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CONGRUENCES OF MODULAR FORMS AND THE IWASAWA λ-INVARIANTS

BY YUICHI HIRANO

ABSTRACT. — In this paper, we show how congruences between cusp forms and Eisenstein series of weight $k \geq 2$ give rise to corresponding congruences between the algebraic parts of the critical values of the associated L-functions. This is a generalization of results of Mazur, Stevens, and Vatsal in the case where $k = 2$. As an application, by proving congruences between the p-adic L-function of a certain cusp form and the product of two Kubota-Leopoldt p-adic L-functions, we prove the Iwasawa main conjecture (up to p-power) for cusp forms at ordinary primes p when the associated residual Galois representations are reducible. This is a generalization of Greenberg and Vatsal in the case where $k = 2$.

RÉSUMÉ (*Congruences de formes modulaires et λ-invariants d'Iwasawa*). — Dans cet article, nous montrons comment les congruences entre formes paraboliques et séries d'Eisenstein de poids $k \geq 2$ donnent lieu à des congruences entre les parties algébriques des valeurs critiques des fonctions L associées. C'est une généralisation des travaux de Mazur, Stevens et Vatsal dans le cas où $k = 2$. Comme application, en prouvant des congruences entre la fonction p-adique L d'une certaine forme parabolique et le produit de deux fonctions de Kubota-Leopoldt p-adiques L, nous prouvons la conjecture principale d'Iwasawa (à puissance p près) pour les formes paraboliques à nombres premiers ordinaires p lorsque les représentations de Galois résiduelles associées sont réductibles. C'est une généralisation des travaux de Greenberg et Vatsal dans le cas où $k = 2$.

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0. Introduction

0.1. Introduction. — The purpose of this paper is to show how congruences between the Fourier coefficients of Hecke eigenforms give rise to corresponding congruences between the special values of the associated L-functions. The study of this topic was initiated by Mazur [25] using the arithmetic of the modular curve $X_0(l)$, where l is a prime number, in order to investigate a weak analog of the Birch and Swinnerton-Dyer conjecture. Mazur's congruence formula was generalized to other congruence subgroups by Stevens [33]. Furthermore, by the theory of higher weight modular symbols, Ash and Stevens [2] have examined congruences between special values of the L-functions of cusp forms of higher weight over $\mathrm{SL}_2(\mathbb{Z})$ and those of the L-functions of cusp forms of weight 2 over $\Gamma_0(l)$. Moreover, Vatsal [39] has proved congruences between special values of the L-functions of two cusp forms of higher weight over $\Gamma_0(N)$, where N is a more general positive integer. Also, he obtained congruences between special values of the L-functions of cusp forms of weight 2 and those of the L-functions of Eisenstein series of weight 2. Moreover, Greenberg and Vatsal [16] used Vatsal's congruences [39] to study the Iwasawa invariants of elliptic curves in towers of cyclotomic fields. In particular, they provided evidence for the Iwasawa main conjecture for elliptic curves. Their work was motivated by Kato's results on the Iwasawa main conjecture for modular forms [21].

In this paper, we present a way to obtain congruences of the special values of the L-functions from congruences between cusp forms and Eisenstein series of weight $k \geq 2$. This is a generalization of the works explained above by Mazur [25], Stevens [33], and Vatsal [39].

Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$ and $\varpi \in \mathcal{O}$ a uniformizer.

THEOREM 0.1 (= Theorem 2.10). — *Let p be an odd prime number, r a positive integer, and k an integer with $2 \leq k \leq p-1$. Let $f = \sum_{n=1}^{\infty} a(n,f)\, e(nz) \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a p-ordinary normalized Hecke eigenform. Assume that the residual Galois representation $\bar{\rho}_f$ associated to f is reducible and of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix},$$

and either ξ_1 or ξ_2 is unramified at p. Assume also that there exists an Eisenstein series $G = E_k(\psi_1, \psi_2) \in M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ (for the definition, see Theorem 3.18) such that G satisfies the assumptions of Theorem 1.9 and $f \equiv G \,(\mathrm{mod}\ \varpi^r)$ (for the definition, see before Theorem 2.10). Then there exist a parity $\alpha \in \{\pm 1\}$ (explicitly given by (A.27)), a complex number $\Omega_f^{\alpha} \in \mathbb{C}^{\times}$, and a p-adic unit $u \in \mathcal{O}^{\times}$ such that, for every primitive Dirichlet character χ whose conductor m_χ is prime to N, the following congruence holds:

(1) *if* $(m_\chi, p) = 1$, *then, for each* j *with* $0 \leq j \leq k-2$ *and* $\alpha = \chi(-1)(-1)^j$,

$$\tau(\bar{\chi})\frac{L(f,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}\Omega_f^\alpha} \equiv u\tau(\bar{\chi})\frac{L(G,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}} \pmod{\varpi^r}.$$

(2) *if* $p|m_\chi$, *we assume that* $m_\chi \in \varpi^r\mathcal{O}$, χ *is non-exceptional (see Definition 2.11), and* $\alpha = \chi(-1)$. *Then*

$$\tau(\bar{\chi})\frac{L(f,\chi,1)}{(2\pi\sqrt{-1})\Omega_f^\alpha} \equiv u\tau(\bar{\chi})\frac{L(G,\chi,1)}{2\pi\sqrt{-1}} \pmod{\varpi^r}.$$

The organization of this paper is as follows.

In §1, we generalize Stevens's results [33, 34]. We construct a desired 1-cocycle π_g associated to a modular form g of weight $k \geq 2$ (Definition 1.2) and prove that π_g is integral, that is, π_g takes values in $L_{k-2}(\mathcal{O})$ under some assumption (Theorem 1.9). In terms of Schoenberg's cocycle, Stevens gave a generalization of the Mazur's congruence formula [25] to general congruence subgroups [33]. Also, he expected that these methods would be generalized to higher weight modular forms and to Hilbert modular forms [33]. The construction of such cocycles π_g associated to modular forms g of weight k has been accomplished so far only in the case of weight $k = 2$ mainly because of certain combinatorial problem arising in the higher weight case $k > 2$. Indeed, a discrete subgroup Γ acts on $L_{k-2}(\mathcal{O})$ trivially only in the case $k = 2$.

In §2, we generalize Vatsal's results [39].

If a Hecke eigenform $f = \sum_{n=1}^\infty a(n,f)e(nz)$ of weight $k \geq 2$ and an Eisenstein series $G = \sum_{n=0}^\infty a(n,G)e(nz)$ of weight $k \geq 2$ are related by a congruence of the Fourier coefficients $a(n,f) \equiv a(n,G) (\mathrm{mod}\ \varpi^r)$ for all $n \geq 0$, we derive congruences between the special values of the associated L-functions (Theorem 2.10). One of the key ingredients in Vatsal's proof [39] is to describe the special values of the L-functions attached to the modular form G as a linear combination of 1-cocycles π_G due to the work of Stevens [33], which allows us to prove congruences between the special values by using cohomological arguments.

In Appendix A, we give a relation between p-adic modular forms and p-adic parabolic cohomologies of Hecke modules in the case the residual Galois representations $\bar{\rho}_f (= \rho_f (\mathrm{mod}\ \varpi))$ associated to a cusp forms f is reducible by using integral p-adic Hodge theory. Our problem on the special values of the L-functions is closely related to a multiplicity-one theorem, which is introduced by Mazur. In the case $\bar{\rho}_f$ is irreducible, $k < p$, and a level N is prime to p, a multiplicity-one theorem is known to be valid by p-adic Hodge theory for open varieties with non-constant coefficients [10]. In particular, Theorem A.12, which may be regarded as p-adic Eichler-Shimura isomorphism, is crucial to define the canonical periods Ω_f^α associated to f and prove congruences between $\pi_f^\alpha/\Omega_f^\alpha$ and π_G^α modulo ϖ^r.

In §3, we generalize Greenberg-Vatsal's results [16]. Using Vatsal's congruences, it is devoted to an application to the Iwasawa main conjecture for elliptic curves under certain assumptions. In the same manner, Theorem 0.1 is used to establish a congruence between a p-adic L-function attached to f and the product of two Kubota-Leopoldt p-adic L-functions (Theorem 3.19). Then, following the work of Kato [21], we will prove the following theorem, which has not been treated by Skinner and Urban [32]:

THEOREM 0.2. — *Let p be an odd prime number and k an integer such that $2 \le k \le p-1$. Let $f \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a p-ordinary normalized Hecke eigenform. We assume that the residual Galois representation $\bar{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\varpi)$ associated to f is reducible and of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix}$$

and that

$$\text{(Assumption)} \qquad \begin{aligned} &\psi \text{ is unramified at } p \text{ and odd, and} \\ &\varphi \text{ is ramified at } p \text{ and even.} \end{aligned}$$

Then $\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}$. In particular, the Iwasawa main conjecture for such f is true up to ϖ-power.

The work of §1, §2, and §3 is based on the author's master thesis at the University of Tokyo in 2010. After I had finished writing this paper, I found a result obtained by Heumann and Vatsal [17], which is almost the same one as Theorem 0.1 (1) (in the case $(m_\chi, p) = 1$) in this paper. We also treat the case $p | m_\chi$ (Theorem 0.1 (2)) and apply Theorem 0.1 (2) to the Iwasawa main conjecture.

0.2. Notation. — In this paper, p and l always denote distinct prime numbers.

We denote by $\mathbb{N}$ the set of natural numbers (that is, positive integers), denote by $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) the ring of rational integers (resp. p-adic integers), and also denote by $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) the rational number field (resp. the p-adic number field). We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and fix embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C},$$

where $\mathbb{C}$ denotes the complex number field.

We assume that every ring is commutative with unity. For a ring R and $n \in \mathbb{N}$, we use the following notation:

$$\begin{aligned} \mathrm{M}_n(R) &= \{(n \times n)\text{-matrices with entries in } R\}, \\ \mathrm{GL}_n(R) &= \{M \in \mathrm{M}_n(R) | M \text{ is an invertible matrix}\}, \\ \mathrm{SL}_n(R) &= \{M \in \mathrm{GL}_n(R) | \det(M) = 1\}. \end{aligned}$$

Moreover, if R is a subring of $\mathbb{R}$, we put

$$\mathrm{GL}_n^+(R) = \{M \in \mathrm{GL}_n(R) | \det(M) > 0\}.$$

Let $\mathfrak{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ be the upper half plane and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ the extended upper half plane obtained by adding the cusps. Then $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathfrak{H}$ by

$$\alpha z = \frac{az+b}{cz+d}$$

for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathfrak{H}$. Let $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

The principal congruence subgroups are the subgroup $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma(N) = \left\{ \alpha \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$

where N is a positive integer. A congruence subgroup is a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ containing a principal congruence group. The smallest integer $N > 0$ for which

$$\Gamma(N) \subset \Gamma$$

is called the level of Γ.

We will be mostly interested in the following special congruence subgroups:

$$\Gamma_0(N) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\},$$

$$\Gamma_1(N) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv d \equiv 1 \bmod N \right\}.$$

Let k be a positive integer ≥ 2. For any function f on $\mathfrak{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, we define the function $f|_k\gamma$ on $\mathfrak{H}$ by

$$f|_k\gamma(z) = \det(\gamma)^{k-1} f(\gamma z)(cz+d)^{-k}.$$

We simply write $f|_k\gamma$ for $f|\gamma$ if there is no risk of confusion. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and N a positive integer such that $\Gamma(N) \subset \Gamma$. Any holomorphic function on $\mathfrak{H}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma(N)$ has the Fourier expansion of the form:

$$\sum_{n=0}^{\infty} a(n, f)\, e\left(\frac{nz}{N}\right),$$

where $e(z) = \exp(2\pi\sqrt{-1}z)$.

We define the space $M_k(\Gamma, \mathbb{C})$ of modular forms of weight k with respect to Γ to be the space of holomorphic functions f on $\mathfrak{H}$ satisfying the following conditions:

(a) $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

(b) $a\,(n, f|_k\alpha) = 0$ if $n < 0$ for each $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Here we note that the function $f|_k\alpha$ is invariant under the action of $\alpha^{-1}\Gamma\alpha$ and hence $f|_k\alpha$ has the Fourier expansion. We define the space $S_k(\Gamma, \mathbb{C})$ of cusp forms to be the subspace of $M_k(\Gamma, \mathbb{C})$ consisting of $f \in M_k(\Gamma, \mathbb{C})$ satisfying the following condition:

(c) $a\,(0, f|_k\alpha) = 0$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Let $\varepsilon\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character modulo N. We put

$$M_k(\Gamma_0(N), \varepsilon, \mathbb{C}) = \Big\{ f \in M_k(\Gamma_1(N), \mathbb{C}) \;\Big|\; f|_k\gamma = \varepsilon(d) f \text{ for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \Big\},$$

$$S_k(\Gamma_0(N), \varepsilon, \mathbb{C}) = M_k(\Gamma_0(N), \varepsilon, \mathbb{C}) \cap S_k(\Gamma_1(N), \mathbb{C}).$$

We remark that $M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ is trivial if $\varepsilon(-1) \neq (-1)^k$.

For a ring R and a non-negative integer n, we denote by $L_n(R)$ the degree n part $\mathrm{Sym}^n_R(RX \oplus RY)$ of the polynomial algebra $R[X, Y]$. Thus, $L_n(R)$ consists of the homogeneous polynomials of degree n in two-variables X and Y, with coefficients in R. The semigroup $\Sigma = \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$ acts on $L_n(R)$ by

$$\gamma \cdot P(X, Y) = P((X, Y)\det(\gamma)^t\gamma^{-1}).$$

If R is a $\mathbb{Q}$-algebra, we also define the action of Σ on $L_n(R)$ by

$$\gamma \star P(X, Y) = P((X, Y)^t\gamma^{-1}).$$

In the similarly way, $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $L_n(R)$ for $\mathbb{Q}$-algebra R and it is denoted by $\star$. We simply denote $\det(\alpha)\alpha^{-1}$ by α^ι for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. We remark that these three actions coincide if they are restricted to $\mathrm{SL}_2(\mathbb{Z})$.

Moreover, for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and a function G on $\mathfrak{H}$, we have the pull-back formula

$$\alpha^*(G(z)(X - zY)^{k-2}dz) = (G|\alpha)(z)\alpha \star (X - zY)^{k-2}dz.$$

Furthermore, for $\alpha, \beta \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\begin{aligned} \alpha^*(G(z)\beta \star (X - zY)^{k-2}dz) &= \beta \star (\alpha^*(G(z)(X - zY)^{k-2}dz) \\ &= (G|\alpha)(z)(\beta\alpha) \star (X - zY)^{k-2}dz. \end{aligned} \tag{0.1}$$

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1. Integrality of 1-cocycles

1.1. Preliminary. — Let $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$. Then $G \in M_k(\Gamma, \mathbb{C})$ has the Fourier expansion of the form

$$G(z) = \sum_{n=0}^{\infty} a\,(n, G)\, e\,(nz)\,.$$

Thus, we may regard $M_k(\Gamma_1(N), \mathbb{C}) = \bigoplus_{\varepsilon:(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}} M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ as a subspace of $\mathbb{C}[[e(z)]]$. For a subring A of $\mathbb{C}$, we put

$$\begin{aligned}
M_k(\Gamma_1(N), A) &= M_k(\Gamma_1(N), \mathbb{C}) \cap A[[e(z)]],\\
S_k(\Gamma_1(N), A) &= S_k(\Gamma_1(N), \mathbb{C}) \cap A[[e(z)]],\\
M_k(\Gamma_0(N), \varepsilon, A) &= M_k(\Gamma_0(N), \varepsilon, \mathbb{C}) \cap A[[e(z)]],\\
S_k(\Gamma_0(N), \varepsilon, A) &= S_k(\Gamma_0(N), \varepsilon, \mathbb{C}) \cap A[[e(z)]].
\end{aligned}$$

Let χ be a Dirichlet character whose conductor m_χ is prime to N. We put

$$(G \otimes \chi)(z) = \sum_{n=0}^{\infty} a\,(n, G)\, \chi(n) e\,(nz)\,.$$

We note that, if $G \in M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$, then $G \otimes \chi \in M_k(\Gamma_0(M), \varepsilon\chi^2, \mathbb{C})$ where $M = \mathrm{lcm}\{N, m_\chi^2, m_\chi m_\varepsilon\}$. The Dirichlet series

$$\sum_{n=1}^{\infty} a(n, G)\chi(n) n^{-s}$$

converges absolutely for $\mathrm{Re}(s) > k$ and extends to a meromorphic function on the complex plane with a possible simple pole at $s = k$. For each $G \in M_k(\Gamma, \mathbb{C})$, let $L(G, \chi, s)$ denote this analytic continuation. If χ is the trivial character, we simply write $L(G, s)$. We define $D(G, \chi, s)$ as

(1.1)

$$\begin{aligned}
D(G, \chi, s) &= \int_0^{\sqrt{-1}\infty} (\widetilde{G \otimes \chi})(z)(X - zY)^{k-2}\mathrm{Im}(z)^{s-1} dz\\
&= \sum_{j=0}^{k-2} \binom{k-2}{j} \sqrt{-1}^{\,j+1} \Gamma(s+j) \left(\frac{1}{2\pi}\right)^{s+j} L(G, \chi, s+j) X^{k-2-j}(-Y)^j,
\end{aligned}$$

where $\widetilde{G}(z) = G(z) - a(0, G)$ (see, for example, [33, Proposition 2.1.2] and the proof of [28, Theorem 4.3.5]). We call $D(G, \chi, s)$ the Mellin transform of G twisted by χ. The integral $D(G, \chi, s)$ converges absolutely for $\mathrm{Re}(s) > k$ and extends to a meromorphic function on the complex plane with simple poles at $s = -(k-2), \ldots, -1, 0$ and $2, 3, \ldots, k$ (see Proposition 1.1 (2)). We are interested in the special values of $L(G,\chi,s)$ at $s=1,\dots,k-1$, that is, in the special value of $D(G,\chi,s)$ at $s=1$.

