eta^{-1}\mathcal{C}_3(g)^{-1} + 1)\|f\|^2_{L^2}, \tag{3.2}$$

where

$$\mathcal{C}_4(g) \stackrel{\text{def}}{=} \frac{\min\{\mathcal{C}_3(g), \|g\|_{L^1_{-s}}\}}{\|g\|_{L^1_s}\eta^{-1}\mathcal{C}_3(g)^{-1} + 2}.$$

Here $\mathcal{C}_3(g) \stackrel{\text{def}}{=} \min\{\mathcal{C}_1(g), \mathcal{C}_2(g), 1\}$ and $\mathcal{C}_1(g)$ and $\mathcal{C}_2(g)$ are defined as follows:

$$\mathcal{C}_1(g) \stackrel{\text{def}}{=} 2\sin^2\varepsilon\bigg[|g|_{L^1} - \frac{|g|_{L^1_1}}{r} - \sup_{|A|<4\varepsilon(2r)^2+\frac{2\varepsilon}{\pi}(2r)^3}\int_A g(v)dv\bigg],$$

where ε and r are chosen in such a way that this quantity is positive, and

$$\mathcal{C}_2(g) \stackrel{\text{def}}{=} 2\vartheta^2\inf_{|\xi|\leq 1}\bigg|\frac{\sin^2(\vartheta|\xi|)}{\vartheta^2|\xi|^2}\bigg|\bigg[|g|_{L^1} - \frac{|g|_{L^1_1}}{r} - \sup_{|A|<4\vartheta(2r)^3(\frac{1}{\pi}+\frac{1}{r})}\int_A g(v)dv\bigg],$$

where ϑ and r are chosen in such a way that this quantity is positive.

If the function g verifies the condition (1.28)*, then due to the definition of $\mathcal{C}_4(g)$, there exists a constant $C(\delta,\lambda,\eta^{-1})$ such that*

$$\mathcal{C}_4(g) \geq C(\delta,\lambda,\eta^{-1}). \tag{3.3}$$

Proof. – By the geometric decomposition (1.35) with $u = r\tau$ and $\varsigma = \frac{\sigma+\tau}{|\sigma+\tau|} \in \mathbb{S}^2$, one has

$$\begin{aligned}\mathcal{E}_g^0(f) &\geq \frac{1}{2}\iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_* b(\cos\theta)\big((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau)\big)^2 d\sigma dv_* du\\ &\quad - \iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_* b(\cos\theta)\big(f(v_*+u^+) - f(v_*+|u|\frac{u^+}{|u^+|})\big)^2 d\sigma dv_* du\\ &\stackrel{\text{def}}{=} \mathcal{E}_1^0 - \mathcal{E}_2^0. \end{aligned} \tag{3.4}$$

Step 1: Estimate of $\mathcal{E}_1^0$. – By change of variables, we have

$$\mathcal{E}_1^0 \gtrsim \iint_{r>0,\tau,\sigma\in\mathbb{S}^2,v_*\in\mathbb{R}^3} g_* b(\sigma\cdot\tau) 1_{\sigma\cdot\tau\geq 0}\big((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau)\big)^2 r^2 d\sigma d\tau dr dv_*.$$

For fixed v_*, $\tau \in \mathbb{S}^2$ and r, if τ is chosen to be the polar direction, one has

$$d\sigma = \sin\theta d\theta d\,\mathbb{S}^1, d\varsigma = \sin\phi d\phi d\,\mathbb{S}^1,$$

where $\theta = 2\phi$. We deduce that

$$d\sigma = 4\cos\phi d\varsigma.$$

Thanks to the facts $b(\tau\cdot\sigma) \sim |\sigma-\tau|^{-(2+2s)}$ and $|\sigma-\tau| \sim |\varsigma-\tau|$, we get

$$\begin{aligned}\mathcal{E}_1^0 &\gtrsim \iint_{v_*\in\mathbb{R}^3,r>0,\tau,\varsigma\in\mathbb{S}^2} g_*|\varsigma-\tau|^{-(2+2s)}\big((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau)\big)^2(4\varsigma\cdot\tau)\\ &\qquad\qquad \times 1_{|\varsigma-\tau|^2\leq 2-\sqrt{2}} r^2 d\varsigma d\tau dr dv_*\\ &\gtrsim \iint_{v_*\in\mathbb{R}^3,r>0,\tau,\varsigma\in\mathbb{S}^2} g_*|\varsigma-\tau|^{-(2+2s)}\big((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau)\big)^2\\ &\qquad\qquad \times 1_{|\varsigma-\tau|^2\leq 2-\sqrt{2}} r^2 d\varsigma d\tau dr dv_* \end{aligned}$$

Thanks to Lemma 5.6 and Lemma 5.10, we obtain that

$$\begin{aligned}\mathcal{E}_1^0 &\gtrsim \int_{\mathbb{R}^3} g_*\|(-\triangle_{\mathbb{S}^2})^{s/2} T_{v_*} f\|_{L^2}^2 dv_* - \|g\|_{L^1}\|f\|_{L^2}^2\\ &\gtrsim \|g\|_{L^1_{-2s}}\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|_{L^2}^2 - \|g\|_{L^1}\|f\|_{H^s}^2.\end{aligned}$$

Step 2: Estimate of $\mathcal{E}_2^0$. – We introduce the dyadic decomposition in the frequency space. Set

$$\begin{aligned}\mathcal{E}_{2,k}^0(g,f) &\stackrel{\text{def}}{=} 2\iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_*\varphi_k(u) b(\cos\theta)\Big(f(v_*+u^+) - f(v_*+|u|\frac{u^+}{|u^+|})\Big)^2 d\sigma dv_* du\\ &= 2\sum_{l,p=-1}^{\infty}\iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_*\varphi_k(u) b(\cos\theta)\Big((\mathfrak{F}_l f)(v_*+u^+) - (\mathfrak{F}_l f)(v_*+|u|\frac{u^+}{|u^+|})\Big)\\ &\quad\times\Big((\mathfrak{F}_p f)(v_*+u^+) - (\mathfrak{F}_p f)(v_*+|u|\frac{u^+}{|u^+|})\Big) d\sigma dv_* du\\ &= 2\Big(\sum_{l\leq p}\mathcal{E}_{l,p} + \sum_{l>p}\mathcal{E}_{l,p}\Big).\end{aligned}$$

By the symmetric property of $\mathcal{E}_{l,p}$, without loss of the generality, we assume $l \leq p$. It is easy to check that

$$\mathcal{E}_{l,p} = \mathcal{E}^1_{l,p} + \mathcal{E}^2_{l,p}, \tag{3.5}$$

where

$$\begin{aligned}
\mathcal{E}^1_{l,p} = & \iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_*\varphi_k(u)b(\cos\theta)\Big((\mathfrak{F}_l f)(v_*+u^+) - (\mathfrak{F}_l f)(v_* + |u|\frac{u^+}{|u^+|})\Big)\\
&\times (\mathfrak{F}_p f)(v_*+u^+)d\sigma dv_* du,\\
\mathcal{E}^2_{l,p} = & \iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_*\varphi_k(u)b(\cos\theta)\Big((\mathfrak{F}_l f)(v_*+u^+) - (\mathfrak{F}_l f)(v_* + |u|\frac{u^+}{|u^+|})\Big)\\
&\times (\mathfrak{F}_p f)(v_* + |u|\frac{u^+}{|u^+|})d\sigma dv_* du.
\end{aligned}$$

Step 2.1: Estimate of $\mathcal{E}^1_{l,p}$. We introduce the function ψ to split $\mathcal{E}^1_{l,p}$ into two parts: $\mathcal{E}^{1,1}_{l,p}$ and $\mathcal{E}^{1,2}_{l,p}$ which are defined by

$$\begin{aligned}
\mathcal{E}^{1,1}_{l,p} = & \iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_*\varphi_k(u)b(\cos\theta)\Big((\mathfrak{F}_l f)(v_*+u^+) - (\mathfrak{F}_l f)(v_* + |u|\frac{u^+}{|u^+|})\Big)\\
&\times (\mathfrak{F}_p f)(v_*+u^+)\psi(2^{k/2}2^{l/2}\sqrt{1-\frac{u}{|u|}\cdot\sigma})d\sigma dv_* du,\\
\mathcal{E}^{1,2}_{l,p} = & \iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_*\varphi_k(u)b(\cos\theta)\Big((\mathfrak{F}_l f)(v_*+u^+) - (\mathfrak{F}_l f)(v_* + |u|\frac{u^+}{|u^+|})\Big)\\
&\times (\mathfrak{F}_p f)(v_*+u^+)\Big(1-\psi(2^{k/2}2^{l/2}\sqrt{1-\frac{u}{|u|}\cdot\sigma})\Big)d\sigma dv_* du.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\Big|(\mathfrak{F}_l f)(v_*+u^+) - (\mathfrak{F}_l f)\big(v_* + |u|\frac{u^+}{|u^+|})\Big|\\
&\lesssim \int_0^1 d\kappa |\nabla(\mathfrak{F}_l f)|(v_* + u^+(\kappa + (1-\kappa)\cos^{-1}(\theta/2)))|u^+||1-\cos^{-1}(\theta/2)|.
\end{aligned} \tag{3.6}$$

We have

$$\begin{aligned}
|\mathcal{E}^{1,1}_{l,p}| \lesssim & \Big(\iint_{\kappa\in[0,1],u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |g_*|\varphi_k(u)b(\cos\theta)|\nabla(\mathfrak{F}_l f)|^2(v_* + u^+(\kappa+(1-\kappa)\cos^{-1}(\theta/2)))\\
&\times|u^+||1-\cos^{-1}\theta|1_{|\theta|\lesssim 2^{-k/2}2^{-l/2}}d\kappa d\sigma dv_* du\Big)^{\frac{1}{2}}\\
&\times\Big(\iint_{\kappa\in[0,1],u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |g_*|\varphi_k(u)b(\cos\theta)|u^+||1-\cos^{-1}(\theta/2)||(\mathfrak{F}_p f)(v_*+u^+)|^2\\
&\times 1_{|\theta|\lesssim 2^{-k/2}2^{-l/2}}d\kappa d\sigma dv_* du\Big)^{\frac{1}{2}}.
\end{aligned}$$

Let $\tilde{u} = u^+(\kappa + (1-\kappa)\cos^{-1}(\theta/2))$. Then by change of the variable from u to $\tilde{u}$, one gets

$$\Big|\frac{d\tilde{u}}{du}\Big| = \Big|\frac{d\tilde{u}}{du^+}\Big|\,\Big|\frac{du^+}{du}\Big| \sim 1. \tag{3.7}$$

Moreover, we have $|u| \sim |\tilde{u}|$. Thanks to this observation, we get

$$
\begin{aligned}
|\mathcal{E}^{1,1}_{l,p}| &\lesssim 2^k \Big(\iint_{\kappa\in[0,1],\tilde{u},v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |g_*||\nabla(\mathfrak{F}_l f)|^2(v_*+\tilde{u}) b(\cos\theta)\theta^2 1_{|\theta|\lesssim 2^{-k/2}2^{-l/2}} d\kappa d\sigma dv_* d\tilde{u} \Big)^{\frac{1}{2}} \\
&\quad \times \Big(\iint_{\kappa\in[0,1],\tilde{u},v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |g_*||(\mathfrak{F}_p f)(v_*+\tilde{u})|^2 b(\cos\theta)\theta^2 1_{|\theta|\lesssim 2^{-k/2}2^{-l/2}} d\kappa d\sigma dv_* d\tilde{u} \Big)^{\frac{1}{2}} \\
&\lesssim 2^{(s-1)k}2^{(s-1)l}2^k \|g\|_{L^1}\|\mathfrak{F}_p f\|_{L^2}\|\nabla\mathfrak{F}_l f\|_{L^2} \\
&\lesssim 2^{(l-p)s/2}2^{ks}\|g\|_{L^1}\|\mathfrak{F}_p f\|_{H^{s/2}}\|\mathfrak{F}_l f\|_{H^{s/2}}.
\end{aligned}
$$

Now we turn to the estimate of $\mathcal{E}^{1,2}_{l,p}$. Following the argument applied to $\mathfrak{D}^{1,2}_k$ in Lemma 2.1, we then have

$$|\mathcal{E}^{1,2}_{l,p}| \lesssim 2^{(l-p)s/2}2^{ks}\|g\|_{L^1}\|\mathfrak{F}_p f\|_{H^{s/2}}\|\mathfrak{F}_l f\|_{H^{s/2}},$$

which implies

$$|\mathcal{E}^{1}_{l,p}| \lesssim 2^{(l-p)s/2}2^{ks}\|g\|_{L^1}\|\mathfrak{F}_p f\|_{H^{s/2}}\|\mathfrak{F}_l f\|_{H^{s/2}}.$$

Step 2.2: Estimate of $\mathcal{E}_{l,p}$. – The similar argument can be applied to $\mathcal{E}^2_{l,p}$ to get

$$|\mathcal{E}^{2}_{l,p}| \lesssim 2^{(l-p)s/2}2^{ks}\|g\|_{L^1}\|\mathfrak{F}_p f\|_{H^{s/2}}\|\mathfrak{F}_l f\|_{H^{s/2}},$$

which yields

$$|\mathcal{E}_{l,p}| \lesssim 2^{(l-p)s/2}2^{ks}\|g\|_{L^1}\|\mathfrak{F}_p f\|_{H^{s/2}}\|\mathfrak{F}_l f\|_{H^{s/2}}.$$

We arrive at

$$|\mathcal{E}^0_{2,k}(g,f)| \le 2(\sum_{l\le p}|\mathcal{E}_{l,p}| + \sum_{l\ge p}|\mathcal{E}_{l,p}|) \lesssim 2^{ks}\|g\|_{L^1}\|f\|^2_{H^{s/2}}.$$

Suppose $|v_*| \sim 2^j$ and $|u| \sim 2^k$. Then thanks to the fact $|u| \sim |u^+|$, we have

- Case 1: $j \le k - N_0$. Then $|v_* + u^+|, |v_* + |u|\frac{u^+}{|u^+|}| \sim 2^k$;
- Case 2: $j \ge k + N_0$. Then $|v_* + u^+|, |v_* + |u|\frac{u^+}{|u^+|}| \sim 2^j$;
- Case 3: $|j-k| < N_0$. Then $|v_* + u^+|, |v_* + |u|\frac{u^+}{|u^+|}| \le 2^{k+N_0}$, $|v'| \le 2^{k+N_0}$.

We get

$$
\begin{aligned}
\mathcal{E}^0_2 &= \sum_{k=-1}^{\infty} \mathcal{E}^0_{2,k}(g,f) \\
&= \sum_{k<j-N_0} \mathcal{E}^0_{2,k}(\mathcal{P}_j g, \tilde{\mathcal{P}}_j f) + \sum_{j<k-N_0} \mathcal{E}^0_{2,k}(\mathcal{P}_j g, \tilde{\mathcal{P}}_k f) + \sum_{k=-1}^{\infty} \mathcal{E}^0_{2,k}(\tilde{\mathcal{P}}_k g, \mathcal{U}_{k+N_0} f).
\end{aligned}
$$

Then

$$
\begin{aligned}
|\mathcal{E}^0_2| &\lesssim \sum_{k<j-N_0} 2^{ks}\|\mathcal{P}_j g\|_{L^1}\|\tilde{\mathcal{P}}_j f\|^2_{H^{s/2}} + \sum_{j<k-N_0} 2^{ks}\|\mathcal{P}_j g\|_{L^1}\|\tilde{\mathcal{P}}_k f\|^2_{H^{s/2}} \\
&\quad + \sum_{k=-1}^{\infty} 2^{ks}\|\tilde{\mathcal{P}}_k g\|_{L^1}\|\mathcal{U}_{k+N_0} f\|^2_{H^{s/2}} \\
&\lesssim \|g\|_{L^1_s}\|f\|^2_{H^{s/2}_{s/2}},
\end{aligned}
$$

where we use Theorem 5.1.

Patch together the estimates of $\mathcal{E}_1^0$ and $\mathcal{E}_2^0$, then we finally get

$$\begin{aligned}\mathcal{E}_g^0(f) &\gtrsim \|g\|_{L^1_{-2s}}\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|^2_{L^2} - \|g\|_{L^1_s}(\|f\|^2_{H^s} + \|f\|^2_{H^{s/2}_{s/2}})\\ &\gtrsim \|g\|_{L^1_{-2s}}\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|^2_{L^2} - \|g\|_{L^1_s}(\eta^{-1}\|f\|^2_{H^s} + \eta\|f\|^2_{L^2_s}).\end{aligned}$$

Thanks to Corollary 3 and Proposition 2 in [1], we deduce that

$$\mathcal{E}_g^0(f) + \|f\|^2_{L^2} \gtrsim \mathcal{C}_3(g)\|f\|^2_{H^s}.$$

Together with the previous lower bound, we are then led to the desired result. □

LEMMA 3.3. – *Suppose the angular function b verifies the conditions*

$$\int_0^{\pi/2} b(\cos\theta)\sin\theta\theta^2 d\theta < \infty$$

and

$$\begin{aligned}(3.8)\quad 1 + \int_{\sigma\in\mathbb{S}^2} b(\tau\cdot\sigma)\min\{|\xi|^2|\tau-\sigma|^2, 1\}d\sigma\\ \sim 1 + \int_{\sigma\in\mathbb{S}^2} b(2(\tau\cdot\sigma)^2-1)\min\{|\xi|^2|\tau-\sigma|^2,1\}d\sigma \sim W^2(\xi),\end{aligned}$$

where $\tau\in\mathbb{S}^2$ and W is a radial function satisfying $W(|\xi||\zeta|) \lesssim W(|\xi|)W(|\zeta|)$ and $W(\xi)\le\langle\xi\rangle$. Then for any smooth function g, it holds

$$|\mathcal{E}_g^0(f)| \lesssim \|g\|_{L^1}\mathcal{E}_\mu^0(f) + \|W^2 g\|_{L^1}\|W(D)f\|^2_{L^2}.$$

If g is a non-negative function verifying the condition (1.28)*, then there exist constants $C(\lambda,\delta)$ and $C(\lambda)$ such that*

$$C(\lambda,\delta)\mathcal{E}_\mu^0(f) - C(\lambda)\|f\|^2_{L^2} \lesssim \mathcal{E}_g^0(f) \lesssim C(\lambda)(\mathcal{E}_\mu^0(f) + \|f\|^2_{L^2}),$$

in other words, $\mathcal{E}_\mu^0(f) + \|f\|^2_{L^2} \sim \mathcal{E}_g^0(f) + \|f\|^2_{L^2}$.

REMARK 3.1. – *We remark that* (3.8) *holds under the assumption* (1.5) *or* (1.7) *or* (1.8)*. Moreover we have*

$$W(\xi) = \begin{cases}\langle\xi\rangle^s, & \text{under the assumption (1.5);}\\ \psi(\epsilon\xi)\langle\xi\rangle^s + \epsilon^{-s}(1-\psi(\epsilon\xi)), & \text{under the assumption (1.7);}\\ \psi(\epsilon\xi)\langle\xi\rangle + \epsilon^{s-1}(1-\psi(\epsilon\xi))\langle\xi\rangle^s, & \text{under the assumption (1.8).}\end{cases}$$

We recall that the function ψ is defined in (1.33)*. It is easy to check that for all the cases, the symbol function W satisfies the properties: $W(|\xi||\zeta|) \lesssim W(|\xi|)W(|\zeta|)$ and $W(\xi)\le\langle\xi\rangle$.*

Proof. – The proof is inspired by [5]. Without loss of generality, we assume that the function g is non-negative. By Bobylev's Formula (2.2), we have

$$\begin{aligned}(3.9)\quad \mathcal{E}_g^0(f) = \frac{1}{(2\pi)^3}\iint_{\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)\Big(\hat{g}(0)|\hat{f}(\xi)-\hat{f}(\xi^+)|^2\\ + 2\mathrm{Re}\big((\hat{g}(0)-\hat{g}(\xi^-))\hat{f}(\xi^+)\bar{\hat{f}}(\xi)\big)\Big)d\xi d\sigma.\end{aligned}$$

We recall that $\xi^- = \frac{\xi-|\xi|\sigma}{2}$ and $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$.

It implies

$$\|\mu\|_{L^1}\,\mathcal{E}^0_g(f) = \|g\|_{L^1}\big(\mathcal{E}^0_\mu(f) - \frac{1}{(2\pi)^3}I_1\big) + \frac{2\|\mu\|_{L^1}}{(2\pi)^3}I_2,$$

where

$$I_1 = \iint_{\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)\mathrm{Re}\big((\hat{\mu}(0)-\hat{\mu}(\xi^-))\hat{f}(\xi^+)\bar{\hat{f}}(\xi)\big)d\sigma d\xi,$$

and

$$\begin{aligned} I_2 &= \iint_{\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)\mathrm{Re}\big((\hat{g}(0)-\hat{g}(\xi^-))(\hat{f}(\xi^+)-\hat{f}(\xi))\bar{\hat{f}}(\xi)\big)d\xi d\sigma\\ &\quad+\iint_{\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)\mathrm{Re}(\hat{g}(0)-\hat{g}(\xi^-))|\hat{f}(\xi)|^2 d\xi d\sigma\\ &\stackrel{\mathrm{def}}{=} I_{2,1}+I_{2,2}. \end{aligned}$$

Thanks to the fact $\hat{\mu}(0)-\hat{\mu}(\xi^-) = \int_{\mathbb{R}^3}(1-\cos(v\cdot\xi^-))\mu(v)dv$, we have

$$\begin{aligned} |I_1| &= \Big|\iint_{v,\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)(1-\cos(v\cdot\xi^-))\mu(v)\mathrm{Re}(\hat{f}(\xi^+)\bar{\hat{f}}(\xi))d\sigma d\xi dv\Big|\\ &\lesssim \Big(\iint_{v,\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)(1-\cos(v\cdot\xi^-))\mu(v)|\hat{f}(\xi^+)|^2 d\sigma d\xi dv\Big)^{1/2}\\ &\quad\times\Big(\iint_{v,\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\frac{\xi}{|\xi|}\cdot\sigma)(1-\cos(v\cdot\xi^-))\mu(v)|\hat{f}(\xi)|^2 d\sigma d\xi dv\Big)^{1/2}. \end{aligned}$$

Observe that

$$(1-\cos(v\cdot\xi^-)) \lesssim |v|^2|\xi^-|^2 \le |v|^2|\xi|^2|\frac{\xi}{|\xi|}-\sigma|^2 \sim |v|^2|\xi^+|^2|\frac{\xi^+}{|\xi^+|}-\sigma|^2$$

and

$$\frac{\xi}{|\xi|}\cdot\sigma = 2(\frac{\xi^+}{|\xi^+|}\cdot\sigma)^2-1.$$

Then by change of the variable from ξ to ξ^+, the assumption (3.8) and the property $W(|\xi||\zeta|)\lesssim W(|\xi|)W(|\zeta|)$, we have

$$\begin{aligned} |I_1| &\lesssim \iint_{v,\xi\in\mathbb{R}^3} W^2(|v||\xi|)|\hat{f}(\xi)|^2\mu(v)dvd\xi\\ &\lesssim \|W^2\mu\|_{L^1}\|W(D)f\|^2_{L^2}. \end{aligned}$$

Notice that $\mathrm{Re}(\hat{g}(0)-\hat{g}(\xi^-)) = \int_{\mathbb{R}^3}(1-\cos(v\cdot\xi^-))g(v)dv$. The similar argument can be applied to get

$$|I_{2,2}| \lesssim \|W^2 g\|_{L^1}\|W(D)f\|^2_{L^2}.$$

Next, by Cauchy-Schwarz inequality, one has

$$|I_{2,1}| \lesssim \Big(\iint_{\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\cos\theta)|\hat{g}(0)-\hat{g}(\xi^-)|^2|\hat{f}(\xi)|^2 d\sigma d\xi\Big)^{\frac{1}{2}}$$
$$\times \Big(\iint_{\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\cos\theta)|\hat{f}(\xi)-\hat{f}(\xi^+)|^2 d\sigma d\xi\Big)^{\frac{1}{2}} \stackrel{\mathrm{def}}{=} (I^1_{2,1})^{\frac{1}{2}}(I^2_{2,1})^{\frac{1}{2}}.$$

Observe that $\hat{g}(0)-\hat{g}(\xi^-) = \int_{\mathbb{R}^3}(1-e^{-iv\cdot\xi^-})g(v)dv$, then it holds

$$I^1_{2,1} \lesssim \iint_{v,w,\xi\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\cos\theta)g(v)g(w)(|1-e^{-iv\cdot\xi^-}|^2+|1-e^{-iw\cdot\xi^-}|^2)|\hat{f}(\xi)|^2 d\sigma d\xi dv dw$$
$$\lesssim \|g\|_{L^1}\|W^2 g\|_{L^1}\|W(D)f\|^2_{L^2}.$$

Thanks to (3.9), we have

$$\frac{1}{(2\pi)^3}\|\mu\|_{L^1} I^2_{2,1} = \mathcal{E}^0_\mu(f) - \frac{1}{(2\pi)^3} I_1,$$

which implies

$$I^2_{2,1} \lesssim \mathcal{E}^0_\mu(f) + \|W^2\mu\|_{L^1}\|W(D)f\|^2_{L^2}.$$

We get

$$|I_{2,1}| \lesssim \eta\|g\|_{L^1}\mathcal{E}^0_\mu(f) + \eta^{-1}(\|W^2 g\|_{L^1}+\|g\|_{L^1})\|W(D)f\|^2_{L^2}.$$

Combining the above estimates, we arrive at

$$\begin{aligned}(3.10)\quad \|\mu\|_{L^1}\mathcal{E}^0_g(f) - C(\eta)\|W^2 g\|_{L^1}\|W(D)f\|^2_{L^2}\\ \lesssim (1-\eta)\|g\|_{L^1}\mathcal{E}^0_\mu(f) \lesssim \|\mu\|_{L^1}\mathcal{E}^o_g(f) + C(\eta)\|W^2 g\|_{L^1}\|W(D)f\|^2_{L^2},\end{aligned}$$

which is enough to derive the first inequality in the lemma. Moreover, if the function g verifies the condition (1.28), then by the computation in [1] and the assumption (3.8), we have

$$\mathcal{E}^0_g(f) + \|f\|^2_{L^2} \gtrsim \mathcal{C}_3(g)\|W(D)f\|^2_{L^2}. \tag{3.11}$$

Together with (3.10), we thus get the equivalence in the lemma. □

In the next lemma, we will show that the lower bound of $\mathcal{E}^\gamma_g(f)$ can be reduced to the lower bound of $\mathcal{E}^0_\mu(W_{\gamma/2}f)$.