PROPOSITION 1.1. — *Let $G\in M_k(\Gamma,\mathbb{C})$, and χ a Dirichlet character whose conductor m_χ is prime to N.*

(1) *If $\alpha=\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\in \mathrm{GL}_2^+(\mathbb{Q})$, then we have*

$$a(0,G|\alpha)=\frac{a^{k-1}}{d}a(0,G),$$
$$\widetilde{G|\alpha}=\widetilde{G}\big|\alpha.$$

(2) *The integral $D(G,\chi,s)$ converges absolutely for $\mathrm{Re}(s)>k$ and extends to a meromorphic function on the complex plane with simple poles at $s=-(k-2),\dots,-1,0$ and $2,3,\dots,k$.*

Proof. — (1) By definition, $(G|\alpha)(z)=\frac{a^{k-1}}{d}G(\alpha z)$. Then we have $a(0,G|\alpha)=\frac{a^{k-1}}{d}a(0,G)$. Moreover, by definition,

$$\begin{aligned}(\widetilde{G|\alpha})(z)&=(G|\alpha)(z)-a(0,G|\alpha)\\ &=\frac{a^{k-1}}{d}(G(\alpha z)-a(0,G))\\ &=\frac{a^{k-1}}{d}\widetilde{G}(\alpha z)\\ &=(\widetilde{G}\big|\alpha)(z).\end{aligned}$$

(2) For $\mathrm{Re}(s)>k$,

$$\begin{aligned}D(G,\chi,s)&=\int_0^{\sqrt{-1}\infty}(\widetilde{G\otimes\chi})(z)(X-zY)^{k-2}\mathrm{Im}(z)^{s-1}dz\\ &=\int_{\sqrt{-1}}^{\sqrt{-1}\infty}(\widetilde{G\otimes\chi})(z)(X-zY)^{k-2}\mathrm{Im}(z)^{s-1}dz\\ &\quad+\int_0^{\sqrt{-1}}(\widetilde{G\otimes\chi})(z)(X-zY)^{k-2}\mathrm{Im}(z)^{s-1}dz.\end{aligned}$$

Now we calculate the second term. We put $y=\mathrm{Im}(z)$. Then we get

$$\begin{aligned}&\int_0^{\sqrt{-1}}(\widetilde{G\otimes\chi})(z)(X-zY)^{k-2}\mathrm{Im}(z)^{s-1}dz\\ &\quad=\int_0^{\sqrt{-1}}(G\otimes\chi)(z)(X-zY)^{k-2}y^{s-1}dz\\ &\qquad-\int_0^{\sqrt{-1}}a(0,G)\chi(0)(X-zY)^{k-2}y^{s-1}dz\end{aligned}$$

$$
\begin{aligned}
&= \int_{\sigma^{-1}0}^{\sigma^{-1}\sqrt{-1}} (G\otimes\chi)(\sigma z)(X-\sigma zY)^{k-2}y^{1-s}d\sigma z \\
&\quad - \int_{0}^{\sqrt{-1}} a(0,G)\chi(0)(X-zY)^{k-2}y^{s-1}dz \\
&= -\int_{\sqrt{-1}}^{\sqrt{-1}\infty} ((G\otimes\chi)|\sigma)(z)\sigma\cdot(X-zY)^{k-2}y^{1-s}dz \\
&\quad - \int_{0}^{\sqrt{-1}} a(0,G)\chi(0)(X-zY)^{k-2}y^{s-1}dz \\
&= -\int_{\sqrt{-1}}^{\sqrt{-1}\infty} ((\widetilde{G\otimes\chi})|\sigma)(z)\sigma\cdot(X-zY)^{k-2}y^{1-s}dz \\
&\quad - \int_{\sqrt{-1}}^{\sqrt{-1}\infty} a(0,(G\otimes\chi)|\sigma)\sigma\cdot(X-zY)^{k-2}y^{1-s}dz \\
&\quad - \int_{0}^{\sqrt{-1}} a(0,G)\chi(0)(X-zY)^{k-2}y^{s-1}dz \\
&= -\int_{\sqrt{-1}}^{\sqrt{-1}\infty} ((\widetilde{G\otimes\chi})|\sigma)(z)\sigma\cdot(X-zY)^{k-2}y^{1-s}dz \\
&\quad - a(0,(G\otimes\chi)|\sigma)\sqrt{-1}\sum_{j=0}^{k-2}\binom{k-2}{j}(\sqrt{-1}X)^{j}Y^{k-2-j}\frac{-1}{j+2-s} \\
&\quad - a(0,G)\chi(0)\sqrt{-1}\sum_{j=0}^{k-2}\binom{k-2}{j}X^{k-2-j}(-\sqrt{-1}Y)^{j}\frac{1}{s+j}.
\end{aligned}
$$

Here the third equality follows from (0.1). By setting $s=1$, the second term is equal to

$$a(0,(G\otimes\chi)|\sigma)\int_{0}^{\sqrt{-1}}\sigma\cdot(X-zY)^{k-2}dz.$$

This proves (2). □

1.2. Construction of 1-cocycles. — In order to define a desired cocycle with good arithmetic and p-adic properties, we need to choose some special coboundary element as in [33].

DEFINITION 1.2. — For a congruence subgroup Γ, let $G\in M_k(\Gamma,\mathbb{C})$. For $\alpha,\beta\in \mathrm{GL}_2^+(\mathbb{Q})$ and $z_0\in\mathfrak{H}$, we define the map

$$\pi_{G,\beta}(z_0)\colon \mathrm{GL}_2^+(\mathbb{Q})\longrightarrow L_{k-2}(\mathbb{C})$$

by

$$\begin{aligned}\pi_{G,\beta}(z_0)(\alpha) = &\int_{z_0}^{\alpha z_0} (G|\beta)(z)\beta\star(X-zY)^{k-2}dz\\ &+\int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\beta\alpha})(z)\beta\alpha\star(X-zY)^{k-2}dz\\ &-a(0,G|\beta\alpha)\int_0^{z_0}\beta\alpha\star(X-zY)^{k-2}dz\\ &-\int_{z_0}^{\sqrt{-1}\infty}(\widetilde{G|\beta})(z)\beta\star(X-zY)^{k-2}dz\\ &+a(0,G|\beta)\int_0^{z_0}\beta\star(X-zY)^{k-2}dz.\end{aligned}$$

We remark that the second and fourth integrals converge absolutely by using Proposition 1.1 (2). If $\beta = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, we simply write π_G instead of $\pi_{G,\beta}$. Then we have

$$\pi_{G,\beta}(z_0)(\alpha) = \beta\star\pi_{G|\beta}(z_0)(\alpha). \tag{1.2}$$

REMARK 1.3. — If G is a cusp form, then, for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, by using (0.1),

$$\pi_G(z_0)(\alpha) = \int_{\sqrt{-1}\infty}^{\alpha\sqrt{-1}\infty} G(z)(X-zY)^{k-2}dz$$

is the usual Eichler-Shimura cocycle.

PROPOSITION 1.4. — *For each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, the value $\pi_{G,\beta}(z_0)(\alpha)$ is independent of $z_0 \in \mathfrak{H}$.*

Proof. — For any $z_0, z_0' \in \mathfrak{H}$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we have

$$\begin{aligned}\pi_{G,\beta}(z_0)(\alpha) - \pi_{G,\beta}(z_0')(\alpha) = &\int_{z_0}^{\alpha z_0}(G|\beta)(z)\beta\star(X-zY)^{k-2}dz\\ &-\int_{z_0'}^{\alpha z_0'}(G|\beta)(z)\beta\star(X-zY)^{k-2}dz\\ &+\int_{z_0}^{\sqrt{-1}\infty}(\widetilde{G|\beta\alpha})(z)\beta\alpha\star(X-zY)^{k-2}dz\\ &-\int_{z_0'}^{\sqrt{-1}\infty}(\widetilde{G|\beta\alpha})(z)\beta\alpha\star(X-zY)^{k-2}dz\\ &-\int_{z_0}^{\sqrt{-1}\infty}(\widetilde{G|\beta})(z)\beta\star(X-zY)^{k-2}dz\end{aligned}$$

$$
\begin{aligned}
&+\int_{z_0'}^{\sqrt{-1}\infty}(\widetilde{G|\beta})(z)\beta\star(X-zY)^{k-2}dz\\
&-a(0,G|\beta\alpha)\int_0^{z_0}\beta\alpha\star(X-zY)^{k-2}dz\\
&+a(0,G|\beta\alpha)\int_0^{z_0'}\beta\alpha\star(X-zY)^{k-2}dz\\
&+a(0,G|\beta)\int_0^{z_0}\beta\star(X-zY)^{k-2}dz\\
&-a(0,G|\beta)\int_0^{z_0'}\beta\star(X-zY)^{k-2}dz.
\end{aligned}
$$

By using the pullback formula (0.1), we get

$$
\begin{aligned}
\pi_{G,\beta}(z_0)(\alpha)-\pi_{G,\beta}(z_0')(\alpha)=&\int_{\alpha z_0'}^{\alpha z_0}(G|\beta)(z)\beta\star(X-zY)^{k-2}dz\\
&+\int_{z_0}^{z_0'}(G|\beta)(z)\beta\star(X-zY)^{k-2}dz\\
&+\int_{z_0}^{z_0'}(\widetilde{G|\beta\alpha})(z)\beta\alpha\star(X-zY)^{k-2}dz\\
&-\int_{z_0}^{z_0'}(\widetilde{G|\beta})(z)\beta\star(X-zY)^{k-2}dz\\
&-a(0,G|\beta\alpha)\int_{z_0'}^{z_0}\beta\alpha\star(X-zY)^{k-2}dz\\
&+a(0,G|\beta)\int_{z_0'}^{z_0}\beta\star(X-zY)^{k-2}dz\\
=&\int_{z_0'}^{z_0}(G|\beta\alpha)(z)\beta\alpha\star(X-zY)^{k-2}dz\\
&-\int_{z_0'}^{z_0}(\widetilde{G|\beta\alpha})(z)\beta\alpha\star(X-zY)^{k-2}dz\\
&-a(0,G|\beta\alpha)\int_{z_0'}^{z_0}\beta\alpha\star(X-zY)^{k-2}dz\\
&-a(0,G|\beta)\int_{z_0}^{z_0'}\beta\star(X-zY)^{k-2}dz\\
&+\int_{z_0}^{z_0'}(G|\beta)(z)\beta\star(X-zY)^{k-2}dz
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{z_0}^{z_0'} (\widetilde{G|\beta})(z)\beta \star (X - zY)^{k-2} dz \\
&= 0.
\end{aligned}
$$
□

By Proposition 1.4, we simply write $\pi_{G,\beta}$ instead of $\pi_{G,\beta}(z_0)$. This map π_G is important for a cohomological treatment of $D(G,\chi,s)$, which we state in the next section. As a preparation for it, the rest of this section is devoted to the proof of some properties of π_G. The proof is based on the method of Stevens [33]. We put

$$D_\alpha(G,s) = \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha})(z)\alpha \star (X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz$$

for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. If $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we simply write $D(G,s)$ instead of $D_\alpha(G,s)$. We remark that $D_\alpha(G,s) = \alpha \star D(G|\alpha, s)$ and hence this integral converges absolutely for $\mathrm{Re}(s) > k$ by using Proposition 1.1 (2).

Proposition 1.5. — (1) *For any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$, we have*

$$D_\beta(G,1) = -\pi_{G,\beta}(\sigma),$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(2) *If* $\tau = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, *then we have*

$$\pi_{G,\alpha}(\sigma) = \pi_{G,\alpha\tau}(\sigma).$$

Proof. — (1) It follows from the proof of Proposition 1.1 (2).

(2) We have

$$
\begin{aligned}
D_{\alpha\tau}(G,s) &= \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha\tau})(z)\alpha\tau \star (X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz \\
&= \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha}\Big|\tau)(z)\alpha\tau \star (X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz \quad \text{(by Prop. 1.1 (1))} \\
&= \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha})(z)\alpha \star (X - zY)^{k-2} \mathrm{Im}(\tau^{-1}z)^{s-1} dz \quad \text{(by (0.1))} \\
&= \left(\frac{v}{u}\right)^{s-1} \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha})(z)\alpha \star (X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz \\
&= \left(\frac{v}{u}\right)^{s-1} D_\alpha(G,s).
\end{aligned}
$$

This proves (2) by setting $s = 1$. □

Proposition 1.6. — *The map π_G has the following properties:*

(1) (cocycle condition) For any $\alpha_1, \alpha_2 \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\pi_G(\alpha_1\alpha_2) = \pi_{G,\alpha_1}(\alpha_2) + \pi_G(\alpha_1).$$

More generally, for any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\pi_{G,\beta}(\alpha_1\alpha_2) = \pi_{G,\beta\alpha_1}(\alpha_2) + \pi_{G,\beta}(\alpha_1).$$

(2) For any $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ with $c \geq 0$,

$$\pi_G(\alpha) = \begin{cases} a(0,G)\displaystyle\int_0^{\frac{a}{c}} (X - zY)^{k-2}dz \\ \quad + a(0,G|\alpha)\displaystyle\int_{-\frac{d}{c}}^{0} \alpha \star (X - zY)^{k-2}dz + \pi_{G,\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}}(\sigma) & \text{if } c > 0, \\ a(0,G)\displaystyle\int_0^{\frac{b}{d}} (X - zY)^{k-2}dz & \text{if } c = 0, \end{cases}$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\delta = \det(\alpha)$.

(3) $\pi_G((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})) = a(0,G)\displaystyle\int_0^1 (X - zY)^{k-2}dz$.

Proof. — (1) For any $\alpha_1, \alpha_2 \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\begin{aligned}
&\pi_{G,\alpha_1}(\alpha_2) + \pi_G(\alpha_1) \\
&= \int_{z_0}^{\alpha_2 z_0} (G|\alpha_1)(z)\alpha_1 \star (X - zY)^{k-2}dz + \int_{z_0}^{\alpha_1 z_0} G(z)(X - zY)^{k-2}dz \\
&\quad + \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\alpha_1\alpha_2})(z)\alpha_1\alpha_2 \star (X - zY)^{k-2}dz \\
&\quad - \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\alpha_1})(z)\alpha_1 \star (X - zY)^{k-2}dz + \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\alpha_1})(z)\alpha_1 \star (X - zY)^{k-2}dz \\
&\quad - a(0,G|\alpha_1\alpha_2)\int_0^{z_0} \alpha_1\alpha_2 \star (X - zY)^{k-2}dz - \int_{z_0}^{\sqrt{-1}\infty} \widetilde{G}(z)(X - zY)^{k-2}dz \\
&\quad + a(0,G|\alpha_1)\int_0^{z_0} \alpha_1 \star (X - zY)^{k-2}dz - a(0,G|\alpha_1)\int_0^{z_0} \alpha_1 \star (X - zY)^{k-2}dz \\
&\quad + a(0,G)\int_0^{z_0} (X - zY)^{k-2}dz \\
&= \pi_G(\alpha_1\alpha_2).
\end{aligned}$$

(2) Let $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}) \right\}$. We use the Bruhat decomposition

$$\mathrm{GL}_2^+(\mathbb{Q}) = B\sigma B \sqcup B.$$

First suppose that $c = 0$. By definition,

$$\begin{aligned}
\pi_G(z_0)(\alpha) = &\int_{z_0}^{\alpha z_0} \widetilde{G}(z)(X - zY)^{k-2} dz + \int_{z_0}^{\alpha z_0} a(0, G)(X - zY)^{k-2} dz \\
&+ \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\alpha})(z)\alpha \star (X - zY)^{k-2} dz - a(0, G|\alpha) \int_0^{z_0} \alpha \star (X - zY)^{k-2} dz \\
&- \int_{z_0}^{\sqrt{-1}\infty} \widetilde{G}(z)(X - zY)^{k-2} dz + a(0, G) \int_0^{z_0} (X - zY)^{k-2} dz.
\end{aligned}$$

When z_0 tends to $\sqrt{-1}\infty$, so does αz_0. Then, the first, third and fifth terms converge to 0. Thus we obtain

$$\begin{aligned}
\pi_G(\alpha) &= \lim_{z_0 \to \sqrt{-1}\infty} \left(a(0, G) \int_0^{\alpha z_0} (X - zY)^{k-2} dz - a(0, G|\alpha) \int_0^{z_0} \alpha \star (X - zY)^{k-2} dz \right) \\
&= \lim_{z_0 \to \sqrt{-1}\infty} \left(a(0, G) \int_0^{\alpha z_0} (X - zY)^{k-2} dz - a(0, G) \int_{\alpha 0}^{\alpha z_0} (X - zY)^{k-2} dz \right) \\
&= a(0, G) \int_0^{\frac{b}{d}} (X - zY)^{k-2} dz.
\end{aligned}$$

Next we consider the case $c > 0$. By Proposition 1.6 (1) and the decomposition

$$\alpha = \begin{pmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix} \sigma \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix},$$

where $\delta = \det(\alpha)$, we get

$$\pi_G(\alpha) = \pi_G(\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}) + \pi_{G, \begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix} \sigma}(\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}) + \pi_{G, \begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}}(\sigma).$$

Here we note that $\pi_{G, \begin{pmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix}} = \pi_G$ and $\pi_G(\begin{pmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix}) = 0$ by the above case. We have already obtained formulas about the first and second terms by the case $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ considered above. Indeed, we have

$$\begin{aligned}
&\pi_{G,\left(\begin{smallmatrix} \delta & a \\ 0 & c \end{smallmatrix}\right)\sigma}\left(\left(\begin{smallmatrix} c & d \\ 0 & 1 \end{smallmatrix}\right)\right)\\
&= a\left(0, G\middle| \left(\begin{smallmatrix} \delta & a \\ 0 & c \end{smallmatrix}\right)\sigma\right) \int_0^d \begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix} \sigma \star (X - zY)^{k-2} dz\\
&= a\left(0, G\middle| \alpha \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} c^{-1} & -c^{-1}d \\ 0 & 1 \end{smallmatrix}\right)\right) \int_0^d \alpha \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} c^{-1} & -c^{-1}d \\ 0 & 1 \end{smallmatrix}\right) \star (X - zY)^{k-2} dz\\
&= a(0, G|\alpha) \int_{-\frac{d}{c}}^0 \alpha \star (X - zY)^{k-2} dz.
\end{aligned}$$

Here the final equality follows from that, for $\beta = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$ and $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, by Proposition 1.1 and (0.1),

$$a\left(0, (G|\alpha)\middle|\beta\right) \alpha\beta \star (X - zY)^{k-2} dz = \beta \star \left(a\left(0, G|\alpha\right) \alpha \star (X - zY)^{k-2} dz\right),$$

and hence

$$a\left(0, (G|\alpha)\middle|\beta\right) \int_0^d \alpha\beta \star (X - zY)^{k-2} dz = a\left(0, G|\alpha\right) \int_{\beta(0)}^{\beta(d)} \alpha \star (X - zY)^{k-2} dz.$$

Thus, we obtain the formula as claimed.

(3) It follows immediately from (2). □

Remark 1.7. — (1) If $\Gamma = \Gamma_1(N)$, then, for any $\beta \in \Gamma$, we have $\pi_{G,\beta} = \beta \star \pi_{G|\beta} = \beta \cdot \pi_G$ by (1.2).

(2) The restriction of π_G to $\Gamma_1(N)$ is a 1-cocycle on $\Gamma_1(N)$, that is, $\pi_G \in Z^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$ (for the definition, see § 2.1).

1.3. Integrality. — The value of $D(G, \chi, s)$ at $s = 1$ is described in terms of π_G as follows.

Lemma 1.8. — *Let $G \in M_k(\Gamma_1(N), \mathbb{C})$ and χ a Dirichlet character whose conductor m_χ is prime to N. Fix $b_1, \dots, b_{\varphi(m_\chi)} \in \mathbb{Z}$ such that $\{\bar{b}_1, \dots, \bar{b}_{\varphi(m_\chi)}\} = (\mathbb{Z}/m_\chi\mathbb{Z})^\times$, where $\bar{b}_i$ is the image of b_i under the natural map $\mathbb{Z} \to \mathbb{Z}/m_\chi\mathbb{Z}$ and φ is the Euler function. Then,*

$$\tau(\bar{\chi}) D(G, \chi, 1) = -\sum_{i=1}^{\varphi(m_\chi)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m_\chi} \\ 0 & 1 \end{pmatrix} \star \pi_{G,\left(\begin{smallmatrix} 1 & b_i \\ 0 & m_\chi \end{smallmatrix}\right)}(\sigma),$$

where $\tau(\bar{\chi}) = \sum_{i=1}^{\varphi(m_\chi)} \bar{\chi}(b_i) e(\frac{b_i}{m_\chi})$ is the Gauss sum of $\bar{\chi}$.