LEMMA 3.4. – *Suppose that the angular function b verifies the same conditions in Lemma 3.3 and g is a non-negative function verifying the condition* (1.28). *Then there exists a constant $C(\lambda,\delta)$ such that*

$$\mathcal{E}^0_\mu(W_{\gamma/2}f) \le C(\lambda,\delta)\big(\|f\|^2_{L^2_{\gamma/2}} + \mathcal{E}^\gamma_g(f)\big).$$

Proof. – Let χ be a radial and smooth function such that $0\le\chi\le 1$, $\chi = 1$ on B_1 and $\mathrm{Supp}\chi \subset B_2$. We set $\chi_R(v) = \chi(v/R)$. We recall the notation: $W_l(v) = \langle v\rangle^l$.

CASE 1: $|v|$ IS SUFFICIENTLY LARGE. — It is easy to check

$$\begin{aligned}\mathcal{E}_g^\gamma(f) &\gtrsim \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |v-v_*|^\gamma (g\chi_{\frac{R}{8}})_* b(\cos\theta)(f'-f)^2(1-\chi_R)^2 d\sigma dv_* dv\\ &\gtrsim \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} W_{\gamma/2}^2 (g\chi_{\frac{R}{8}})_* b(\cos\theta)(f'-f)^2(1-\chi_R)^2 d\sigma dv_* dv.\end{aligned}$$

Thanks to the inequality $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$, we obtain that

$$\begin{aligned}\mathcal{E}_g^\gamma(f) \geq \frac{1}{2}\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} &(g\chi_{\frac{R}{8}})_* b(\cos\theta)\big((W_{\gamma/2}(1-\chi_R)f)' - W_{\gamma/2}(1-\chi_R)f\big)^2 d\sigma dv_* dv\\ -2\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} &(g\chi_{\frac{R}{8}})_* b(\cos\theta) f'^2\big((W_{\gamma/2}(1-\chi_R))' - W_{\gamma/2}(1-\chi_R)\big)^2 d\sigma dv_* dv.\end{aligned}$$

Suppose that $\kappa(v) = v + \kappa(v'-v)$ with $\kappa \in [0,1]$. It is easy to check that $\frac{\sqrt{2}}{2}|v-v_*| \leq |v'-v_*| \leq |\kappa(v)-v_*| \leq |v-v_*|$. Since now $|v_*| \leq R/4$, then if $|v| \geq R$, we have $|v| \sim |\kappa(v)| \sim |v-v_*|$. Similarly if $|v'| \geq R$, we have $|v'| \sim |\kappa(v)| \sim |v-v_*|$. In both cases, we then have $|v'| \sim |v-v_*| \sim |\kappa(v)|$. By the mean value theorem, we get

$$\begin{aligned}&\left|\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} (g\chi_{\frac{R}{8}})_* b(\cos\theta) f'^2\big((W_{\gamma/2}(1-\chi_R))' - W_{\gamma/2}(1-\chi_R)\big)^2 d\sigma dv_* dv\right|\\ &\lesssim \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} (g\chi_{\frac{R}{8}})_* b(\cos\theta) f'^2 \langle\kappa(v)\rangle^{\gamma-2}|v-v_*|^2\theta^2 1_{|v'|\sim|v-v_*|\sim|\kappa(v)|} d\sigma dv_* dv\\ &\lesssim \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} (g\chi_{\frac{R}{8}})_* b(\cos\theta)\theta^2 f'^2 W_\gamma' d\sigma dv_* dv\\ &\lesssim \|g\|_{L^1}\|f\|_{L^2_{\gamma/2}}^2.\end{aligned}$$

Thus we arrive at

$$\begin{aligned}\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} (g\chi_{\frac{R}{8}})_* b(\cos\theta)\big((W_{\gamma/2}(1-\chi_R)f)' - W_{\gamma/2}(1-\chi_R)f\big)^2 d\sigma dv_* dv&\\ \lesssim \|g\|_{L^1}\|f\|_{L^2_{\gamma/2}}^2 + \mathcal{E}_g^\gamma(f)&,\end{aligned}$$

that is,

$$\mathcal{E}^0_{g\chi_{\frac{R}{8}}}((1-\chi_R)W_{\gamma/2}f) \lesssim \|g\|_{L^1}\|f\|_{L^2_{\gamma/2}}^2 + \mathcal{E}_g^\gamma(f). \tag{3.12}$$

CASE 2: $|v|$ IS BOUNDED. — Let A, B be the subsets in B_{3R}. We denote χ_A and χ_B by the mollified characteristic functions corresponding to the sets A and B. Then it yields

$$\begin{aligned}&\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\cos\theta)(g\chi_B)_* f'^2(\chi_A'-\chi_A)^2 d\sigma dv_* dv\\ &\quad\lesssim \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\cos\theta)(g\chi_B)_* f'^2(\chi_A'-\chi_A)^2 1_{|v-v_*|\leq 8R}1_{|v'|\leq 8R} d\sigma dv_* dv\\ &\quad\lesssim \|\nabla(\chi_A)\|_{L^\infty}^2 \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2\, a} b(\cos\theta)(g\chi_B)_* f'^2|v-v_*|^2\theta^2 1_{|v-v_*|\leq 8R}\\ &\qquad 1_{|v'|\leq 8R} d\sigma dv_* dv\end{aligned}$$

$$\lesssim R^2\|\nabla(\chi_A)\|^2_{L^\infty}\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} b(\cos\theta)(g\chi_B)_* f'^2 1_{|v'|\leq 8R}\theta^2 d\sigma dv_* dv$$

$$\lesssim \|\nabla(\chi_A)\|^2_{L^\infty} R^2\max\{R^{-\gamma},1\}\|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}}. \tag{3.13}$$

With the help of Lemma 2.1 in [1] and replacing (35) in [1] by (3.13), we conclude that if $\gamma<0$,

$$R^\gamma\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2}(g\chi_{\frac{R}{8}})_* b(\cos\theta)\big((\chi_R f)'-\chi_R f\big)^2 d\sigma dv_* dv\lesssim \|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}}+\mathcal{E}^\gamma_g(f),$$

and if $\gamma>0$,

$$r_0^\gamma\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2}(g\chi_{B_j})_* b(\cos\theta)\big((\chi_{A_j} f)'-\chi_{A_j} f\big)^2 d\sigma dv_* dv$$
$$\lesssim r_0^{-2}R^2\|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}}+\mathcal{E}^\gamma_g(f),$$

where $\chi_{A_j}=\chi(\frac{v-v_j}{r_0})$, $\chi_{B_j}=\chi_{3R}-\chi(\frac{v-v_j}{3r_0})$ with $v_j\in B_{2R}$ and r_0 will be chosen later. Notice that

$$(\chi_A f)'-\chi_A f=\big((\chi_A W_{\gamma/2}f)'-(\chi_A W_{\gamma/2}f)\big)W_{-\gamma/2}+(\chi_A W_{\gamma/2}f)'\big((W_{-\gamma/2})'-W_{-\gamma/2}\big).$$

By a slight modification, we may derive that if $\gamma<0$,

$$\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2}(g\chi_{\frac{R}{8}})_* b(\cos\theta)\big((\chi_R W_{\gamma/2}f)'-(\chi_R W_{\gamma/2}f)\big)^2 d\sigma dv_* dv$$
$$\lesssim R^{2-2\gamma}\|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}}+R^{-\gamma}\mathcal{E}^\gamma_g(f), \tag{3.14}$$

and if $\gamma>0$,

$$\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2}(g\chi_{B_j})_* b(\cos\theta)\big((\chi_{A_j} W_{\gamma/2}f)'-(\chi_{A_j} W_{\gamma/2}f)\big)^2 d\sigma dv_* dv \tag{3.15}$$
$$\lesssim r_0^{-2}R^{4+\gamma}\|g\|_{L^1}\|f\|^2_{L^2_{\gamma/2}}+R^\gamma r_0^{-\gamma}\mathcal{E}^\gamma_g(f).$$

By finite covering theorem, there exists an integer N such that

$$B_{2R}\subset\bigcup_{j=1}^N\{|v-v_j|\leq r_0\}\quad\text{and}\quad N\sim\left(\frac{R}{r_0}\right)^3, \tag{3.16}$$

where $v_j\in B_{2R}$. Observe that

$$\|g\chi_{R/8}\|_{L^1}\geq\|g\|_{L^1}-R^{-1}\|g\|_{L^1_1}$$

and

$$\|g\chi_{B_j}\|_{L^1}\geq\|g\|_{L^1}-(3R)^{-1}\|g\|_{L^1_1}-M(6r_0)^3-(\log M)^{-1}\|g\|_{L\log L}.$$

Then by choosing $R=\frac{4\lambda}{3\delta}+1$, $M=e^{4\lambda/\delta}$ and $r_0=\frac16 e^{-4\lambda/(3\delta)}$, we get

$$N\sim 6^3(\frac{4\lambda}{3\delta}+1)^3 e^{4\lambda/\delta}\stackrel{\text{def}}{=}C_1(\delta,\lambda)$$

and

$$\|g\chi_{R/8}\|_{L^1}\geq\delta/4,\ \|g\chi_{B_j}\|_{L^1}\geq\delta/4.$$

Then there exists a constant $C(\lambda,\delta)$ such that

$$\mathcal{C}_3(g\chi_{R/8}) \geq C(\lambda,\delta), \quad \mathcal{C}_3(g\chi_{B_j}) \geq C(\lambda,\delta).$$

Thanks to (3.10) and (3.11) in the proof of Lemma 3.3, we may rewrite (3.12)-(3.15) as:

$$\mathcal{E}^0_\mu((1-\chi_R)W_{\gamma/2}f) \lesssim C_2(\lambda,\delta)\big(\|f\|^2_{L^2_{\gamma/2}} + \mathcal{E}^\gamma_g(f)\big), \tag{3.17}$$

$$\mathcal{E}^0_\mu(\chi_{A_j}W_{\gamma/2}f) \lesssim C_3(\lambda,\delta)\big(\|f\|^2_{L^2_{\gamma/2}} + \mathcal{E}^\gamma_g(f)\big), \quad \text{if } \gamma > 0, \tag{3.18}$$

$$\mathcal{E}^0_\mu(\chi_R W_{\gamma/2}f) \lesssim C_4(\lambda,\delta)\big(\|f\|^2_{L^2_{\gamma/2}} + \mathcal{E}^\gamma_g(f)\big), \quad \text{if } \gamma < 0. \tag{3.19}$$

We conclude that (3.17) and (3.19) yield the desired result for soft potentials. For $\gamma > 0$, thanks to the facts

$$\begin{aligned}\mathcal{E}^0_\mu(\chi_{A_j}W_{\gamma/2}f) \geq \frac{1}{2}\iint_{\sigma\in\mathbb{S}^2,v,v_*\in\mathbb{R}^3} \mu_* b(\cos\theta)\chi^2_{A_j}\big((W_{\gamma/2}f)' - (W_{\gamma/2}f)\big)^2 d\sigma dv_* dv \\ - \iint_{\sigma\in\mathbb{S}^2,v,v_*\in\mathbb{R}^3} \mu_* b(\cos\theta)(\chi'_{A_j} - \chi_{A_j})^2\big((W_{\gamma/2}f)'\big)^2 d\sigma dv_* dv,\end{aligned}$$

and

$$\mu_*(\chi'_{A_j} - \chi_{A_j})^2(1_{|v_*|\leq 8R} + 1_{|v_*|\geq 8R}) \lesssim \mu_*(\chi'_{A_j} - \chi_{A_j})^2(1_{|v-v_*|\leq 15R} + 1_{|v_*-v|\sim|v_*|}),$$

we have

$$\begin{aligned}\iint_{\sigma\in\mathbb{S}^2,v,v_*\in\mathbb{R}^3} \mu_* b(\cos\theta)\chi^2_{A_j}\big((W_{\gamma/2}f)' - (W_{\gamma/2}f)\big)^2 d\sigma dv_* dv \\ \lesssim C_5(\lambda,\delta)\big(\|f\|^2_{L^2_{\gamma/2}} + \mathcal{E}^\gamma_g(f)\big).\end{aligned} \tag{3.20}$$

From which together with (3.16) and (3.17), we are led to the desired result for hard potentials. We complete the proof of the lemma. □

We are now in a position to complete the proof of Theorem 1.2.

Proof. – The desired results are easily derived from (3.1), Lemma 3.1, Lemma 3.2 and Lemma 3.4. □

3.2. Proof of Theorem 1.3

Finally we give the proof to Theorem 1.3.

Proof. – Following the computation in [1], we first have if $\gamma \geq 0$,

$$\begin{aligned}D_B(f) &= \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta) f_*(f\ln\frac{f}{f'} - f + f')dvdv_*d\sigma \\ &\quad - \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta)(f - f')dvdv_*d\sigma \\ &\geq \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta) f_*(\sqrt{f'} - \sqrt{f})^2 dvdv_*d\sigma \\ &\quad - \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |v-v_*|^\gamma b(\cos\theta) f_*(f - f')dvdv_*d\sigma\end{aligned}$$

$$\geq \mathcal{E}^{\gamma}_{f}(\sqrt{f}) - \|f\|_{L^1}\|f\|_{L^1_2} \gtrsim \mathcal{E}^{0}_{\mu}(W_{\gamma/2}\sqrt{f}) - \|f\|^2_{L^1_2},$$

where we use the inequality $x \ln \frac{x}{y} - x + y \geq (\sqrt{x} - \sqrt{y})^2$ and Lemma 3.4.

For soft potentials ($\gamma < 0$), we observe that

$$\begin{aligned}
D_B(f) &= \frac{1}{4}\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |v-v_*|^{\gamma} b(\cos\theta)(f'f'_* - ff_*)\ln\frac{f'_*f'}{f_*f}dvdv_*d\sigma \\
&\geq \frac{1}{4}\iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} \langle v-v_*\rangle^{\gamma} b(\cos\theta)(f'f'_* - ff_*)\ln\frac{f'_*f'}{f_*f}dvdv_*d\sigma \\
&\geq \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} \langle v-v_*\rangle^{\gamma} b(\cos\theta) f_*(\sqrt{f'} - \sqrt{f})^2 dvdv_*d\sigma \\
&\quad - \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} \langle v-v_*\rangle^{\gamma} b(\cos\theta) f_*(f - f')dvdv_*d\sigma \\
&\gtrsim \mathcal{E}^{0}_{\mu}(W_{\gamma/2}\sqrt{f}) - \|f\|^2_{L^1}.
\end{aligned}$$

The last inequality is deduced from the proof of Lemma 3.4. We complete the proof of the theorem with the help of Lemma 3.2. □

3.3. Proof of Theorem 1.4

Now we are ready to give the proof to Theorem 1.4.

Proof. – To get sharp bounds for the Boltzmann collision operator in anisotropic spaces, we only need to give the new estimates to $\mathfrak{W}^2_{k,l}$ and $\mathfrak{W}^3_{k,p}$ due to the geometric decomposition (1.35) for $k \geq 0$. Recalling that $u = r\tau$ and $\varsigma = \frac{\sigma+\tau}{|\sigma+\tau|} \in \mathbb{S}^2$, we have

$$\mathfrak{W}^3_{k,p} = \mathfrak{W}^{3,1}_{k,p} + \mathfrak{W}^{3,2}_{k,p},$$

where

$$\begin{aligned}
\mathfrak{W}^{3,1}_{k,p} &= \iint_{\sigma\in\mathbb{S}^2,v_*,u\in\mathbb{R}^3} \Phi^{\gamma}_{k}(|u|)b(\sigma\cdot\tau)(\mathfrak{F}_p g)_*(T_{v_*}\tilde{\mathfrak{F}}_p h)(r\tau) \\
&\quad \times \big((T_{v_*}\tilde{\mathfrak{F}}_p f)(r\varsigma) - (T_{v_*}\tilde{\mathfrak{F}}_p f)(r\tau)\big)d\sigma du dv_*, \\
\mathfrak{W}^{3,2}_{k,p} &= \iint_{\sigma\in\mathbb{S}^2,v_*,u\in\mathbb{R}^3} \Phi^{\gamma}_{k}(|v-v_*|)b(\cos\theta)(\mathfrak{F}_p g)_*(\tilde{\mathfrak{F}}_p h) \\
&\quad \times \big((\tilde{\mathfrak{F}}_p f)(v_* + u^+) - (\tilde{\mathfrak{F}}_p f)(v_* + |u|\frac{u^+}{|u^+|})\big)d\sigma dv_* du.
\end{aligned}$$

For the term $\mathfrak{W}^{3,1}_{k,p}$, by change of the variable from σ to ς, we have

$$\begin{aligned}
\mathfrak{W}^{3,1}_{k,p} &= \iint_{\varsigma,\tau\in\mathbb{S}^2,v_*\in\mathbb{R}^3,r>0} \Phi^{\gamma}_{k}(r)b(2(\varsigma\cdot\tau)^2 - 1)(\mathfrak{F}_p g)_*(T_{v_*}\tilde{\mathfrak{F}}_p h)(r\tau) \\
&\quad \times \big((T_{v_*}\tilde{\mathfrak{F}}_p f)(r\varsigma) - (T_{v_*}\tilde{\mathfrak{F}}_p f)(r\tau)\big)r^2 4(\varsigma\cdot\tau)d\varsigma dv_* d\tau dr \\
&= \iint_{v_*\in\mathbb{R}^3,r>0} dr dv_* \Phi^{\gamma}_{k}(r)(\mathfrak{F}_p g)_* r^2 \iint_{\varsigma,\tau\in\mathbb{S}^2} b\big(2(\varsigma\cdot\tau)^2 - 1\big)4(\varsigma\cdot\tau) \\
&\quad \times (T_{v_*}\tilde{\mathfrak{F}}_p h)(r\tau)\big((T_{v_*}\tilde{\mathfrak{F}}_p f)(r\varsigma) - (T_{v_*}\tilde{\mathfrak{F}}_p f)(r\tau)\big)d\varsigma d\tau.
\end{aligned}$$

Thanks to Corollary 5.1 and Lemma 5.10, for $a, b \in [0, 2s]$ with $a + b = 2s$, we get

$$
\begin{aligned}
|\mathfrak{W}^{3,1}_{k,p}| &\lesssim 2^{\gamma k} \int_{\mathbb{R}^3} |(\mathfrak{F}_p g)_*| \|(1-\triangle_{\mathbb{S}^2})^{a/2}(T_{v_*}\tilde{\mathfrak{F}}_p h)\|_{L^2} \|(1-\triangle_{\mathbb{S}^2})^{b/2}(T_{v_*}\tilde{\mathfrak{F}}_p f)\|_{L^2} dv_* \\
&\lesssim 2^{\gamma k} \|\mathfrak{F}_p g\|_{L^1_{2s}} (\|(-\triangle_{\mathbb{S}^2})^{a/2}\tilde{\mathfrak{F}}_p h\|_{L^2} + \|\tilde{\mathfrak{F}}_p h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2}\tilde{\mathfrak{F}}_p f\|_{L^2} + \|\tilde{\mathfrak{F}}_p f\|_{H^b}).
\end{aligned}
$$

Hence, together with Lemma 5.8, we deduce that

$$
\sum_{p=-1}^{\infty} |\mathfrak{W}^{3,1}_{k,p}| \lesssim 2^{\gamma k} \|g\|_{L^1_{2s}} (\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f\|_{L^2} + \|f\|_{H^b}).
$$

For the term $\mathfrak{W}^{3,2}_{k,p}$, we may follow the argument used to bound $\mathcal{E}_{l,p}$ (see (3.5)) to get for $k \geq 0$,

$$
|\mathfrak{W}^{3,2}_{k,p}| \lesssim 2^{(\gamma+s)k} 2^{sp} \|g\|_{L^1} \|\tilde{\mathfrak{F}}_p h\|_{L^2} \|\tilde{\mathfrak{F}}_p f\|_{L^2},
$$

which implies

$$
\sum_{p=-1}^{\infty} |\mathfrak{W}^{3,2}_{k,p}| \lesssim 2^{(\gamma+s)k} \|g\|_{L^1} \|h\|_{H^{a_1}} \|f\|_{H^{b_1}},
$$

where $a_1, b_1 \in \mathbb{R}$ with $a_1 + b_1 = s$.

We finally arrive at for $k \geq 0$,

$$
\begin{aligned}
\sum_{p=-1}^{\infty} |\mathfrak{W}^{3}_{k,p}| + \sum_{l=-1}^{\infty} |\mathfrak{W}^{2}_{k,l}| &\lesssim 2^{\gamma k} \|g\|_{L^1_{2s}} (\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f\|_{L^2} + \|f\|_{H^b}) \\
&\quad + 2^{(\gamma+s)k} \|g\|_{L^1} \|h\|_{H^{a_1}} \|f\|_{H^{b_1}}.
\end{aligned}
$$

Thanks to Lemma 2.1 and Lemma 2.2, we also have for $k \geq 0$,

$$
\sum_{l \leq p-N_0} |\mathfrak{W}^{1}_{k,p,l}| + \sum_{m < p-N_0} |\mathfrak{W}^{4}_{k,p,m}| \lesssim 2^{\gamma k} \|g\|_{L^1} \|h\|_{H^a} \|f\|_{H^b}.
$$

Now we are in a position to prove the sharp bounds. We conclude that for $k \geq 0$,

$$
\begin{aligned}
|\langle Q_k(g,h), f\rangle_v| &\lesssim 2^{(\gamma+s)k} \|g\|_{L^1} \|h\|_{H^{a_1}} \|f\|_{H^{b_1}} \\
&\quad + 2^{\gamma k} \|g\|_{L^1_{2s}} (\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f\|_{L^2} + \|f\|_{H^b}),
\end{aligned} \tag{3.21}
$$

and

$$
|\langle Q_{-1}(g,h), f\rangle_v| \lesssim (\|g\|_{L^1} + \|g\|_{L^2}) \|h\|_{H^a} \|f\|_{H^b}. \tag{3.22}
$$

Recalling (2.1), we rewrite it by

$$
\begin{aligned}
\langle Q(g,h), f\rangle_v &= \sum_{k \geq N_0-1} \langle Q_k(\mathcal{U}_{k-N_0} g, \tilde{\mathcal{P}}_k h), \tilde{\mathcal{P}}_k f\rangle_v + \sum_{j \geq k+N_0} \langle Q_k(\mathcal{P}_j g, \tilde{\mathcal{P}}_j h), \tilde{\mathcal{P}}_j f\rangle_v \\
&\quad + \sum_{|j-k| \leq N_0} \langle Q_k(\mathcal{P}_j g, \mathcal{U}_{k+N_0} h), \mathcal{U}_{k+N_0} f\rangle_v \\
&= \mathfrak{U}_4 + \mathfrak{U}_5 + \mathfrak{U}_6.
\end{aligned}
$$

Thanks to (3.21) and (3.22), we can give the estimates term by term .

Suppose $w_1, w_2 \in \mathbb{R}$ with $w_1 + w_2 = \gamma + s$. It is not difficult to check

$$|\mathfrak{U}_4| \lesssim \sum_{k \geq N_0 - 1} \Big(2^{(\gamma+s)k} \|\mathcal{U}_{k-N_0} g\|_{L^1} \|\tilde{\mathcal{P}}_k h\|_{H^{a_1}} \|\tilde{\mathcal{P}}_k f\|_{H^{b_1}} + 2^{\gamma k} \|\mathcal{U}_{k-N_0} g\|_{L^1_{2s}} \times (\|(-\triangle_{\mathbb{S}^2})^{a/2} \tilde{\mathcal{P}}_k h\|_{L^2} + \|\tilde{\mathcal{P}}_k h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} \tilde{\mathcal{P}}_k f)\|_{L^2} + \|\tilde{\mathcal{P}}_k f\|_{H^b}) \Big).$$

Hence, together with Theorem 5.1, we get

$$|\mathfrak{U}_4| \lesssim \|g\|_{L^1_{2s}} \Big((\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}}) \times (\|(-\triangle_{\mathbb{S}^2})^{b/2} f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}}) + \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{b_1}_{w_2}} \Big).$$

For the term $\mathfrak{U}_5$, it holds

$$|\mathfrak{U}_5| \lesssim \sum_{j \geq k+N_0, k \geq 0} \Big(2^{(\gamma+s)k} \|\mathcal{P}_j g\|_{L^1} \|\tilde{\mathcal{P}}_j h\|_{H^{a_1}} \|\tilde{\mathcal{P}}_j f\|_{H^{b_1}} + 2^{\gamma k} \|\mathcal{P}_j g\|_{L^1_{2s}} \times (\|(-\triangle_{\mathbb{S}^2})^{a/2} \tilde{\mathcal{P}}_j h\|_{L^2} + \|\tilde{\mathcal{P}}_j h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} \tilde{\mathcal{P}}_j f)\|_{L^2} + \|\tilde{\mathcal{P}}_j f\|_{H^b}) \Big) + \sum_{j \geq N_0 - 1} (\|\mathcal{P}_j g\|_{L^1} + \|\mathcal{P}_j g\|_{L^2}) \|\tilde{\mathcal{P}}_j h\|_{H^a} \|\tilde{\mathcal{P}}_j f\|_{H^b}.$$

Thanks to Theorem 5.1, we obtain that

1. if $\gamma > 0$

$$|\mathfrak{U}_5| \lesssim (\|g\|_{L^1_{2s}} + \|g\|_{L^2}) \Big((\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}}) \times (\|(-\triangle_{\mathbb{S}^2})^{b/2} f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}}) + \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{b_1}_{w_2}} \Big).$$

2. If $\gamma = 0$, for any $\delta > 0$,

$$|\mathfrak{U}_5| \lesssim (\|g\|_{L^1_{2s+\delta}} + \|g\|_{L^2}) \Big((\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a}) \times (\|(-\triangle_{\mathbb{S}^2})^{b/2} f)\|_{L^2} + \|f\|_{H^b}) + \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{a_2}_{w_2}} \Big).$$

3. If $\gamma < 0$,

$$|\mathfrak{U}_5| \lesssim (\|g\|_{L^1_{-\gamma+2s}} + \|g\|_{L^2_{-\gamma}}) \Big((\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}}) \times (\|(-\triangle_{\mathbb{S}^2})^{b/2} f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}}) + \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{b_1}_{w_2}} \Big).$$

Finally we turn to the estimate of $\mathfrak{U}_6$. One has

$$
\begin{aligned}
|\mathfrak{U}_6| \lesssim \sum_{|j-k|\le N_0, k\ge 0} \Big(& 2^{(\gamma+s)k}\|\mathcal{P}_j g\|_{L^1}\|\mathcal{U}_{k+N_0}h\|_{H^{a_1}}\|\mathcal{U}_{k+N_0}f\|_{H^{b_1}} + 2^{\gamma k}\|\mathcal{P}_j g\|_{L^1_{2s}} \\
& \times (\|(-\triangle_{\mathbb{S}^2})^{a/2}\mathcal{U}_{k+N_0}h\|_{L^2} + \|\mathcal{U}_{k+N_0}h\|_{H^a}) \\
& \times (\|(-\triangle_{\mathbb{S}^2})^{b/2}\mathcal{U}_{k+N_0}f)\|_{L^2} + \|\mathcal{U}_{k+N_0}f\|_{H^b})\Big) \\
& +(\|\tilde{\mathcal{P}}_{-1}g\|_{L^1} + \|\tilde{\mathcal{P}}_{-1}g\|_{L^2})\|\mathcal{U}_{N_0}h\|_{H^a}\|\mathcal{U}_{N_0}f\|_{H^b}.
\end{aligned}
$$

Then by Lemma 5.8 and (2.16), we have

1. if $\gamma > 0$

$$
\begin{aligned}
|\mathfrak{U}_6| \lesssim (\|g\|_{L^1_{\gamma+2s}} + \|g\|_{L^1_{\gamma+s+(-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2})\Big(& (\|(-\triangle_{\mathbb{S}^2})^{a/2}h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}}) \\
\times (\|(-\triangle_{\mathbb{S}^2})^{b/2}f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}}) + \|h\|_{H^{a_1}_{w_1}}\|f\|_{H^{b_1}_{w_2}}\Big).
\end{aligned}
$$

2. If $\gamma = 0$,

$$
\begin{aligned}
|\mathfrak{U}_6| \lesssim (\|g\|_{L^1_{2s}} + \|g\|_{L^1_{s+(-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2})\Big(& (\|(-\triangle_{\mathbb{S}^2})^{a/2}h\|_{L^2} + \|h\|_{H^a}) \\
\times (\|(-\triangle_{\mathbb{S}^2})^{b/2}f)\|_{L^2} + \|f\|_{H^b}) + \|h\|_{H^{a_1}_{w_1}}\|f\|_{H^{b_1}_{w_2}}\Big).
\end{aligned}
$$

3. If $\gamma < 0$,

$$
\begin{aligned}
|\mathfrak{U}_6| \lesssim (\|g\|_{L^1_{-\gamma+2s}} + \|g\|_{L^1_{\gamma+s+(-w_1)^+ + (-w_2)^+}} + \|g\|_{L^2})\Big(& (\|(-\triangle_{\mathbb{S}^2})^{a/2}h\|_{L^2_{\gamma/2}} + \|h\|_{H^a_{\gamma/2}}) \\
\times (\|(-\triangle_{\mathbb{S}^2})^{b/2}f)\|_{L^2_{\gamma/2}} + \|f\|_{H^b_{\gamma/2}}) + \|h\|_{H^{a_1}_{w_1}}\|f\|_{H^{b_1}_{w_2}}\Big).
\end{aligned}
$$

The theorem is obtained by patching together all the estimates to $\mathfrak{U}_4, \mathfrak{U}_5$ and $\mathfrak{U}_6$. We complete the proof of the theorem. ☐

4. Sharp bounds for the Landau collision operator via grazing collision limit

In this section, we will show that the strategy used to handle the Boltzmann collision operator is robust. It can be applied to capture the intrinsic structure of the collision operator in the process of the grazing collision limit. Before giving the estimates, we first introduce the special function W^ϵ defined by

$$
W^\epsilon(x) = \psi(\epsilon x)\langle x\rangle + \epsilon^{s-1}(1-\psi(\epsilon x))\langle x\rangle^s, \tag{4.1}
$$

which characterizes the symbol of the collision operator in the process of the limit. We emphasize that the function ψ is defined in (1.33).

We begin with a technical lemma which describes the behavior of the fractional Laplace-Beltrami operator in the limit. We postpone the proof to the end of Section 5.4.

LEMMA 4.1. – *Suppose* $0 < s < 1$. *For any smooth function* f *defined in* $\mathbb{S}^2$, *the following equivalences hold:*

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{S}^2)} &+ \epsilon^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}1_{|\sigma-\tau|\le\epsilon}d\sigma d\tau \\ &\sim \|f\|^2_{L^2(\mathbb{S}^2)} + \|(-\triangle_{\mathbb{S}^2})^{1/2}\mathbb{P}_{\le\frac{1}{\epsilon}}f\|^2_{L^2(\mathbb{S}^2)} + \epsilon^{2s-2}\|(-\triangle_{\mathbb{S}^2})^{s/2}\mathbb{P}_{>\frac{1}{\epsilon}}f\|^2_{L^2(\mathbb{S}^2)} \\ &\sim \|f\|^2_{L^2(\mathbb{S}^2)} + \|\big((-\triangle_{\mathbb{S}^2})^{1/2}\mathbb{P}_{\le\frac{1}{\epsilon}} + \epsilon^{s-1}(-\triangle_{\mathbb{S}^2})^{s/2}\mathbb{P}_{>\frac{1}{\epsilon}}\big)f\|^2_{L^2(\mathbb{S}^2)},\end{aligned}$$

where the projection operators $\mathbb{P}_{\le\frac{1}{\epsilon}}$ *and* $\mathbb{P}_{>\frac{1}{\epsilon}}$ *are defined as follows: if* $f(\sigma) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m Y_l^m(\sigma)$, *then*

$$\big(\mathbb{P}_{\le\frac{1}{\epsilon}}f\big)(\sigma) \stackrel{\text{def}}{=} \sum_{[l(l+1)]^{\frac{1}{2}}\le\frac{1}{\epsilon}}\sum_{m=-l}^{l} f_l^m Y_l^m(\sigma),$$

$$\big(\mathbb{P}_{>\frac{1}{\epsilon}}f\big)(\sigma) \stackrel{\text{def}}{=} \sum_{[l(l+1)]^{\frac{1}{2}}>\frac{1}{\epsilon}}\sum_{m=-l}^{l} f_l^m Y_l^m(\sigma). \tag{4.2}$$

REMARK 4.1. – *We remark that the projection operators* $\mathbb{P}_{\le\frac{1}{\epsilon}}$ *and* $\mathbb{P}_{>\frac{1}{\epsilon}}$ *commutate with the fractional Laplace-Beltrami operator. Moreover, since the Laplace-Beltrami operator is a self-adjoint operator with orthogonal basis of the eigenfunctions, the spectrum theorem yields that*

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{S}^2)} + \|\big((-\triangle_{\mathbb{S}^2})^{1/2}\mathbb{P}_{\le\frac{1}{\epsilon}} + \epsilon^{s-1}(-\triangle_{\mathbb{S}^2})^{s/2}\mathbb{P}_{>\frac{1}{\epsilon}}\big)f\|^2_{L^2(\mathbb{S}^2)} \\ \sim \|W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})f\|^2_{L^2(\mathbb{S}^2)} + \|f\|^2_{L^2(\mathbb{S}^2)}.\end{aligned}$$

4.1. Proof of Theorem 1.5

We are in a position to prove Theorem 1.5.

Proof. – In [11] , it is proved that for any smooth functions g, h and f, there holds

$$\lim_{\epsilon\to 0}\langle Q^\epsilon(g,h), f\rangle = \langle Q_L(g,h), f\rangle,$$

where Q^ϵ is a collision operator with the kernel B^ϵ under the assumption (B1). Then the bounds of the Landau operator can be reduced to the uniform bounds of the operator Q^ϵ with respect to the parameter ϵ. Since Q^ϵ is still the Boltzmann collision operator, we may copy the argument used in the proof of Theorem 1.1 and Theorem 1.4 to get the desired results.

Let us follow the same notations used in Theorem 1.1 and Theorem 1.4. In the next we only point out the difference. In order to cancel the singularity caused by the kernel and get the uniform estimates with respect to the parameter ϵ, we have to make use of the fact:

$$\int_0^{\pi/2} b^\epsilon(\cos\theta)\sin\theta\theta^2 d\theta \sim 1.$$

Therefore there is no need to introduce the function ψ to make the decomposition for the term $\mathfrak{D}_k^1$ in the proof of Lemma 2.1, the terms $\mathfrak{C}_k^1$ and $\mathfrak{C}_k^2$ in the proof of Lemma 2.2, the term $\mathfrak{M}^3_{k,p}$ in the proof of Lemma 2.3 and the term $\mathcal{E}_{l,p}$ in the proof of Lemma 3.2. Keeping it in mind and following almost the same calculation, we get that the results stated in Lemma 2.1, Lemma 2.2 and Lemma 2.3 are valid for Q^ϵ with $s = 1$. In particular, we have

$$|\mathcal{E}_{l,p}| \lesssim 2^{(l-p)/2}2^k \|g\|_{L^1}\|\mathfrak{F}_p f\|_{H^{1/2}}\|\mathfrak{F}_l f\|_{H^{1/2}}, \tag{4.3}$$

where $\mathcal{E}_{l,p}$ is defined in (3.5). With these in hand, following the same argument used in the proof of Theorem 1.1 will yield the desired results (1.30)–(1.32).

Next we turn to the upper bounds of the Landau operator in anisotropic spaces. Let $a, b \in [0, 2]$ with $a + b = 2$ and $a_1, b_1 \in \mathbb{R}$ with $a_1 + b_1 = 1$. We first give the bounds to $\mathfrak{W}^1_{k,p,l}, \mathfrak{W}^4_{k,p,m}, \mathfrak{W}^3_{k,p}$ and $\mathfrak{W}^2_{k,l}$. Thanks to the facts that Lemma 2.1, Lemma 2.2 and Lemma 2.3 are valid for Q^ϵ with $s = 1$, we deduce that for $k \geq 0$,

$$\sum_{l\leq p-N_0} |\mathfrak{W}^1_{k,p,l}| + \sum_{m<p-N_0} |\mathfrak{W}^4_{k,p,m}| \lesssim \|g\|_{L^1}\|h\|_{L^2}\|f\|_{L^2}$$

and

$$|\langle Q^\epsilon_{-1}(g,h), f\rangle_v| \lesssim (\|g\|_{L^1} + \|g\|_{L^2})\|h\|_{H^a}\|f\|_{H^b}. \tag{4.4}$$

Thanks to Lemma 4.1, we have

$$\epsilon^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2} \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\leq\epsilon} d\sigma d\tau \lesssim \|f\|^2_{L^2(\mathbb{S}^2)} + \|(-\triangle_{\mathbb{S}^2})^{1/2} f\|^2_{L^2(\mathbb{S}^2)},$$

which implies that for smooth functions g and h,

$$\begin{aligned}
&\epsilon^{2s-2}\int_{\mathbb{S}^2\times\mathbb{S}^2} \big(g(\sigma)-g(\tau)\big)h(\tau)|\sigma-\tau|^{-(2+2s)} 1_{|\sigma-\tau|\leq\epsilon} d\sigma d\tau\\
&\lesssim \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g^m_l h^m_l (\|(-\triangle_{\mathbb{S}^2})^{1/2} Y^m_l\|_{L^2(\mathbb{S}^2)} + 1)(\|(-\triangle_{\mathbb{S}^2})^{1/2} Y^m_l\|_{L^2(\mathbb{S}^2)} + 1)\\
&\lesssim \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g^m_l h^m_l (l(l+1)+1)^2\\
&\lesssim \|(1-\triangle_{\mathbb{S}^2})^{a/2} g\|_{L^2(\mathbb{S}^2)}\|(1-\triangle_{\mathbb{S}^2})^{b/2} h\|_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

Hence, together with the strategy explained in Section 1.3 and the fact $b^\epsilon(\sigma\cdot\tau) \sim |\sigma - \tau|^{-(2+2s)} 1_{|\sigma-\tau|\leq\epsilon}$, we have

$$\begin{aligned}
|\mathfrak{W}^{3,1}_{k,p}| &\lesssim 2^{\gamma k}\int_{\mathbb{R}^3} |(\mathfrak{F}_p g)_*| \|(1-\triangle_{\mathbb{S}^2})^{a/2}(T_{v_*}\tilde{\mathfrak{F}}_p h)\|_{L^2}\|(1-\triangle_{\mathbb{S}^2})^{b/2}(T_{v_*}\tilde{\mathfrak{F}}_p f)\|_{L^2} dv_*\\
&\lesssim 2^{\gamma k}\|\mathfrak{F}_p g\|_{L^1_{\frac{1}{2}}}(\|(-\triangle_{\mathbb{S}^2})^{a/2}\tilde{\mathfrak{F}}_p h\|_{L^2} + \|\tilde{\mathfrak{F}}_p h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2}\tilde{\mathfrak{F}}_p f\|_{L^2} + \|\tilde{\mathfrak{F}}_p f\|_{H^b}).
\end{aligned}$$

Hence, together with Lemma 5.8, one has

$$\sum_{p=-1}^{\infty} |\mathfrak{W}^{3,1}_{k,p}| \lesssim 2^{\gamma k}\|g\|_{L^1_{\frac{1}{2}}}(\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f\|_{L^2} + \|f\|_{H^b}).$$

Observing that the term $\mathfrak{W}^{3,2}_{k,p}$ enjoys the similar structure as that of the term $\mathcal{E}_{l,p}$, (4.3) indicates that it is not difficult to derive

$$|\mathfrak{W}^{3,2}_{k,p}| \lesssim 2^{(\gamma+1)k}2^p\|g\|_{L^1}\|\tilde{\mathfrak{F}}_p h\|_{L^2}\|\tilde{\mathfrak{F}}_p f\|_{L^2},$$

which implies that

$$\sum_{p=-1}^{\infty} |\mathfrak{W}_{k,p}^{3,2}| \lesssim 2^{(\gamma+1)k} \|g\|_{L^1} \|h\|_{H^{a_1}} \|f\|_{H^{b_1}}.$$

Therefore we have

$$\sum_{p=-1}^{\infty} |\mathfrak{W}_{k,p}^{3}| \lesssim 2^{\gamma k} \|g\|_{L^1_2} (\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f\|_{L^2} + \|f\|_{H^b}) \\ + 2^{(\gamma+1)k} \|g\|_{L^1} \|h\|_{H^{a_1}} \|f\|_{H^{b_1}}.$$

Because $\mathfrak{W}_{k,l}^2$ enjoys the similar structure as that of $\mathfrak{W}_{k,p}^3$, we finally arrive at, for $k \geq 0$,

$$\sum_{p=-1}^{\infty} |\mathfrak{W}_{k,p}^{3}| + \sum_{l=-1}^{\infty} |\mathfrak{W}_{k,l}^{2}| \lesssim 2^{\gamma k} \|g\|_{L^1_2} (\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f\|_{L^2} + \|f\|_{H^b}) \\ + 2^{(\gamma+1)k} \|g\|_{L^1} \|h\|_{H^{a_1}} \|f\|_{H^{b_1}}.$$

Due to (2.3), we conclude that for $k \geq 0$,

$$|\langle Q_k^{\epsilon}(g,h), f\rangle_v| \lesssim 2^{(\gamma+1)k} \|g\|_{L^1} \|h\|_{H^{a_1}} \|f\|_{H^{b_1}} \\ + 2^{\gamma k} \|g\|_{L^1_2} (\|(-\triangle_{\mathbb{S}^2})^{a/2} h\|_{L^2} + \|h\|_{H^a})(\|(-\triangle_{\mathbb{S}^2})^{b/2} f\|_{L^2} + \|f\|_{H^b}).$$

Hence, together with (4.4), it is enough to derive the upper bounds in anisotropic spaces by the argument used in the proof of Theorem 1.4. This completes the proof of the theorem. □

4.2. Proof of Theorem 1.6

We give the proof to Theorem 1.6.

Proof. – We first focus on the lower bound of the functional $\langle -Q^{\epsilon}(g,f), f\rangle$ where g satisfies the condition (1.28). Observe that

$$\langle -Q^{\epsilon}(g,f), f\rangle = \frac{1}{2}\mathcal{E}_g^{\gamma,\epsilon}(f) - \frac{1}{2}\mathcal{L}_g^{\epsilon}(f),$$

where

$$\mathcal{E}_g^{\gamma,\epsilon}(f) \stackrel{\text{def}}{=} \iint_{v,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} |v-v_*|^{\gamma} g_* b^{\epsilon}(\cos\theta)(f'-f)^2 d\sigma dv_* dv,$$

$$\mathcal{L}_g^{\epsilon}(f) \stackrel{\text{def}}{=} \int_{\mathbb{R}^6} dv_* dv \int_{\mathbb{S}^2} B^{\epsilon}(|v-v_*|,\sigma) g_* (f'^2 - f^2) d\sigma.$$

By the Cancellation Lemma, it holds

$$|\mathcal{L}_g^{\epsilon}(f)| \lesssim \mathcal{R}.$$

We note that this term is controlled by Lemma 3.1.

Next we concentrate on the term $\mathcal{E}_g^{\gamma,\epsilon}(f)$. Due to Lemma 3.4, it suffices to consider the lower bound of $\mathcal{E}_g^{0,\epsilon}(f)$. Thanks to the geometric decomposition (1.35), we have

$$\mathcal{E}_g^{0,\epsilon}(f) \geq \mathcal{E}_{1,g}^{0,\epsilon}(f) - \mathcal{E}_{2,g}^{0,\epsilon}(f),$$

where

$$\mathcal{E}^{0,\epsilon}_{1,g}(f) \stackrel{\text{def}}{=} \frac{1}{2}\iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_* b^\epsilon(\cos\theta)\big((T_{v_*}f)(r\varsigma)-(T_{v_*}f)(r\tau)\big)^2 d\sigma dv_* du,$$

$$\mathcal{E}^{0,\epsilon}_{2,g}(f) \stackrel{\text{def}}{=} 2\iint_{u,v_*\in\mathbb{R}^3,\sigma\in\mathbb{S}^2} g_* b^\epsilon(\cos\theta)\big(f(v_*+u^+)-f(v_*+|u|\frac{u^+}{|u^+|})\big)^2 d\sigma dv_* du.$$

Step 1: Estimate of $\mathcal{E}^{0,\epsilon}_{1,g}(f)$. – By the strategy explained in Section 1.3, the fact $b^\epsilon(\sigma\cdot\tau)\sim|\sigma-\tau|^{-(2+2s)}1_{|\sigma-\tau|\le\epsilon}$ and Lemma 4.1, we first get the lower bound of $\mathcal{E}^{0,\epsilon}_{1,g}(f)$, that is, if $g\ge 0$, then

$$\mathcal{E}^{0,\epsilon}_{1,g}(f)+\|g\|_{L^1}\|f\|^2_{L^2}\sim\int_{\mathbb{R}^3} g_*\|W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})T_{v_*}f\|^2_{L^2}dv_*+\|g\|_{L^1}\|f\|^2_{L^2}.$$

Step 2: Estimate of $\mathcal{E}^{0,\epsilon}_{2,g}(f)$. – By a slight modification of the estimate to $\mathcal{E}^0_2$(defined in (3.4)) in the proof of Lemma 3.2, we can get

$$|\mathcal{E}^{0,\epsilon}_{2,g}(f)|\lesssim\|g\|_{L^1_1}\|f\|^2_{H^{1/2}_{1/2}}.$$

We point out that it is a consequence of (4.3).

We finally arrive at, for $\eta>0$,

$$\begin{aligned}\mathcal{E}^{0,\epsilon}_{g}(f)+\|g\|_{L^1}\|f\|^2_{L^2} &\gtrsim \int_{\mathbb{R}^3} g_*\|W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})T_{v_*}f\|^2_{L^2}dv_*-\|g\|_{L^1_1}(\eta^{-1}\|f\|^2_{H^1}+\eta\|f\|^2_{L^2_1})\\ &\gtrsim \int_{\mathbb{R}^3} g_*\|W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})T_{v_*}f\|^2_{L^2}dv_*-\|g\|_{L^1_1}(\eta^{-1}\|\psi(\epsilon D)f\|^2_{H^1}\\ &\quad+\eta^{-1}\|(1-\psi(\epsilon D))f\|^2_{H^1}+\eta\|f\|^2_{L^2_1}).\end{aligned}$$

Thanks to the condition (1.28), we also have

$$\mathcal{E}^{0,\epsilon}_{g}(f)+\|g\|_{L^1}\|f\|^2_{L^2}\ge C(\lambda,\delta)\|W^\epsilon(D)f\|^2_{L^2},$$

which was proven in [11]. Combining these two inequalities, we obtain that

$$\begin{aligned}&\mathcal{E}^{0,\epsilon}_{g}(f)+\|g\|_{L^1_1}(\eta^{-1}\|(1-\psi(\epsilon D))f\|^2_{H^1}+\eta^{-1}\|f\|^2_{L^2}+\eta\|f\|^2_{L^2_1})\\ &\qquad\ge C(\lambda,\delta,\eta)\big(\|W^\epsilon(D)f\|^2_{L^2}+\int_{\mathbb{R}^3} g_*\|W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})T_{v_*}f\|^2_{L^2}dv_*\big).\end{aligned}$$

Thanks to the condition (1.28) and Lemma 3.4, we have

$$\begin{aligned}&C(\lambda,\delta,\eta)\big(\int_{\mathbb{R}^3}\mu_*\|W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})T_{v_*}W_{\gamma/2}f\|^2_{L^2}dv_*+\|W^\epsilon(D)W_{\gamma/2}f\|^2_{L^2}\big)\\ &\lesssim C(\delta,\lambda)\big(\mathcal{E}^{\gamma,\epsilon}_{g}(f)+\eta^{-1}(\|f\|^2_{L^2_{\gamma/2}}+\|(1-\psi(\epsilon D))W_{\gamma/2}f\|^2_{H^1})+\eta\|f\|^2_{L^2_{\gamma/2+1}}\big).\end{aligned}\tag{4.5}$$

Noticing that for any smooth function f, we have

$$\langle -Q_L(g,f),f\rangle_v=\lim_{\epsilon\to 0}\langle -Q^\epsilon(g,f),f\rangle\ge\lim_{\epsilon\to 0}\frac{1}{2}\mathcal{E}^{\gamma,\epsilon}_{g}(f)-C\mathcal{R}.$$

Then the results are easily obtained by Fatou Lemma, (4.5), Lemma 3.1 and Lemma 5.10. □

4.3. Proof of Theorem 1.7

Finally we give the proof to Theorem 1.7.

Proof. – It is proved in [18] (see (55) in [20]) that for any smooth function f, $\lim_{\epsilon\to 0} D_B^\epsilon(f) = D_L(f)$. By the proof of Theorem 1.3 and Lemma 3.4, we have

$$D_B^\epsilon(f) \gtrsim \mathcal{E}_\mu^{0,\epsilon}(W_{\gamma/2}\sqrt{f}) - \|f\|_{L^1_2}.$$

From the proof of Theorem 1.6, we have that, for any $\eta > 0$,

$$\begin{aligned}\mathcal{E}_\mu^{0,\epsilon}(W_{\gamma/2}\sqrt{f}) &+ (\eta^{-1}\|(1-\psi(\epsilon D))W_{\gamma/2}\sqrt{f}\|_{H^1}^2 + \eta^{-1}\|W_{\gamma/2}\sqrt{f}\|_{L^2}^2 + \eta\|W_{\gamma/2}\sqrt{f}\|_{L^2_1}^2)\\ &\gtrsim C(\lambda,\delta,\eta)\big(\|W^\epsilon(D)W_{\gamma/2}\sqrt{f}\|_{L^2}^2 + \int_{\mathbb{R}^3}\mu_*\|W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})T_{v_*}W_{\gamma/2}\sqrt{f}\|_{L^2}^2 dv_*\big).\end{aligned}$$

In other words,

$$\begin{aligned}D_B^\epsilon(f) &+ \|f\|_{L^1_{\gamma+2}} + \|f\|_{L^1_2} + \|(1-\psi(\epsilon D))W_{\gamma/2}\sqrt{f}\|_{H^1}^2\\ &\gtrsim C(\lambda,\delta)\big(\|W^\epsilon(D)W_{\gamma/2}\sqrt{f}\|_{L^2}^2 + \int_{\mathbb{R}^3}\mu_*\|W^\epsilon((-\triangle_{\mathbb{S}^2})^{1/2})T_{v_*}W_{\gamma/2}\sqrt{f}\|_{L^2}^2 dv_*\big).\end{aligned}$$

Thanks to Fatou Lemma and Lemma 5.10, the theorem is easily obtained by passing the limit $\epsilon \to 0$. □

5. Toolbox: weighted Sobolev spaces, interpolation theory and L^2 profile of the Laplace-Beltrami operator

In this section, we will first give new profiles of the weighted Sobolev Spaces. Then we will state a new version of interpolation theory which slightly relaxes the assumption that operators are needed to be commutated with each other. Then in the next we will list some basic properties of the real spherical harmonics and introduce the definition of the Laplace-Beltrami operator. After giving the L^2 profile of the fractional Laplace-Beltrami operator, we will give a detailed proof to (1.38) which is crucial to capture the anisotropic structure of the collision operator. We want to point out that results in this section have independent interest.