Proof. — We put $m = m_\chi$. We have

$$\tau(\bar{\chi})(G \otimes \chi)(z) = \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) G(z + \frac{b_i}{m}).$$

Thus,

$$\begin{aligned}
&\tau(\bar{\chi}) D(G, \chi, s) \\
&= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \int_0^{\sqrt{-1}\infty} \widetilde{G(z + \frac{b_i}{m})} (X - zY)^{k-2} y^{s-1} dz \qquad (y = \mathrm{Im}(z)) \\
&= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}^{-1} \star \int_0^{\sqrt{-1}\infty} \widetilde{G(z + \frac{b_i}{m})} \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star (X - zY)^{k-2} y^{s-1} dz \\
&= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \int_0^{\sqrt{-1}\infty} \left(\widetilde{G \Big| \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}} \right)(z) \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star (X - zY)^{k-2} y^{s-1} dz.
\end{aligned}$$

We remark that

$$D_{\begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}}(G, s) = \int_0^{\sqrt{-1}\infty} \left(\widetilde{G \Big| \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}} \right)(z) \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star (X - zY)^{k-2} y^{s-1} dz.$$

By Proposition 1.5 (1), we have

$$D_{\begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}}(G, 1) = -\pi_{G, \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}}(\sigma). \tag{1.3}$$

Therefore, we obtain

$$\tau(\bar{\chi}) D(G, \chi, 1) = -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \pi_{G, \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}}(\sigma).$$

In addition, by using Proposition 1.5 (2), we have

$$\begin{aligned}
\tau(\bar{\chi}) D(G, \chi, 1) &= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \pi_{G, \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}}(\sigma) \\
&= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \pi_{G, \begin{pmatrix} 1 & b_i \\ 0 & m \end{pmatrix}}(\sigma).
\end{aligned}$$

We have proved the lemma. □

We fix a rational odd prime number p such that $(p, N) = 1$. Let S be a set of rational prime numbers satisfying the following properties:

(1) both $(m, pN) = 1$ and $(\varphi(m), p) = 1$ hold for all $m \in S$;

(2) S has non-empty intersection with every arithmetic progression of the form $\{d + cpNe | e \in \mathbb{Z}\}$ for all pair $(c, d) \in \mathbb{Z}^2$ such that $c > 0$, $(p, cd) = 1$, $d \not\equiv 1 \pmod p$, and $(d, cN) = 1$

(for example, $S = \{m | m$ is a prime number such that $(m, pN) = 1$ and $m \not\equiv 1 \pmod p\}$).

We remark that $p \notin S$. For such a set S, let $\mathfrak{X}_S$ denote the set of Dirichlet characters χ whose conductor m_χ belongs to S. For $m \in S$, we fix $b_1, \ldots, b_{\varphi(m)} \in \mathbb{Z}$ such that $\{b_1, \ldots, b_{\varphi(m)}\} = (\mathbb{Z}/m\mathbb{Z})^\times$.

THEOREM 1.9 (Integrality). — *Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$. Suppose that $G \in M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ and that the following conditions hold:*

(1) $k < p + 1$;
(2) $a(0, G|\alpha) \in \mathcal{O}$ *for each* $\alpha \in \mathrm{SL}_2(\mathbb{Z})$;
(3) $D_{\begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}}(G, 1) \in L_{k-2}(\mathcal{O})$ *for each* $m \in S$ *and* i;
(4) $\pi_G(\sigma) \in L_{k-2}(\mathcal{O})$.

Then π_G is integral, that is, $\pi_G(\Gamma_0(N)) \subset L_{k-2}(\mathcal{O})$.

Proof. — We put $\Gamma = \Gamma_0(N)$ and

$$\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma.$$

In the case where $c = 0$, we have $\pi_G(\gamma) \in L_{k-2}(\mathcal{O})$ by Proposition 1.6 (2) and the assumptions (1) and (2). In the case where $c \neq 0$, we may assume that $c > 0$. Indeed, we have $\pi_G(-\gamma) = \pi_G(\gamma)$ by using Proposition 1.6 (1), (2), and (1.2). Then, by Proposition 1.6 (2), we have

$$\pi_G(\gamma) = a(0, G) \int_0^{\frac{a}{cN}} (X - zY)^{k-2} dz + a(0, G|\gamma) \int_{-\frac{d}{cN}}^{0} \gamma \cdot (X - zY)^{k-2} dz + \pi_{G, \begin{pmatrix} 1 & a \\ 0 & cN \end{pmatrix}}(\sigma).$$

We prove that $\pi_G(\gamma)$ is integral in two cases.

Case 1. — Assume that $(p, c) = 1$.

It is enough to prove that $\pi_G(\gamma)$ is integral in the case where $(p, d) = 1$ and $d \not\equiv 1 \pmod p$. Indeed, if $p | d$ or $d \equiv 1 \pmod p$, then we put

$$\gamma' := \gamma \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ab' + b \\ cN & cNb' + d \end{pmatrix} \in \Gamma.$$

Since $(p, cN) = 1$, note that $cNb' + d \not\equiv 0, 1 \pmod p$ for some $b' \in \mathbb{Z}$. Then, by applying the cocycle condition (Proposition 1.6 (1)) for π_G to the element γ', we get

$$\pi_G(\gamma') = \pi_G(\gamma) + \varepsilon(d)\gamma \cdot \pi_G(\left(\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right)).$$

Now the integrality of $\pi_G(\gamma)$ follows from the integrality of $\pi_G(\gamma')$.

We remark that $(cN)^{-1} \in \mathcal{O}$ by assumption. Then, for the proof of the integrality of $\pi_G(\gamma)$, by the formula above and the assumptions (1) and (2), it is enough to show that $\pi_{G,\left(\begin{smallmatrix} 1 & a \\ 0 & cN \end{smallmatrix}\right)}(\sigma) \in L_{k-2}(\mathcal{O})$. Therefore, for the proof in this case, it suffices to show that $\pi_G(\gamma')$ is integral by choosing $b', d' \in \mathbb{Z}$ such that $\gamma' = \begin{pmatrix} a & b' \\ cN & d' \end{pmatrix} \in \Gamma$. Indeed, we have $\pi_G(\gamma') \equiv \pi_{G,\left(\begin{smallmatrix} 1 & a \\ 0 & cN \end{smallmatrix}\right)}(\sigma) \pmod{L_{k-2}(\mathcal{O})}$. Since $(p, cd) = 1$, $d \not\equiv 1 \pmod{p}$, and $(d, cN) = 1$, there exists $e \in \mathbb{Z}$ such that $d + cpNe \in S$. We put $m = d + cpNe$ and

$$\gamma' = \gamma \begin{pmatrix} 1 & ep \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b' \\ cN & m \end{pmatrix} \in \Gamma,$$

where $b' = aep + b$. By applying the cocycle condition (Proposition 1.6 (1)) for π_G to the element

$$\gamma'\sigma = \begin{pmatrix} a & b' \\ cN & m \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b' & -a \\ m & -cN \end{pmatrix},$$

we get

$$\pi_G(\gamma'\sigma) = \pi_G(\gamma') + \varepsilon(m)\gamma' \cdot \pi_G(\sigma).$$

Since $\gamma' \cdot \pi_G(\sigma)$ is integral by the assumption (4), for the proof of the integrality of $\pi_G(\gamma')$, it suffices to show that $\pi_G(\gamma'\sigma)$ is integral. Using Proposition 1.6 (2), we have

$$\begin{aligned} \pi_G(\gamma'\sigma) = a(0, G) &\int_0^{\frac{b'}{m}} (X - zY)^{k-2} dz \\ &+ a(0, G|\gamma'\sigma) \int_{\frac{cN}{m}}^0 \gamma'\sigma \cdot (X - zY)^{k-2} dz + \pi_{G,\left(\begin{smallmatrix} 1 & b' \\ 0 & m \end{smallmatrix}\right)}(\sigma). \end{aligned}$$

Therefore, by the assumptions (1) and (2), it is enough to show that the final term is integral. It follows from (1.3) and the assumption (3).

Case 2. — Assume that $p|c$.

We put

$$\gamma' := \gamma \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = \begin{pmatrix} a + bN & b \\ (c+d)N & d \end{pmatrix} \in \Gamma.$$

Then, by applying the cocycle condition (Proposition 1.6 (1)) for π_G to the element γ', we get

$$\pi_G(\gamma') = \pi_G(\gamma) + \varepsilon(d)\gamma \cdot \pi_G(\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)).$$

Since $(p, c+d) = 1$ and $(p, N) = 1$, we see that both $\pi_G(\gamma')$ and $\pi_G((\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}))$ are integral by Case 1, and therefore so is $\pi_G(\gamma)$. Now this completes the proof of the theorem. □

REMARK 1.10. — Theorem 1.9 is a partial generalization of [34, Theorem 1.3].

2. Congruences for L-functions

2.1. Group cohomology. — To state our theorem, we need to recall some properties about group cohomology. We define an action of $\mathrm{GL}_2(\mathbb{Q})$ on the upper half complex plane $\mathfrak{H}$ as follows. For $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ with $\det(\alpha) > 0$, α act on $\mathfrak{H}$ by the usual linear fractional transformation. For $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, τ act on $\mathfrak{H}$ by $\tau z = -\bar{z}$. If $\det(\alpha) < 0$, then we define $\alpha(z) = (\alpha\tau)(\tau(z))$. This action is associative and so is well-defined. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

DEFINITION 2.1 (The standard $R[\Gamma]$-free resolution of R). — Let R be a commutative ring and M a left $R[\Gamma]$-module. We define $F_q = (R[\Gamma])^{\otimes(q+1)}$ and regard it as an $R[\Gamma]$-module via the multiplication of $R[\Gamma]$ on the first factor. Then F_q is a free $R[\Gamma]$-module with a basis $\{[\gamma_1, \ldots, \gamma_q] = 1 \otimes \gamma_1 \otimes \cdots \otimes \gamma_q | \gamma_i \in \Gamma\}$. We define the $R[\Gamma]$-linear boundary map $\partial_q \colon F_q \to F_{q-1}$ by $\partial_1[\gamma] = \gamma - 1$ and

$$\begin{aligned}\partial_q[\gamma_1, \ldots, \gamma_q] &= \gamma_1[\gamma_2, \ldots, \gamma_q] \\ &\quad + \sum_{j=1}^{q-1}(-1)^j[\gamma_1, \ldots, \gamma_j\gamma_{j+1}, \ldots, \gamma_q] + (-1)^q[\gamma_1, \ldots, \gamma_{q-1}]\end{aligned}$$

for $q > 1$. It is well known that (F_*, ∂_*) is a $R[\Gamma]$-free resolution of R. Let $C^i = C^i(\Gamma, M)$ be the space of functions on Γ^i with values in M for $i \geq 1$, and M for $i = 0$. Note that $\mathrm{Hom}_{R[\Gamma]}(F_q, M) \cong C^q$. Then the differential map $d^i \colon C^i \to C^{i+1}$ induced by ∂_* on F_* is given by $d^0u(\gamma) = (\gamma - 1)u$ for $u \in M$ if $i = 0$, and if $i > 0$,

$$\begin{aligned}d^iu(\gamma_1, \ldots, \gamma_{i+1}) &= \gamma_1 u(\gamma_2, \ldots, \gamma_{i+1}) \\ &\quad + \sum_{j=1}^{i}(-1)^j u(\gamma_1, \ldots, \gamma_j\gamma_{j+1}, \ldots, \gamma_{i+1}) + (-1)^{i+1}u(\gamma_1, \ldots, \gamma_i).\end{aligned}$$

The associated i-th cohomology group of Γ with coefficients in M is given by

$$H^i(\Gamma, M) = Z^i(\Gamma, M)/B^i(\Gamma, M),$$

where

$$Z^i(\Gamma, M) = \ker(d^i \colon C^i \to C^{i+1}) \quad \text{and} \quad B^i(\Gamma, M) = \mathrm{im}(d^{i-1} \colon C^{i-1} \to C^i).$$

We fix a base point $z_0 \in \mathfrak{H}$. For $G \in M_k(\Gamma_1(N), \mathbb{C})$ and $\gamma \in \Gamma_1(N)$, we define $\omega_G(z_0) \in C^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$ by

$$\omega_G(z_0)(\gamma) = \int_{z_0}^{\gamma z_0} G(z)(X - zY)^{k-2} dz.$$

Then we have $\omega_G(z_0) \in Z^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$.

Also we have $\pi_G \in Z^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$ by Proposition 1.6 (1) and (1.2). Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$ be a character and

$$\Delta = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \;\middle|\; \det(\alpha) \neq 0, c \equiv 0 \pmod{N}, (a, N) = 1 \right\}.$$

We define an $R[\Delta]$-module $L_{k-2}(\varepsilon, R)$ as follows: let $L_{k-2}(\varepsilon, R)$ be the R-module $L_{k-2}(R)$ with left $R[\Delta]$-action by

$$\gamma \bullet P(X, Y) = \varepsilon(d)\gamma \cdot P(X, Y)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta$. For $G \in M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ and $\gamma \in \Gamma_0(N)$, we define $\omega_G(z_0) \in C^1(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$ by

$$\omega_G(z_0)(\gamma) = \int_{z_0}^{\gamma z_0} G(z)(X - zY)^{k-2} dz.$$

Then we have $\omega_G(z_0) \in Z^1(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$.

Also we have $\pi_G \in Z^1(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$ by Proposition 1.6 (1) and (1.2).

For each cusp $s \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, let Γ_s denote the stabilizer of s in Γ, and let π_s be a generator of Γ_s:

$$\Gamma_s = \{\alpha \in \Gamma | \alpha s = s\} = \{\pm \pi_s^m \in \Gamma | m \in \mathbb{Z}\}.$$

Let $Z(\Gamma)$ be a representative set for Γ-equivalence classes of cusps, which is a finite set. Then we note that for each cusp $s \in \mathbb{P}^1(\mathbb{Q})$, we can find $\gamma \in \Gamma$ and $s_0 \in Z(\Gamma)$ such that $\gamma s = s_0$. We consider the set of all conjugates of π_s in Γ for all $s \in Z(\Gamma)$, which is denoted by P. The parabolic cohomology group of Γ with coefficients in M is given by

$$H^1_{\mathrm{par}}(\Gamma, M) = Z^1_{\mathrm{par}}(\Gamma, M)/B^1(\Gamma, M),$$

where

$$Z^1_{\mathrm{par}}(\Gamma, M) = \{u \in Z^1(\Gamma, M) | u(\pi) \in (\pi - 1)M \text{ for all } \pi \in P\}.$$

If $f \in S_k(\Gamma_1(N), \mathbb{C})$ (resp. $f \in S_k(\Gamma_0(N), \varepsilon, \mathbb{C})$), we have $\omega_f(z_0), \pi_f \in Z^1_{\mathrm{par}}(\Gamma_1(N), L_{k-2}(\mathbb{C}))$ (resp. $\omega_f(z_0), \pi_f \in Z^1_{\mathrm{par}}(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$).

2.2. The Hecke eigenvalues. — We recall the definitions of the Hecke operators on group cohomology and the space of modular forms. Let $\Gamma, \Gamma' < \mathrm{SL}_2(\mathbb{Z})$ be congruence subgroups. For any $\alpha \in \mathrm{GL}_2(\mathbb{Q})$, we have a decomposition $\Gamma\alpha\Gamma' = \coprod_i \Gamma\alpha_i$ as a disjoint union. We denote $\det(\alpha)\alpha^{-1}$ simply by α^ι for any $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. Let $\langle \Gamma, \Gamma', \alpha^\iota \rangle$ be the semi-group in $\mathrm{GL}_2(\mathbb{Q})$ generated by α^ι for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ and two congruences subgroups Γ and Γ'. For any $\langle \Gamma, \Gamma', \alpha^\iota \rangle$-module M, we define the Hecke operator $[\Gamma\alpha\Gamma']$ as follows. For each $\gamma \in \Gamma'$, we can write $\alpha_i\gamma = \gamma_i\alpha_j$ for a unique j with $\gamma_i \in \Gamma$. For each cocycle $u : \Gamma \to M \in Z^1(\Gamma, M)$, we define $v = u\big|[\Gamma\alpha\Gamma']$ by $v(\gamma) = \sum_i \alpha_i^\iota u(\gamma_i)$. The operator $[\Gamma\alpha\Gamma']$ is a well-defined linear operator from $H^1(\Gamma, M)$ into $H^1(\Gamma', M)$. Also $[\Gamma\alpha\Gamma']$ sends $H^1_{\mathrm{par}}(\Gamma, M)$ into $H^1_{\mathrm{par}}(\Gamma', M)$.

We consider the case $\Gamma = \Gamma' = \Gamma_0(N)$ or $\Gamma_1(N)$. If $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$ for a prime number l, we abbreviate $[\Gamma\alpha\Gamma]$ to $T(l)$. We have the following lemma ([28, Lemma 4.5.6 (1)]):

Lemma 2.2. — *An explicit left coset decomposition is given by*

$$\Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & l^e \end{pmatrix}\Gamma_0(N) = \begin{cases} \displaystyle\coprod_{\substack{0\le f\le e,\\ 0\le r<l^f \text{ with } (r,\, l^f,\, l^{e-f})=1}} \Gamma_0(N)\begin{pmatrix} l^{e-f} & r \\ 0 & l^f \end{pmatrix} & \text{if } (l,N)=1, \\ \displaystyle\coprod_{0\le r<l^e} \Gamma_0(N)\begin{pmatrix} 1 & r \\ 0 & l^e \end{pmatrix} & \text{if } l|N, \end{cases}$$

as a disjoint union.

Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$. We define the Hecke operator $T(l)$ on $M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ for a prime number l. We put the disjoint decompositions $\Gamma\alpha\Gamma = \coprod_i \Gamma\alpha_i$, where $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$. Then we define

$$f|T(l) = \sum_i \varepsilon(\alpha_i) f|\alpha_i,$$

where $\varepsilon(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \varepsilon(a)$. Here we note that it is independent of the choice of $\{\alpha_i\}_i$ and $f|T(l) \in M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$. Moreover, we define the Hecke operator $T(l^e)$ for a prime number l and an integer $e \geq 1$ inductively by

$$T(l^{e+1}) = \begin{cases} T(l)T(l^e) - \varepsilon(l)l^{k-1}T(l^{e-1}) & \text{if } (l,N)=1, \\ T(l)^{e+1} & \text{if } l|N, \end{cases}$$

where we define $T(1)$ to be the identity map. More generally, we can define the Hecke operator $T(m)$ by

$$T(l)T(l') = T(l')T(l),$$
$$T(m) = \prod_l T(l^{e_l})$$

for different primes l and l' and each positive integer $m = \prod_l l^{e_l}$ for primes l.

Using Lemma 2.2, for $A = \mathcal{O}$ or $\mathbb{C}$, the Hecke operators on $H^1(\Gamma_0(N), L_{k-2}(\varepsilon, A))$ and $M_k(\Gamma_0(N), \varepsilon, A)$ can be described explicitly. We prove that the map from $M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ to $H^1(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$ sending G to the class of π_G is Hecke equivariant (see (2.2) below). In order to do it, we make the following calculations. We abbreviate $\Gamma_0(N)$ to Γ. We fix $G \in M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$. For a prime number l, we put $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$, and $G' = G|[\Gamma\alpha\Gamma] \in M_k(\Gamma, \varepsilon, \mathbb{C})$. By the pull-back formula (0.1), for any $\gamma \in \Gamma$,

$$\begin{aligned} \omega_{G'}(z_0)(\gamma) &= \int_{z_0}^{\gamma z_0} G'(z)(X - zY)^{k-2} dz \\ &= \sum_i \varepsilon(\alpha_i) \int_{z_0}^{\gamma z_0} (G|\alpha_i)(z)(X - zY)^{k-2} dz \\ &= \sum_i \alpha_i^{\iota} \bullet \int_{\alpha_i z_0}^{\alpha_i \gamma z_0} G(z)(X - zY)^{k-2} dz. \end{aligned}$$

For any $\gamma \in \Gamma$, by the definition of π_G,

$$\pi_G(\gamma) = \int_{z_0}^{\gamma z_0} G(z)(X - zY)^{k-2} dz + (\gamma - 1) \bullet I_G(X, Y), \tag{2.1}$$

where

$$I_G(X, Y) = \int_{z_0}^{\sqrt{-1}\infty} \widetilde{G}(z)(X - zY)^{k-2} dz - a(0, G) \int_0^{z_0} (X - zY)^{k-2} dz.$$

We simply write the above equation for

$$\pi_G(\gamma) = \omega_G(z_0)(\gamma) + (\gamma - 1) \bullet I_G(X, Y).$$

Further, for any $w \in \mathfrak{H}$, we define $F(z_0)(w) = \int_{z_0}^{w} G(z)(X - zY)^{k-2} dz - I_G(X, Y)$. For any $\gamma \in \Gamma$, we put $u(z_0)(w)(\gamma) = F(z_0)(\gamma w) - \gamma \bullet F(z_0)(w)$.