5.1. Weighted Sobolev spaces

Before stating the results, we list some basic facts which will be used in the proof of the new profiles of the weighted Sobolev spaces.

Lemma 5.1 (Bernstein inequalities). – *There exists a constant C independent of j and f such that*

1) *For any $s \in \mathbb{R}$ and $j \geq 0$,*

$$C^{-1}2^{js}\|\mathfrak{F}_j f\|_{L^2(\mathbb{R}^3)} \leq \|\mathfrak{F}_j f\|_{H^s(\mathbb{R}^3)} \leq C2^{js}\|\mathfrak{F}_j f\|_{L^2(\mathbb{R}^3)}. \quad (5.1)$$

2) *For integers $j,k \geq 0$ and $p,q \in [1,\infty]$, the Bernstein inequalities are shown as*

$$\sup_{|\alpha|=k} \|\partial^\alpha \mathfrak{F}_j f\|_{L^q(\mathbb{R}^3)} \lesssim 2^{jk} 2^{3j(\frac{1}{p}-\frac{1}{q})} \|\mathfrak{F}_j f\|_{L^p(\mathbb{R}^3)},$$
$$2^{jk}\|\mathfrak{F}_j f\|_{L^p(\mathbb{R}^3)} \lesssim \sup_{|\alpha|=k} \|\partial^\alpha \mathfrak{F}_j f\|_{L^p(\mathbb{R}^3)} \lesssim 2^{jk}\|\mathfrak{F}_j f\|_{L^p(\mathbb{R}^3)}. \tag{5.2}$$

3) *For any $f \in H^s$, it holds that*

$$\|f\|^2_{H^s(\mathbb{R}^3)} \sim \sum_{k=-1}^{\infty} 2^{2ks} \|\mathfrak{F}_k f\|^2_{L^2(\mathbb{R}^3)}. \tag{5.3}$$

LEMMA 5.2 (See [13]). – *Let $s,r \in \mathbb{R}$ and $a(v), b(v) \in C^\infty$ satisfy for any $\alpha \in \mathbb{Z}^3_+$,*

$$|\partial^\alpha_v a(v)| \leq C_{1,\alpha}\langle v\rangle^{r-|\alpha|}, |\partial^\alpha_\xi b(\xi)| \leq C_{2,\alpha}\langle \xi\rangle^{s-|\alpha|}$$

for constants $C_{1,\alpha}, C_{2,\alpha}$. Then there exists a constant C depending only on s,r and finite numbers of $C_{1,\alpha}, C_{2,\alpha}$ such that for any Schwarz function f,

$$\|a(\cdot)b(D)f\|_{L^2} \leq C\|\langle D\rangle^s W_r f\|_{L^2},$$
$$\|b(D)a(\cdot)f\|_{L^2} \leq C\|W_r\langle D\rangle^s f\|_{L^2}.$$

REMARK 5.1. – *As a direct consequence, we get*

$$\|\langle D\rangle^m W_l f\|_{L^2} \sim \|W_l\langle D\rangle^m f\|_{L^2} \sim \|f\|_{H^m_l}.$$

DEFINITION 5.1. – *A smooth function $a(v,\xi)$ is said to be a symbol of type $S^m_{1,0}$ if $a(v,\xi)$ verifies that for any multi-indices α and β,*

$$|(\partial^\alpha_\xi \partial^\beta_v a)(v,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|},$$

where $C_{\alpha,\beta}$ is a constant depending only on α and β.

LEMMA 5.3. – *Let $l,s,r \in \mathbb{R}$, $M(\xi) \in S^r_{1,0}$ and $\Phi(v) \in S^l_{1,0}$. Then there exists a constant C such that*

$$\|[M(D_v),\Phi]f\|_{H^s} \leq C\|f\|_{H^{r+s-1}_{l-1}}.$$

Proof. – We prove it in the spirit of [13]. Thanks to the expansion of the pseudo-differential operator, it holds that for any $N \in \mathbb{N}$,

$$M(D_v)\Phi = \Phi M(D_v) + \sum_{1\leq|\alpha|<N} \frac{1}{\alpha!}\Phi_\alpha M^\alpha(D_v) + r_N(v,D_v), \tag{5.4}$$

where $\Phi_\alpha(v) = \partial^\alpha_v \Phi$, $M^\alpha(\xi) = \partial^\alpha_\xi M(\xi)$ and

$$r_N(v,\xi) = N\sum_{|\alpha|=N}\int_0^1 \frac{(1-\tau)^{N-1}}{\alpha!} r_{N,\tau,\alpha}(v,\xi)d\tau.$$

Here

$$r_{N,\tau,\alpha}(v,\xi) = \int \big[(1-\triangle_y)^n\Phi_\alpha(v+y)\big]I(\xi;y)\langle y\rangle^{-2m}dy$$

with $2m > N-l+3$, $2n > N-r+3$ and

$$I(\xi,y) = \frac{1}{(2\pi)^3}\int e^{-iy\eta}(1-\triangle_\eta)^m\Big[\langle\eta\rangle^{-2n}M^{(\alpha)}(\xi+\tau\eta)\Big]d\eta.$$

It is not difficult to check that it holds uniformly with respect to $\tau \in [0,1]$,

$$\left|\partial_v^{\beta}\partial_{\xi}^{\beta'} r_{N,\tau,\alpha}(v,\xi)\right| \le C_{\beta,\beta'}\langle\xi\rangle^{r-N-|\beta'|}\langle v\rangle^{l-N-|\beta|}.$$

Then (5.4) and Lemma 5.2 imply the lemma with $s = 0$. The case $s \neq 0$ can be treated similarly and we skip the proof here. $\square$

Now we are in a position to give the new profiles of the weighted Sobolev spaces.

THEOREM 5.1. – *Let $m, l \in \mathbb{R}$. Then for $f \in H_l^m$, we have*

$$\sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2 \sim \|f\|_{H_l^m}^2.$$

Proof. – We first observe that $2^{k(l+1)}\varphi(\frac{v}{2^k})$ verifies the condition in Lemma 5.2. Then we have

$$\begin{aligned} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2 &= 2^{-2k}\|2^{k(l+1)}\mathcal{P}_k f\|_{H^m}^2 \\ &\lesssim 2^{-2k}\|\langle D\rangle^m W_{l+1}\mathcal{P}_k f\|_{L^2}^2 \\ &\lesssim 2^{-2k}\left[\|W_{l+1}\mathcal{P}_k\langle D\rangle^m f\|_{L^2}^2 + \|f\|_{H_l^{m-1}}^2\right], \end{aligned}$$

where we use Lemma 5.3 in the last step. This implies

$$\begin{aligned} \sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2 &\lesssim \sum_{k=-1}^{\infty}\|W_l\mathcal{P}_k\langle D\rangle^m f\|_{L^2}^2 + \|f\|_{H_l^m}^2 \\ &\lesssim \|f\|_{H_l^m}^2. \end{aligned}$$

To prove the inverse inequality, we first treat with the case $m \ge 0$. Thanks to Remark 5.1, we have

$$\begin{aligned} \|f\|_{H_l^m}^2 &\sim \sum_{k=-1}^{\infty}\|\mathcal{P}_k W_l\langle D\rangle^m f\|_{L^2}^2 \sim \sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k\langle D\rangle^m f\|_{L^2}^2 \\ &\lesssim \sum_{k=-1}^{\infty}\left(2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2 + 2^{-k}\|[2^{k(l+\frac{1}{2})}\mathcal{P}_k, \langle D\rangle^m]f\|_{L^2}^2\right) \\ &\lesssim \sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2 + \|f\|_{H_{l-1/2}^{m-1}}^2, \end{aligned}$$

where we use Lemma 5.3 in the last two steps. Then by iterated argument, we obtain that for any $N \in \mathbb{N}$,

$$\|f\|_{H_l^m}^2 \lesssim \sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2 + \|f\|_{H_{l-N/2}^{m-N}}^2.$$

Thanks to the fact that for $m \ge 0$, it holds

$$\sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2 \gtrsim \|f\|_{L_l^2}^2.$$

Choosing N sufficiently large, we get

$$\|f\|_{H^m_l}^2 \lesssim \sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2,$$

which gives the proof to desired result with $m \geq 0$.

Next we will use the duality method to deal with the case $m < 0$. Notice that

$$\begin{aligned}\int_{\mathbb{R}^3} fg dv &= \sum_{k=-1}^{\infty} \int_{\mathbb{R}^3} \mathcal{P}_k f \tilde{\mathcal{P}}_k g dv \lesssim \sum_{k=-1}^{\infty} \|\mathcal{P}_k f\|_{H^m} \|\tilde{\mathcal{P}}_k g\|_{H^{-m}} \\ &\lesssim \Big(\sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2\Big)^{\frac{1}{2}} \|g\|_{H^{-m}_{-l}}.\end{aligned}$$

Then for any Schwarz function g,

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} \langle v\rangle^l (\langle D\rangle^m f) g dv\Big| &\lesssim \Big(\sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2\Big)^{\frac{1}{2}} \|\langle D\rangle^m W_l g\|_{H^{-m}_{-l}} \\ &\lesssim \Big(\sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2\Big)^{\frac{1}{2}} \|g\|_{L^2},\end{aligned}$$

which implies

$$\|f\|_{H^m_l}^2 \lesssim \sum_{k=-1}^{\infty} 2^{2kl}\|\mathcal{P}_k f\|_{H^m}^2.$$

We complete the proof of the lemma. □

5.2. Interpolation theory

The couple of Banach spaces (X, Y) is said to be an interpolation couple if both X and Y are continuously embedding in a Hausdorff topological vector space. Let (X, Y) be a real interpolation couple, then the real interpolation space $(X, Y)_{\theta,p}$ with $\theta \in (0, 1)$ and $p \in [1, \infty]$ is defined as follows:

$$(X, Y)_{\theta,p} \stackrel{\text{def}}{=} \left\{ x \in X + Y \,\middle|\, \|x\|_{\theta,p} \stackrel{\text{def}}{=} \left\| t^{-\theta} K(t, x) \right\|_{L^p_*(0,\infty)} < \infty \right\},$$

where $K(t, x) = \inf\limits_{x=a+b, a\in X, b\in Y} (\|a\|_X + t\|b\|_Y)$ and $L^p_*(0, \infty)$ is a Lebesgue space L^p with respect to the measure dt/t.

Let X be a real Banach space with norm $\|\cdot\|$. Let T be a closed operator: $\mathcal{D}(T) \subset X \to X$ satisfying there exists a constant M such that for any $\lambda > 0$,

$$(0, \infty) \subset \rho(T), \|\lambda R(\lambda, T)\|_{L(X)} \leq M, \tag{5.5}$$

where $\rho(T)$ denotes the resolvent set of the operator T and

$$R(\lambda, T) \stackrel{\text{def}}{=} (\lambda I - T)^{-1}, \|T\|_{L(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=1} \|Tx\|.$$

Then $\mathcal{D}(T)$ is a Banach space with the graph norm $\|x\|_{\mathcal{D}(T)} = \|x\| + \|Tx\|$ for $x \in \mathcal{D}(T)$.

PROPOSITION 5.1 (see [16]). – *Let A satisfy* (5.5). *If we set* $\mathcal{D}_A(\theta,p) \overset{\text{def}}{=} (X,\mathcal{D}(A))_{\theta,p}$, *then*

$$\mathcal{D}_A(\theta,p) = \left\{x \in X \middle| \left\| \lambda^\theta \|AR(\lambda,A)x\| \right\|_{L^p_*(0,\infty)} < \infty\right\}.$$

Let A, B be two closed operators satisfying (5.5). We recall that $[A,B] = AB - BA$. In general, if $[A,B] \neq 0$, it is not easy to derive

$$(X,\mathcal{D}(A)\cap\mathcal{D}(B))_{\theta,2} = \mathcal{D}_A(\theta,2)\cap\mathcal{D}_B(\theta,2). \tag{5.6}$$

The aim of this subsection is to show that under some special conditions on the operators A and B, the real interpolation space $(X,\mathcal{D}(A)\cap\mathcal{D}(B))_{\theta,2}$ still verifies (5.6). We will use this fact to prove (1.38).

Let us give the typical examples of the operators which verify the condition (5.5). Let $\Omega_{ij} = x_i\partial_j - x_j\partial_i$ with $1 \le i < j \le 3$ and the domain of the operator is defined as

$$\mathcal{D}(\Omega_{ij}) = \left\{f \in L^2(\mathbb{R}^3_x) | \exists\, g \in L^2(\mathbb{R}^3_x), \forall h \in C^\infty_c(\mathbb{R}^3_x), \int_{\mathbb{R}^3} \big(\Omega_{ij}h\big) f dx = -\int_{\mathbb{R}^3} hg dx\right\}.$$

From which, we give the definition: $g \overset{\text{def}}{=} \Omega_{ij} f$. Then Ω_{ij} is a closed operator and verifies the condition (5.5). Another example is the partial derivative operator ∂_k with $1 \le k \le 3$. We mention that in this case the domain of the operator ∂_k is defined by

$$\mathcal{D}(\partial_k) = \left\{f \in L^2(\mathbb{R}^3_x) | \exists\, g \in L^2(\mathbb{R}^3_x), \forall h \in C^\infty_c(\mathbb{R}^3_x), \int_{\mathbb{R}^3} \big(\partial_k h\big) f dx = -\int_{\mathbb{R}^3} hg dx\right\}.$$

The new interpolation theory can be stated as follows:

THEOREM 5.2. – *Let A, B_1, B_2 and B_3 be closed operators satisfying the condition* (5.5) *and*

$$[B_i,B_j] = 0, [A,B_1] = -B_2, [A,B_2] = -B_1, [A,B_3] = 0. \tag{5.7}$$

If we set $\mathcal{D}(B) = \bigcap\limits_{i=1}^{3} \mathcal{D}(B_i)$ *and* $\|x\|_{\mathcal{D}(B)} = \|x\| + \sum_{i=1}^3 \|B_i x\|$, *then*

$$(X,D)_{\theta,2} = \mathcal{D}_A(\theta,2)\cap\mathcal{D}_B(\theta,2),$$

where $D = \mathcal{D}(A)\cap\mathcal{D}(B)$.

Proof. – By the definition of the real interpolation space, it is easy to check

$$(X,D)_{\theta,2} \subset \mathcal{D}_A(\theta,2)\cap\mathcal{D}_B(\theta,2).$$

Therefore we only need to prove the inverse conclusion. In other words, we only need to prove

$$\mathcal{D}_A(\theta,2)\cap\mathcal{D}_B(\theta,2) \subset (X,D)_{\theta,2}.$$

By the definition of real interpolation space $(X,D)_{\theta,2}$, it is reduced to prove that for $f \in \mathcal{D}_A(\theta,2)\cap\mathcal{D}_B(\theta,2)$,

$$\|t^{-\theta}K(t,f)\|_{L^2_*(0,\infty)} < \infty, \tag{5.8}$$

where $K(t,f) = \inf_{f=a+b, a\in X, b\in D}(\|a\| + t\|b\|_D)$ and $\|b\|_D \stackrel{\text{def}}{=} \|b\| + \|Ab\| + \sum_{i=1}^3 \|B_i b\|$. We remark that for $t \geq 1$ and $f \in \mathscr{D}_A(\theta,2) \cap \mathscr{D}_B(\theta,2)$, it holds

$$K(t,f) \leq \|f\|,$$

which yields

$$\|t^{-\theta} K(t,f)\|_{L^2_*(1,\infty)} \lesssim \|f\|.$$

Now it suffices to give the bound for $\|t^{-\theta}K(t,f)\|_{L^2_*(0,1)}$. In order to do that, we perform the following decomposition:

$$f = f - V(\lambda) + V(\lambda),$$

where

$$V(\lambda) = \lambda^8 R(\lambda, B_3)[R(\lambda, B_1)R(\lambda, B_2)]^2 R(\lambda, A)R(\lambda, B_1)R(\lambda, B_2)f.$$

Step 1: Estimate of $\|f - V(\lambda)\|$. – From the fact

$$\lambda R(\lambda, T) = I + TR(\lambda, T), \tag{5.9}$$

it holds

$$\begin{aligned} V(\lambda) - f = & -f + \lambda^7 R(\lambda, B_3)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, A)R(\lambda, B_1)f \\ & + \lambda^7 R(\lambda, B_3)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, A)R(\lambda, B_1)B_2R(\lambda, B_2)f. \end{aligned}$$

Using the condition (5.5), we get

$$\begin{aligned} &\|\lambda^7 R(\lambda, B_3)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, A)R(\lambda, B_1)B_2R(\lambda, B_2)f\| \\ &\qquad\qquad \lesssim \|B_2R(\lambda, B_2)f\|. \end{aligned}$$

It gives

$$\begin{aligned} \|V(\lambda) - f\| \lesssim & \|-f + \lambda^7 R(\lambda, B_3)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, A)R(\lambda, B_1)f\| \\ & + \|B_2R(\lambda, B_2)f\|. \end{aligned}$$

Using (5.9) again and following the similar argument, we derive that

$$\begin{aligned} &\|-f + \lambda^7 R(\lambda, B_3)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, A)R(\lambda, B_1)f\| \\ &\lesssim \|-f + \lambda^7 R(\lambda, B_3)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_1)R(\lambda, B_2)R(\lambda, A)f\| + \|B_1R(\lambda, B_1)f\|. \end{aligned}$$

Then by the inductive method, we obtain that

$$\|V(\lambda) - f\| \lesssim \|AR(\lambda, A)f\| + \sum_{i=1}^3 \|B_iR(\lambda, B_i)f\|. \tag{5.10}$$

5.2.1. *Step 2: Estimate of* $\|V(\lambda)\|_D$. – Thanks to the condition (5.5), we have

$$\|V(\lambda)\| \lesssim \|f\|.$$

Observe that if $[T_i, T_j] = 0$, one has

$$R(\lambda, T_i)R(\lambda, T_j) = R(\lambda, T_i)R(\lambda, T_j), T_i R(\lambda, T_j) = R(\lambda, T_j)T_i.$$

Hence, together with the condition (5.5) and $[B_i, B_3] = [A, B_3] = 0$, we deduce that

$$\|B_3 V(\lambda)\| \lesssim \lambda \|B_3 R(\lambda, B_3) f\|.$$

Due to the condition (5.7), the standard computation for the resolvent will give the following three facts:

$$\begin{aligned}
&[R(\lambda, B_1)R(\lambda, B_2), R(\lambda, A)] \\
&\quad = R(\lambda, A)R(\lambda, B_1)R(\lambda, B_2)[\lambda(B_1 + B_2) - B_1^2 - B_2^2] \\
&\qquad \times R(\lambda, B_1)R(\lambda, B_2)R(\lambda, A),
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
&[R(\lambda, B_1)R(\lambda, B_2)^2, R(\lambda, A)] \\
&\quad = R(\lambda, A)R(\lambda, B_1)R(\lambda, B_2)^2[-\lambda^2(B_2 + 2B_1) + 2\lambda(B_1 B_2 + B_1^2 + B_2^2)
\end{aligned} \tag{5.12}$$

$$\qquad + B_2^3 - 2B_1^2 B_2]R(\lambda, B_1)R(\lambda, B_2)^2 R(\lambda, A), \tag{5.13}$$

and

$$\begin{aligned}
&[(R(\lambda, B_1)R(\lambda, B_2))^2, R(\lambda, A)] \\
&\quad = R(\lambda, A)[R(\lambda, B_1)R(\lambda, B_2)]^2[2\lambda^3(B_1 + B_2) - 4\lambda^2(B_1 B_2 + B_1^2 + B_2^2) \\
&\qquad + 2\lambda(B_1^3 + B_2^3 + 2B_1^2 B_3 + 2B_2^2 B_1) - 2B_1 B_2^3 \\
&\qquad - 2B_2 B_1^3][R(\lambda, B_1)R(\lambda, B_2)]^2 R(\lambda, A).
\end{aligned} \tag{5.14}$$

Now we start to estimate $\|AV(\lambda)\|$ and $\|B_1 V(\lambda)\|$. It is easy to check

$$\begin{aligned}
AV(\lambda) &= \lambda^8 AR(\lambda, B_3)[R(\lambda, B_1)R(\lambda, B_2)]^2 R(\lambda, A)R(\lambda, B_1)R(\lambda, B_2)f \\
&= \lambda^8 AR(\lambda, A)[R(\lambda, B_1)R(\lambda, B_2)]^3 R(\lambda, B_3)f \\
&\quad + \lambda^8 A[(R(\lambda, B_1)R(\lambda, B_2))^2, R(\lambda, A)]R(\lambda, B_1)R(\lambda, B_2)R(\lambda, B_3)f \\
&\stackrel{\text{def}}{=} R_1 + R_2.
\end{aligned}$$

By (5.9) and the condition (5.5), we get

$$\|TR(\lambda, T)\|_{L(X)} \lesssim 1. \tag{5.15}$$

Then, together with (5.14) and the condition (5.5), we obtain that

$$\|R_2\| \lesssim \|f\|.$$

Notice that

$$\begin{aligned} R_1 &= \lambda^8(-I+\lambda R(\lambda,A))[R(\lambda,B_1)R(\lambda,B_2)]^3 R(\lambda,B_3)f \\ &= \lambda^8[R(\lambda,B_1)R(\lambda,B_2)]^3 R(\lambda,B_3)(-I)f \\ &\quad + \lambda^9[R(\lambda,A),(R(\lambda,B_1)R(\lambda,B_2))^2]R(\lambda,B_1)R(\lambda,B_2)R(\lambda,B_3)f \\ &\quad + \lambda^9[R(\lambda,B_1)R(\lambda,B_2)]^2[R(\lambda,A),R(\lambda,B_1)R(\lambda,B_2)]R(\lambda,B_3)f \\ &\quad + \lambda^9[R(\lambda,B_1)R(\lambda,B_2)]^3 R(\lambda,B_3)R(\lambda,A)f \\ &= \lambda^8[R(\lambda,B_1)R(\lambda,B_2)]^3 R(\lambda,B_3)AR(\lambda,A)f \\ &\quad + \lambda^9[R(\lambda,A),(R(\lambda,B_1)R(\lambda,B_2))^2]R(\lambda,B_1)R(\lambda,B_2)R(\lambda,B_3)f \\ &\quad + \lambda^9[R(\lambda,B_1)R(\lambda,B_2)]^2[R(\lambda,A),R(\lambda,B_1)R(\lambda,B_2)]R(\lambda,B_3)f \\ &\stackrel{\text{def}}{=} R_3+R_4+R_5. \end{aligned}$$

Thanks to (5.5), (5.15) , (5.11) and (5.14), we get

$$\|R_3\| \lesssim \lambda\|AR(\lambda,A)f\|, \|R_4\|+\|R_5\| \lesssim \|f\|,$$

which implies that

$$\|AV(\lambda)\| \lesssim \|f\|+\lambda\|AR(\lambda,A)f\|.$$

Similarly we have

$$\begin{aligned} B_1V(\lambda) &= \lambda^8(-I+\lambda R(\lambda,B_1))R(\lambda,B_1)R(\lambda,B_2)^2R(\lambda,A)R(\lambda,B_1)R(\lambda,B_2)R(\lambda,B_3) \\ &= \lambda^8 R(\lambda,A)R(\lambda,B_2)^3R(\lambda,B_1)^2R(\lambda,B_3)B_1R(\lambda,B_1)f \\ &\quad - \lambda^8[R(\lambda,B_1)R(\lambda,B_2)^2,R(\lambda,A)]R(\lambda,B_1)R(\lambda,B_2)R(\lambda,B_3)f \\ &\quad + \lambda^9[R(\lambda,B_1)^2R(\lambda,B_2)^2,R(\lambda,A)]R(\lambda,B_1)R(\lambda,B_2)R(\lambda,B_3)f. \end{aligned}$$

Thanks to (5.5), (5.15) , (5.12) and (5.14), we have

$$\|B_1V(\lambda)\| \lesssim \|f\|+\lambda\|B_1R(\lambda,B_1)f\|.$$

By the same argument, we can get

$$\|B_2V(\lambda)\| \lesssim \|f\|+\lambda\|B_2R(\lambda,B_2)f\|.$$

Patching together all the estimates, we finally get

$$\|V(\lambda)\|_D \lesssim \|f\|+\lambda\|AR(\lambda,A)f\|+\sum_{i=1}^3\lambda\|B_iR(\lambda,B_i)f\|. \tag{5.16}$$

Then for $\lambda \geq 1$, one has

$$\lambda^\theta(\|V(\lambda)-f\|+\lambda^{-1}\|V(\lambda)\|_D) \lesssim \lambda^\theta(\|AR(\lambda,A)f\|+\sum_{i=1}^3\|B_iR(\lambda,B_i)f\|)+\lambda^{\theta-1}\|f\|.$$

Thanks to the condition $[B_i,B_j]=0$, by Proposition 3.1 in [16], one has

$$\mathcal{D}_B(\theta,2)=\bigcap_{i=1}^3\mathcal{D}_{B_i}(\theta,2).$$

Then if $f\in\mathcal{D}_A(\theta,2)\cap\mathcal{D}_B(\theta,2)$, by Proposition 5.1, we have for $\lambda>0$,

$$\lambda^\theta\|AR(\lambda,A)f\|, \lambda^\theta\|B_iR(\lambda,B_i)f\| \in L^2_*(0,\infty). \tag{5.17}$$

It means that

$$\left\|\lambda^{\theta}(\|V(\lambda)-f\|+\lambda^{-1}\|V(\lambda)\|_{D})\right\|_{L^2_*(1,\infty)}\lesssim\|f\|+\|f\|_{\mathscr{D}_A(\theta,2)}+\|f\|_{\mathscr{D}_B(\theta,2)}.$$

In other words, we get

$$\begin{aligned}\|t^{-\theta}K(t,f)\|_{L^2_*(0,1)}&\leq\left\|t^{-\theta}(\|V(t^{-1})-f\|+t\|V(t^{-1})\|_{D})\right\|_{L^2_*(0,1)}\\&\lesssim\|f\|+\|f\|_{\mathscr{D}_A(\theta,2)}+\|f\|_{\mathscr{D}_B(\theta,2)}.\end{aligned}$$

We complete the proof to (5.8) and this ends the proof of the theorem. □

5.3. Spherical harmonics

In this subsection, we introduce the definition and basic properties of the real spherical harmonics.