Then, for any $\gamma \in \Gamma$,

$$\begin{aligned}
&u(z_0)(w)(\gamma)\\
&= \int_{z_0}^{\gamma w} G(z)(X-zY)^{k-2}dz - I_G(X,Y)\\
&\quad - \gamma \bullet \int_{z_0}^{w} G(z)(X-zY)^{k-2}dz + \gamma \bullet I_G(X,Y)\\
&= \int_{z_0}^{\gamma w} G(z)(X-zY)^{k-2}dz - \int_{\gamma z_0}^{\gamma w} G(z)(X-zY)^{k-2}dz + (\gamma-1)\bullet I_G(X,Y)\\
&= \int_{z_0}^{\gamma z_0} G(z)(X-zY)^{k-2}dz + (\gamma-1)\bullet I_G(X,Y)\\
&= \omega_G(z_0)(\gamma) + (\gamma-1)\bullet I_G(X,Y)\\
&= \pi_G(z_0)(\gamma).
\end{aligned}$$

This value is independent of the choice of $w \in \mathfrak{H}$ and hence we simply write $u(z_0)(\gamma)$ instead of $u(z_0)(w)(\gamma)$. By the definition of $F(z_0)(w)$ and the above calculations, for any $\gamma \in \Gamma$, we get

$$\begin{aligned}
\omega_{G'}(z_0)(\gamma) &= \sum_i \alpha_i^\iota \bullet (F(z_0)(\alpha_i\gamma z_0) - F(z_0)(\alpha_i z_0))\\
&= \sum_i \alpha_i^\iota \bullet (F(z_0)(\gamma_i\alpha_j z_0) - F(z_0)(\alpha_i z_0))\\
&= \sum_i \alpha_i^\iota \bullet (u(z_0)(\gamma_i) + \gamma_i \bullet F(z_0)(\alpha_j z_0)) - \sum_i \alpha_i^\iota \bullet F(z_0)(\alpha_i z_0)\\
&= u(z_0)\big|[\Gamma\alpha\Gamma](\gamma) + (\gamma-1)\bullet \left(\sum_i \alpha_i^\iota \bullet F(z_0)(\alpha_i z_0)\right).
\end{aligned}$$

Then, by the above calculations, for any $\gamma \in \Gamma$, we obtain

$$\begin{aligned}
\pi_{G'}(\gamma) &= \omega_{G'}(z_0)(\gamma) + (\gamma-1)\bullet I_{G'}(X,Y)\\
&= \pi_G\big|[\Gamma\alpha\Gamma](\gamma) + (\gamma-1)\bullet \left(I_{G'}(X,Y) + \sum_i \alpha_i^\iota \bullet F(z_0)(\alpha_i z_0)\right).
\end{aligned} \tag{2.2}$$

We now prove the following proposition.

Proposition 2.3. — (1) *Suppose $G' = G|[\Gamma\alpha\Gamma] = \lambda(l,G)G$ for $\lambda(l,G) \in \mathbb{C}$. Then we have $\pi_{G'} = \lambda(l,G)\pi_G$.*

(2) *$I_{G'}(X,Y) + \sum_i \alpha_i^\iota \bullet F(z_0)(\alpha_i z_0) = \sum_i \alpha_i^\iota \bullet \pi_G(\alpha_i)$.*

(3) *Let $G \in M_k(\Gamma_0(N),\varepsilon,\mathcal{O})$ be a Hecke eigenform, $\lambda(m,G)$ the eigenvalue of $T(m)$, $\varpi \in \mathcal{O}$ a uniformizer, and r a non-negative integer. Assume that $k < p$, $a(0,G) \equiv 0 \pmod{\varpi^r\mathcal{O}}$, and π_G is integral, that is, $\pi_G(\Gamma_0(N)) \subset L_{k-2}(\varepsilon,\mathcal{O})$. Then $[\pi_G]\big|T(m) = \lambda(m,G)[\pi_G]$ in $H^1(\Gamma_0(N), L_{k-2}(\varepsilon,\mathcal{O}/\varpi^r))$ for any positive integer m.*

Proof. — (1) It follows from (2.1).

(2) First we calculate $I_{G'}(X,Y)$. By the definition of $I_{G'}(X,Y)$, we have

$$\begin{aligned}
I_{G'}(X,Y) &= \int_{z_0}^{\sqrt{-1}\infty} \widetilde{G'}(z)(X-zY)^{k-2}dz - a(0,G')\int_0^{z_0}(X-zY)^{k-2}dz \\
&= \sum_i \varepsilon(\alpha_i)\int_{z_0}^{\sqrt{-1}\infty} \widetilde{(G|\alpha_i)}(z)(X-zY)^{k-2}dz \\
&\quad - \sum_i \varepsilon(\alpha_i)a(0,G|\alpha_i)\int_0^{z_0}(X-zY)^{k-2}dz \\
&= \sum_i \alpha_i^{\iota}\bullet\int_{z_0}^{\sqrt{-1}\infty} \widetilde{(G|\alpha_i)}(z)\alpha_i\star(X-zY)^{k-2}dz \\
&\quad - \sum_i \alpha_i^{\iota}\bullet a(0,G|\alpha_i)\int_0^{z_0}\alpha_i\star(X-zY)^{k-2}dz.
\end{aligned}$$

Therefore, by the definition of $\pi_G(\alpha_i)$, we have

$$\begin{aligned}
&I_{G'}(X,Y)+\sum_i \alpha_i^{\iota}\bullet F(z_0)(\alpha_i z_0) \\
&\quad = I_{G'}(X,Y)+\sum_i \alpha_i^{\iota}\bullet\Big[\int_{z_0}^{\alpha_i z_0} G(z)(X-zY)^{k-2}dz \\
&\qquad - \int_{z_0}^{\sqrt{-1}\infty}\widetilde{G}(z)(X-zY)^{k-2}dz + a(0,G)\int_0^{z_0}(X-zY)^{k-2}dz\Big] \\
&\quad = \sum_i \alpha_i^{\iota}\bullet\pi_G(\alpha_i),
\end{aligned}$$

as required.

(3) We fix a prime number l. Using Lemma 2.2, (2.2) and (2), we obtain

$$\begin{aligned}
\pi_{G'}(\gamma)-\pi_G\big|[\Gamma\alpha\Gamma](\gamma) &= (\gamma-1)\bullet\Big(I_{G'}(X,Y)+\sum_i \alpha_i^{\iota}\bullet F(z_0)(\alpha_i z_0)\Big) \quad \text{(by (2.2))} \\
&= (\gamma-1)\bullet\Big(\sum_i \alpha_i^{\iota}\bullet\pi_G(\alpha_i)\Big) \qquad \text{(by (2))} \\
&= (\gamma-1)\bullet\Big(\sum_i \alpha_i^{\iota}\bullet a(0,G)\int_0^{\frac{r_i}{l^{f_i}}}(X-zY)^{k-2}dz\Big) \\
&\equiv 0 \pmod{\varpi^r\mathcal{O}}
\end{aligned}$$

for any $\gamma\in\Gamma$. Here the third equality follows from Proposition 1.6 (2) and the last congruence follows from an explicit calculation with $\sum_{r=1}^{p-1} r^{j+1}\equiv 0 \pmod p$

for any non-negative integer j such that $j+1<p-1$ if $l=p$. Therefore, by using (1), we prove (3). $\square$

2.3. Canonical periods. — We put $\Gamma_0=\Gamma_0(N)$ and $\Gamma_1=\Gamma_1(N)$. Let $f\in S_k(\Gamma_0,\varepsilon,\mathcal{O})$ be a normalized Hecke eigenform. We assume that if $k>2$, then $(p,N)=1$ and $k-2<p$. Let Y denote the modular curve $\Gamma_1\backslash\mathfrak{H}$ with Γ_1-structure. Let $\mathcal{L}_{k-2}(\mathcal{O})$ be the local system on Y corresponding to the Γ_1-module $L_{k-2}(\mathcal{O})$. For a prime number l, we simply write $T_l=T(l)$ if $(l,N)=1$ and $U_l=T(l)$ if $l|N$. We denote by $\mathfrak{M}_f$ a maximal ideal of the Hecke algebra generated by ϖ, $T_l-a(l,f)$ (for $(l,N)=1$), $U_l-a(l,f)$ (for $l|N$), and $\langle d\rangle-\varepsilon(d)$. For $\tau=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, we consider the complex conjugation $[\Gamma_1\tau\Gamma_1]$ on $H^1_{\mathrm{par}}(\Gamma_1,L_{k-2}(\mathcal{O}))$ defined in §2.2. We note that the complex conjugation $[\Gamma_1\tau\Gamma_1]$ commutes with T_l and U_l for any prime number l because $\Gamma_1\tau\Gamma_1=\Gamma_1\tau$ and $\Gamma_1\begin{pmatrix}1&0\\0&l\end{pmatrix}\Gamma_1=\coprod_i\Gamma_1\alpha_i=\coprod_i\Gamma_1\tau^{-1}\alpha_i\tau$.

Proposition 2.4. — *For each parity $\alpha\in\{\pm1\}$,*

$$\text{the } \alpha\text{-eigenspace } H^1_{\mathrm{c}}(Y,\mathcal{L}_{k-2}(\mathcal{O}))^{\alpha}_{\mathfrak{M}_f} \text{ is free of rank } 1 \text{ over } \mathcal{O}.$$

Proof. — The Eichler-Shimura isomorphism and the q-expansion principle over $\mathbb{C}$ imply that

$$H^1_{\mathrm{c}}(Y,\mathcal{L}_{k-2}(\mathbb{C}))^{\alpha}_{\mathfrak{M}_f}\simeq H^1_{\mathrm{par}}(Y,\mathcal{L}_{k-2}(\mathbb{C}))^{\alpha}_{\mathfrak{M}_f}, \tag{2.3}$$

whose dimension over $\mathbb{C}$ is equal to 1. Then it suffices to show that $H^1_{\mathrm{c}}(Y,\mathcal{L}_{k-2}(\mathcal{O}))$ is torsion-free. First suppose that $k=2$. By considering the exact sequence $0\to\mathcal{O}\xrightarrow{\times\varpi}\mathcal{O}\to\mathcal{O}/\varpi\to0$ and taking its cohomology, we see that $H^1_{\mathrm{c}}(Y,\mathcal{O})$ is torsion-free. Next suppose that $k>2$. We note that, if $(p,N)=1$ and $k-2<p$, then

$$H^0(Y,\mathcal{L}_{k-2}(A))\simeq H^0(\Gamma_1,L_{k-2}(A))=0 \text{ for } A=\mathcal{O},\ \mathcal{O}/\varpi \tag{2.4}$$

because $\begin{pmatrix}1&1\\0&1\end{pmatrix},\begin{pmatrix}1&0\\N&1\end{pmatrix}\in\Gamma_1$. Thus, by considering the exact sequence $0\to\mathcal{O}\xrightarrow{\times\varpi}\mathcal{O}\to\mathcal{O}/\varpi\to0$ and taking its cohomology, we see that $H^1(Y,\mathcal{L}_{k-2}(\mathcal{O}))$ is torsion-free. In particular, $H^1_{\mathrm{par}}(Y,\mathcal{L}_{k-2}(\mathcal{O}))$ is torsion-free. Thus it suffices to show that the kernel of $H^1_{\mathrm{c}}(Y,\mathcal{L}_{k-2}(\mathcal{O}))\twoheadrightarrow H^1_{\mathrm{par}}(Y,\mathcal{L}_{k-2}(\mathcal{O}))$ is torsion-free. The Gysin sequence with the help of (2.4) implies that the kernel is identified with the boundary cohomology of degree 0 and hence it is torsion-free as desired. $\square$

Proposition 2.5. — *For each parity $\alpha\in\{\pm1\}$, the canonical morphism induces an isomorphism*

$$H^1_{\mathrm{c}}(Y,\mathcal{L}_{k-2}(\mathcal{O}))^{\alpha}_{\mathfrak{M}_f}\simeq H^1_{\mathrm{par}}(Y,\mathcal{L}_{k-2}(\mathcal{O}))^{\alpha}_{\mathfrak{M}_f}.$$

Proof. — It suffices to show the injectivity of this morphism. As mentioned in the proof of Proposition 2.4, both $H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))$ and $H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))$ are torsion-free. Hence the injectivity follows from the isomorphism (2.3). $\square$

For each parity $\alpha \in \{\pm 1\}$, we define the canonical period Ω_f^α. We choose a generator $[\delta_f]_{\mathrm{c}}^\alpha$ (resp. $[\delta_f]^\alpha$) of $H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))^\alpha_{\mathfrak{M}_f}$ (resp. $H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))^\alpha_{\mathfrak{M}_f}$).

Let $\Omega^\bullet(Y, \mathbb{C})$ denote the complex of $\mathbb{C}$-valued C^∞-differential Γ_1-invariant forms in $\mathfrak{H}$. Moreover, let $\Omega^\bullet_{\mathrm{c}}(Y, \mathbb{C})$ denote the complex of forms in $\Omega^\bullet(Y, \mathbb{C})$ which, together with their exterior differentials, are fast decreasing at each cusp $s \in Z(\Gamma_1)$. By [4, Theorem 5.2], we have

$$H^1_{\mathrm{dR}}(Y, \Omega^\bullet_{\mathrm{c}}(Y, \mathbb{C}) \otimes_{\mathbb{C}} L_{k-2}(\mathbb{C})) \simeq H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathbb{C})).$$

Let $[\omega_f]_{\mathrm{dR}} \in H^1_{\mathrm{dR}}(Y, \Omega^\bullet_{\mathrm{c}}(Y, \mathbb{C}) \otimes_{\mathbb{C}} L_{k-2}(\mathbb{C}))$ be the de Rham cohomology class attached to f. Let $[\omega_f]_{\mathrm{c}} \in H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathbb{C}))$ (resp. $[\omega_f] \in H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathbb{C}))$) be the image of $[\omega_f]_{\mathrm{dR}}$. We note that, by (2.1), the cocycle π_f defines the same cohomology class as $\omega_f(z_0)$ and also $[\omega_f] = [\omega_f(z_0)]$ via the comparison theorem between Betti cohomology and group cohomology (cf. [3, Proposition 2.5]). By using the proof of Proposition 2.3, the Hecke eigenvalues of the cohomology classes $[\omega_f]_{\mathrm{c}}$ and $[\omega_f]$ are the same as those of f. We write $[\omega_f]_{\mathrm{c}}^\alpha$ and $[\omega_f]^\alpha$ for the projections to the α-parts. Thus, by Proposition 2.4 and Proposition 2.5, there exist complex numbers $\Omega^\alpha_{f,\mathrm{c}}$, $\Omega^\alpha_f \in \mathbb{C}^\times$ such that

$$\begin{aligned} [\omega_f]_{\mathrm{c}}^\alpha &= \Omega^\alpha_{f,\mathrm{c}} [\delta_f]^\alpha_{\mathrm{c}}, \\ [\omega_f]^\alpha &= \Omega^\alpha_f [\delta_f]^\alpha. \end{aligned} \tag{2.5}$$

We note that, by the definition, $\Omega^\alpha_{f,\mathrm{c}}$ is equal to Ω^α_f up to $\mathcal{O}^\times$.

Proposition 2.6. — *For each parity $\alpha \in \{\pm 1\}$, let*

$$\pi_f^\alpha = \frac{1}{2}\left(\pi_f + \alpha \pi_f | [\Gamma_1 \tau \Gamma_1]\right).$$

Then the image of Γ_0 under the map $\pi_f^\alpha / \Omega_f^\alpha$ and $\pi_f^\alpha(\sigma)/\Omega^\alpha_f \in L_{k-2}(\mathbb{C})$ are contained in $L_{k-2}(\mathcal{O})$.

Proof. — By the proof of Theorem 1.9, it suffices to show the integrality of the coefficients of $X^{k-2-j}Y^j$ in $D(f,1)/\Omega_f^\alpha \in L_{k-2}(\mathbb{C})$ for each j with $\alpha = (-1)^j$ and $D_{\left(\begin{smallmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{smallmatrix}\right)}(f,1)/\Omega_f^\alpha \in L_{k-2}(\mathbb{C})$ for each $m \in S$, i, and j with $\alpha = \chi(-1)(-1)^j$. Here we note that $\pi_f(\sigma) = -D(f,1)$ by Proposition 1.5 (1). In order to prove this integrality, we give a cohomological treatment of the special values of the L-functions.

Let χ be the trivial character or a Dirichlet character with conductor $m \in S$. We note that m is prime to p. Fix a representative set $\{b_i\}_i$ of $(\mathbb{Z}/m\mathbb{Z})^\times$ in $\mathbb{Z}$.

For each b_i, we consider the following subset H_{b_i} of $\mathfrak{H}$:

$$H_{b_i} = \frac{b_i}{m} + \sqrt{-1}\mathbb{R}_+^\times = \left\{ \frac{b_i}{m} + \sqrt{-1}y \;\middle|\; y \in \mathbb{R} \text{ with } y > 0 \right\}.$$

Then we have $H_{b_i} \to Y$ and it induces

$$H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(A)) \to H^1_{\mathrm{c}}(H_{b_i}, \mathcal{L}_{k-2}(A)) \tag{2.6}$$

for $A = \mathcal{O}$, K, $\mathbb{C}$. Then, for each j with $0 \leq j \leq k-2$, we define the evaluation map

$$\mathrm{ev}^j_{b_i,A} : H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(A)) \to A \tag{2.7}$$

by the composition of

$$H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(A)) \xrightarrow{(2.6)} H^1_{\mathrm{c}}(H_{b_i}, \mathcal{L}_{k-2}(A)) \xrightarrow{\begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix}} H^1_{\mathrm{c}}(H_{b_i}, \mathcal{L}_{k-2}(A)) \tag{2.8}$$

and

$$H^1_{\mathrm{c}}(H_{b_i}, \mathcal{L}_{k-2}(A)) \xrightarrow{\text{coeff. of } X^{k-2-j}Y^j} H^1_{\mathrm{c}}(H_{b_i}, A) \xrightarrow{\text{trace}} A. \tag{2.9}$$

Here the second arrow of (2.8) is induced by

$$L_{k-2}(A) \to L_{k-2}(A)\,;\; P(X,Y) \mapsto \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star P(X,Y)$$

because m is prime to p, the first arrow of (2.9) is induced by

$$L_{k-2}(A) \to A\,;\; \sum_{j=0}^{k-2} a_j X^{k-2-j} Y^j \mapsto a_j,$$

and the second arrow of (2.9) is the trace map:

$$\omega \mapsto \int_{\sqrt{-1}\infty}^{\frac{b_i}{m}} \omega.$$

.

PROPOSITION 2.7. — *Let χ be the trivial character or a character with conductor $m \in S$. Then*

$$-\tau(\bar{\chi}) D(f, \chi, 1) = \sum_{j=0}^{k-2} \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]_{\mathrm{c}}).$$

Proof. — Direct calculation shows that

$$\begin{aligned}\tau(\bar{\chi})D(f,\chi,1) &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix} \star \int_0^{\sqrt{-1}\infty} \left(f\Big|\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}\right)(z)\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix} \star (X-zY)^{k-2}dz\\ &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix} \star \int_{\frac{b_i}{m}}^{\sqrt{-1}\infty} f(z)(X-zY)^{k-2}dz\\ &= -\sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]_{\mathrm{c}}).\end{aligned}$$

Here the first equality follows from the proof of Lemma 1.8, the second equality follows from the pull-back formula (0.1), and the last equality follows from the definition of the evaluation map $\mathrm{ev}^j_{b_i,\mathbb{C}}$. □

We also treat the anti-holomorphic case.