Let $\sigma=(\cos\phi\sin\theta,\sin\phi\sin\theta,\cos\theta)\in\mathbb{S}^2$ with $\theta\in[0,\pi]$ and $\phi\in[0,2\pi)$. The real spherical harmonics $Y_l^m(\sigma)$ with $l\in\mathbb{N}$, $-l\leq m\leq l$, are defined as $Y_0^0(\sigma)=(4\pi)^{-1/2}$ and for any $l\geq1$,

$$Y_l^m(\sigma)=\begin{cases}\left(\dfrac{2l+1}{4\pi}\right)^{1/2}P_l(\cos\theta),\,\text{if}\,m=0,\\\left(\dfrac{2l+1}{2\pi}\dfrac{(l-m)!}{(l+m)!}\right)^{1/2}P_l^m(\cos\theta)\cos(m\phi),\ \text{if}\ m=1,\dots,l,\\\left(\dfrac{2l+1}{2\pi}\dfrac{(l+m)!}{(l-m)!}\right)^{1/2}P_l^{-m}(\cos\theta)\sin(-m\phi),\ \text{if}\ m=-l,\ \dots,\ -1,\end{cases}$$

where P_l denotes the l-th Legendre polynomial and P_l^m denotes the associated Legendre functions of order l and degree m. It is well-known that

$$(-\triangle_{\mathbb{S}^2})Y_l^m=l(l+1)Y_l^m.$$

We remark that the family $(Y_l^m)_{l,m}$ is an orthonormal basis of the space $L^2(\mathbb{S}^2,d\sigma)$ with $d\sigma$ being the surface measure on $\mathbb{S}^2$. Thus if $f\in L^2(\mathbb{S}^2)$, then we have

$$f(\sigma)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}f_l^mY_l^m(\sigma),$$

where $f_l^m=\int_{\mathbb{S}^2}f(\sigma)Y_l^m(\sigma)d\sigma$. Then for $s\in\mathbb{R}$, the fractional Laplace-Beltrami operator $(-\triangle_{\mathbb{S}^2})^{s/2}$ is defined by

$$\big((-\triangle_{\mathbb{S}^2})^{s/2}f\big)(\sigma)\stackrel{\text{def}}{=}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l(l+1))^{s/2}f_l^mY_l^m(\sigma).\tag{5.18}$$

Similarly we have

$$\big((1-\triangle_{\mathbb{S}^2})^{s/2}f\big)(\sigma)\stackrel{\text{def}}{=}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1+l(l+1))^{s/2}f_l^mY_l^m(\sigma).\tag{5.19}$$

Next we denote $\mathscr{A}_l$ by the space of solid spherical harmonics of degree l, that is, the set of all homogeneous polynomials of degree l on $\mathbb{R}^3$ that are harmonic. Let $\mathscr{H}_l$ be the space of spherical harmonics of degree l. Then we define $\mathcal{D}_l$ to be a space of all linear combinations of functions of the form $f(r)P(x)$, where f ranges over the radial functions and P over the solid spherical harmonics of degree l, in such a way that $f(r)P(x)$ belongs to $L^2(\mathbb{R}^3)$. We have

THEOREM 5.3. – *We have*

$$L^2(\mathbb{R}^3) = \sum_{l=0}^{\infty} \bigoplus \mathcal{D}_l.$$

Moreover, for $Y \in \mathcal{H}_l$, there exists a function Ψ defined on the $[0,\infty)$ such that for $w > 0$,

$$\int_{\mathbb{S}^2} e^{-2\pi i w\sigma\cdot\tau} Y(\sigma)d\sigma = \Psi(w)Y(\tau), \tag{5.20}$$

which means that the Fourier transform maps $\mathcal{D}_l$ into itself.

Suppose f is a Schwarz function. Thanks to Theorem 5.3, we have

$$f(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\sigma) f_l^m(r),$$

where $x = r\sigma$ and $\sigma \in \mathbb{S}^2$. Then for $s \in \mathbb{R}$,

$$\big((-\triangle_{\mathbb{S}^2})^{s/2} f\big)(x) \stackrel{\text{def}}{=} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (l(l+1))^{s/2} Y_l^m(\sigma) f_l^m(r). \tag{5.21}$$

Similarly we have

$$\big((1-\triangle_{\mathbb{S}^2})^{s/2} f\big)(x) \stackrel{\text{def}}{=} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (1+l(l+1))^{s/2} Y_l^m(\sigma) f_l^m(r). \tag{5.22}$$

We recall the statement of the addition theorem:

THEOREM 5.4 (Addition Theorem). – *Suppose that σ and τ are two unit vectors. Then*

$$P_l(\sigma\cdot\tau) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\sigma) Y_l^m(\tau).$$

Now we want to prove

LEMMA 5.4. – *Suppose $H(x) \in L^2([-1,1])$. Then we have*

$$\int_{\mathbb{S}^2\times\mathbb{S}^2} \big(g(\sigma)-g(\tau)\big)h(\sigma)H(\sigma\cdot\tau)d\sigma d\tau$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l^m h_l^m \int_{\mathbb{S}^2\times\mathbb{S}^2} \big(Y_l^m(\sigma)-Y_l^m(\tau)\big)Y_l^m(\sigma)H(\sigma\cdot\tau)d\sigma d\tau.$$

Here we use the notation: $f_l^m \stackrel{\text{def}}{=} \int_{\mathbb{S}^2} f(\sigma)Y_l^m(\sigma)d\sigma$.

Proof. – Thanks to the fact the family $\{P_n\}_{n\geq 0}$ is an orthogonal basis of the space $L^2[-1,1]$, we have

$$H(x)=\sum_{n\geq 0}a_nP_n(x),$$

where $a_n(x)=(n+\frac{1}{2})\int_{-1}^{1}H(x)P_n(x)dx$. In particular, it gives

$$\begin{aligned}\int_{\mathbb{S}^2\times\mathbb{S}^2}g(\tau)h(\sigma)H(\sigma\cdot\tau)d\sigma d\tau&=\sum_{n=0}^{\infty}a_n\int_{\mathbb{S}^2\times\mathbb{S}^2}g(\tau)h(\sigma)P_n(\sigma\cdot\tau)d\sigma d\tau\\&=\sum_{n=0}^{\infty}\sum_{q=-n}^{n}a_n\frac{4\pi}{2n+1}\int_{\mathbb{S}^2\times\mathbb{S}^2}g(\tau)h(\sigma)Y_n^q(\sigma)Y_n^q(\tau)d\sigma d\tau\\&=\sum_{n=0}^{\infty}\sum_{q=-n}^{n}g_n^qh_n^qa_n\frac{4\pi}{2n+1}\\&=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g_l^mh_l^m\int_{\mathbb{S}^2\times\mathbb{S}^2}Y_l^m(\tau)Y_l^m(\sigma)H(\sigma\cdot\tau)d\sigma d\tau,\end{aligned}$$

where we use Theorem 5.4 in the second and the last equalities. On the other hand,

$$\begin{aligned}\int_{\mathbb{S}^2\times\mathbb{S}^2}g(\sigma)h(\sigma)H(\sigma\cdot\tau)d\sigma d\tau&=\int_{\mathbb{S}^2}g(\sigma)h(\sigma)d\sigma\int_{\mathbb{S}^2}H(\sigma\cdot\tau)d\tau\\&=(\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g_l^mh_l^m)\iint_{\theta\in[0,\pi],\phi\in[0,2\pi]}H(\cos\theta)\sin\theta d\theta d\phi\\&=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g_l^mh_l^m\int_{\mathbb{S}^2\times\mathbb{S}^2}Y_l^m(\sigma)Y_l^m(\sigma)H(\sigma\cdot\tau)d\sigma d\tau.\end{aligned}$$

Combine these two equalities and then we get the desired result. □

5.4. L^2 profile of the fractional Laplace-Beltrami operator

In this subsection, we first show the L^2 profile of the fractional Laplace-Beltrami operator. Then we show that in the whole space the fractional Laplace-Beltrami operator has strong connection to the rotation vector fields.

LEMMA 5.5. – *Suppose that f is a smooth function defined in $\mathbb{S}^2$. Then if $0<s<1$, it holds*

$$\|f\|^2_{L^2(\mathbb{S}^2)}+\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}d\sigma d\tau\sim\|f\|^2_{L^2(\mathbb{S}^2)}+\|(-\triangle_{\mathbb{S}^2})^{s/2}f\|^2_{L^2(\mathbb{S}^2)}.$$

Proof. – Let $\omega_1,\omega_2,\omega_3\in C_c^\infty(\mathbb{R})$ be non-negative functions. Assume that $\omega_1(x)=1$ in the Ball $B_{\frac{2}{3}}$ with compact support in the Ball $B_{\frac{3}{4}}$, $\omega_2(x)=1$ in the Ball $B_{\frac{3}{4}}$ with compact support in the Ball $B_{\frac{4}{5}}$ and $\omega_3(x)=1$ in the Ball $B_{\frac{4}{5}}$ with compact support in the Ball $B_{\frac{5}{6}}$. Let χ be a smooth function verifying

$$\chi(x)=\begin{cases}1, & \text{if}\quad x\geq 0;\\ 0, & \text{if}\quad x<-\dfrac{1}{10}.\end{cases}$$

Suppose $u=(u_1,u_2,u_3)\in\mathbb{R}^3$. Then it is easy to check for $k\in\mathbb{N}$ with $1\le k\le 3$ and $u\neq 0$,

$$\sum_{i=1}^{3}\omega_k\Big(\sum_{j\neq i}\frac{u_j^2}{|u|^2}\Big)\ge 1.$$

Then we suppose that for $1\le m\le 3$,

$$\vartheta_{km+}(u)\stackrel{\text{def}}{=}\frac{\omega_k(\sum_{j\neq m}\frac{u_j^2}{|u|^2})}{\sum_{i=1}^{3}\omega_k(\sum_{j\neq i}\frac{u_j^2}{|u|^2})}\chi\Big(\frac{u_m}{|u|}\Big)\quad\text{and}\quad\vartheta_{km-}(u)\stackrel{\text{def}}{=}\frac{\omega_k(\sum_{j\neq m}\frac{u_j^2}{|u|^2})}{\sum_{i=1}^{3}\omega_k(\sum_{j\neq i}\frac{u_j^2}{|u|^2})}\chi\Big(-\frac{u_m}{|u|}\Big).$$

We conclude that for $u\in\mathbb{S}^2$,

$$\sum_{m=1}^{3}\big[\vartheta_{km+}(u)+\vartheta_{km-}(u)\big]=1. \tag{5.23}$$

Observe that

$$\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}d\sigma d\tau\sim\sum_{m=1}^{3}\Big[\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}[\vartheta_{1m+}^2(\sigma)+\vartheta_{1m-}^2(\sigma)]d\sigma d\tau. \tag{5.24}$$

Then due to the symmetric structure, we only need to focus on the estimate of

$$\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)d\sigma d\tau.$$

Notice that

$$\begin{aligned}&\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)d\sigma d\tau\\&\quad=\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)\vartheta_{33+}(\sigma)\vartheta_{33+}(\tau)d\sigma d\tau\\&\quad+\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)\vartheta_{33+}(\sigma)[1-\vartheta_{33+}(\tau)]d\sigma d\tau.\end{aligned}$$

From which together with the fact that $|\sigma-\tau|\ge\frac12-\frac{1}{\sqrt5}$ if $\sigma\in\operatorname{Supp}\vartheta_{13+}$ and $\tau\in\operatorname{Supp}(1-\vartheta_{33+})$, we deduce that

$$\begin{aligned}&\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)d\sigma d\tau+\|f\|_{L^2(\mathbb{S}^2)}^2\\&\quad\sim\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)\vartheta_{33+}(\sigma)\vartheta_{33+}(\tau)d\sigma d\tau+\|f\|_{L^2(\mathbb{S}^2)}^2\\&\quad\sim\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|(\vartheta_{13+}f)(\sigma)-(\vartheta_{13+}f)(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{33+}(\sigma)\vartheta_{33+}(\tau)d\sigma d\tau+\|f\|_{L^2(\mathbb{S}^2)}^2.\end{aligned}$$

Suppose $\sigma=(x_1,x_2,x_3)\in\mathbb{S}^2$, $x=(x_1,x_2)$. Let $F_{13}^+(x)\stackrel{\text{def}}{=}(\vartheta_{13+}f)(x_1,x_2,\sqrt{1-x_1^2-x_2^2})$ and $\Theta_{i3}^+(x)=\vartheta_{i3+}(x_1,x_2,\sqrt{1-x_1^2-x_2^2})(i=1,2,3)$. Then by change of variables, we have

$$\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|(\vartheta_{13+}f)(\sigma)-(\vartheta_{13+}f)(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{33+}(\sigma)\vartheta_{33+}(\tau)d\sigma d\tau$$
$$=\int_{|x|,|y|\le\sqrt{\frac{5}{6}}}\frac{|F_{13}^+(x)-F_{13}^+(y)|^2}{\left(|x-y|^2+|\sqrt{1-x_1^2-x_2^2}-\sqrt{1-y_1^2-y_2^2}|^2\right)^{1+s}}$$
$$\times\,\Theta_{33}^+(x)\Theta_{33}^+(y)\frac{1}{\sqrt{1-x_1^2-x_2^2}}\frac{1}{\sqrt{1-y_1^2-y_2^2}}dxdy$$
$$\gtrsim\int_{|x|,|y|\le\sqrt{\frac{4}{5}}}\frac{|F_{13}^+(x)-F_{13}^+(y)|^2}{|x-y|^{2+2s}}dxdy,$$

which yields

$$\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)d\sigma d\tau+\|f\|_{L^2(\mathbb{S}^2)}^2\gtrsim\|F_{13}^+\|_{H^s(B_{\frac{2}{\sqrt{5}}})}^2. \tag{5.25}$$

On the other hand, one has

$$\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|(\vartheta_{13+}f)(\sigma)-(\vartheta_{13+}f)(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{33+}(\sigma)\vartheta_{33+}(\tau)d\sigma d\tau$$
$$\lesssim\int_{|x-y|\le\eta}\frac{|F_{13}^+(x)-F_{13}^+(y)|^2}{|x-y|^{2+2s}}\Theta_{33}^+(x)\Theta_{33}^+(y)dxdy+C_\eta\|F_{13}^+\|_{L^2(B_{\frac{2}{\sqrt{5}}})}^2.$$

Choose η sufficiently small such that

$$1_{|x-y|\le\eta}F_{13}^+(x)^2\Theta_{33}^+(x)\Theta_{33}^+(y)=1_{|x-y|\le\eta}F_{13}^+(x)^2\Theta_{23}^+(x)\Theta_{23}^+(y),$$
$$\text{and}\quad 1_{|x-y|\le\eta}F_{13}^+(x)F_{13}^+(y)\Theta_{33}^+(x)\Theta_{33}^+(y)=1_{|x-y|\le\eta}F_{13}^+(x)F_{13}^+(y)\Theta_{23}^+(x)\Theta_{23}^+(y).$$

Then we get

$$\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|(\vartheta_{13+}f)(\sigma)-(\vartheta_{13+}f)(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{33+}(\sigma)\vartheta_{33+}(\tau)d\sigma d\tau$$
$$\lesssim\int_{|x|,|y|\le\sqrt{\frac{4}{5}}}\frac{|F_{13}^+(x)-F_{13}^+(y)|^2}{|x-y|^{2+2s}}\Theta_{23}^+(x)\Theta_{23}^+(y)dxdy+C_\eta\|F_{23}^+\|_{L^2(B_{\frac{2}{\sqrt{5}}})}^2$$
$$\lesssim\|F_{13}^+\|_{H^s(B_{\frac{2}{\sqrt{5}}})}^2,$$

which implies

$$\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)d\sigma d\tau\lesssim\|f\|_{L^2(\mathbb{S}^2)}^2+\|F_{13}^+\|_{H^s(B_{\frac{2}{\sqrt{5}}})}^2.$$

Together with (5.25), we have

$$\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}}\vartheta_{13+}^2(\sigma)d\sigma d\tau+\|f\|_{L^2(\mathbb{S}^2)}^2\sim\|F_{13}^+\|_{H^s(B_{\frac{2}{\sqrt{5}}})}^2+\|f\|_{L^2(\mathbb{S}^2)}^2. \tag{5.26}$$

Observe that

$$\|(-\triangle_{\mathbb{S}^2})^{\frac{1}{2}}(\vartheta_{13+}f)\|_{L^2(\mathbb{S}^2)}^2=\int_{\mathbb{S}^2}\big((-\triangle_{\mathbb{S}^2})(\vartheta_{13+}f)\big)(\sigma)(\vartheta_{13+}f)(\sigma)d\sigma.$$

Thanks to the fact $(-\triangle_{\mathbb{S}^2} f)(\sigma) = -\sum_{1\le i<j\le 3} (\Omega_{ij}^2 f)(x_1,x_2,x_3)$ with $\sigma = (x_1,x_2,x_3)$ and $\Omega_{ij} = x_i\partial_j - x_j\partial_i$, by change of variables, we obtain that

$$\begin{aligned}&\|(-\triangle_{\mathbb{S}^2})^{\frac12}(\vartheta_{13+}f)\|^2_{L^2(\mathbb{S}^2)}\\ &= \int_{|x|\le\sqrt{\frac45}} -\sum_{1\le i<j\le 3}\Big(\Omega_{ij}^2(\vartheta_{13+}f)\Big)(x,\sqrt{1-|x|^2})(\vartheta_{13+}f)(x,\sqrt{1-|x|^2})\frac{1}{\sqrt{1-|x|^2}}dx.\end{aligned}$$

It is easy to see that for $i = 1,2$,

$$\partial_i F^+_{13}(x_1,x_2) = \frac{1}{\sqrt{1-|x|^2}}\Big(-\Omega_{i3}(\vartheta_{13+}f)\Big)(x,\sqrt{1-|x|^2}),$$

which implies that

$$\Big(\Omega_{i3}^2(\vartheta_{13+}f)\Big)(x,\sqrt{1-|x|^2}) = \Big((\sqrt{1-|x|^2}\partial_i)^2F^+_{13}\Big)(x).$$

Then by direct calculation, it yields

$$\begin{aligned}&\|(-\triangle_{\mathbb{S}^2})^{\frac12}(\vartheta_{13+}f)\|^2_{L^2(\mathbb{S}^2)}\\ &\quad= -\int_{|x|\le\sqrt{\frac45}}\big(\partial_1(\sqrt{1-|x|^2}\partial_1)F^+_{13}\big)F^+_{13}dx - \int_{|x|\le\sqrt{\frac45}}\big(\partial_2(\sqrt{1-|x|^2}\partial_2)F^+_{13}\big)F^+_{13}dx\\ &\qquad- \int_{|x|\le\sqrt{\frac45}}(\Omega_{12})^2F^+_{13}F^+_{13}\frac{1}{\sqrt{1-|x|^2}}dx.\end{aligned}$$

Thus we have

$$\begin{aligned}(5.27)\quad &\|(-\triangle_{\mathbb{S}^2})^{\frac12}(\vartheta_{13+}f)\|^2_{L^2(\mathbb{S}^2)} + \|\vartheta_{13+}f\|^2_{L^2(\mathbb{S}^2)}\\ &\qquad\sim \|F^+_{13}\|^2_{H^1(B_{\frac{2}{\sqrt5}})} + \|\Omega_{12}F^+_{13}\|^2_{L^2(B_{\frac{2}{\sqrt5}})} \sim \|F^+_{13}\|^2_{H^1(B_{\frac{2}{\sqrt5}})},\end{aligned}$$

where we use the fact $\|\vartheta_{13+}f\|_{L^2(\mathbb{S}^2)} \sim \|F^+_{13}\|_{L^2(B_{\frac{2}{\sqrt5}})}$.

By the real interpolation method, we obtain that for $0\le s\le 1$,

$$(5.28)\qquad \|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+}f)\|_{L^2(\mathbb{S}^2)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{S}^2)} \sim \|F^+_{13}\|_{H^s(B_{\frac{2}{\sqrt5}})}.$$

Next we claim that for $0\le s\le 2$,

$$\begin{aligned}(5.29)\quad &\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{1m+}f)\|_{L^2(\mathbb{S}^2)} + \|\vartheta_{1m+}f\|_{L^2(\mathbb{S}^2)}\\ &\qquad\lesssim \|(-\triangle_{\mathbb{S}^2})^{s/2}f\|_{L^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}.\end{aligned}$$

This is easily followed by the real interpolation method since

$$\begin{aligned}&\|(-\triangle_{\mathbb{S}^2})(\vartheta_{1m+}f)\|_{L^2(\mathbb{S}^2)} \lesssim \|(-\triangle_{\mathbb{S}^2})f\|_{L^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)},\\ \text{and}\quad &\|\vartheta_{1m+}f\|_{L^2(\mathbb{S}^2)} \lesssim \|f\|_{L^2(\mathbb{S}^2)}.\end{aligned}$$

Thanks to (5.23) and (5.29), we deduce that for $0 \leq s \leq 2$,

$$\begin{aligned}(5.30)\quad \sum_{m=1}^{3}\Big[&\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{1m+}f)\|_{L^2(\mathbb{S}^2)} + \|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{1m-}f)\|_{L^2(\mathbb{S}^2)}\\ &+ \|\vartheta_{1m+}f\|_{L^2(\mathbb{S}^2)} + \|\vartheta_{1m-}f\|_{L^2(\mathbb{S}^2)}\Big] \sim \|(-\triangle_{\mathbb{S}^2})^{s/2}f\|_{L^2(\mathbb{S}^2)} + \|f\|_{L^2(\mathbb{S}^2)}.\end{aligned}$$

Then (5.26)-(5.28) and (5.30) imply the lemma. □

As a consequence of Lemma 5.4 and Lemma 5.5, we get the following estimate:

Corollary 5.1. – *Suppose that g and h are smooth functions defined in $\mathbb{S}^2$. Then for $a, b \in \mathbb{R}$ with $a + b = 2s$,*

$$\Big|\int_{\mathbb{S}^2\times\mathbb{S}^2}\big(g(\sigma) - g(\tau)\big)h(\tau)H(\sigma\cdot\tau)d\sigma d\tau\Big| \lesssim \|(1-\triangle_{\mathbb{S}^2})^{a/2}g\|_{L^2(\mathbb{S}^2)}\|(1-\triangle_{\mathbb{S}^2})^{b/2}h\|_{L^2(\mathbb{S}^2)},$$

where $H(\sigma\cdot\tau) = |\sigma - \tau|^{-(2+2s)}$.

Proof. – Let $\lambda > 0$. Then by Lemma 5.4 and the notation: $f_l^m \stackrel{\text{def}}{=} \int_{\mathbb{S}^2} f(\sigma)Y_l^m(\sigma)d\sigma$, we have

$$\begin{aligned}&\int_{\mathbb{S}^2\times\mathbb{S}^2}\big(g(\sigma) - g(\tau)\big)h(\tau)H(\sigma\cdot\tau)d\sigma d\tau\\ &= \lim_{\lambda\to 0}\int_{\mathbb{S}^2\times\mathbb{S}^2}\big(g(\sigma) - g(\tau)\big)h(\tau)H(\sigma\cdot\tau)1_{|\sigma\cdot\tau|\geq\lambda}d\sigma d\tau\\ &= \lim_{\lambda\to 0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l^m h_l^m\int_{\mathbb{S}^2\times\mathbb{S}^2}(Y_l^m(\sigma) - Y_l^m(\tau))Y_l^m(\tau)H(\sigma\cdot\tau)1_{|\sigma\cdot\tau|\geq\lambda}d\sigma d\tau\\ &= \frac{1}{2}\lim_{\lambda\to 0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l^m h_l^m\int_{\mathbb{S}^2\times\mathbb{S}^2}(Y_l^m(\sigma) - Y_l^m(\tau))(Y_l^m(\sigma) - Y_l^m(\tau))H(\sigma\cdot\tau)1_{|\sigma\cdot\tau|\geq\lambda}d\sigma d\tau,\end{aligned}$$

where we use the symmetric property of the integral in the last step. Applying the Cauchy-Schwarz inequality and Lemma 5.5, we obtain that

$$\begin{aligned}&\int_{\mathbb{S}^2\times\mathbb{S}^2}\big(g(\sigma) - g(\tau)\big)h(\tau)H(\sigma\cdot\tau)d\sigma d\tau\\ &\quad\lesssim \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l^m h_l^m(\|(-\triangle_{\mathbb{S}^2})^{s/2}Y_l^m\|_{L^2(\mathbb{S}^2)} + 1)(\|(-\triangle_{\mathbb{S}^2})^{s/2}Y_l^m\|_{L^2(\mathbb{S}^2)} + 1)\\ &\quad\lesssim \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l^m h_l^m(l(l+1)+1)^s \lesssim \|(1-\triangle_{\mathbb{S}^2})^{a/2}g\|_{L^2(\mathbb{S}^2)}\|(1-\triangle_{\mathbb{S}^2})^{b/2}h\|_{L^2(\mathbb{S}^2)},\end{aligned}$$

which completes the proof of the lemma. □

Next we show the strong connection between the Laplace-Beltrami operator and the rotation vector fields.