Proposition 2.8. — *Under the same notation of Proposition 2.7,*

$$-\chi(-1)\tau^\iota \bullet \tau(\bar{\chi})D(f,\chi,1) = \sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]_{\mathrm{c}}|[\Gamma_1\tau\Gamma_1]).$$

Proof. — We note that $[\omega_f]_{\mathrm{c}}|[\Gamma_1\tau\Gamma_1]$ corresponds to the de Rham cohomology class

$$\tau^\iota \bullet f(-\bar{z})(X-(-\bar{z})Y)^{k-2}d(-\bar{z}) = -f(-\bar{z})(X-\bar{z}Y)^{k-2}d\bar{z}$$

via the de Rham theorem (cf. [18, §6.4]). Thus,

$$\begin{aligned}&\sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]_{\mathrm{c}}|[\Gamma_1\tau\Gamma_1])\\ &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix} \tau^\iota \bullet \int_{\sqrt{-1}\infty}^{\frac{b_i}{m}} f(-\bar{z})(X-(-\bar{z})Y)^{k-2}d(-\bar{z})\\ &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix} \tau^\iota \bullet \int_{\sqrt{-1}\infty}^{-\frac{b_i}{m}} f(z)(X-zY)^{k-2}dz\\ &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\tau^\iota\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix} \bullet \int_{\sqrt{-1}\infty}^{-\frac{b_i}{m}} f(z)(X-zY)^{k-2}dz\\ &= \chi(-1)\tau^\iota \bullet \sum_{i=1}^{\varphi(m)} \bar{\chi}(-b_i)\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix} \star \int_{\sqrt{-1}\infty}^{-\frac{b_i}{m}} f(z)(X-zY)^{k-2}dz\\ &= -\chi(-1)\tau^\iota \bullet \tau(\bar{\chi})D(f,\chi,1).\end{aligned}$$

Here the third equality follows from

$$\begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \tau^\iota = \tau^\iota \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}$$

and the last equality follows from the proof of Lemma 1.8. □

PROPOSITION 2.9. — *For each $\alpha \in \{\pm 1\}$, under the same notation of Proposition 2.7,*

$$\sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]^\alpha_{\mathrm{c}}) = \tau(\bar{\chi})\sum_{j=0}^{k-2}\left(\frac{1+\alpha\chi(-1)(-1)^j}{2}\right)\binom{k-2}{j} \cdot j!\left(\frac{1}{2\pi\sqrt{-1}}\right)^{j+1} L(f,\chi,j+1)X^{k-2-j}Y^j.$$

Proof. — Direct calculation shows that

$$\begin{aligned}\sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]^\alpha_{\mathrm{c}}) &= \frac{1}{2}\sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}\left([\omega_f]_{\mathrm{c}} + \alpha[\omega_f]_{\mathrm{c}}|[\Gamma_1\tau\Gamma_1]\right) \\ &= -\frac{1}{2}\tau(\bar{\chi})D(f,\chi,1) - \frac{1}{2}\alpha\chi(-1)\tau^\iota \bullet \tau(\bar{\chi})D(f,\chi,1).\end{aligned}$$

Here the last equality follows from Proposition 2.7 and Proposition 2.8. Hence, our proposition follows from (1.1):

$$D(f,\chi,1) = -\sum_{j=0}^{k-2}\binom{k-2}{j} j!\left(\frac{1}{2\pi\sqrt{-1}}\right)^{j+1} L(f,\chi,j+1)X^{k-2-j}Y^j. \quad \square$$

Therefore, Proposition 2.6 follows from Proposition 2.9, the functoriality of the evaluation map $\mathrm{ev}^j_{b_i,A}$ for A, the integrality of $[\omega_f]^\alpha_{\mathrm{c}}/\Omega^\alpha_f \in H^1_{\mathrm{c}}(Y,\mathcal{L}_{k-2}(\mathcal{O}))$, Lemma 1.8, and (1.3). □

2.4. Congruences of special values. — For modular forms $f, g \in M_k(\Gamma,\mathcal{O})$ and a positive integer $r \in \mathbb{Z}$, we define a congruence of modular forms $f \equiv g \pmod{\varpi^r}$ by $a(m,f) \equiv a(m,g) \pmod{\varpi^r}$ for any integer $m \in \mathbb{Z}$.

THEOREM 2.10. — *Let p be an odd prime number, r a positive integer, and k an integer with $2 \le k \le p-1$. Let $f = \sum_{n=1}^\infty a(n,f)\ e(nz) \in S_k(\Gamma_0(N),\varepsilon,\mathcal{O})$ be a p-ordinary normalized Hecke eigenform. Assume that the residual Galois representation $\bar{\rho}_f$ associated to f is reducible of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix},$$

and either ξ_1 or ξ_2 is unramified at p. Assume also that there exists an Eisenstein series $G = E_k(\psi_1,\psi_2) \in M_k(\Gamma_0(N),\varepsilon,\mathcal{O})$ (for the definition, see Theorem 3.18) such that G satisfies the assumptions of Theorem 1.9 and $f \equiv$

$G \pmod{\varpi^r}$. *Then there exist a parity* $\alpha \in \{\pm 1\}$ *(explicitly given by* (A.27)*), a complex number* $\Omega_f^\alpha \in \mathbb{C}^\times$*, and a p-adic unit* $u \in \mathcal{O}^\times$ *such that, for every primitive Dirichlet character* χ *whose conductor* m_χ *is prime to* N*, the following congruence holds:*

(1) *if* $(m_\chi, p) = 1$*, then, for each* j *with* $0 \le j \le k-2$ *and* $\alpha = \chi(-1)(-1)^j$*,*

$$\tau(\bar{\chi})\frac{L(f,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}\Omega_f^\alpha} \equiv u\tau(\bar{\chi})\frac{L(G,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}} \pmod{\varpi^r};$$

(2) *if* $p|m_\chi$*, we assume that* $m_\chi \in \varpi^r\mathcal{O}$*,* χ *is non-exceptional (see Definition 2.11) and* $\alpha = \chi(-1)$*. Then*

$$\tau(\bar{\chi})\frac{L(f,\chi,1)}{(2\pi\sqrt{-1})\Omega_f^\alpha} \equiv u\tau(\bar{\chi})\frac{L(G,\chi,1)}{2\pi\sqrt{-1}} \pmod{\varpi^r}.$$

Proof. — We put $\Gamma = \Gamma_0(N)$. By Proposition 2.3 (3), we get $[\pi_G]^\alpha \big| T(m) \equiv a(m,G)[\pi_G]^\alpha \pmod{\varpi^r}$ and $[\delta_f]^\alpha \big| T(m) \equiv a(m,f)[\delta_f]^\alpha \pmod{\varpi^r}$ for any positive integer m. We will see that $[\pi_G]^\alpha$ is non-trivial in $H^1_{\mathrm{par}}(\Gamma, L_{k-2}(\varepsilon, \mathcal{O}/\varpi))$ by a mod p non-vanishing theorem ([14, Lemma 3, page 430 (cf. the remark at the end of the proof, page 432)]) and (2.10) represented as below. Therefore, by Theorem A.12, there exists a p-adic unit $u \in \mathcal{O}^\times$ such that $[\delta_f]^\alpha = u[\pi_G]^\alpha$ in $H^1_{\mathrm{par}}(\Gamma_1(N), L_{k-2}(\mathcal{O}/\varpi^r))^\alpha[\mathfrak{M}_f] \simeq \mathcal{O}/\varpi^r$. Let $\delta_f^\alpha = \pi_f^\alpha/\Omega_f^\alpha$ which is integral by Proposition 2.6 and represents $[\delta_f]^\alpha$. Hence, for some $Q(X,Y) \in L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$, we obtain $\delta_f^\alpha - u\pi_G^\alpha = \partial Q(X,Y)$ in $Z^1(\Gamma, L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r))$.

Let χ be a non-trivial primitive Dirichlet character, whose conductor is denoted by m_χ. We fix $b_1, \ldots, b_{\varphi(m_\chi)} \in \mathbb{Z}$ such that $\{\bar{b}_1, \ldots, \bar{b}_{\varphi(m_\chi)}\} = (\mathbb{Z}/m_\chi\mathbb{Z})^\times$.

We consider in two cases.

(i) We treat the case $(p, m_\chi) = 1$.

We put $m = m_\chi$. For each b_i, we put

$$\gamma_{b_i} = \begin{pmatrix} a_i & b_i p^h \\ c_i p^h & m \end{pmatrix} \in \Gamma$$

for some choice of $a_i, c_i, h \in \mathbb{Z}$ with $p^h \in \varpi^r\mathcal{O}$. An explicit calculation with the cocycle condition (Proposition 1.6 (1)) and Theorem 1.9 shows that

$$\pi_G(\gamma_{b_i}\sigma) = \pi_G(\gamma_{b_i}) + \gamma_{b_i} \bullet \pi_G(\sigma) \in L_{k-2}(\varepsilon, \mathcal{O}).$$

Here we recall that $\pi_G(\sigma) = -D(G,1) \in L_{k-2}(\varepsilon, \mathcal{O})$ by Proposition 1.5 (1). By the choice of h, the action of γ_{b_i} on $L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$ is given by

$$\gamma_{b_i} \bullet P(X,Y) \equiv \varepsilon(m)P(mX, m^{-1}Y) \pmod{\varpi^r}.$$

We remark that the action of γ_{b_i} on $L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$ is independent of b_i. On the other hand, by using Proposition 1.6 (2) with $\pi_G(\gamma_{b_i}\sigma) \in L_{k-2}(\varepsilon, \mathcal{O})$ and our assumption, we get

$$\pi_G(\gamma_{b_i}\sigma) \equiv \pi_{G,\left(\begin{smallmatrix}1 & b_ip^h\\ 0 & m\end{smallmatrix}\right)}(\sigma) \pmod{\varpi^r}.$$

Here we remark that $a(0, G|\gamma_{b_i}\sigma) \in \mathcal{O}$. Therefore, by Lemma 1.8, computing modulo ϖ^r, we obtain

$$\begin{aligned}
\tau(\bar{\chi})D(G,\chi,1) &= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\begin{pmatrix}1 & -\frac{b_ip^h}{m}\\ 0 & 1\end{pmatrix} \star \pi_{G,\left(\begin{smallmatrix}1 & b_ip^h\\ 0 & m\end{smallmatrix}\right)}(\sigma)\\
&\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\begin{pmatrix}1 & -\frac{b_ip^h}{m}\\ 0 & 1\end{pmatrix} \star \pi_G(\gamma_{b_i}\sigma)\\
&\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\pi_G(\gamma_{b_i}\sigma)\\
&= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\{\pi_G(\gamma_{b_i}) + \gamma_{b_i} \bullet \pi_G(\sigma)\}\\
&\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\pi_G(\gamma_{b_i}) \pmod{\varpi^r}.
\end{aligned}$$

By definition, we recall that

$$\pi_G|[\Gamma\tau\Gamma](\gamma_{b_i}) = \tau^{\iota} \bullet \pi_G(\gamma'_{b_i}),$$

where

$$\gamma'_{b_i} = \tau\gamma_{b_i}\tau^{-1} = \begin{pmatrix}a_i & -b_ip^h\\ -c_ip^h & m\end{pmatrix} \in \Gamma.$$

In a similar way as above, we get

$$\chi(-1)\tau(\bar{\chi})D(G,\chi,1) \equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\pi_G(\gamma'_{b_i}) \pmod{\varpi^r}.$$

Therefore, computing modulo ϖ^r, we obtain

$$\begin{aligned}
\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\pi_G^{\alpha}(\gamma_{b_i}) &= \frac{1}{2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\left(\pi_G(\gamma_{b_i}) + \alpha\tau^{\iota} \bullet \pi_G(\gamma'_{b_i})\right) \qquad (2.10)\\
&\equiv -\frac{1}{2}\left(1 + \alpha\chi(-1)\tau^{\iota}\right) \bullet \tau(\bar{\chi})D(G,\chi,1)\\
&= \tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\left(\frac{1+\alpha\chi(-1)(-1)^j}{2}\right)\\
&\quad \cdot j!\left(\frac{1}{2\pi\sqrt{-1}}\right)^{j+1} L(G,\chi,j+1)X^{k-2-j}Y^j.
\end{aligned}$$

Here the last equality follows from (1.1). We put

$$\tau(\bar{\chi})D(G,\chi,1)^{\alpha} = -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right) \bullet \tau(\bar{\chi})D(G,\chi,1).$$

By the cocycle condition (Proposition 1.6 (1)), we have

$$\begin{aligned} \pi_f(\gamma_{b_i}\sigma) &= \pi_f(\gamma_{b_i}) + \gamma_{b_i} \bullet \pi_f(\sigma), \\ (\pi_f|[\Gamma\tau\Gamma])(\gamma_{b_i}\sigma) &= \tau^{\iota} \bullet \pi_f(\gamma'_{b_i}(-\sigma)) \\ &= \tau^{\iota} \bullet \left(\pi_f(\gamma'_{b_i}) + \gamma'_{b_i} \bullet \pi_f(-\sigma)\right) \\ &= (\pi_f|[\Gamma\tau\Gamma])(\gamma_{b_i}) + \gamma_{b_i} \bullet (\pi_f|[\Gamma\tau\Gamma])(\sigma). \end{aligned}$$

Thus we get

$$\delta_f^{\alpha}(\gamma_{b_i}\sigma) = \delta_f^{\alpha}(\gamma_{b_i}) + \gamma_{b_i} \bullet \delta_f^{\alpha}(\sigma) \in L_{k-2}(\varepsilon,\mathcal{O}),$$

where the integrality follows from Proposition 2.6. On the other hand, we have

$$\begin{aligned} \delta_f^{\alpha}(\gamma_{b_i}\sigma) &= \frac{1}{2\Omega_f^{\alpha}}(\pi_f(\gamma_{b_i}\sigma) + \alpha(\pi_f|[\Gamma\tau\Gamma])(\gamma_{b_i}\sigma)) \\ &= \frac{1}{2\Omega_f^{\alpha}}\left(\pi_f(\gamma_{b_i}\sigma) + \alpha\tau^{\iota} \bullet \pi_f(\gamma'_{b_i}(-\sigma))\right) \\ &= \frac{1}{2\Omega_f^{\alpha}}\left(\pi_{f,\begin{pmatrix} 1 & b_ip^h \\ 0 & m \end{pmatrix}}(\sigma) + \alpha\tau^{\iota} \bullet \pi_{f,\begin{pmatrix} 1 & -b_ip^h \\ 0 & m \end{pmatrix}}(\sigma)\right). \end{aligned}$$

Here the last equality follows from that $\pi_f(\gamma'_{b_i}(-\sigma)) = \pi_f(\gamma'_{b_i}\sigma)$ and Proposition 1.6 (2). Therefore, by Lemma 1.8, computing modulo ϖ^r, we obtain

$$\begin{aligned} \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\delta_f^{\alpha}(\gamma_{b_i}) &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\left(\delta_f^{\alpha}(\gamma_{b_i}\sigma) - \gamma_{b_i} \bullet \delta_f^{\alpha}(\sigma)\right) \\ &\equiv \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\delta_f^{\alpha}(\gamma_{b_i}\sigma) \equiv \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\begin{pmatrix} 1 & -\frac{b_ip^h}{m} \\ 0 & 1 \end{pmatrix} \star \delta_f^{\alpha}(\gamma_{b_i}\sigma) \\ &= -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right) \bullet \tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}} \\ &= \tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\left(\frac{1+\alpha\chi(-1)(-1)^j}{2}\right) \\ &\quad \cdot j!\left(\frac{1}{2\pi\sqrt{-1}}\right)^{j+1}\frac{L(f,\chi,j+1)}{\Omega_f^{\alpha}}X^{k-2-j}Y^j. \end{aligned}$$

We put

$$\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}^{\alpha} = -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right) \bullet \tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}.$$

Since $\delta_f^\alpha - u\pi_G^\alpha = \partial Q(X,Y)$ for some $Q(X,Y) \in L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$, we have

$$\begin{aligned}
\tau(\bar\chi)\frac{D(f,\chi,1)}{\Omega_f^\alpha}^\alpha &- u\tau(\bar\chi)D(G,\chi,1)^\alpha \pmod{\varpi^r}\\
&= \sum_{i=1}^{\varphi(m)} \bar\chi(b_i p^h)(\gamma_{b_i}-1)\bullet Q(X,Y)\\
&= \sum_{i=1}^{\varphi(m)} \bar\chi(b_i p^h)\{\varepsilon(m)Q(mX, m^{-1}Y) - Q(X,Y)\}\\
&= 0
\end{aligned}$$

if χ is non-trivial, as required.

(ii) We consider the case $p|m_\chi$.
This case is more difficult because the relation $\pi_G(\gamma_b\sigma) \equiv \pi_{G,\left(\begin{smallmatrix}1 & b\\ 0 & m_\chi\end{smallmatrix}\right)}(\sigma) \pmod{\varpi^r}$ does not hold, since m_χ is not invertible in $\mathcal{O}$. Thus this case is more delicate. In order to obtain the congruence for special values of L-functions, we will make the substitution $Y=0$.

We put $m = m_\chi$. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}[\chi]$.

Definition 2.11. — We say that a Dirichlet character χ is non-exceptional at $\mathfrak{p}$ if χ satisfies the following three conditions:
(a) $p|m$;
(b) for each $j \in \{k-2, k-1\}$, $\bar\chi(x) \not\equiv x^j \pmod{\mathfrak{p}}$ for some $x \in \mathbb{Z}$;
(c) $\chi(x) \not\equiv x \pmod{\mathfrak{p}}$ for some $x \in \mathbb{Z}$.

Sublemma. — *Assume that χ is non-exceptional at $\mathfrak{p}$. Then,*

$$A(\chi) = \sum_{i=1}^{\varphi(m)} \bar\chi(b_i)\frac{b_i}{m}, \quad A_j(\chi) = \sum_{i=1}^{\varphi(m)} \chi(b_i)\left(\frac{b_i}{m}\right)^j \; (j \in \{k-2, k-1\})$$

are $\mathfrak{p}$-integral.