LEMMA 5.6. – *Let f be a smooth function defined in $\mathbb{R}^3$. Suppose $f(u) = f(u_1, u_2, u_3)$ with $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $\Omega_{ij} f \stackrel{\text{def}}{=} (u_i \partial_{u_j} - u_j \partial_{u_i}) f$. Then if $0 < s < 1$, it holds*

$$\int_{\sigma,\tau\in\mathbb{S}^2, r>0} \frac{|f(r\sigma) - f(r\tau)|^2}{|\sigma-\tau|^{2+2s}} r^2 d\sigma d\tau dr + \|f\|^2_{L^2(\mathbb{R}^3)}$$
$$\sim \|(-\triangle_{\mathbb{S}^2})^{s/2} f\|^2_{L^2} + \|f\|^2_{L^2} \sim \sum_{1\le i<j\le 3} \|f\|^2_{\mathcal{D}_{\Omega_{ij}}(s,2)}.$$

Moreover for $s \in [0,2]$, we have

$$\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|^2_{L^2} + \|f\|^2_{L^2} \sim \sum_{1\le i<j\le 3} \|f\|^2_{\mathcal{D}_{\Omega^2_{ij}}(s/2,2)}. \tag{5.31}$$

Here we use notations $\mathcal{D}_{\Omega_{ij}}(s,2) \stackrel{\text{def}}{=} (L^2, \mathcal{D}(\Omega_{ij}))_{s,2}$ and $\mathcal{D}_{\Omega^2_{ij}}(s/2,2) \stackrel{\text{def}}{=} (L^2, \mathcal{D}(\Omega^2_{ij}))_{s/2,2}$.

Proof. – For $r > 0$ and $x = (x_1, x_2)$, we set

$$\bar{F}^+_{13}(r,x) \stackrel{\text{def}}{=} r(\vartheta_{13+} f)(rx_1, rx_2, r\sqrt{1-x_1^2-x_2^2})$$

and

$$\widetilde{F^+_{13}}(r,x) \stackrel{\text{def}}{=} \begin{cases} r(\vartheta_{13+} f)(rx_1, rx_2, r\sqrt{1-x_1^2-x_2^2}), & \text{if} \quad |x| \le \sqrt{\frac{4}{5}}; \\ 0, \quad \text{if} \quad |x| \ge \sqrt{\frac{4}{5}}, \end{cases}$$

where we use the fact $\vartheta_{13+}(rx_1, rx_2, r\sqrt{1-x_1^2-x_2^2}) = \vartheta_{13+}(x_1, x_2, \sqrt{1-x_1^2-x_2^2})$. Thanks to (5.28), one has

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+} f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+} f\|_{L^2(\mathbb{R}^3)} \sim \|\bar{F}^+_{13}\|_{L^2((0,\infty);H^s(B_{\frac{2}{\sqrt{5}}}))}.$$

Let $T_1 : L^2((0,\infty)\times B_{\frac{2}{\sqrt{5}}}) \mapsto L^2((0,\infty)\times\mathbb{R}^2)$ be a linear operator defined by

$$(T_1 f)(r,x) \stackrel{\text{def}}{=} \begin{cases} \vartheta_{23+}(x_1, x_2, \sqrt{1-x_1^2-x_2^2}) f(r,x), & \text{if} \quad |x| \le \sqrt{\frac{4}{5}}; \\ 0, \quad \text{if} \quad |x| \ge \sqrt{\frac{4}{5}}. \end{cases} \tag{5.32}$$

Then we have

$$\|T_1 f\|_{L^2((0,\infty);H^1(\mathbb{R}^2))} \lesssim \|f\|_{L^2((0,\infty);H^1(B_{\frac{2}{\sqrt{5}}}))},$$
$$\|T_1 f\|_{L^2((0,\infty);L^2(\mathbb{R}^2))} \lesssim \|f\|_{L^2((0,\infty);L^2(B_{\frac{2}{\sqrt{5}}}))}.$$

Then by real interpolation, we obtain that

$$\|T_1 f\|_{L^2((0,\infty);H^s(\mathbb{R}^2))} \lesssim \|f\|_{L^2((0,\infty);H^s(B_{\frac{2}{\sqrt{5}}}))}.$$

By the definition of $\bar{F}^+_{13}$, we have $\mathrm{Supp}\bar{F}^+_{13}(r,x) \subset (0,\infty)\times B_{\frac{\sqrt{3}}{2}}$. Thus if we take $f = \bar{F}^+_{13}$, then we get

$$\|\widetilde{F^+_{13}}\|_{L^2((0,\infty);H^s(\mathbb{R}^2))} \lesssim \|\bar{F}^+_{13}(r,x)\|_{L^2((0,\infty);H^s(B_{\frac{2}{\sqrt{5}}}))}.$$

Let $T_2 : L^2((0,\infty)\times\mathbb{R}^2) \mapsto L^2((0,\infty)\times B_{\frac{2}{\sqrt{5}}})$ be a linear operator defined by

$$(T_2 f)(r,x) \stackrel{\text{def}}{=} \vartheta_{23+}(x_1,x_2,\sqrt{1-x_1^2-x_2^2})f(r,x). \tag{5.33}$$

Then by the similar argument, we may obtain that

$$\|T_2 f\|_{L^2((0,\infty);H^s(B_{\frac{2}{\sqrt{5}}}))} \lesssim \|f\|_{L^2((0,\infty);H^s(\mathbb{R}^2))}.$$

Thus if we take $f = \widetilde{F_{13}^+}$, then we get

$$\|\bar{F}_{13}^+\|_{L^2((0,\infty);H^s(B_{\frac{2}{\sqrt{5}}}))} \lesssim \|\widetilde{F_{13}^+}\|_{L^2((0,\infty);H^s(\mathbb{R}^2))}.$$

Therefore, we are led to

$$\|\bar{F}_{13}^+\|_{L^2((0,\infty);H^s(B_{\frac{2}{\sqrt{5}}}))} \sim \|\widetilde{F_{13}^+}\|_{L^2((0,\infty);H^s(\mathbb{R}^2))},$$

which implies that

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)} \sim \|\widetilde{F_{13}^+}\|_{L^2((0,\infty);H^s(\mathbb{R}^2))}. \tag{5.34}$$

In what follows, we use the notation $\|\widetilde{F_{13}^+}\|_{L^2_r H^s_x} \stackrel{\text{def}}{=} \|\widetilde{F_{13}^+}\|_{L^2((0,\infty);H^s(\mathbb{R}^2))}$. It is easy to see that

$$\|\widetilde{F_{13}^+}\|_{L^2_r H^s_x} \sim \|\widetilde{F_{13}^+}\|_{L^2_r L^2_{x_1} H^s_{x_2}} + \|\widetilde{F_{13}^+}\|_{L^2_r L^2_{x_2} H^s_{x_1}}.$$

Notice that

$$\|\widetilde{F_{13}^+}\|_{L^2_r L^2_{x_2} H^1_{x_1}} \sim \|\Omega_{13}(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)},$$
$$\|\widetilde{F_{13}^+}\|_{L^2_r L^2_{x_2} L^2_{x_1}} \sim \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)}.$$

By real interpolation, one has

$$\|\widetilde{F_{13}^+}\|_{L^2_r L^2_{x_1} H^s_{x_2}} \sim \|\vartheta_{13+}f\|_{\mathscr{D}_{\Omega_{13}}(s,2)}.$$

Similarly

$$\|\widetilde{F_{13}^+}\|_{L^2_r L^2_{x_2} H^s_{x_1}} \sim \|\vartheta_{13+}f\|_{\mathscr{D}_{\Omega_{23}}(s,2)}.$$

Then, together with (5.34), we have

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)} \\ \sim \|\widetilde{F_{13}^+}\|_{L^2((0,\infty);H^s(\mathbb{R}^2))} \sim \|\vartheta_{13+}f\|_{\mathscr{D}_{\Omega_{13}}(s,2)} + \|\vartheta_{13+}f\|_{\mathscr{D}_{\Omega_{23}}(s,2)}.\end{aligned} \tag{5.35}$$

Observe that

$$\big(\Omega_{12}(\vartheta_{13+}f)\big)(u) = (x_1\partial_{x_2} - x_2\partial_{x_1})\big(r^{-1}\bar{F}_{13}^+(r,x)\big),$$

where $u = (rx_1, rx_2, r\sqrt{1-|x|^2})$. It yields

$$\|\Omega_{12}(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)} \lesssim \|\widetilde{F_{13}^+}\|_{L^2_r H^1_x}.$$

Therefore by real interpolation, we deduce that

$$\|\vartheta_{13+}f\|_{\mathscr{D}_{\Omega_{12}}(s,2)} \lesssim \|\widetilde{F_{13}^+}\|_{L^2_r H^s_x} \lesssim \|\vartheta_{13+}f\|_{\mathscr{D}_{\Omega_{13}}(s,2)} + \|\vartheta_{13+}f\|_{\mathscr{D}_{\Omega_{23}}(s,2)}$$

which yields

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)} \sim \sum_{1\le i<j\le 3} \|\vartheta_{13+}f\|_{\mathscr{D}_{\Omega_{ij}}(s,2)}. \tag{5.36}$$

With the help of (5.30), we are led to

$$\sum_{m=1}^{3}\sum_{1\le i<j\le 3}[\|\vartheta_{1m+}f\|_{\mathscr{D}_{\Omega_{ij}}(s,2)} + \|\vartheta_{1m-}f\|_{\mathscr{D}_{\Omega_{ij}}(s,2)}] \sim \|(-\triangle_{\mathbb{S}^2})^{s/2}f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}. \tag{5.37}$$

Due to the fact

$$\|\Omega_{ij}(\vartheta_{1m+}f)\|_{L^2(\mathbb{R}^3)} \lesssim \|\Omega_{ij}f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)},$$

we have

$$\|\vartheta_{1m+}f\|_{\mathscr{D}_{\Omega_{ij}}(s,2)} \lesssim \|f\|_{\mathscr{D}_{\Omega_{ij}}(s,2)}.$$

Together with (5.37), we then derive that

$$\sum_{1\le i<j\le 3}\|f\|_{\mathscr{D}_{\Omega_{ij}}(s,2)} \sim \|(-\triangle_{\mathbb{S}^2})^{s/2}f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^2(\mathbb{R}^3)}.$$

We complete the proof to the first equivalence.

The interpolation theory indicates

$$\sum_{1\le i<j\le 3}\|f\|^2_{\mathscr{D}_{\Omega_{ij}}(s,2)} \sim \sum_{1\le i<j\le 3}\|f\|^2_{\mathscr{D}_{\Omega^2_{ij}}(s/2,2)},$$

which implies the second equivalence in the case of $0\le s\le 1$. Next we want to prove that the result still holds for $1<s\le 2$.

We first show

$$\|(-\triangle_{\mathbb{S}^2})(\vartheta_{13+}f)\|^2_{L^2(\mathbb{S}^2)} + \|\vartheta_{13+}f\|^2_{L^2(\mathbb{S}^2)} \sim \|F^+_{13}\|^2_{H^2(B_{\frac{2}{\sqrt{5}}})}. \tag{5.38}$$

It derives from the fact that

$$(-\triangle_{\mathbb{S}^2})(\vartheta_{13+}f) = \mathbf{L}F^+_{13}, \tag{5.39}$$

where $\mathbf{L} = -(1-x_1^2)\partial_1^2 - (1-x_2^2)\partial_2^2 + 2x_1\partial_1 + 2x_2\partial_2$. Since $\mathbf{L}$ is a uniformly elliptic in $B_{\frac{2}{\sqrt{5}}}$ and F^+_{13} vanishes in the boundary of $B_{\frac{2}{\sqrt{5}}}$, the standard elliptic estimate implies that

$$\begin{aligned}\|F^+_{13}\|_{H^2(B_{\frac{2}{\sqrt{5}}})} &\lesssim \|F^+_{13}\|_{L^2(B_{\frac{2}{\sqrt{5}}})} + \|\mathbf{L}F^+_{13}\|_{L^2(B_{\frac{2}{\sqrt{5}}})}\\ &\lesssim \|(-\triangle_{\mathbb{S}^2})(\vartheta_{13+}f)\|_{L^2(\mathbb{S}^2)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{S}^2)},\end{aligned}$$

which gives the proof to (5.38) since the inverse inequality is obviously valid recalling the definition of $-\triangle_{\mathbb{S}^2}$. By real interpolation, (5.38) yields that (5.34) holds for $0\le s\le 2$.

Due to the fact

$$(\sqrt{1-|x|^2}\partial_{x_1})^2\Big(r^{-1}\bar{F}^+_{13}(r,x)\Big) = \big(\Omega^2_{13}(\vartheta_{13+}f)\big)(u),$$

where $u=(rx_1, rx_2, r\sqrt{1-|x|^2})$, we derive

$$\|\widetilde{F_{13}^{+}}\|_{L_r^2 L_{x_2}^2 H_{x_1}^2} \sim \|\Omega_{13}^2(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)}.$$

By real interpolation, one has that for $0 \le s \le 2$

$$\|\widetilde{F_{13}^{+}}\|_{L_r^2 L_{x_1}^2 H_{x_2}^s} \sim \|\vartheta_{13+}f\|_{\mathcal{D}_{\Omega_{13}^2}(s/2,2)}.$$

Similarly for $0 \le s \le 2$, it holds

$$\|\widetilde{F_{13}^{+}}\|_{L_r^2 L_{x_1}^2 H_{x_2}^s} \sim \|\vartheta_{13+}f\|_{\mathcal{D}_{\Omega_{23}^2}(s/2,2)}.$$

We have then together with (5.34), for $0 \le s \le 2$,

$$\begin{aligned}(5.40)\quad &\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)}\\ &\sim \|\widetilde{F_{13}^{+}}\|_{L^2((0,\infty);H^s(\mathbb{R}^2))} \sim \|\vartheta_{13+}f\|_{\mathcal{D}_{\Omega_{13}^2}(s/2,2)} + \|\vartheta_{13+}f\|_{\mathcal{D}_{\Omega_{23}^2}(s/2,2)}.\end{aligned}$$

Observe that

$$\big(\Omega_{12}(\vartheta_{13+}f)\big)(u) = (x_1\partial_{x_2} - x_2\partial_{x_1})\big(r^{-1}\bar{F}_{13}^{+}(r,x)\big),$$

where $u=(rx_1, rx_2, r\sqrt{1-|x|^2})$. It yields

$$\|\Omega_{12}^2(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)} \lesssim \|\widetilde{F_{13}^{+}}\|_{L_r^2 H_x^2}.$$

Therefore by real interpolation, we deduce that

$$\|\vartheta_{13+}f\|_{\mathcal{D}_{\Omega_{12}^2}(s/2,2)} \lesssim \|\widetilde{F_{13}^{+}}\|_{L_r^2 H_x^s} \lesssim \|\vartheta_{13+}f\|_{\mathcal{D}_{\Omega_{13}^2}(s/2,2)} + \|\vartheta_{13+}f\|_{\mathcal{D}_{\Omega_{23}^2}(s/2,2)}$$

which yields

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+}f)\|_{L^2(\mathbb{R}^3)} + \|\vartheta_{13+}f\|_{L^2(\mathbb{R}^3)} \sim \sum_{1\le i<j\le 3} \|\vartheta_{13+}f\|_{\mathcal{D}_{\Omega_{ij}^2}(s/2,2)}.$$

Then, together with (5.30), we thus get the equivalence (5.31). □

As a consequence, we show that the L^2 norm of the fractional Laplace-Beltrami operator can be bounded by the weighted Sobolev norm. It explains why the additional weights are needed in Theorem 1.1.

LEMMA 5.7. – *Suppose $f \in H_s^s(\mathbb{R}^3)$ with $s \ge 0$. Then it holds*

$$\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|_{L^2} \lesssim \|f\|_{H_s^s}.$$

Proof. – Suppose $0 \le s \le 2m$ with $m \in \mathbb{N}$. Then we have

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^m f\|_{L^2}^2 &= \sum_{k=-1}^{\infty} \|(-\triangle_{\mathbb{S}^2})^m \mathcal{P}_k f\|_{L^2}^2\\ &= \sum_{k=-1}^{\infty} \|(\sum_{1\le i<j\le 3} \Omega_{ij}^2)^m \mathcal{P}_k f\|_{L^2}^2\\ &\lesssim \sum_{k=-1}^{\infty} 2^{4mk} \|\mathcal{P}_k f\|_{H^{2m}}^2.\end{aligned}$$

Since it holds

$$\|(-\triangle_{\mathbb{S}^2})^m \tilde{\mathcal{P}}_k f\|_{L^2} \lesssim 2^{2mk}\|f\|_{H^{2m}}, \|\tilde{\mathcal{P}}_k f\|_{L^2} \lesssim \|f\|_{L^2},$$

by real interpolation, we get

$$\|(-\triangle_{\mathbb{S}^2})^{s/2} \tilde{\mathcal{P}}_k f\|_{L^2} \lesssim 2^{ks}\|f\|_{H^s}.$$

In particular, it yields

$$\|(-\triangle_{\mathbb{S}^2})^{s/2} \mathcal{P}_k f\|_{L^2} \lesssim 2^{ks}\|\mathcal{P}_k f\|_{H^s}.$$

We finally get

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|_{L^2}^2 &= \sum_{k=-1}^{\infty} \|(-\triangle_{\mathbb{S}^2})^{s/2} \mathcal{P}_k f\|_{L^2}^2 \\ &\lesssim \sum_{k=-1}^{\infty} 2^{2ks}\|\mathcal{P}_k f\|_{H^s}^2 \lesssim \|f\|_{H^s_s}^2.\end{aligned}$$

This completes the proof of the Lemma. □

Now we are in a position to give the proof to Lemma 4.1.

Proof. – The result can be reduced to prove

$$\begin{aligned}&\epsilon^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2} \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\leq\epsilon} d\sigma d\tau + \|f\|_{L^2(\mathbb{S}^2)}^2 \\ &\qquad \sim \sum_{[l(l+1)]^{1/2}\leq\epsilon^{-1}} \sum_{m=-l}^{l} l(l+1)|f_l^m|^2 + \epsilon^{2s-2} \\ &\qquad \times \sum_{[l(l+1)]^{1/2}>\epsilon^{-1}} \sum_{m=-l}^{l} [l(l+1)]^s |f_l^m|^2 + \|f\|_{L^2(\mathbb{S}^2)}^2,\end{aligned}$$

where $f_l^m = \int_{\mathbb{S}^2} f(\sigma) Y_l^m(\sigma) d\sigma$. Thanks to Lemma 5.4, we have

$$\begin{aligned}&\epsilon^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2} \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\leq\epsilon} d\sigma d\tau \\ &\qquad = \epsilon^{2s-2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} |f_l^m|^2 \int_{\sigma,\tau\in\mathbb{S}^2} \frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\leq\epsilon} d\sigma d\tau.\end{aligned}$$

To prove the result, it suffices to estimate the quantity A_l defined by

$$A_l \stackrel{\text{def}}{=} \epsilon^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2} \frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\leq\epsilon} d\sigma . d\tau$$

We divide the estimate of A_l into three cases. We will follow the notations used in Lemma 5.5.

CASE 1: l IS SMALL. — We first claim that

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{1/2} f\|^2_{L^2(\mathbb{S}^2)} &+ \|f\|^2_{L^2(\mathbb{S}^2)} - \epsilon^2 \|(-\triangle_{\mathbb{S}^2}) f\|^2_{L^2(\mathbb{S}^2)} \\ &\lesssim \epsilon^{2s-2} \int_{\sigma,\tau\in\mathbb{S}^2} \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\le\epsilon} d\sigma d\tau + \|f\|^2_{L^2(\mathbb{S}^2)} \\ &\lesssim \|(-\triangle_{\mathbb{S}^2})^{1/2} f\|^2_{L^2(\mathbb{S}^2)} + \|f\|^2_{L^2(\mathbb{S}^2)} + \epsilon^2 \|(-\triangle_{\mathbb{S}^2}) f\|^2_{L^2(\mathbb{S}^2)}.\end{aligned}$$

In particular, if we choose $f = Y_l^m$, then there exist universal constants c_1 and c_2 such that

$$(1 - c_1[l(l+1)]\epsilon^2)[l(l+1)] + 1 \lesssim A_l + 1 \lesssim (1 + c_2[l(l+1)]\epsilon^2)[l(l+1)] + 1. \tag{5.41}$$

Then we arrive to the fact that if $[l(l+1)]^{1/2} \le (2c_1)^{-1/2}\epsilon^{-1}$,

$$A_l \sim l(l+1) + 1. \tag{5.42}$$

To prove the claim, we set

$$I \stackrel{\text{def}}{=} \epsilon^{2s-2} \int_{\sigma,\tau\in\mathbb{S}^2} \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}} \vartheta^2_{13+}(\sigma) 1_{|\sigma-\tau|\le\epsilon} d\sigma d\tau + \|f\|^2_{L^2(\mathbb{S}^2)}.$$

Then it is easy to check

$$I \sim \epsilon^{2s-2} \int_{\sigma,\tau\in\mathbb{S}^2} \frac{|(\vartheta_{13+}f)(\sigma) - (\vartheta_{13+}f)(\tau)|^2}{|\sigma-\tau|^{2+2s}} \vartheta_{33+}(\sigma)\vartheta_{33+}(\tau) 1_{|\sigma-\tau|\le\epsilon} d\sigma d\tau + \|f\|^2_{L^2(\mathbb{S}^2)}.$$

Suppose $\sigma = (x_1, x_2, x_3) \in \mathbb{S}^2$ and $x = (x_1, x_2)$. Let $F^+_{13}(x) \stackrel{\text{def}}{=} (\vartheta_{13+}f)(x_1, x_2, \sqrt{1-x_1^2-x_2^2})$ and $\Theta^+_{33}(x) = \vartheta_{33+}(x_1, x_2, \sqrt{1-x_1^2-x_2^2})$. Then by change of variables, we have

$$\begin{aligned}\epsilon^{2s-2} &\int_{\sigma,\tau\in\mathbb{S}^2} \frac{|(\vartheta_{13+}f)(\sigma) - (\vartheta_{13+}f)(\tau)|^2}{|\sigma-\tau|^{2+2s}} \vartheta_{33+}(\sigma)\vartheta_{33+}(\tau) 1_{|\sigma-\tau|\le\epsilon} d\sigma d\tau \\ &\sim \epsilon^{2s-2} \int_{|x|,|y|\le\sqrt{\frac{5}{6}}} \frac{|F^+_{13}(x) - F^+_{13}(y)|^2}{\left(|x-y|^2 + |\sqrt{1-x_1^2-x_2^2} - \sqrt{1-y_1^2-y_2^2}|^2\right)^{1+s}} \\ &\quad \times \Theta^+_{33}(x)\Theta^+_{33}(y) \frac{1}{\sqrt{1-x_1^2-x_2^2}} \frac{1}{\sqrt{1-y_1^2-y_2^2}} 1_{|x-y|\lesssim\epsilon} dx dy \\ &\gtrsim \epsilon^{2s-2} \int_{|x|,|y|\le\sqrt{\frac{4}{5}}} \frac{|F^+_{13}(x) - F^+_{13}(y)|^2}{|x-y|^{2+2s}} 1_{|x-y|\le\epsilon} dx dy \\ &\gtrsim \epsilon^{2s-2} \int_{|x|\le\sqrt{\frac{3}{4}}, |x-y|\le\epsilon} \frac{|F^+_{13}(x) - F^+_{13}(y)|^2}{|x-y|^{2+2s}} dx dy.\end{aligned}$$

Thanks to the Taylor expansion, it yields that

$$\begin{aligned}I &\gtrsim \epsilon^{2s-2} \int_{|x|\le\sqrt{\frac{3}{4}}, |x-y|\le\epsilon} \frac{|\nabla F^+_{13}(x)\cdot(y-x)|^2}{|x-y|^{2+2s}} dx dy \\ &\quad - \epsilon^{2s-2} \int_0^1 \int_{|x|\le\sqrt{\frac{3}{4}}, |x-y|\le\epsilon} \frac{|\nabla^2 F^+_{13}(x+\kappa(y-x))|^2 |x-y|^4}{|x-y|^{2+2s}} dx dy d\kappa.\end{aligned}$$

Note that

$$\epsilon^{2s-2}\int_{|x|\le\sqrt{\frac{3}{4}},|x-y|\le\epsilon}\frac{|\nabla F_{13}^{+}(x)\cdot(y-x)|^2}{|x-y|^{2+2s}}dxdy$$
$$=\int_{|x|\le\sqrt{\frac{3}{4}}}|\nabla F_{13}^{+}(x)|^2\int_{|h|\le\epsilon}\epsilon^{2s-2}\frac{\left|\frac{\nabla F_{13}^{+}(x)}{|\nabla F_{13}^{+}(x)|}\cdot\frac{h}{|h|}\right|^2}{|h|^{2s}}dhdx$$

and

$$\epsilon^{2s-2}\int_0^1\int_{|x|\le\sqrt{\frac{3}{4}},|x-y|\le\epsilon}\frac{|\nabla^2 F_{13}^{+}(x+\kappa(y-x))|^2||x-y|^4}{|x-y|^{2+2s}}dxdyd\kappa$$
$$\lesssim\epsilon^2\|F_{13}^{+}\|^2_{H^2(B_{\frac{2}{\sqrt{5}}})}.$$

Then by the definition of ϑ_{13+}, we derive that

$$I\gtrsim\|F_{13}^{+}\|^2_{H^1(B_{\frac{2}{\sqrt{5}}})}-\epsilon^2\|F_{13}^{+}\|^2_{H^2(B_{\frac{2}{\sqrt{5}}})}. \tag{5.43}$$

On the other hand, following the similar argument, we may get

$$I\lesssim\|F_{13}^{+}\|^2_{H^1(B_{\frac{2}{\sqrt{5}}})}+\epsilon^2\|F_{13}^{+}\|^2_{H^2(B_{\frac{2}{\sqrt{5}}})}. \tag{5.44}$$

Thanks to the facts (5.27) and (5.38), (5.43) and (5.44) imply that

$$\|(-\triangle_{\mathbb{S}^2})^{1/2}(\vartheta_{13+}f)\|^2_{L^2(\mathbb{S}^2)}+\|\vartheta_{13+}f\|^2_{L^2(\mathbb{S}^2)}-\epsilon^2\|(-\triangle_{\mathbb{S}^2})(\vartheta_{13+}f)\|^2_{L^2(\mathbb{S}^2)}$$
$$\lesssim I\lesssim\|(-\triangle_{\mathbb{S}^2})^{1/2}(\vartheta_{13+}f)\|^2_{L^2(\mathbb{S}^2)}+\|\vartheta_{13+}f\|^2_{L^2(\mathbb{S}^2)}+\epsilon^2\|(-\triangle_{\mathbb{S}^2})(\vartheta_{13+}f)\|^2_{L^2(\mathbb{S}^2)}.$$

Due to the decomposition (5.23), we finally conclude the claim.