Proof. — We treat the case $A_j(\chi)$ (the case $A(\chi)$ is similar). Let $x \in \mathbb{Z}$ such that $\bar\chi(x) - x^j$ is a $\mathfrak{p}$-unit of $\mathcal{O}[\chi]$. If $(m,x) = 1$, then $\{b_i x\}_i$ is a set of representatives of $(\mathbb{Z}/m\mathbb{Z})^\times$ and hence we get

$$\begin{aligned}
(\bar\chi(x) - x^j)A_j(\chi) &= \sum_{i=1}^{\varphi(m)} \chi(b_i)\bar\chi(x)\left(\frac{b_i}{\varphi(m)}\right)^j - \sum_{i=1}^{\varphi(m)} \chi(b_i)\left(\frac{b_i x}{m}\right)^j\\
&\equiv \sum_{i=1}^{\varphi(m)} \chi(b_i)\left(\frac{b_i x}{m}\right)^j - \sum_{i=1}^{\varphi(m)} \chi(b_i)\left(\frac{b_i x}{m}\right)^j \mod \mathcal{O}[\chi]\\
&\equiv 0 \mod \mathcal{O}[\chi].
\end{aligned}$$

Suppose that $d = (m, x) \neq 1$. We put $m = dm'$ and $x = dx'$. Since χ is primitive, we have

$$\begin{aligned} x^j A_j(\chi) &= \sum_{i=1}^{\varphi(m)} \chi(b_i) \left(\frac{b_i x'}{m'}\right)^j \\ &\equiv \sum_{s=1}^{\varphi(m')} \left\{ \sum_{b_i \equiv s (\bmod m')} \chi(b_i) \right\} \left(\frac{s x'}{m'}\right)^j \mod \mathcal{O}[\chi] \\ &\equiv 0 \mod \mathcal{O}[\chi]. \end{aligned}$$ □

We define

$$\gamma_{b_i} = \begin{pmatrix} a_i p^h & b_i \\ c_i & m \end{pmatrix} \in \Gamma$$

for some choice of $a_i, c_i, h \in \mathbb{Z}$ with $p^h \in \varpi^r \mathcal{O}$. By the choice of h and our assumption that $m \in \varpi^r \mathcal{O}$, the action of γ_{b_i} on $L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$ is given by

$$\gamma_{b_i} \bullet P(X, Y) \equiv \varepsilon(m) P(-b_i Y, -c_i X) \pmod{\varpi^r}.$$

By the definition of γ_{b_i},

$$\gamma_{b_i} \sigma = \begin{pmatrix} a_i p^h & b_i \\ c_i & m \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b_i & -a_i p^h \\ m & -c_i \end{pmatrix}.$$

By the cocycle condition (Proposition 1.6 (1)) and Theorem 1.9, we have

$$\pi_G(\gamma_{b_i}\sigma) = \pi_G(\gamma_{b_i}) + \gamma_{b_i} \bullet \pi_G(\sigma) \in L_{k-2}(\varepsilon, \mathcal{O}).$$

On the other hand, by using Proposition 1.6 (2), we get

$$\begin{aligned} \pi_G(\gamma_{b_i}\sigma) = a(0, G) &\int_0^{\frac{b_i}{m}} (X - zY)^{k-2} dz \\ &+ \varepsilon(m) a(0, G|\sigma) \int_{\frac{c_i}{m}}^0 \gamma_{b_i}\sigma \star (X - zY)^{k-2} dz + \pi_{G, \begin{pmatrix} 1 & b_i \\ 0 & m \end{pmatrix}}(\sigma). \end{aligned}$$

Therefore, by Lemma 1.8, we obtain

$$\begin{aligned} \tau(\bar{\chi}) D(G, \chi, 1) &= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \pi_{G, \begin{pmatrix} 1 & b_i \\ 0 & m \end{pmatrix}}(\sigma) \\ = -\sum_{i=1}^{\varphi(m)} &\bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \pi_G(\gamma_{b_i}\sigma) \\ -\sum_{i=1}^{\varphi(m)} &\bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \left\{ -a(0, G) \int_0^{\frac{b_i}{m}} (X - zY)^{k-2} dz \right\} \end{aligned}$$

$$-\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star\left\{-\varepsilon(m)a(0,G|\sigma)\int_{\frac{c_i}{m}}^{0}\gamma_{b_i}\sigma\star(X-zY)^{k-2}dz\right\}.$$

Then an explicit calculation shows that the coefficients of X^{k-2} in the second and final terms are

$$a(0,G)\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_i)\frac{b_i}{m},\quad \varepsilon(m)a(0,G|\sigma)\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(-1)^{k-j-1}}{j+1}\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_i)\frac{c_i^{k-1}}{m},$$

respectively. Thus they are integral and congruent to 0 modulo ϖ^r, since both $A(\chi)$ and $A_{k-1}(\chi)$ are integral by Sublemma and both $a(0,G)$ and $a(0,G|\sigma)$ belongs to $\varpi^r\mathcal{O}$ by our assumptions. Therefore, in the same way as the case (i), computing modulo ϖ^r, we obtain

$$\begin{aligned}\tau(\bar{\chi})D(G,\chi,1)\Big|_{Y=0} &\equiv -\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star\pi_G(\gamma_{b_i}\sigma)\Big|_{Y=0}\\ &\equiv -\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_i)\pi_G(\gamma_{b_i}\sigma)\Big|_{Y=0}\\ &\equiv -\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_i)\{\pi_G(\gamma_{b_i})+\gamma_{b_i}\bullet\pi_G(\sigma)\}\Big|_{Y=0}\\ &\equiv -\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_i)\pi_G(\gamma_{b_i})\Big|_{Y=0}\pmod{\varpi^r}.\end{aligned}$$

Here the last equality follows from that, for any $P(X,Y)\in L_{k-2}(\mathcal{O})$,

$$\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_i)\gamma_{b_i}\bullet P(X,Y)\Big|_{Y=0}\equiv 0\pmod{\varpi^r},\tag{2.11}$$

which is obtained by $m\in\varpi^r\mathcal{O}$ and $A_{k-2}(\chi)$ is integral by Sublemma. Similarly as above by substituting $-b_i$ and γ'_{b_i} for b_i and γ_{b_i} respectively, we have

$$\chi(-1)\tau(\bar{\chi})D(G,\chi,1)\Big|_{Y=0}\equiv -\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_i)\pi_G(\gamma'_{b_i})\Big|_{Y=0}\pmod{\varpi^r},$$

where

$$\gamma'_{b_i}=\tau\gamma_{b_i}\tau^{-1}=\begin{pmatrix}a_ip^h & -b_i\\ -c_i & m\end{pmatrix}\in\Gamma.$$

Therefore, computing modulo ϖ^r, we obtain

$$\begin{aligned}
(2.12)\quad \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\pi_G^{\alpha}(\gamma_{b_i})\Big|_{Y=0} &= \frac{1}{2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\left(\pi_G(\gamma_{b_i}) + \alpha\tau^{\iota}\bullet\pi_G(\gamma'_{b_i})\right)\Big|_{Y=0} \\
&\equiv -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet\tau(\bar{\chi})D(G,\chi,1)\Big|_{Y=0} \\
&= \tau(\bar{\chi})\left(\frac{1+\alpha\chi(-1)}{2}\right)\frac{L(G,\chi,1)}{2\pi\sqrt{-1}}X^{k-2}.
\end{aligned}$$

Here the last equality follows from (1.1). We put

$$\tau(\bar{\chi})D(G,\chi,1)^{\alpha} = -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet\tau(\bar{\chi})D(G,\chi,1)\Big|_{Y=0}.$$

In the same way as the case (i) with the help of (2.11), computing modulo ϖ^r, we obtain

$$\begin{aligned}
\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\delta_f^{\alpha}(\gamma_{b_i})\Big|_{Y=0} &\equiv -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}\Big|_{Y=0} \\
&= \tau(\bar{\chi})\left(\frac{1+\alpha\chi(-1)}{2}\right)\frac{L(f,\chi,1)}{(2\pi\sqrt{-1})\Omega_f^{\alpha}}X^{k-2}
\end{aligned}$$

and put

$$\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}^{\alpha} = -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}\Big|_{Y=0}.$$

Since $\delta_f^{\alpha} - u\pi_G^{\alpha} = \partial Q(X,Y)$ for some $Q(X,Y)\in L_{k-2}(\varepsilon,\mathcal{O}/\varpi^r)$, we have

$$\begin{aligned}
\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}^{\alpha} - u\tau(\bar{\chi})D(G,\chi,1)^{\alpha} &\equiv \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)(\gamma_{b_i}-1)\bullet Q(X,Y)\Big|_{Y=0} \\
&\equiv \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\{\varepsilon(m)Q(-b_iY,-c_iX) - Q(X,Y)\}\Big|_{Y=0} \\
&\equiv 0 \pmod{\varpi^r} \quad (\text{by } (2.11))
\end{aligned}$$

if χ is non-trivial and non-exceptional. We have completed the proof of our theorem. □

3. Application to the Iwasawa invariants

In this section, we first compare the Iwasawa invariant of non-primitive Selmer groups associated to modular forms with that of Selmer groups associated to Dirichlet characters. Next, in order to provide evidence for the Iwasawa main conjecture, we prove congruences between the p-adic L-function of a certain cusp form and a product of two Kubota-Leopoldt p-adic L-functions.

3.1. Iwasawa modules. — In this subsection, we summarize basic results on Iwasawa modules to define the Iwasawa invariants. We refer the reader to [42] for proofs. Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$, ϖ a uniformizer, and $\Lambda = \mathcal{O}[[T]]$ the power series ring in one variable T over $\mathcal{O}$.

Definition 3.1. — A polynomial $P(T) \in \mathcal{O}[T]$ is said to be distinguished if $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$ with $a_i \in \varpi\mathcal{O}$ for $0 \leq i \leq n-1$.

Theorem 3.2 (Weierstrass Preparation Theorem). — *If $f(T) \in \Lambda$ is non-zero, then we may uniquely write*

$$f(T) = \varpi^{\mu} P(T) U(T),$$

where $U(T) \in \Lambda$ is a unit, $P(T)$ is a distinguished polynomial, and μ is a non-negative integer.

For a non-zero element $f(T) \in \Lambda$, we define the Iwasawa λ-invariant and the Iwasawa μ-invariant of $f(T)$ by

$$\lambda(f(T)) = \deg(P(T)), \ \ \mu(f(T)) = \mu,$$

respectively.

Definition 3.3. — Two Λ-modules M and M' are said to be pseudo-isomorphic and we write $M \sim M'$, if there is a homomorphism $M \to M'$ with finite kernel and cokernel.

Theorem 3.4. — *Let M be a finitely generated Λ-module. Then*

$$M \sim \Lambda^{\oplus r} \oplus \left(\bigoplus_{i=1}^{s} \Lambda/(\varpi^{m_i}) \right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(f_j(T)^{n_j}) \right)$$

for some non-negative integers r, s, t, m_i, n_j, and distinguished and irreducible polynomials $f_j(T)$ for $1 \leq j \leq t$.

We say that a Λ-module is a torsion Λ-module if every element is annihilated by some power of the maximal ideal (ϖ, T). If M is a finitely generated torsion Λ-module, then $r = 0$. We define the Iwasawa λ-invariant, the Iwasawa μ-invariant, and the characteristic ideal of M by

$$\lambda(M) = \sum_{j=1}^{t} \deg(f_j(T)^{n_j}), \ \ \mu(M) = \sum_{i=1}^{s} m_i, \ \ \mathrm{Char}_{\Lambda}(M) = \left(\prod_{i=1}^{s} \varpi^{m_i} \prod_{j=1}^{t} f_j(T)^{n_j} \right),$$

respectively.

For a number field K, let K_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of K and $\Lambda = \mathbb{Z}_p[[T]] \simeq \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n]$. Then $\mathrm{Gal}(K_\infty/K)$ is a finitely generated torsion Λ-module.

THEOREM 3.5 (Ferrero-Washington [11]). — *Let K be a finite abelian extension of $\mathbb{Q}$ and p a prime number. Then $\mu(\mathrm{Gal}(K_\infty/K))$ for K is equal to zero.*

3.2. Selmer groups. — We will recall general results on Selmer groups. We omit details, which can be found in [15], [16]. Let Σ be a finite set of primes of $\mathbb{Q}$ containing p and ∞, and let $\mathbb{Q}_\Sigma$ be the maximal extension of $\mathbb{Q}$ which is unramified outside Σ. Let F_p be a finite extension of $\mathbb{Q}_p$ and V_p a finite dimensional F_p-vector space endowed with a continuous F_p-linear action of $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$. We put $d = \dim_{F_p}(V_p)$. Let $\mathcal{O}$ denote the ring of integers of F_p. Choose a $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$-stable $\mathcal{O}$-lattice T_p in V_p. We put $A = V_p/T_p$. Then A is a discrete $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$-module which is isomorphic to $(F_p/\mathcal{O})^d$ as an $\mathcal{O}$-module. We denote by $d^\pm$ the dimension of the (± 1)-eigenspaces of complex conjugation acting on V_p, respectively. Then we have $d = d^+ + d^-$. Since we have fixed an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we can identify $G_{\mathbb{Q}_p}$ with a decomposition group for some prime of $\overline{\mathbb{Q}}$ above p. We will assume that V_p is ordinary at p, that is, V_p contains an F_p-vector subspace F^+V_p of dimension d^+ which is stable under the action of $G_{\mathbb{Q}_p}$. Let F^+A denote the image of F^+V_p in A under the canonical map $V_p \to A$.

$$\begin{tikzcd}
0 \arrow[r] & F^+V_p \arrow[r] \arrow[d] & V_p \arrow[d] & & \\
0 \arrow[r] & F^+A \arrow[r] & A = V_p/T_p \arrow[r] & A/F^+A \arrow[r] & 0.
\end{tikzcd}$$

For a pair (A, F^+A), we define the Selmer group of A in the sense of Greenberg [15] by

$$S_A(\mathbb{Q}_\infty) = S(\mathbb{Q}_\infty; A, F^+A) = \ker\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \to \prod_{l\in\Sigma} \mathcal{H}_l(\mathbb{Q}_\infty, A)\right),$$

where $\mathcal{H}_l(\mathbb{Q}_\infty, A)$ is defined as follows: if $l \neq p$, we let

$$\mathcal{H}_l(\mathbb{Q}_\infty, A) = \prod_{\eta | l} H^1((\mathbb{Q}_\infty)_\eta, A),$$

where the product is taken over the finite set of primes η of $\mathbb{Q}_\infty$ lying above l. There is a unique prime η_p of $\mathbb{Q}_\infty$ lying above p. Let I_{η_p} denote the inertia subgroup of $G_{(\mathbb{Q}_\infty)_{\eta_p}}$. We define

$$\mathcal{H}_p(\mathbb{Q}_\infty, A) = \mathcal{H}_p(\mathbb{Q}_\infty; A, F^+A) = \mathrm{im}\left(H^1((\mathbb{Q}_\infty)_{\eta_p}, A) \to H^1(I_{\eta_p}, A/F^+A)\right).$$

We define the Iwasawa algebra Λ by $\Lambda = \mathcal{O}[[\Gamma]]$, where $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. We know that the groups $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A)$, $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A)$, $\mathcal{H}_l(\mathbb{Q}_\infty, A)$, and $S_A(\mathbb{Q}_\infty)$ are discrete $\mathcal{O}$-modules with a natural continuous action of Γ. Hence these groups are regarded as Λ-modules and are known to be cofinitely generated, that is, their Pontryagin duals are finitely generated Λ-modules. The following corank formulas follow from the results in [15, §3, §4]:

Proposition 3.6. — *The following statements hold:*

(1) $\mathrm{corank}_\Lambda(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A)) = d^- + \mathrm{corank}_\Lambda(H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A))$.
(2) $\mathrm{corank}_\Lambda(\mathcal{H}_p(\mathbb{Q}_\infty, A)) = d^-$.
(3) $\mathrm{corank}_\Lambda(\mathcal{H}_l(\mathbb{Q}_\infty, A)) = 0$ *if* $l \neq p$.

We always assume that $S_A(\mathbb{Q}_\infty)$ is Λ-cotorsion in the rest of §3.2. Put $A^* = \mathrm{Hom}(T_p, \mu_{p^\infty})$. This is also a discrete $\mathcal{O}$-module equipped with a continuous action of $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$. The next proposition, which is proved in [16, Proposition 2.1], is important in this paper.

Proposition 3.7. — *Assume that $S_A(\mathbb{Q}_\infty)$ is Λ-cotorsion and $H^0(\mathbb{Q}_\infty, A^*)$ is finite. Then the following sequence is exact:*

$$0 \to S_A(\mathbb{Q}_\infty) \to H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \to \prod_{l\in\Sigma} \mathcal{H}_l(\mathbb{Q}_\infty, A) \to 0.$$

Next we recall the non-primitive Selmer groups of A in the sense of Greenberg. Let Σ_0 be any finite subset of Σ which does not contain neither p nor ∞. The non-primitive Selmer groups for (A, F^+A) and Σ_0 is defined by

$$S_A^{\Sigma_0}(\mathbb{Q}_\infty) = S^{\Sigma_0}(\mathbb{Q}_\infty; A, F^+A) = \ker\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \to \prod_{l\in\Sigma\setminus\Sigma_0} \mathcal{H}_l(\mathbb{Q}_\infty, A)\right).$$

We have $S_A(\mathbb{Q}_\infty) \subset S_A^{\Sigma_0}(\mathbb{Q}_\infty)$ by the definition. We denote by $M^\vee$ the Pontryagin dual of any locally compact $\mathbb{Z}_p$-module M. We obtain the following corollary of Proposition 3.6 (3), Proposition 3.7, and [15, Proposition 2], which is proved in [16, Corollary 2.3].

Corollary 3.8. — *Under the assumption as in Proposition 3.7, we have*

$$S_A^{\Sigma_0}(\mathbb{Q}_\infty)/S_A(\mathbb{Q}_\infty) \cong \prod_{l\in\Sigma_0} \mathcal{H}_l(\mathbb{Q}_\infty, A)$$

as Λ-modules. In particular, $S_A^{\Sigma_0}(\mathbb{Q}_\infty)$ is Λ-cotorsion, and the following equalities hold:

$$\mathrm{corank}_\mathcal{O}(S_A^{\Sigma_0}(\mathbb{Q}_\infty)) = \mathrm{corank}_\mathcal{O}(S_A(\mathbb{Q}_\infty)) + \sum_{l\in\Sigma_0} \mathrm{corank}_\mathcal{O}(\mathcal{H}_l(\mathbb{Q}_\infty, A)),$$

$$\mu(S_A^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = \mu(S_A(\mathbb{Q}_\infty)^\vee).$$

Next, in order to compare $\mathrm{corank}_{\mathcal{O}}(S_A^{\Sigma_0}(\mathbb{Q}_\infty))$ with $\mathrm{corank}_{\mathcal{O}}(S_A(\mathbb{Q}_\infty))$, we would like to find a generator of $\mathcal{H}_l(\mathbb{Q}_\infty, A)^\vee$. The following proposition is the result in [16, Proposition 2.4].

PROPOSITION 3.9. — *Let l be a prime number with $l \neq p$. Put $P_l(X) = \det((1 - \mathrm{Frob}_l X)|_{(V_p)_{I_l}}) \in \mathcal{O}[X]$ and $\mathcal{P}_l = P_l(l^{-1}\gamma_l) \in \Lambda = \mathcal{O}[[\Gamma]]$, where γ_l denotes the Frobenius automorphism corresponding to the prime l in $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. The characteristic ideal of the Λ-module $\mathcal{H}_l(\mathbb{Q}_\infty, A)^\vee$ is generated by $\mathcal{P}_l$.*

Let ϖ be a uniformizer of $\mathcal{O}$. Let $A[\varpi]$ denote the ϖ-torsion of A. We now define a Selmer group of $A[\varpi]$. For any subset Σ_0 of $\Sigma - \{p, \infty\}$, we define

$$\begin{aligned} S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty) &= S^{\Sigma_0}(\mathbb{Q}_\infty; A[\varpi], F^+A[\varpi]) \\ &= \ker\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A[\varpi]) \to \prod_{l \in \Sigma \setminus \Sigma_0} \mathcal{H}_l(\mathbb{Q}_\infty, A[\varpi])\right), \end{aligned}$$

where $\mathcal{H}_l(\mathbb{Q}_\infty, A[\varpi])$ is defined by

$$\mathcal{H}_l(\mathbb{Q}_\infty, A[\varpi]) = \begin{cases} \prod_{\eta | l} H^1(I_\eta, A[\varpi]) & \text{if } l \neq p, \\ H^1(I_{\eta_p}, A[\varpi]/F^+A[\varpi]) & \text{if } l = p. \end{cases}$$

Under certain hypotheses, the next proposition obtained by [16, Proposition 2.8] allows us to describe $\lambda(S_A^{\Sigma_0}(\mathbb{Q}_\infty))$ in terms of the Galois module $A[\varpi]$. We put $\mathrm{Ram}(A) = \{l | l \neq p, \infty$ and the action of $G_{\mathbb{Q}_l}$ on A is ramified$\}$.

PROPOSITION 3.10. — *Let p be an odd prime number and Σ_0 a subset of $\Sigma - \{p, \infty\}$ containing $\mathrm{Ram}(A)$. Assume that I_{η_p} acts trivially on A/F^+A and $H^0(\mathbb{Q}_\infty, A) = 0$. Then we have*

$$S_A^{\Sigma_0}(\mathbb{Q}_\infty)[\varpi] \cong S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty).$$

Consequently, $S_A(\mathbb{Q}_\infty)$ is Λ-cotorsion, and has μ-invariant is zero if and only if $S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty)$ is finite. If this is the case,

$$\lambda(S_A^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = \dim_{\mathcal{O}/\varpi\mathcal{O}}(S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty)).$$

Now we apply the general theory recalled above to the Iwasawa main conjecture of modular forms. Let $f = \sum_{n=1}^\infty a(n,f)e(nz) \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a normalized Hecke eigenform and

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}(T_f) \simeq \mathrm{GL}_2(\mathcal{O})$$

the associated Galois representation, which satisfies

(1) ρ_f is unramified at all primes $l \nmid Np$,
(2) $\mathrm{Tr}(\rho_f(\mathrm{Frob}_l)) = a(l,f)$ for $l \nmid Np$,
(3) $\det(\rho_f(\mathrm{Frob}_l)) = \varepsilon(l) l^{k-1}$ for $l \nmid Np$,

(4) ρ_f is odd.

We write κ for the residue field of $\mathcal{O}$. Let $A_f = T_f \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)$ denote the cofree $\mathcal{O}$-module of corank 2 with $G_{\mathbb{Q}}$-action via ρ_f. We assume that

(RR) the residual representation $\bar{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\kappa)$ is reducible.

Then $\bar{\rho}_f$ is of the form

$$\bar{\rho}_f \sim \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix},$$

that is, there exists an exact sequence

$$0 \to \Phi \to A_f[\varpi] \to \Psi \to 0 \tag{3.1}$$

of $\kappa[G_{\mathbb{Q}}]$-modules, where $G_{\mathbb{Q}}$ acts on Ψ via the character $\psi\colon G_{\mathbb{Q}} \to \kappa^\times$, and on Φ via the character $\varphi\colon G_{\mathbb{Q}} \to \kappa^\times$.

Hereafter we assume that f is p-ordinary. From the result of [27] and [43, Theorem 2.1.4], the restriction of ρ_f to the decomposition group D_p is of the form

$$\rho_f\big|_{D_p} \sim \begin{pmatrix} \chi_p^{k-1}\rho_1 & * \\ 0 & \rho_2 \end{pmatrix},$$

where $\rho_1, \rho_2\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ are unramified characters such that ρ_2 sends the arithmetic Frobenius to a unit-root of $X^2 - a(p,f)X + \varepsilon(p)p^{k-1} = 0$ and χ_p is the p-adic cyclotomic character. Then F^+A_f is defined by the following exact sequence of $\mathcal{O}[G_{\mathbb{Q}_p}]$-modules:

$$0 \to F^+A_f \to A_f \to A_f/F^+A_f \to 0, \tag{3.2}$$

where $G_{\mathbb{Q}_p}$ acts on F^+A_f via the character $\chi_p^{k-1}\rho_1\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$, and on A_f/F^+A_f via the character $\rho_2\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$. We can define the Selmer group of (A_f, F^+A_f) by

$$S_{A_f}(\mathbb{Q}_\infty) = S(\mathbb{Q}_\infty; A_f, F^+A_f).$$

Let $\Sigma_0 = \{l \in \mathbb{N} | l \text{ is a prime number such that } l|N\}$ and $\Sigma = \Sigma_0 \cup \{p, \infty\}$ a finite set of places of $\mathbb{Q}$. Then the non-primitive Selmer group of A_f is defined by

$$S_{A_f}^{\Sigma_0}(\mathbb{Q}_\infty) = S^{\Sigma_0}(\mathbb{Q}_\infty; A_f, F^+A_f).$$

We assume that $2 \le k \le p-1$ and

(Assumption) ψ is unramified at p and odd, and
φ is ramified at p and even.

Hence $\psi(\mathrm{Frob}_p) \equiv a(p,f) (\mathrm{mod}\ \varpi)$.

Lemma 3.11. — *We assume that φ and ψ are as above and p is odd. Then we have*

$$H^0(\mathbb{Q}, A_f[\varpi]) = 0.$$

Proof. — Since φ is ramified at p, $H^0(\mathbb{Q}, \Phi) \subset H^0(G_{\mathbb{Q}_p}, \Phi) = 0$. Since ψ is odd and p is odd, $H^0(\mathbb{Q}, \Psi) \subset H^0(\langle c \rangle, \Psi) = 0$, where $c \in G_{\mathbb{Q}}$ is the complex conjugation. □

LEMMA 3.12. — *Suppose that ψ is odd and p is odd. Then,*

$$H^0(\mathbb{Q}_\infty, \Psi) = 0.$$

Proof. — Since ψ is odd and p is odd, $H^0(\mathbb{Q}_\infty, \Psi) \subset H^0(\langle c \rangle, \Psi) = 0$, where $c \in G_{\mathbb{Q}_\infty}$ is the complex conjugation. □

LEMMA 3.13. — *Assume that p is odd. Then,*

$$H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, \Phi) = 0.$$

Proof. — For a Galois module $A \cong F_p/\mathcal{O}$ via the character $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty) \xrightarrow{\varphi} \kappa^\times \hookrightarrow \mathcal{O}^\times$, we have

$$0 \to \Phi \to A \xrightarrow{\varpi} A \to 0.$$

Therefore, in order to prove the lemma, it is enough to show that

$$\text{(i) } H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \text{ is divisible, and}$$
$$\text{(ii) } H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) = 0.$$

Indeed, we have an exact sequence

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \xrightarrow{\varpi} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \to H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, \Phi) \to H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A)$$

as a part of the cohomology long exact sequence. The proof of (i) and (ii) can be found in [16, p.46] just after the equation (16) under the assumptions that φ is even and non-trivial. □

Therefore, using Lemma 3.12 and Lemma 3.13, we have an exact sequence

$$0 \to H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, \Phi) \xrightarrow{\alpha} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A_f[\varpi]) \xrightarrow{\beta} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, \Psi) \to 0.$$

By this exact sequence and the definition of $S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty)$, $S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty)$, and $S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty)$, we have

$$S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty)/S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty) = S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty).$$

Here, by the definition, $S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty) = \ker(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, \Psi) \to H^1(I_{\eta_p}, \Psi))$ and $S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty) = H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, \Phi)$. Hence we have

$$\dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty)) = \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty)) + \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty)).$$

We compute the Selmer groups for one-dimensional representations V_p with some assumptions. The Galois group $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$ acts on V_p via a continuous homomorphism $\theta\colon \mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}) \to \mathcal{O}^\times$. Then, θ factors through $G = \mathrm{Gal}(K_\infty/\mathbb{Q})$, where K_∞ is a certain finite extension of $\mathbb{Q}_\infty$ such that K_∞ is an abelian extension over $\mathbb{Q}$. We put $\Delta = \mathrm{Gal}(K_\infty/\mathbb{Q}_\infty)$ and assume that

$(p, \sharp\Delta) = 1$. We can identify Γ with a subgroup of G such that $G = \Delta \times \Gamma$. This decomposition is unique for our case $(p, \sharp\Delta) = 1$. We have $\mathbb{Z}_p[[G]] = \mathbb{Z}_p[\Delta][[\Gamma]]$.

Let $X_\infty = \mathrm{Gal}(M_\infty/K_\infty)$ and $Y_\infty = \mathrm{Gal}(L_\infty/K_\infty)$. Here M_∞ denotes the maximal abelian pro-p extension of K_∞ which is unramified outside $\{p, \infty\}$, and L_∞ denotes the maximal abelian pro-p extension of K_∞ which is unramified everywhere. Let $\xi = \theta|_\Delta$ be the restriction of θ to Δ and $\Sigma_0 = \Sigma - \{p, \infty\}$. If θ is even (resp. odd), then $d^+ = 1$ (resp. $d^+ = 0$) and we have $F^+V_p = V_p$ (resp. $F^+V_p = 0$).

Proposition 3.14 ([16], p.45, 46). — *The Selmer groups for one-dimensional representations have the following properties.*

(1)

$$S_A(\mathbb{Q}_\infty) \simeq \begin{cases} \mathrm{Hom}_{\mathcal{O}}((X_\infty \otimes_{\mathbb{Z}_p} \mathcal{O})^\xi, A) \text{ if } \theta \text{ is even,} \\ \mathrm{Hom}_{\mathcal{O}}((Y_\infty \otimes_{\mathbb{Z}_p} \mathcal{O})^\xi, A) \ \text{ if } \theta \text{ is odd.} \end{cases}$$

(2) *The Λ-modules $S_A(\mathbb{Q}_\infty)$ and $S_A^{\Sigma_0}(\mathbb{Q}_\infty)$ are cotorsion, and we have*

$$\mu(S_A(\mathbb{Q}_\infty)^\vee) = \mu(S_A^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = 0.$$

.

(3) *Assume that ξ is non-trivial if θ is even, and $\xi \neq \omega$ if θ is odd. Then we have*

$$\begin{aligned}\dim_{\mathcal{O}/\varpi\mathcal{O}}(S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty)) &= \mathrm{corank}_{\mathcal{O}}(S_A^{\Sigma_0}(\mathbb{Q}_\infty)) \\ &= \mathrm{corank}_{\mathcal{O}}(S_A(\mathbb{Q}_\infty)) + \sum_{l \in \Sigma_0} \mathrm{corank}_{\mathcal{O}}(\mathcal{H}_l(\mathbb{Q}_\infty, A)).\end{aligned}$$

In particular, $S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty)$ is finite.

We can apply these results to $(A, \theta) = (A_\varphi, \widetilde{\varphi})$ (resp. $(A_\psi, \widetilde{\psi})$) for a Galois module $A_\varphi \cong F_p/\mathcal{O}$ (resp. $A_\psi \cong F_p/\mathcal{O}$) via the character $\widetilde{\varphi} = \chi_p^{k-1}\varepsilon\widetilde{\psi}^{-1} : G_\mathbb{Q} \to \kappa^\times \hookrightarrow \mathcal{O}^\times$ (resp. $\widetilde{\psi} : G_\mathbb{Q} \to \kappa^\times \hookrightarrow \mathcal{O}^\times$). We remark that $A_\varphi[\varpi] = \Phi$ and $A_\psi[\varpi] = \Psi$.

(i) We consider $S_\Phi^{\Sigma_0}(\mathbb{Q}_\infty) = S^{\Sigma_0}(\mathbb{Q}_\infty; \Phi, \Phi)$.
Since φ is even and ramified at p by our assumption, it is non-trivial. Therefore we have $\mu(S_{A_\varphi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = 0$ and

$$\dim_{\mathcal{O}/\varpi\mathcal{O}}(S_\Phi^{\Sigma_0}(\mathbb{Q}_\infty)) = \mathrm{corank}_{\mathcal{O}}(S_{A_\varphi}^{\Sigma_0}(\mathbb{Q}_\infty)).$$

(ii) We consider $S_\Psi^{\Sigma_0}(\mathbb{Q}_\infty) = S^{\Sigma_0}(\mathbb{Q}_\infty; \Psi, 0)$.
Since ψ is odd and unramified at p, we have $\psi \neq \omega$. Therefore, we have $\mu(S_{A_\psi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = 0$ and

$$\dim_{\mathcal{O}/\varpi\mathcal{O}}(S_\Psi^{\Sigma_0}(\mathbb{Q}_\infty)) = \mathrm{corank}_{\mathcal{O}}(S_{A_\psi}^{\Sigma_0}(\mathbb{Q}_\infty)).$$

We define the Iwasawa λ-invariants by

$$\lambda_{\varphi,\Sigma_0} = \mathrm{corank}_{\mathcal{O}}(S^{\Sigma_0}_{A_\varphi}(\mathbb{Q}_\infty)),\ \lambda_{\psi,\Sigma_0} = \mathrm{corank}_{\mathcal{O}}(S^{\Sigma_0}_{A_\psi}(\mathbb{Q}_\infty)).$$

By Proposition 3.14 (3), using the exact sequence

$$0 \to S^{\Sigma_0}_{A_\varphi}(\mathbb{Q}_\infty) \to S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty) \to S^{\Sigma_0}_{A_\psi}(\mathbb{Q}_\infty) \to 0,$$

$S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty)$ is finite. Therefore, by combining these results, Proposition 3.10 and Lemma 3.11, we see that $S_{A_f}(\mathbb{Q}_\infty)$ and $S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty)$ are Λ-cotorsion. Thus we can define the algebraic Iwasawa invariants by

$$\begin{aligned}
\lambda_f^{\mathrm{alg}} &= \lambda(S_{A_f}(\mathbb{Q}_\infty)^\vee) = \lambda(S(\mathbb{Q}_\infty; A_f, F^+A_f)^\vee) = \deg(f^{\mathrm{alg}}(T)),\\
\mu_f^{\mathrm{alg}} &= \mu(S_{A_f}(\mathbb{Q}_\infty)^\vee) = \mu(S(\mathbb{Q}_\infty; A_f, F^+A_f)^\vee),\\
\lambda_{f,\Sigma_0}^{\mathrm{alg}} &= \lambda(S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty)^\vee) = \lambda(S^{\Sigma_0}(\mathbb{Q}_\infty; A_f, F^+A_f)^\vee) = \deg(f^{\mathrm{alg}}_{\Sigma_0}(T)),\\
\mu_{f,\Sigma_0}^{\mathrm{alg}} &= \mu(S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty)^\vee) = \mu(S^{\Sigma_0}(\mathbb{Q}_\infty; A_f, F^+A_f)^\vee),
\end{aligned}$$

where $f^{\mathrm{alg}}(T)$ (resp. $f^{\mathrm{alg}}_{\Sigma_0}(T)$) is the distinguished polynomial corresponding to $S_{A_f}(\mathbb{Q}_\infty)^\vee$ (resp. $S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty)^\vee$) via the Weierstrass preparation theorem. Again by using Proposition 3.10 and Lemma 3.11, we obtain

$$\mu_f^{\mathrm{alg}} = \mu_{f,\Sigma_0}^{\mathrm{alg}} = 0 \tag{3.3}$$

and

$$\begin{aligned}
\lambda_{f,\Sigma_0}^{\mathrm{alg}} &= \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty))\\
&= \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty)) + \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty))\\
&= \lambda_{\varphi,\Sigma_0} + \lambda_{\psi,\Sigma_0}.
\end{aligned} \tag{3.4}$$

3.3. *p*-adic *L*-functions. — We recall p-adic L-functions of modular forms. These functions have been constructed by Amice-Vélu [1], Vishik [40], Mazur, Tate, and Teitelbaum [26]. Also, we recall non-primitive p-adic L-functions of modular forms in the sense of Greenberg. Let K be an abelian number field, $2 \leq k \leq p-1$, and $f(z) = \sum_{n=1}^\infty a(n,f)e(nz) \in S_k(\Gamma_0(N),\varepsilon,\mathcal{O})$ a p-ordinary normalized Hecke eigenform which satisfies (RR), (3.1), and (Assumption). We assume that K is unramified at all primes dividing the level N, and tamely ramified at p. Put $G = \mathrm{Gal}(K/\mathbb{Q})$, and fix a character χ of G. We write $\Gamma = \mathrm{Gal}(K_\infty/K)$, where K_∞ denotes the cyclotomic $\mathbb{Z}_p$-extension of K. We can identify Γ with the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let γ denote a fixed topological generator of Γ. Put $\Lambda = \mathcal{O}[\chi][[\Gamma]] \cong \mathcal{O}[\chi][[T]]\,;\ \gamma \mapsto 1+T$. For a finite order character $\rho\colon \Gamma \to \mathbb{C}^\times$, we define $\zeta \in \mu_{p^\infty}$ by $\zeta = \rho(\gamma)$. The p-adic L-function $\mathscr{L}_p(f,\chi,T) \in \Lambda$ is the power series characterized by the following interpolation property: for every non-trivial p-adic character $\rho\colon \Gamma \to \overline{\mathbb{Q}}_p^{\times}$ of finite order with conductor p^{ν_ρ},

$$\mathscr{L}_p(f,\chi,\zeta-1) = \tau(\chi^{-1}\rho^{-1})\alpha(p,f)^{-\nu_\rho}\frac{L(f,\chi\rho,1)}{(-2\pi\sqrt{-1})\Omega_f^{\alpha}} \in \mathcal{O}[\chi,\rho],$$

where $\tau(\chi^{-1}\rho^{-1})$ is the Gauss sum of $\chi^{-1}\rho^{-1}$, $\alpha(p,f)$ is a unit root of $X^2 - a(p,f)X + \varepsilon(p)p^{k-1} = 0$, Ω_f^{α} is the canonical period defined by (2.5), and $\alpha = \chi(-1)$. By the Weierstrass preparation theorem, this interpolation property characterizes $\mathscr{L}_p(f,\chi,T)$. Also, for any finite set of primes Σ_0 with $p \notin \Sigma_0$, the non-primitive p-adic L-function $\mathscr{L}_p^{\Sigma_0}(f,\chi,T) \in \Lambda$ is characterized by the interpolation property

$$\mathscr{L}_p^{\Sigma_0}(f,\chi,\zeta-1) = \tau(\chi^{-1}\rho^{-1})\alpha(p,f)^{-\nu_\rho}\frac{L_{\Sigma_0}(f,\chi\rho,1)}{(-2\pi\sqrt{-1})\Omega_f^{\alpha}} \in \mathcal{O}[\chi,\rho],$$

where $L(f,\chi,s)\prod_{l\in\Sigma_0} E_l(f,\chi,s) = L_{\Sigma_0}(f,\chi,s)$. Here, $E_l(f,\chi,s)$ is the Euler factor of $L(f,\chi,s)$ at l. Then, putting $\chi =$ trivial character, we have

$$\mathscr{L}_p^{\Sigma_0}(f,T) = \mathscr{L}_p(f,T)\prod_{l\in\Sigma_0}\mathcal{P}_l(T),$$

where $\mathcal{P}_l(T)$ is defined by Proposition 3.9. We define the analytic Iwasawa invariants by

$$\begin{aligned}
\lambda_f^{\mathrm{anal}} &= \lambda(\mathscr{L}_p(f,T)) = \deg(f^{\mathrm{anal}}(T)),\\
\mu_f^{\mathrm{anal}} &= \mu(\mathscr{L}_p(f,T)),\\
\lambda_{f,\Sigma_0}^{\mathrm{anal}} &= \lambda(\mathscr{L}_p^{\Sigma_0}(f,T)) = \deg(f_{\Sigma_0}^{\mathrm{anal}}(T)),\\
\mu_{f,\Sigma_0}^{\mathrm{anal}} &= \mu(\mathscr{L}_p^{\Sigma_0}(f,T)),
\end{aligned}$$

where $f^{\mathrm{anal}}(T)$ (resp. $f_{\Sigma_0}^{\mathrm{anal}}(T)$) is the distinguished polynomial corresponding to $\mathscr{L}_p$ (resp. $\mathscr{L}_p^{\Sigma_0}$) via the Weierstrass preparation theorem.