CASE 2: l IS SUFFICIENTLY LARGE. — Observe that

$$A_l=\epsilon^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}}d\sigma d\tau-\epsilon^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}}1_{|\sigma-\tau|\ge\epsilon}d\sigma d\tau.$$

Thanks to Lemma 5.5, there exist universal constants c_3 and c_4 such that

$$\epsilon^{2s-2}\big([l(l+1)]^s-c_3\epsilon^{-2s}\big)\lesssim A_l\lesssim\epsilon^{2s-2}\big([l(l+1)]^s+c_4\epsilon^{-2s}\big).$$

It implies that if $[l(l+1)]^{1/2}\ge 2c_3^{1/2s}\epsilon^{-1}$,

$$A_l\sim\epsilon^{2s-2}[l(l+1)]^s+1. \tag{5.45}$$

CASE 3: $[l(l+1)]^{1/2}\sim\epsilon^{-1}$. — We claim that in this case, $A_l\sim l(l+1)+1$. Observe that for any $N\in\mathbb{N}$,

$$A_l\ge N^{2s-2}(\epsilon/N)^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}}1_{|\sigma-\tau|\le\epsilon/N}d\sigma d\tau$$
$$\ge N^{2s-2}\bigg((\epsilon/N)^{2s-2}\int_{\sigma,\tau\in\mathbb{S}^2}\frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}}1_{|\sigma-\tau|\le\epsilon/N}d\sigma d\tau\bigg).$$

Then, together with (5.41) and (5.42), we derive that if $[l(l+1)]^{1/2}\le(2c_1)^{-1/2}N\epsilon^{-1}$,

$$N^{2s-2}\big(l(l+1)+1\big)\lesssim A_l\lesssim\big(l(l+1)+1\big). \tag{5.46}$$

Choose $N \geq 2\sqrt{2}c_3^{1/(2s)}c_1^{1/2}$, then (5.42), (5.45) and (5.46) yield the claim.

We are in a position to prove the lemma. It is easily obtained from the behavior of A_l. We complete the proof of the lemma. □

5.5. Proof of (1.38)

To prove (1.38), we first give several estimates to the commutator between the Laplace-Beltrami operator and the standard derivatives.

LEMMA 5.8. – *Suppose $a, b \in \mathbb{R}$ and ϕ to be a radial function. Then we have*

$$\mathscr{F}(-\triangle_{\mathbb{S}^2})^{a/2} = (-\triangle_{\mathbb{S}^2})^{a/2}\mathscr{F}$$

and

$$\phi(|D|)(-\triangle_{\mathbb{S}^2})^{a/2} = (-\triangle_{\mathbb{S}^2})^{a/2}\phi(|D|).$$

In particular, it holds

$$\|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{H^b}^2 \sim \sum_{p\geq -1} 2^{2pb}\|(1-\triangle_{\mathbb{S}^2})^{a/2}\mathfrak{F}_p f\|_{L^2}^2.$$

Proof. – Suppose f is a smooth function. Thanks to Theorem 5.3, we have

$$\big((-\triangle_{\mathbb{S}^2})^{a/2}f\big)(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l(l+1))^{a/2}Y_l^m(\sigma)f_l^m(r),$$

where $x = r\sigma$ and $\sigma \in \mathbb{S}^2$. Then if $\xi = \rho\tau$ with $\tau \in \mathbb{S}^2$, then

$$\begin{aligned}\mathscr{F}\big((-\triangle_{\mathbb{S}^2})^{a/2}f\big)(\xi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l(l+1))^{a/2}\mathscr{F}(Y_l^m f_l^m)(\xi)\\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l(l+1))^{a/2}Y_l^m(\tau)W_l^m(\rho),\end{aligned}$$

where we use (5.20) to assume that $\mathscr{F}(Y_l^m f_l^m)(\xi) = Y_l^m(\tau)W_l^m(\rho)$.

Using the same notation, we have

$$(\mathscr{F}f)(\xi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_l^m(\tau)W_l^m(\rho),$$

which implies

$$(-\triangle_{\mathbb{S}^2})^{a/2}(\mathscr{F}f)(\xi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l(l+1))^{a/2}Y_l^m(\tau)W_l^m(\rho) = \mathscr{F}\big((-\triangle_{\mathbb{S}^2})^{a/2}f\big)(\xi).$$

This gives the first equality.

Observe that

$$\begin{aligned}\mathscr{F}\phi(|D|)(-\triangle_{\mathbb{S}^2})^{a/2} &= \phi\mathscr{F}(-\triangle_{\mathbb{S}^2})^{a/2}\\ &= \phi(-\triangle_{\mathbb{S}^2})^{a/2}\mathscr{F} = (-\triangle_{\mathbb{S}^2})^{a/2}\phi\mathscr{F},\end{aligned}$$

where we use the fact that ϕ is a radial function in the last equality.

Moreover, we have

$$\mathscr{F}(-\triangle_{\mathbb{S}^2})^{a/2}\phi(|D|) = (-\triangle_{\mathbb{S}^2})^{a/2}\mathscr{F}\phi(|D|) = (-\triangle_{\mathbb{S}^2})^{a/2}\phi\,\mathscr{F},$$

which is enough to yield the second equality in the lemma.

Finally we give the proof to the last equivalence. It is derived from the facts

$$\|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{H^b}^2 \sim \sum_{p\geq -1} 2^{2pb}\|\mathfrak{F}_p(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{L^2}^2$$

and

$$\begin{aligned}\mathscr{F}(\mathfrak{F}_p(1-\triangle_{\mathbb{S}^2})^{a/2}) &= \varphi(2^{-p}\cdot)\mathscr{F}(1-\triangle_{\mathbb{S}^2})^{a/2} = \varphi(2^{-p}\cdot)(1-\triangle_{\mathbb{S}^2})^{a/2}\mathscr{F}\\ &= (1-\triangle_{\mathbb{S}^2})^{a/2}\varphi(2^{-p}\cdot)\mathscr{F} = (1-\triangle_{\mathbb{S}^2})^{a/2}\mathscr{F}\mathfrak{F}_p = \mathscr{F}(1-\triangle_{\mathbb{S}^2})^{a/2}\mathfrak{F}_p,\end{aligned}$$

where $\varphi(2^{-p}\cdot)(\xi) \overset{\text{def}}{=} \varphi(2^{-p}\xi) = \varphi(2^{-p}|\xi|)$, the multiplier of the operator $\mathfrak{F}_p$. We complete the proof of the lemma. □

LEMMA 5.9. – *Suppose $a, b \geq 0$, $m \in \mathbb{N}$ and f is a smooth function. Then we have*

$$\sum_{1\leq i<j\leq 3}\|\Omega_{ij}f\|_{H^a} \sim \|(-\triangle_{\mathbb{S}^2})^{1/2}f\|_{H^a}, \tag{5.47}$$

$$\sum_{1\leq i<j\leq 3}\|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij}f\|_{H^m} + \|f\|_{H^m} \sim \|(1-\triangle_{\mathbb{S}^2})^{(a+1)/2}f\|_{H^m}, \tag{5.48}$$

and

$$\|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{H^m} \sim \sum_{|\alpha|\leq m}\|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2}. \tag{5.49}$$

Moreover, it holds

$$\|(-\triangle_{\mathbb{S}^2})^{a/2}f\|_{H^b} \lesssim \|(-\triangle_{\mathbb{S}^2})^{(a+b)/2}f\|_{L^2} + \|f\|_{H^{a+b}}. \tag{5.50}$$

Proof. – (i). We first give the proof to the last inequality. Thanks to Theorem 5.3, we have for $f \in L^2$,

$$f(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\mathscr{B}_l^m f)(x),$$

where $x = r\sigma$ with $r \geq 0$ and $\sigma \in \mathbb{S}^2$ and $\mathscr{B}_l^m(x) \overset{\text{def}}{=} f_l^m(r)Y_l^m(\sigma)$. Then one has

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{a/2}f\|_{H^b}^2 &\sim \sum_{k\geq -1} 2^{2kb}\|\mathfrak{F}_k(-\triangle_{\mathbb{S}^2})^{a/2}f\|_{L^2}^2\\ &\sim \sum_{k\geq -1} 2^{2kb}\|(-\triangle_{\mathbb{S}^2})^{a/2}\mathfrak{F}_k f\|_{L^2}^2\\ &\sim \sum_{k\geq -1}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} 2^{2kb}(l(l+1))^a\|\mathscr{B}_l^m(\mathfrak{F}_k f)\|_{L^2}^2\\ &\lesssim \sum_{k\geq -1}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(2^{2k(a+b)} + (l(l+1))^{a+b})\|\mathscr{B}_l^m(\mathfrak{F}_k f)\|_{L^2}^2\\ &\lesssim \|(-\triangle_{\mathbb{S}^2})^{(a+b)/2}f\|_{L^2}^2 + \|f\|_{H^{a+b}}^2.\end{aligned}$$

(ii). Now we turn to the proof of (5.47). Thanks to the fact that $\mathscr{J}\Omega_{ij} = -\Omega_{ij}\mathscr{J}$ if $i \neq j$, we deduce that if $i \neq j$,

$$\mathfrak{F}_k\Omega_{ij} = -\Omega_{ij}\mathfrak{F}_k.$$

Then we have

$$\sum_{1\le i<j\le 3}\|\Omega_{ij}f\|^2_{H^a(\mathbb{R}^3)} \sim \sum_{k\ge -1}\sum_{1\le i<j\le 3} 2^{2ka}\|\Omega_{ij}\mathfrak{F}_k f\|^2_{L^2},$$

which yields

$$\begin{aligned}\sum_{1\le i<j\le 3}\|\Omega_{ij}f\|^2_{H^a(\mathbb{R}^3)} &\sim \sum_{k\ge -1} 2^{2ka}\|(-\triangle_{\mathbb{S}^2})^{1/2}\mathfrak{F}_k f\|^2_{L^2}\\ &\sim \sum_{k\ge -1} 2^{2ka}\|\mathfrak{F}_k(-\triangle_{\mathbb{S}^2})^{1/2} f\|^2_{L^2} \sim \|(-\triangle_{\mathbb{S}^2})^{1/2} f\|^2_{H^a}.\end{aligned}$$

This gives (5.47).

(iii). We divide the proof of (5.48) into two steps.

Step 1: $m = 0$. – We want to prove

$$\sum_{1\le i<j\le 3}\|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij}f\|_{L^2} + \|f\|_{L^2} \sim \|(1-\triangle_{\mathbb{S}^2})^{(a+1)/2}f\|_{L^2}. \tag{5.51}$$

We begin with the case $0 \le a \le 1$. Observing that

$$\begin{aligned}&\langle\Omega_{mn}\Omega_{ij}f,\Omega_{mn}\Omega_{ij}f\rangle - \langle\Omega_{mn}\Omega_{mn}f,\Omega_{ij}\Omega_{ij}f\rangle\\ &\qquad= \langle[\Omega_{mn},\Omega_{ij}f],\Omega_{mn}\Omega_{ij}f\rangle + \langle\big[[\Omega_{ij},\Omega_{mn}],\Omega_{mn}\big]f,\Omega_{ij}f\rangle\\ &\qquad\quad - \langle[\Omega_{ij},\Omega_{mn}]f,\Omega_{mn}\Omega_{ij}f\rangle\end{aligned} \tag{5.52}$$

and

$$[\Omega_{mn},\Omega_{ij}]f = \delta_{ni}\Omega_{mj} + \delta_{jn}\Omega_{im} - \delta_{jm}\Omega_{in} - \delta_{mi}\Omega_{nj},$$

we deduce that

$$\sum_{1\le m<n\le 3}\sum_{1\le i<j\le 3}\|\Omega_{mn}\Omega_{ij}f\|_{L^2} \lesssim \|(-\triangle_{\mathbb{S}^2})^{1/2}f\|_{L^2} + \|(-\triangle_{\mathbb{S}^2})f\|_{L^2}. \tag{5.53}$$

Due to the fact

$$\|(-\triangle_{\mathbb{S}^2})^{1/2}\Omega_{ij}f\|_{L^2} \sim \sum_{1\le m<n\le 3}\|\Omega_{mn}\Omega_{ij}f\|_{L^2},$$

we get

$$\sum_{1\le i<j\le 3}\|(-\triangle_{\mathbb{S}^2})^{1/2}\Omega_{ij}f\|_{L^2} \lesssim \|\triangle_{\mathbb{S}^2}f\|_{L^2} + \|f\|_{L^2}. \tag{5.54}$$

On the other hand, by (5.52), we obtain

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})f\|_{L^2} &\lesssim \sum_{1\le m<n\le 3}\sum_{1\le i<j\le 3}\|\Omega_{mn}\Omega_{ij}f\|_{L^2} + \|(-\triangle_{\mathbb{S}^2})^{1/2}f\|_{L^2}\\ &\lesssim \sum_{1\le i<j\le 3}\|(-\triangle_{\mathbb{S}^2})^{1/2}\Omega_{ij}f\|_{L^2} + \eta\|(-\triangle_{\mathbb{S}^2})f\|_{L^2} + C_\eta\|f\|_{L^2},\end{aligned}$$

where η is a small constant. Together with (5.54), we then obtain (5.51) with $a = 1$.

We turn to the case $0 < a < 1$. Due to Lemma 5.6, for smooth functions g and f, we have

$$\|(-\triangle_{\mathbb{S}^2})^{a/2}(fg)\|_{L^2(\mathbb{S}^2)} \lesssim (\|\nabla_{\mathbb{S}^2} g\|_{L^\infty(\mathbb{S}^2)} + \|g\|_{L^\infty(\mathbb{S}^2)})\|(1-\triangle_{\mathbb{S}^2})^{a/2} f\|_{L^2(\mathbb{S}^2)}.$$

It implies

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{a/2}\vartheta_{13+}\Omega_{ij} f\|_{L^2} &= \|(-\triangle_{\mathbb{S}^2})^{a/2}[\Omega_{ij}(\vartheta_{13+} f) - (\Omega_{ij}\vartheta_{13+})f]\|_{L^2}\\ &\gtrsim \|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij}(\vartheta_{13+} f)\|_{L^2} - \|(1-\triangle_{\mathbb{S}^2})^{a/2} f\|_{L^2}.\end{aligned}\tag{5.55}$$

Using the notations introduced in Lemma 5.6, we have

$$\Omega_{ij}(\vartheta_{13+} f)(u) = r^{-1}\mathscr{A}_k \bar{F}^+_{13}(r, x_1, x_2),$$

with $u = (rx_1, rx_2, r\sqrt{1-|x|^2})$ and $1 \le k \le 3$. Here the operator $\mathscr{A}_k$ is defined by

$$\mathscr{A}_1 \stackrel{\text{def}}{=} \sqrt{1-|x|^2}\partial_{x_1},\ \mathscr{A}_2 \stackrel{\text{def}}{=} \sqrt{1-|x|^2}\partial_{x_2},\ \mathscr{A}_3 \stackrel{\text{def}}{=} x_1\partial_{x_2} - x_2\partial_{x_1}.$$

Therefore, by (5.34), we obtain that

$$\begin{aligned}&\textstyle\sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij}(\vartheta_{13+} f)\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{1/2}(\vartheta_{13+} f)\|_{L^2}\\ &\quad\sim \sum_{1\le k\le 3} \|\mathscr{A}_k \widetilde{F^+_{13}}\|_{L^2_r H^s_x} + \|\widetilde{F^+_{13}}\|_{L^2_r H^1_x}\\ &\quad\sim \|\widetilde{F^+_{13}}\|_{L^2_r H^{1+s}_x} \sim \|(1-\triangle_{\mathbb{S}^2})^{(a+1)/2}(\vartheta_{13+} f)\|_{L^2},\end{aligned}$$

where (5.40) is used in the last equivalence. Then, together with (5.55) and

$$\begin{aligned}&\|\Omega_{ij} f\|_{L^2} + \|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij} f\|_{L^2}\\ &\quad\sim \sum_{m=1}^{3}\big(\|(-\triangle_{\mathbb{S}^2})^{a/2}\vartheta_{1m+}\Omega_{ij} f\|_{L^2} + \|(-\triangle_{\mathbb{S}^2})^{a/2}\vartheta_{1m-}\Omega_{ij} f\|_{L^2}\big) + \|\Omega_{ij} f\|_{L^2},\end{aligned}$$

we have

$$\sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij} f\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{1/2} f\|_{L^2} \sim \|(1-\triangle_{\mathbb{S}^2})^{(a+1)/2} f\|_{L^2},$$

which implies (5.51) with $0 < a < 1$. It completes the proof to (5.51) for $a \in [0, 1]$.

Next we prove that (5.51) holds for $1 < a \le 2$. Suppose $a = 1 + s$ with $0 < s \le 1$. Thanks to the fact that (5.51) holds for $0 \le a \le 1$, we have

$$\begin{aligned}\sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{a/2}\,\Omega_{ij} f\|_{L^2} + \|f\|_{L^2} &\sim \sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{s+1/2}\Omega_{ij} f\|_{L^2} + \|f\|_{L^2}\\ &\sim \sum_{1\le i<j\le 3}\Big(\sum_{1\le m<n\le 3} \|(-\triangle_{\mathbb{S}^2})^{s/2}\Omega_{mn}\Omega_{ij} f\|_{L^2} + \|\Omega_{ij} f\|_{L^2}\Big) + \|f\|_{L^2}\\ &\sim \sum_{1\le i<j\le 3}\sum_{1\le m<n\le 3}\sum_{p=1}^{3}\big(\|(-\triangle_{\mathbb{S}^2})^{s/2}\big(\vartheta_{1p+}\Omega_{mn}\Omega_{ij} f\big)\|_{L^2}\\ &\qquad + \|(-\triangle_{\mathbb{S}^2})^{s/2}\big(\vartheta_{1p-}\Omega_{mn}\Omega_{ij} f\big)\|_{L^2}\big) + \|(1-\triangle_{\mathbb{S}^2})^{1/2} f\|_{L^2}.\end{aligned}$$

Notice that

$$\begin{aligned}(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+}\Omega_{mn}\Omega_{ij} f) &= (-\triangle_{\mathbb{S}^2})^{s/2}\Omega_{mn}\Omega_{ij}(\vartheta_{13+} f) - (-\triangle_{\mathbb{S}^2})^{s/2}\big((\Omega_{mn}\vartheta_{13+})(\Omega_{ij} f)\big)\\ &\quad - (-\triangle_{\mathbb{S}^2})^{s/2}\big((\Omega_{ij}\vartheta_{13+})(\Omega_{mn} f)\big) - (-\triangle_{\mathbb{S}^2})^{s/2}\big((\Omega_{mn}\Omega_{ij}\vartheta_{13+}) f\big),\end{aligned}$$

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}\big((\Omega_{mn}\vartheta_{13+})(\Omega_{ij}f)\big)\|_{L^2} \lesssim \|(1-\triangle_{\mathbb{S}^2})^{s/2}\Omega_{ij}f\|_{L^2} \lesssim \|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{L^2}$$

and

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}\big((\Omega_{mn}\Omega_{ij}\vartheta_{13+})f\big)\|_{L^2} \lesssim \|(1-\triangle_{\mathbb{S}^2})^{s/2}f\|_{L^2},$$

then we get

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}(\vartheta_{13+}\Omega_{mn}\Omega_{ij}f)\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{L^2} \sim \|(-\triangle_{\mathbb{S}^2})^{s/2}\Omega_{mn}\Omega_{ij}(\vartheta_{13+}f)\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{L^2}.$$

Thus we have

$$\begin{aligned}\textstyle\sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij}f\|_{L^2} + \|f\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{L^2} \\ \sim \sum_{1\le i<j\le 3}\sum_{1\le m<n\le 3}\sum_{p=1}^{3}\big(\|(-\triangle_{\mathbb{S}^2})^{s/2}\big(\Omega_{mn}\Omega_{ij}\vartheta_{1p+}f\big)\|_{L^2} \\ + \|(-\triangle_{\mathbb{S}^2})^{s/2}\big(\Omega_{mn}\Omega_{ij}\vartheta_{1p-}f\big)\|_{L^2}\big) + \|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{L^2}.\end{aligned}$$

If we show

$$\sum_{1\le i<j\le 3}\sum_{1\le m<n\le 3}\|(-\triangle_{\mathbb{S}^2})^{s/2}\Omega_{mn}\Omega_{ij}(\vartheta_{13+}f)\|_{L^2} + \|\vartheta_{13+}f\|_{L^2} \sim \|(1-\triangle_{\mathbb{S}^2})^{(s+2)/2}(\vartheta_{13+}f)\|_{L^2}, \tag{5.56}$$

then by Young inequality,

$$\|(1-\triangle_{\mathbb{S}^2})^{a/2}f\|_{L^2} \le \eta\|(1-\triangle_{\mathbb{S}^2})^{(a+1)/2}f\|_{L^2} + C_\eta\|f\|_{L^2},$$

and we conclude the equivalence (5.51) with $1 \le a \le 2$. It remains to prove (5.56). On the one hand, we observe that

$$\begin{aligned}\sum_{1\le i<j\le 3}\sum_{1\le m<n\le 3}\big(\|(-\triangle_{\mathbb{S}^2})^{s/2}\Omega_{mn}\Omega_{ij}(\vartheta_{13+}f)\|_{L^2} + \|\Omega_{mn}\Omega_{ij}(\vartheta_{13+}f)\|_{L^2}\big) + \|\vartheta_{13+}f\|_{L^2} \\ \sim \sum_{k=1}^{3}\sum_{p=1}^{3}\|\mathscr{A}_k\mathscr{A}_p\widetilde{F}_{13+}\|_{L^2_rH^s_x} + \|\widetilde{F}_{13+}\|_{L^2_rH^2_x} \\ \sim \|\widetilde{F}_{13+}\|_{L^2_rH^{2+s}_x}.\end{aligned}$$

On the other hand, it is easy to check that

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{(s+2)/2}(\vartheta_{13+}f)\|_{L^2} + \|\vartheta_{13+}f\|_{L^2} &\sim \|(-\triangle_{\mathbb{S}^2})^{s/2}(-\triangle_{\mathbb{S}^2})(\vartheta_{13+}f)\|_{L^2} + \|\vartheta_{13+}f\|_{L^2} \\ &\sim \|\mathbf{L}\widetilde{F}_{13+}\|_{L^2_rH^s_x} + \|\widetilde{F}_{13+}\|_{L^2_rL^2_x} \\ &\sim \|\widetilde{F}_{13+}\|_{L^2_rH^{2+s}_x},\end{aligned}$$

where we use (5.38) and (5.39). We end the proof to (5.56) by these two equivalences. Then we get (5.51) with $a \in [0,2]$.

To complete the proof, we first use inductive method to show that for $a \ge 0$,

$$\|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij}f\|_{L^2} \lesssim \|(1-\triangle_{\mathbb{S}^2})^{a/2}(-\triangle_{\mathbb{S}^2})^{1/2}f\|_{L^2}. \tag{5.57}$$

Since (5.57) holds for $0 \le a \le 2$, we assume (5.57) holds for $a \le m$ with $m \ge 2$. Suppose $a \in [m, m+1]$, then we have

$$\begin{aligned}
\|(-\triangle_{\mathbb{S}^2})^{a/2}\ \Omega_{ij}\ f\|_{L^2} &= \|(-\triangle_{\mathbb{S}^2})^{a/2-1} \sum_{1\le m<n\le 3} \Omega_{mn}^2 \Omega_{ij} f\|_{L^2}\\
&\lesssim \|(-\triangle_{\mathbb{S}^2})^{a/2-1}\Omega_{ij}(-\triangle_{\mathbb{S}^2}) f\|_{L^2} + \|(-\triangle_{\mathbb{S}^2})^{a/2-1}[\sum_{1\le m<n\le 3} \Omega_{mn}^2, \Omega_{ij}] f\|_{L^2}\\
&\lesssim \|(-\triangle_{\mathbb{S}^2})^{a/2-1}\Omega_{ij}(-\triangle_{\mathbb{S}^2}) f\|_{L^2} + \sum_{1\le m<n\le 3}\sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{a/2-1}\Omega_{mn}\Omega_{ij} f\|_{L^2}\\
&\lesssim \|(1-\triangle_{\mathbb{S}^2})^{a/2}(-\triangle_{\mathbb{S}^2})^{1/2} f\|_{L^2},
\end{aligned}$$

where we use the inductive assumption in the last inequality and the fact

$$[\sum_{1\le m<n\le 3} \Omega_{mn}^2, \Omega_{ij}] = \sum_{1\le m<n\le 3} \big(\Omega_{mn}[\Omega_{mn}, \Omega_{ij}] + [\Omega_{mn}, \Omega_{ij}]\Omega_{mn}\big).$$

We complete the inductive argument to derive (5.57).

Now we are ready to prove (5.51). We may assume that (5.51) holds for $a \le m$ with $m \ge 1$. Suppose $a \in [m, m+1]$. We have

$$\begin{aligned}
&\sum_{1\le i<j\le 3} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij} f\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{1/2} f\|_{L^2}\\
\sim &\sum_{1\le i<j\le 3} \|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2+1/2}\Omega_{ij} f\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{1/2} f\|_{L^2}\\
\sim &\sum_{1\le i<j\le 3}\sum_{1\le m<n\le 3} \|(-\triangle_{\mathbb{S}^2})^{(a-1)/2}\Omega_{mn}\Omega_{ij} f\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{1/2}\Omega_{ij} f\|_{L^2} + \|(1-\triangle_{\mathbb{S}^2})^{1/2} f\|_{L^2}.
\end{aligned}$$

Due the fact

$$\begin{aligned}
\|(-\triangle_{\mathbb{S}^2})^{(a+1)/2} f\|_{L^2} &= \|(-\triangle_{\mathbb{S}^2})^{(a-1)/2}(-\triangle_{\mathbb{S}^2}) f\|_{L^2}\\
&\le \sum_{1\le i<j\le 3}\sum_{1\le m<n\le 3} \|(-\triangle_{\mathbb{S}^2})^{(a-1)/2}\Omega_{mn}\Omega_{ij} f\|_{L^2}\\
&\lesssim \|(1-\triangle_{\mathbb{S}^2})^{a/2}(-\triangle_{\mathbb{S}^2})^{1/2} f\|_{L^2} \le \|(1-\triangle_{\mathbb{S}^2})^{(a+1)/2} f\|_{L^2},
\end{aligned}$$

where we use (5.57) in the second inequality, we finally derive the desired result and end the inductive argument to the equivalence (5.51).

5.5.1. *Step 2: $m \in \mathbb{N}$.* – By Lemma 5.8 and (5.51), we have that for $a \ge 1$,

$$\begin{aligned}
\|(1-\triangle_{\mathbb{S}^2})^{a/2} f\|_{H^m}^2 &\sim \sum_{k\ge -1} 2^{2mk}\|(1-\triangle_{\mathbb{S}^2})^{a/2}\mathfrak{F}_k f\|_{L^2}^2\\
&\sim \sum_{k\ge -1} 2^{2mk}\Big(\sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{(a-1)/2}\Omega_{ij}\mathfrak{F}_k f\|_{L^2}^2 + \|\mathfrak{F}_k f\|_{L^2}^2\Big)\\
&\sim \sum_{k\ge -1} 2^{2mk}\Big(\sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{(a-1)/2}\mathfrak{F}_k\Omega_{ij} f\|_{L^2}^2 + \|\mathfrak{F}_k f\|_{L^2}^2\Big)\\
&\sim \sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{(a-1)/2}\Omega_{ij} f\|_{H^m}^2 + \|f\|_{H^m}^2,
\end{aligned}$$

where we use the fact that $\Omega_{ij}\,\mathscr{F} = -\mathscr{F}\,\Omega_{ij}$ if $i \ne j$. This completes the proof of (5.48).

(iv). The proof of (5.49) has three steps.

Step 1: Proof of (5.49) *with* $a = 0$ *and* $m = 1$. – We want to prove that for $0 \leq a \leq 1$,

$$\|(1-\triangle_{\mathbb{S}^2})^{a/2} f\|_{H^1} \sim \sum_{|\alpha|\leq 1} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2}. \tag{5.58}$$

Obviously (5.58) holds for $a = 0$. Then we separate the proof of (5.58) into two cases.

CASE 1: $0 < a < 1$. — Indeed, by Plancherel theorem and Lemma 5.8, we have

$$\sum_{1\leq i\leq 3} \big(\|(-\triangle_{\mathbb{S}^2})^{a/2}\partial_i f\|_{L^2} + \|\partial_i f\|_{L^2}\big) = \sum_{1\leq i\leq 3} \big(\|(-\triangle_{\mathbb{S}^2})^{a/2}\xi_i \mathscr{F} f\|_{L^2} + \|\xi_i \mathscr{F} f\|_{L^2}\big).$$

Due to Lemma 5.6 and by setting $\xi = r\sigma = (r\sigma_1, r\sigma_2, r\sigma_3)$, we have

$$\begin{aligned}\sum_{1\leq i\leq 3} &\|(-\triangle_{\mathbb{S}^2})^{a/2}\partial_i f\|_{L^2}^2 + \|f\|_{H^1}^2 \\ &\sim \sum_{1\leq i\leq 3}\int_{\sigma,\tau\in\mathbb{S}^2, r>0} \frac{|r\sigma_i(\mathscr{F} f)(r\sigma) - r\tau_i(\mathscr{F} f)(r\tau)|^2}{|\sigma-\tau|^{2+2a}} r^2 d\sigma d\tau dr + \|f\|_{H^1}^2.\end{aligned}$$

Thanks to the observation

$$\begin{aligned}\tfrac{1}{2}|\tau_i|^2|r\ (\mathscr{F} f)(r\sigma) - r(\mathscr{F} f)(r\tau)|^2 &- 2|\sigma_i-\tau_i|^2|r(\mathscr{F} f)(r\sigma)|^2 \\ &\leq |r\sigma_i(\mathscr{F} f)(r\sigma) - r\tau_i(\mathscr{F} f)(r\tau)|^2 \\ &\lesssim |\tau_i|^2|r(\mathscr{F} f)(r\sigma) - r(\mathscr{F} f)(r\tau)|^2 + |\sigma_i-\tau_i|^2|r(\mathscr{F} f)(r\sigma)|^2,\end{aligned}$$

we deduce that

$$\begin{aligned}\sum_{1\leq i\leq 3} &\|(-\triangle_{\mathbb{S}^2})^{a/2}\ \partial_i f\|_{L^2}^2 + \|f\|_{H^1}^2 \\ &\sim \int_{\sigma,\tau\in\mathbb{S}^2, r>0} \frac{|r(\mathscr{F} f)(r\sigma) - r(\mathscr{F} f)(r\tau)|^2}{|\sigma-\tau|^{2+2a}} r^2 d\sigma d\tau dr + \|f\|_{H^1}^2,\end{aligned}$$

which implies

$$\sum_{1\leq i\leq 3} \|(-\triangle_{\mathbb{S}^2})^{a/2}\partial_i f\|_{L^2} + \|f\|_{H^1} \sim \|(-\triangle_{\mathbb{S}^2})^{a/2}(|D|f)\|_{L^2} + \|f\|_{H^1}. \tag{5.59}$$

This is enough to get (5.58) for $0 < a < 1$.

CASE 2: $a = 1$. — It is not difficult to check

$$\begin{aligned}\sum_{|\alpha|\leq 1} \|(-\triangle_{\mathbb{S}^2})^{1/2}\partial^\alpha f\|_{L^2} + \|f\|_{H^1} &\sim \sum_{|\alpha|\leq 1}\sum_{1\leq m<n\leq 3} \|\Omega_{mn}\partial^\alpha f\|_{L^2} + \|f\|_{H^1} \\ &\sim \sum_{|\alpha|\leq 1}\sum_{1\leq m<n\leq 3} \|\partial^\alpha \Omega_{mn} f\|_{L^2} + \|f\|_{H^1} \\ &\sim \sum_{1\leq m<n\leq 3} \|\Omega_{mn} f\|_{H^1} + \|f\|_{H^1} \\ &\sim \|(1-\triangle_{\mathbb{S}^2})^{1/2} f\|_{H^1},\end{aligned}$$

where we use (5.47) in the last equivalence. This completes the proof of (5.58).

Step 2: Proof of (5.49) *with* $a \in [0,1]$ *and* $m \in \mathbb{N}$. Thanks to (5.58), we assume (5.47) holds for $m \le N-1$ with $N \ge 2$. Recall that

$$\sum_{|\alpha|\le N} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2} \sim \sum_{|\alpha|\le N-1} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2} + \sum_{|\alpha|=N-1}\sum_{i=1}^{3} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial_i\partial^\alpha f\|_{L^2}.$$

Due to the result in *Step 1*, we have

$$\begin{aligned}\sum_{|\alpha|\le N-1} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2} &+ \sum_{|\alpha|=N-1}\sum_{i=1}^{3} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial_i\partial^\alpha f\|_{L^2}\\ &\sim \sum_{|\alpha|\le N-1} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2} + \sum_{|\alpha|=N-1} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\langle D\rangle\partial^\alpha f\|_{L^2}.\end{aligned}$$

Hence, together with the assumption that (5.47) holds for $N-1$, we have

$$\begin{aligned}&\sum_{|\alpha|\le N} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2}\\ &\qquad\sim \|(1-\triangle_{\mathbb{S}^2})^{a/2}\langle D\rangle^{N-2}\langle D\rangle f\|_{L^2} + \sum_{|\alpha|=N-1} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha\langle D\rangle f\|_{L^2}.\end{aligned}$$

We deduce that

$$\begin{aligned}&\sum_{|\alpha|\le N} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2}\\ &\quad\sim \sum_{|\alpha|\le N-2} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha\langle D\rangle f\|_{L^2} + \sum_{|\alpha|=N-1} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha\langle D\rangle f\|_{L^2}\\ &\quad\sim \|(1-\triangle_{\mathbb{S}^2})^{a/2} f\|_{H^N},\end{aligned}$$

which completes the inductive argument to derive (5.49) with $a \in [0,1]$.

Step 3: Proof of (5.49) *with* $a \ge 0$ *and* $m \in \mathbb{N}$. *–* We still use the inductive method. Suppose (5.49) holds for $a \le N$ with $N \ge 1$. Suppose now $a \in [N, N+1]$. Due to (5.48) and the inductive assumption, we have

$$\begin{aligned}\|(1-\triangle_{\mathbb{S}^2})^{a/2} f\|_{H^m} &\sim \sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{(a-1)/2}\Omega_{ij} f\|_{H^m} + \|f\|_{H^m}\\ &\sim \sum_{1\le i<j\le 3}\sum_{|\alpha|\le m} \|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2}\partial^\alpha\Omega_{ij} f\|_{L^2} + \|f\|_{H^m}.\end{aligned}$$

Thanks to (5.48), we also have

$$\sum_{|\alpha|\le m}\sum_{1\le i<j\le 3} \|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2}\Omega_{ij}\partial^\alpha f\|_{L^2} + \|f\|_{H^m} \sim \sum_{|\alpha|\le m} \|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2}.$$

Notice that

$$\begin{aligned}
&\sum_{|\alpha|\le m}\sum_{1\le i<j\le 3}\|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2}\Omega_{ij}\partial^\alpha f\|_{L^2}-\sum_{|\beta|\le m}\|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2}\partial^\beta f\|_{L^2}\\
&\quad\lesssim \sum_{|\alpha|\le m}\sum_{1\le i<j\le 3}\|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2}\partial^\alpha\Omega_{ij} f\|_{L^2}\\
&\quad\lesssim \sum_{|\alpha|\le m}\sum_{1\le i<j\le 3}\|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2}\Omega_{ij}\partial^\alpha f\|_{L^2}+\sum_{|\beta|\le m}\|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2}\partial^\beta f\|_{L^2},
\end{aligned}$$

we derive that

$$\begin{aligned}
&\sum_{|\alpha|\le m}\|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2}-C_1\|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2} f\|_{H^m}^2\\
&\quad\lesssim \|(1-\triangle_{\mathbb{S}^2})^{a/2} f\|_{H^m}^2\\
&\quad\lesssim \sum_{|\alpha|\le m}\|(1-\triangle_{\mathbb{S}^2})^{a/2}\partial^\alpha f\|_{L^2}+C_2\|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2} f\|_{H^m}^2.
\end{aligned}\tag{5.60}$$

Observe that

$$\begin{aligned}
\|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2} f\|_{H^m} &\sim \|(1-\triangle_{\mathbb{S}^2})^{(a-1)/2}\langle D\rangle^m f\|_{L^2}\\
&\le \eta\|(1-\triangle_{\mathbb{S}^2})^{a/2}\langle D\rangle^m f\|_{L^2}+C_\eta\|\langle D\rangle^m f\|_{L^2}\\
&\le \eta\|(1-\triangle_{\mathbb{S}^2})^{a/2} f\|_{H^m}+C_\eta\|f\|_{H^m}.
\end{aligned}$$

Then (5.60) yields the desired result and we complete the inductive argument to (5.49). This ends the proof of the lemma. □

We are in a position to prove (1.38).

LEMMA 5.10. – *If* $T_h f(v)\stackrel{\text{def}}{=} f(v+h)$*, then for* $s\ge 0$*, it holds*

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}T_h f\|_{L^2(\mathbb{R}^3)}\lesssim \langle h\rangle^s\big(\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|_{L^2(\mathbb{R}^3)}+\|f\|_{H^s(\mathbb{R}^3)}\big).$$

Proof. – We begin with the case $0\le s\le 1$. Thanks to Lemma 5.6, we have

$$\|(-\triangle_{\mathbb{S}^2})^{s/2}T_h f\|_{L^2(\mathbb{R}^3)}\lesssim \sum_{1\le i<j\le 3}\|T_h f\|_{\mathcal{D}_{\Omega_{ij}}(s,2)}.$$

Since

$$\|\Omega_{ij}T_h f\|_{L^2}\lesssim \langle h\rangle(\|f\|_{H^1}+\|\Omega_{ij} f\|_{L^2}),\qquad \|T_h f\|_{L^2}=\|f\|_{L^2},$$

then applying Lemma 5.2 with $A=\Omega_{ij}$ and $B_k=\partial_k$, we get

$$\|T_h f\|_{\mathcal{D}_{\Omega_{ij}}(s,2)}\lesssim \langle h\rangle^s(\|f\|_{H^s}+\|f\|_{\mathcal{D}_{\Omega_{ij}}(s,2)}).\tag{5.61}$$

Together with Lemma 5.6, we thus obtain the desired result.

Next we turn to the case $1<s\le 2$. Suppose $s=1+a$. Then by Lemma 5.9, we have

$$\begin{aligned}
\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|_{L^2(\mathbb{R}^3)} &\lesssim \sum_{1\le i<j\le 3}(\|f\|_{L^2}+\|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij} f\|_{L^2}),\\
&\lesssim \sum_{1\le i<j\le 3}\big(\|f\|_{L^2}+\|\Omega_{ij} f\|_{\mathcal{D}_{\Omega_{ij}}(a,2)}\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} &\lesssim \|f\| + \sum_{1\le i<j\le 3} \|\Omega_{ij} T_h f\|_{\mathscr{D}_{\Omega_{ij}}(a,2)}\\ &\lesssim \|f\| + \sum_{1\le i<j\le 3} \big(\|T_h \Omega_{ij} f\|_{\mathscr{D}_{\Omega_{ij}}(a,2)}\\ &\quad + \langle h\rangle(\|T_h\partial_i f\|_{\mathscr{D}_{\Omega_{ij}}(a,2)} + \|T_h\partial_j f\|_{\mathscr{D}_{\Omega_{ij}}(a,2)})\big).\end{aligned}$$

Thanks to (5.61) and Lemma 5.6, we are led to

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} &\lesssim \langle h\rangle^s \sum_{1\le i<j\le 3} \big(\|(-\triangle_{\mathbb{S}^2})^{a/2}\Omega_{ij} f\|_{L^2(\mathbb{R}^3)} + \|\Omega_{ij} f\|_{H^a(\mathbb{R}^3)} + \|f\|_{H^s}\\ &\quad + \|(-\triangle_{\mathbb{S}^2})^{a/2}\partial_i f\|_{L^2(\mathbb{R}^3)} + \|(-\triangle_{\mathbb{S}^2})^{a/2}\partial_j f\|_{L^2(\mathbb{R}^3)}\big).\end{aligned}$$

Due to Lemma 5.9, we deduce

$$\|(-\triangle_{\mathbb{S}^2})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} \lesssim \langle h\rangle^s(\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|_{L^2(\mathbb{R}^3)} + \|f\|_{H^s}).$$

Finally we use the inductive method to handle the case $s > 2$. We assume the result holds for $s \le 2N$. Suppose $s \in (2N, 2N+2]$. Then

$$\|(-\triangle_{\mathbb{S}^2})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} = \sum_{1\le i<j\le 3} \|(-\triangle_{\mathbb{S}^2})^{s/2-1}\Omega_{ij}^2 T_h f\|_{L^2(\mathbb{R}^3)}.$$

It is easy to check that

$$\begin{aligned}\Omega_{ij}^2 T_h f &= T_h(\Omega_{ij}^2 f) - h_i T_h(\partial_j\Omega_{ij} f) + h_j T_h(\partial_i\Omega_{ij} f) - h_i T_h(\Omega_{ij}\partial_j f)\\ &\quad + h_i^2 T_h(\partial_j^2 f) - h_j^2 T_h(\partial_i^2 f) + h_j T_h(\Omega_{ij}\partial_i f).\end{aligned}$$

Then by the inductive assumption and Lemma 5.9, we deduce that

$$\begin{aligned}\|(-\triangle_{\mathbb{S}^2})^{s/2} T_h f\|_{L^2(\mathbb{R}^3)} &\lesssim \langle h\rangle^s \sum_{1\le i,j\le 3} \big(\|(-\triangle_{\mathbb{S}^2})^{s/2-1}\Omega_{ij}^2 f\|_{L^2(\mathbb{R}^3)}\\ &\quad + \|\Omega_{ij}^2 f\|_{H^{s-2}} + \|(-\triangle_{\mathbb{S}^2})^{s/2-1}\partial_j\Omega_{ij} f\|_{L^2(\mathbb{R}^3)}\\ &\quad + \|\partial_j\Omega_{ij} f\|_{H^{s-2}} + \|(-\triangle_{\mathbb{S}^2})^{(s-1)/2}\partial_i f\|_{L^2(\mathbb{R}^3)}\\ &\quad + \|\Omega_{ij}\partial_i f\|_{H^{s-2}}\big) + \|(-\triangle_{\mathbb{S}^2})^{(s-2)/2}\partial_i^2 f\|_{L^2(\mathbb{R}^3)} + \|\partial_i^2 f\|_{H^{s-2}}\big)\\ &\lesssim \langle h\rangle^s(\|(-\triangle_{\mathbb{S}^2})^{s/2} f\|_{L^2(\mathbb{R}^3)} + \|f\|_{H^s}),\end{aligned}$$

which completes the inductive argument to get the result. □

6. Conclusions and Perspectives

In this paper, by making full use of two types of the decomposition performed in phase and frequency spaces and the geometric decomposition, we successfully establish several lower and upper bounds for Boltzmann collision operator in weighted Sobolev spaces and in anisotropic spaces. By comparing with the behavior of the linearized operator, we show that all the bounds are sharp. We further show that the strategy of the proof is so robust that we can apply it to the rescaled Boltzmann collision operator (see assumption (B1)). Finally we obtain sharp bounds for the Landau collision operator by so-called grazing collision limit.

It is very interesting to see whether our method used here can be applied or not to capture the exact behavior of the operator under the assumption (1.7) or (1.8). In Section 4, we make an attempt to analyze the Boltzmann collision operator in the process of the grazing collision limit (see Lemma 4.1). We conjecture that if $\mathscr{L}_B^\epsilon$ is the linearized Boltzmann collision operator with the rescaled kernel under the assumption (B1), then

$$\langle \mathscr{L}_B^\epsilon f, f\rangle_v + \|f\|^2_{L^2_{\gamma/2}} \sim \|W^\epsilon(D)f\|^2_{L^2_{\gamma/2}} + \|W^\epsilon((-\triangle_{\mathbb{S}^2})^{\frac{1}{2}})f\|^2_{L^2_{\gamma/2}} + \|W^\epsilon f\|^2_{L^2_{\gamma/2}}, \tag{6.1}$$

where the symbol function W^ϵ is defined in (4.1). Based on the conjecture, we may ask:

1. How to establish a unified framework to solve the Boltzmann and Landau equations in the perturbation regime and prove the asymptotic Formula (1.11);
2. How to describe the behavior of the spectrum of the operator $\mathscr{L}_B^\epsilon$ in the limit $\epsilon \to 0$ for $\gamma \in [-2, -2s)$; we recall that the spectrum gap exists for $\mathscr{L}_B$ if and only if $\gamma \geq -2s$ while it exists for $\mathscr{L}_L$ if and only if $\gamma \geq -2$.

The similar conjecture can be questioned on the operator with the assumption (1.7) or with the Coulomb potential. Then the asymptotics of the Boltzmann equation from short-range interactions to long-range interactions and the Landau approximation for Coulomb potential can be investigated.

Acknowledgments. – Ling-Bing He is supported by NSF of China under Grant 11001149 and 11171173 and the Importation and Development of High-Caliber Talents Project of Beijing Municipal Institutions. The author would like to express his gratitude to colleagues Xu-Guang Lu and Pin Yu for profitable discussions.

BIBLIOGRAPHY

[1] R. Alexandre, L. Desvillettes, C. Villani, B. Wennberg, Entropy dissipation and long-range interactions, *Arch. Ration. Mech. Anal.* **152** (2000), 327–355.

[2] R. Alexandre, F. Herau, W.-X. Li, Global hypoelliptic and symbolic estimates for the linearized boltzmann operator without angular cutoff, preprint arXiv:1212.4632.

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, Regularizing effect and local existence for the non-cutoff Boltzmann equation, *Arch. Ration. Mech. Anal.* **198** (2010), 39–123.

[4] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, The Boltzmann equation without angular cutoff in the whole space: II, Global existence for hard potential, *Anal. Appl. (Singap.)* **9** (2011), 113–134.

[5] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential, *J. Funct. Anal.* **262** (2012), 915–1010.

[6] R. Alexandre, C. Villani, On the Boltzmann equation for long-range interactions, *Comm. Pure Appl. Math.* **55** (2002), 30–70.

[7] Y. Chen, L. He, Smoothing estimates for Boltzmann equation with full-range interactions: spatially homogeneous case, *Arch. Ration. Mech. Anal.* **201** (2011), 501–548.

[8] P. T. Gressman, R. M. Strain, Global classical solutions of the Boltzmann equation without angular cut-off, *J. Amer. Math. Soc.* **24** (2011), 771–847.

[9] P. T. Gressman, R. M. Strain, Sharp anisotropic estimates for the Boltzmann collision operator and its entropy production, *Adv. Math.* **227** (2011), 2349–2384.

[10] L. He, Well-posedness of spatially homogeneous Boltzmann equation with full-range interaction, *Comm. Math. Phys.* **312** (2012), 447–476.

[11] L. He, Asymptotic analysis of the spatially homogeneous Boltzmann equation: grazing collisions limit, *J. Stat. Phys.* **155** (2014), 151–210.

[12] L. He, J.-C. Jiang, Well-posedness of the inhomogeneous Boltzmann equation with Hard potentials and Maxwellian molecules, in preparation.

[13] Z. Huo, Y. Morimoto, S. Ukai, T. Yang, Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff, *Kinet. Relat. Models* **1** (2008), 453–489.

[14] J. C. Jiang, T. P. Liu, Boltzmann collision operator for the infinite range potential, personal communication.

[15] N. Lerner, Y. Morimoto, K. Pravda-Starov, C.-J. Xu, Phase space analysis and functional calculus for the linearized Landau and Boltzmann operators, *Kinet. Relat. Models* **6** (2013), 625–648.

[16] A. Lunardi, *Interpolation theory*, Appunti. Scuola Normale Superiore di Pisa, Edizioni della Normale, Pisa, 2009.

[17] Y. Morimoto, S. Sakamoto, Global solutions in the critical Besov space for the non-cutoff Boltzmann equation, *J. Differential Equations* **261** (2016), 4073–4134.

[18] C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations, *Arch. Rational Mech. Anal.* **143** (1998), 273–307.

[19] C. Villani, On the spatially homogeneous Landau equation for Maxwellian molecules, *Math. Models Methods Appl. Sci.* **8** (1998), 957–983.

[20] C. Villani, Cercignani's conjecture is sometimes true and always almost true, *Comm. Math. Phys.* **234** (2003), 455–490.

[21] C. S. Wang Chang, G. E. Uhlenbeck, *On the propagation of sound in monoatomic gases*, Univ. of Michigan Press, 1952, reprinted in 1970 in *Studies in Statistical Mechanics*. Vol. V.

(Manuscrit reçu le 9 mai 2016 ;
accepté, après révision, le 13 avril 2017.)

Ling-Bing He
Department of Mathematical Sciences
Tsinghua University
Beijing 100084, P. R. China.
E-mail: lbhe@math.tsinghua.edu.cn