3.4. The Iwasawa main conjecture. — In this subsection, we assume that $2 \le k \le p-1$ and a normalized Hecke eigenform $f \in S_k(\Gamma_0(N),\varepsilon,\mathcal{O})$ is p-ordinary and satisfies (RR), (3.1), and (Assumption). Let $\mathrm{Char}_\Lambda(S_{A_f}(\mathbb{Q}_\infty)^\vee)$ be the characteristic ideal of the Pontryagin dual of the Selmer group $S_{A_f}(\mathbb{Q}_\infty)$. The following main conjecture for ρ_f is formulated by Greenberg.

CONJECTURE 3.15. — *We have*

$$\mathrm{Char}_\Lambda(S_{A_f}(\mathbb{Q}_\infty)^\vee) = (\mathscr{L}_p(f,T)) \text{ in } \Lambda.$$

Kato has proven the following deep theorem in [21].

THEOREM 3.16. — *We have*

$$\mathrm{Char}_\Lambda(S_{A_f}(\mathbb{Q}_\infty)^\vee) \supset (\mathscr{L}_p(f,T)) \text{ in } \Lambda\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

Therefore, in order to confirm the Iwasawa main conjecture, we will show that

$$\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}.$$

Non-primitive objects $S_{A_f}^{\Sigma_0}(\mathbb{Q}_\infty)$ and $\mathscr{L}_p^{\Sigma_0}(f,T)$ will behave well under congruences, where $\Sigma_0 = \{l \text{ is a prime number} | l | N\}$. Then the following theorem obtained by [16, Theorem 1.5] is crucial for our proof.

THEOREM 3.17. — *The following statements hold:*
(1) $\mu_f^{\mathrm{alg}} = \mu_f^{\mathrm{anal}}$ *if and only if* $\mu_{f,\Sigma_0}^{\mathrm{alg}} = \mu_{f,\Sigma_0}^{\mathrm{anal}}$.
(2) $\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}$ *if and only if* $\lambda_{f,\Sigma_0}^{\mathrm{alg}} = \lambda_{f,\Sigma_0}^{\mathrm{anal}}$.
(3) $f^{\mathrm{alg}}(T) = f^{\mathrm{anal}}(T)$ *if and only if* $f_{\Sigma_0}^{\mathrm{alg}}(T) = f_{\Sigma_0}^{\mathrm{anal}}(T)$.

Now, we analogously define the p-adic L-functions for the Galois representations A_φ and A_ψ appearing in the previous subsection.

(i) The p-adic L-function $\mathscr{L}_p(A_\varphi, T) \in \Lambda$ is defined by the interpolation property

$$\mathscr{L}_p(A_\varphi, \zeta - 1) = L(\varepsilon\psi^{-1}\rho, 2-k) = L(\chi_p^{k-1}\varepsilon\psi^{-1}\rho, 1)$$

for every non-trivial p-adic character $\rho : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order and $\zeta = \rho(\gamma)$. Here we remark that, by (Assumption), $\omega^{k-1}\varepsilon\psi^{-1}$ is non-trivial character and hence $L(\chi_p^{k-1}\varepsilon\psi^{-1}\rho, s)$ is holomorphic for $s \in \mathbb{C}$. Then, $\mathscr{L}_p(A_\varphi, T)$ is related to the Kubota-Leopoldt p-adic L-function by

$$L_p(\varepsilon\chi_p^{k-2}\omega\psi^{-1}, s) = \mathscr{L}_p(A_\varphi, \kappa(\gamma)^{-s} - 1)$$

for any $s \in \mathbb{Z}_p$. Here, $\kappa(\gamma)$ is the element of $1+p\mathbb{Z}_p$ which induces the action of γ on μ_{p^∞} when we identify Γ with $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$. The Ferrero-Washington theorem (Theorem 3.5) and the Mazur-Wiles theorem assert that $\mathscr{L}_p(A_\varphi, T) \notin \varpi\Lambda$ and the λ-invariant of $\mathscr{L}_p(A_\varphi, T)$ is equal to $\mathrm{corank}_{\mathcal{O}}(S_{A_\varphi}(\mathbb{Q}_\infty))$, which is denoted by $\lambda_{\varepsilon\omega^{k-1}\psi^{-1}} = \lambda_\varphi$. In addition, the non-primitive p-adic L-function $\mathscr{L}_p^{\Sigma_0}(A_\varphi, T)$ is defined by

$$\mathscr{L}_p^{\Sigma_0}(A_\varphi, T) = \mathscr{L}_p(A_\varphi, T) \prod_{l \in \Sigma_0} (1 - \varepsilon\psi^{-1}(l) l^{k-2}(1+T)^{f_l}).$$

Here $f_l \in \mathbb{Z}_p$ is determined by $\gamma_l = \gamma^{f_l}$, where γ_l is the Frobenius element corresponding to the prime l in Γ.

(ii) The p-adic L-function $\mathscr{L}_p(A_\psi, T) \in \Lambda$ is defined by the interpolation property

$$\mathscr{L}_p(A_\psi, \zeta - 1) = \tau(\psi^{-1}\rho^{-1}) \frac{L(\psi\rho, 1)}{2\pi\sqrt{-1}} = \frac{1}{2} L(\psi^{-1}\rho^{-1}, 0)$$

for every non-trivial p-adic character $\rho : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order and $\zeta = \rho(\gamma)$. Here we remark that, by (Assumption), $\omega\psi^{-1}$ is non-trivial character. Then $\mathscr{L}_p(A_\psi, T)$ is related to the Kubota-Leopoldt p-adic L-function by

$$L_p(\omega\psi^{-1}, s) = \frac{1}{2}\mathscr{L}_p(A_\psi, \kappa(\gamma)^s - 1)$$

for any $s \in \mathbb{Z}_p$. The μ-invariant of $\mathscr{L}_p(A_\psi, T)$ is again zero and its λ-invariant is $\lambda_{\omega\psi^{-1}} = \lambda_\psi$, which is equal to $\mathrm{corank}_{\mathcal{O}}(S_{A_\psi}(\mathbb{Q}_\infty))$ by the Mazur-Wiles theorem. In addition, the non-primitive p-adic L-function $\mathscr{L}_p^{\Sigma_0}(A_\psi, T)$ is defined by

$$\mathscr{L}_p^{\Sigma_0}(A_\psi, T) = \mathscr{L}_p(A_\psi, T) \prod_{l \in \Sigma_0} (1 - \psi(l) l^{-1} (1+T)^{f_l}).$$

To state our theorem, we need to recall some facts about Eisenstein series. The following theorem is obtained from the results in [28, Theorem 4.7.1].

THEOREM 3.18 (Lifting). — *Let ε_i be a primitive Dirichlet character modulo M_i for $i = 1, 2$. We put $M = M_1 M_2$ and $\varepsilon = \varepsilon_1 \varepsilon_2$. If $\varepsilon(-1) = (-1)^k$, there exists an Eisenstein series $G = E_k(\varepsilon_1, \varepsilon_2) \in M_k(\Gamma_0(M), \varepsilon, \mathbb{C})$ such that*

$$L(G, s) = L(\varepsilon_1, s) L(\varepsilon_2, s - k + 1).$$

Moreover, $a(0, G) = 0$ if $k \neq 1$ and ε_1 is non-trivial.

Let $G = \sum_{n=0}^\infty a(n, G) e(nz) \in M_k(\Gamma_0(M), \varepsilon, \mathcal{O})$ be the Eisenstein series of weight k determined by

$$L(G, s) = L_{\Sigma_0}(\psi, s) L_{\Sigma_0}(\varepsilon\psi^{-1}, s - k + 1).$$

Note that Theorem 3.18 assures the existence of such G.

We define the p-adic L-function $\mathscr{L}_p(G, T)$ by the interpolation property

$$\begin{aligned}\mathscr{L}_p(G, \zeta - 1) &= \tau(\psi^{-1}\rho^{-1}) \frac{L(G, \rho, 1)}{2\pi\sqrt{-1}} \\ &= L_{\Sigma_0}(\varepsilon\psi^{-1}\rho, 2 - k) \tau(\psi^{-1}\rho^{-1}) \frac{L_{\Sigma_0}(\psi\rho, 1)}{2\pi\sqrt{-1}}\end{aligned}$$

for every non-trivial p-adic character $\rho : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order and $\zeta = \rho(\gamma)$. Then clearly we have

$$\mathscr{L}_p(G, T) = \mathscr{L}_p^{\Sigma_0}(A_\varphi, T) \mathscr{L}_p^{\Sigma_0}(A_\psi, T).$$

Therefore, the μ-invariant of $\mathscr{L}_p(G, T)$ is zero and the λ-invariant of $\mathscr{L}_p(G, T)$ is equal to $\lambda_{\varphi, \Sigma_0} + \lambda_{\psi, \Sigma_0}$.

We define an eigenform $g(z) \in S_k(\Gamma_0(M), \varepsilon, \mathcal{O})$ by

$$(f \otimes \mathbf{1}_N)(z) = \sum_{(n,N)=1} a(n, f) e(nz),$$

where $\mathbf{1}_N$ denotes the trivial character on $(\mathbb{Z}/N\mathbb{Z})^\times$.

THEOREM 3.19. — *With the notation and the assumptions above, we have the congruence*

$$\frac{\Omega_f^+}{\Omega_g^+}\mathscr{L}_p^{\Sigma_0}(f,T) \equiv u(1+T)^{-n_\psi}\mathscr{L}_p(G,T) \bmod \varpi\Lambda,$$

where u is a unit in $\mathcal{O}$ and $(1+T)^{n_\psi} \in \Lambda^\times$ is the image of the conductor m_ψ under $\mathbb{Z}_p \twoheadrightarrow 1+p\mathbb{Z}_p \simeq \Gamma \hookrightarrow \Lambda$. Here α (explicitly given by (A.27)*) is equal to* $+1$ *by our assumption of φ and ψ.*

Proof. — We remark that

$$L(f,s) = \prod_{l\nmid N}(1-a(l,f)l^{-s}+\varepsilon(l)l^{k-1-2s})^{-1} \times \sum_{(n,N)\neq 1} a(n,f)n^{-s},$$
$$L(g,s) = \prod_{l\nmid N}(1-a(l,f)l^{-s}+\varepsilon(l)l^{k-1-2s})^{-1}.$$

Thus, we have $\mathscr{L}_p(g,\zeta-1) = \frac{\Omega_f^+}{\Omega_g^+}\mathscr{L}_p^{\Sigma_0}(f,\zeta-1)$ for every $\zeta \neq 1$ and hence

$$\mathscr{L}_p(g,T) = \frac{\Omega_f^+}{\Omega_g^+}\mathscr{L}_p^{\Sigma_0}(f,T).$$

For any l with $l \nmid Np$,

$$\begin{aligned} a(l,g) = a(l,f) &= \mathrm{Tr}(\rho_f(\mathrm{Frob}_l)) \\ &\equiv \psi(\mathrm{Frob}_l) + \varphi(\mathrm{Frob}_l) \\ &\equiv \psi(\mathrm{Frob}_l) + \det(\rho_f)\psi^{-1}(\mathrm{Frob}_l) \\ &= \psi(\mathrm{Frob}_l) + \varepsilon\chi_p^{k-1}\psi^{-1}(\mathrm{Frob}_l) \\ &= \psi(l) + \varepsilon(l)l^{k-1}\psi^{-1}(l) \\ &= a(l,G) \pmod{\varpi}. \end{aligned}$$

Also, by (3.1), (3.2), and (Assumption), we obtain

$$a(p,g) = a(p,f) \equiv \psi(\mathrm{Frob}_p) \equiv a(p,G) \pmod{\varpi}.$$

Therefore we have

$$g \equiv G \pmod{\varpi}.$$

We have $a(0,G) = 0$ since ψ is non-trivial. Since $(p,M) = 1$, the assumption (2) of Theorem 1.9 follows immediately from the q-expansion principle. Next we will check the assumptions (3) and (4) of Theorem 1.9. We claim that, under the assumption $k < p$,

$$\tau(\bar\chi)D(G,\chi,1) = \sum_{i=1}^{\varphi(m)} \bar\chi(b_i)P_i \tag{3.5}$$

for some $P_i \in L_{k-2}(\mathcal{O})$ depending only on the parity $\chi(-1)$.

For the moment, we admit the claim (3.5). For $\delta \in \{0,1\}$, note that

$$\sum_{\substack{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \\ \chi(-1)=(-1)^\delta}} \chi(c) = \begin{cases} \varphi(m)/2 & \text{if } c \equiv 1 \pmod{m}, \\ (-1)^\delta \varphi(m)/2 & \text{if } c \equiv -1 \pmod{m}, \\ 0 & \text{otherwise.} \end{cases}$$

By (3.5), we have

$$\begin{aligned} \sum_{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times}} \chi(b_i)\tau(\bar{\chi})D(G,\chi,1) &= \sum_{\substack{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \\ \chi(-1)=1}} \chi(b_i)\tau(\bar{\chi})D(G,\chi,1) \\ &\quad + \sum_{\substack{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \\ \chi(-1)=-1}} \chi(b_i)\tau(\bar{\chi})D(G,\chi,1) \\ &= \varphi(m)P_i \in L_{k-2}(\mathcal{O}). \end{aligned}$$

On the other hand, by Lemma 1.8, we have

$$\sum_{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times}} \chi(b_i)\tau(\bar{\chi})D(G,\chi,1) = \varphi(m) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star D_{\begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}}(G,1).$$

Therefore, the assumptions (3) and (4) of Theorem 1.9 are satisfied.

Thus it remains to prove the claim (3.5). The coefficient of $X^{k-2-j}Y^j$ in $\tau(\bar{\chi})D(G,\chi,1)$ is equal to

$$\begin{aligned} &-\tau(\bar{\chi}) \binom{k-2}{j} j! \left(\frac{1}{2\pi\sqrt{-1}}\right)^{j+1} L(G,\chi,j+1) \\ &\qquad = -\binom{k-2}{j} \tau(\bar{\chi}) \frac{j! L_{\Sigma_0}(\psi\chi, j+1)}{(2\pi\sqrt{-1})^{j+1}} \cdot L_{\Sigma_0}(\varepsilon\chi\psi^{-1}, 1-(k-j-1)). \end{aligned}$$

Then the existence of such polynomials P_i in $L_{k-2}(F_p)$ follows from the functional equation for $L_{\Sigma_0}(\psi\chi, s)$ (see, for example, [28, Theorem 3.3.1, page 93] or [18, Theorem 2, page 47]) and the Siegel-Klingen theorem. We prove that P_i belongs to $L_{k-2}(\mathcal{O})$. In order to do it, we show that $\tau(\overline{\chi})D(G,\chi,1) \in L_{k-2}(\mathcal{O}[\mathfrak{X}_S])$. For any Dirichlet character χ and any positive integer n, we have

$$L(\chi, 1-n) = -\frac{1}{n}\sum_{a=1}^{m_\chi} \chi(a) m_\chi^{n-1} B_n\left(\frac{a}{m_\chi}\right).$$

Here recall that the n-th Bernoulli polynomial $B_n(X)$ is characterized by

$$B_n(X) = \sum_{j=0}^{n} \binom{n}{j} B_j X^{n-j},$$

where B_j is the j-th Bernoulli number. The von Staudt-Clausen theorem implies that, for a positive integer j,

$$B_j + \sum_{(l-1)|j} \frac{1}{l} \in \mathbb{Z}.$$

Here the sum runs over prime numbers l such that $l-1$ divides j. Hence, for each $1 \leq n < p-1$ and Dirichlet character χ with $(p, m_\chi) = 1$,

$$L(\chi, 1-n) \in \mathbb{Z}_p[\chi]. \tag{3.6}$$

Hence the integrality $\tau(\overline{\chi})D(G,\chi,1) \in L_{k-2}(\mathcal{O}[\mathfrak{X}_S])$ follows from (3.6) and the functional equation for $L_{\Sigma_0}(\psi\chi, s)$, where we use the assumption that the conductor of $\psi\chi$ is prime to p. Therefore, the integrality $P_i \in L_{k-2}(\mathcal{O})$ follows from $\varphi(m)P_i \in L_{k-2}(\mathcal{O}[\mathfrak{X}_S])$, which is obtained by the same argument mentioned after (3.5), and $(\varphi(m), p) = 1$.

Therefore, by applying the proof of Theorem 2.10 to the triple (g, G, ρ) instead of (f, G, χ), there exists a p-adic unit u' in $\mathcal{O}^\times$ such that $[\delta_g]^+ = u'[\pi_G]^+$ in $H^1_{\mathrm{par}}(\Gamma_1(M), L_{k-2}(\mathcal{O}/\varpi))$ (by the same argument mentioned at the beginning of the proof), and it gives the congruence for L-functions.

$$\tau(\bar{\rho})\frac{L(g,\rho,1)}{(2\pi\sqrt{-1})\Omega_g^+} \equiv u'\tau(\bar{\rho})\frac{L(G,\rho,1)}{2\pi\sqrt{-1}} \pmod{\varpi}$$

for every non-trivial, and non-exceptional p-adic character $\rho : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order whose conductor $m_\rho = p^{\nu_\rho}$. An explicit calculation with $(m_\rho, m_\psi) = 1$ shows that

$$\tau(\bar{\psi})\tau(\bar{\rho}) = \bar{\psi}(m_\rho)^{-1}\bar{\rho}(m_\psi)^{-1}\tau(\bar{\psi}\bar{\rho}).$$

We remark that $\bar{\psi}(m_\rho) \equiv \alpha(p,g)^{-\nu_\rho} \pmod{\varpi}$. Therefore we obtain

$$\begin{aligned}
\mathscr{L}_p(g, \zeta-1) &= \tau(\bar{\rho})\alpha(p,g)^{-\nu_\rho}\frac{L(g,\rho,1)}{(2\pi\sqrt{-1})\Omega_g^+} \\
&\equiv u'\tau(\bar{\rho})\alpha(p,g)^{-\nu_\rho}\frac{L(G,\rho,1)}{2\pi\sqrt{-1}} \\
&\equiv u'\tau(\bar{\psi})^{-1}\bar{\rho}(m_\psi)^{-1}\tau(\bar{\psi}\bar{\rho})\frac{L(G,\rho,1)}{2\pi\sqrt{-1}} \\
&\equiv u'\tau(\bar{\psi})^{-1}(1+T)^{-n_\psi}\mathscr{L}_p(G,\zeta-1) \pmod{\varpi},
\end{aligned}$$

for every $\zeta = \rho(\gamma) \neq 1$. This proves the theorem. □

Finally, we prove Theorem 0.2. By Theorem 3.19 and $\mu(\mathscr{L}_p(G,T)) = 0$, we obtain

$$\lambda^{\mathrm{anal}}_{f,\Sigma_0} = \lambda(\mathscr{L}_p(G,T)) = \lambda(\mathscr{L}_p^{\Sigma_0}(A_\varphi, T)) + \lambda(\mathscr{L}_p^{\Sigma_0}(A_\psi, T)).$$

By the definition,

$$\lambda(\mathscr{L}_p^{\Sigma_0}(\tilde{\varphi},T)) = \lambda(\mathscr{L}_p(A_\varphi,T)) + \sum_{l\in\Sigma_0} \lambda(1-\tilde{\varphi}(l)l^{-1}(1+T)^{f_l}),$$
$$\lambda(\mathscr{L}_p^{\Sigma_0}(\tilde{\psi},T)) = \lambda(\mathscr{L}_p(A_\psi,T)) + \sum_{l\in\Sigma_0} \lambda(1-\tilde{\psi}(l)l^{-1}(1+T)^{f_l}).$$

On the other hand, by Proposition 3.14 (3), we have

$$\lambda(S_{A_\varphi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = \lambda(S_{A_\varphi}(\mathbb{Q}_\infty)^\vee) + \sum_{l\in\Sigma_0} \lambda(\mathcal{H}_l(\mathbb{Q}_\infty, A_\varphi)),$$
$$\lambda(S_{A_\psi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = \lambda(S_{A_\psi}(\mathbb{Q}_\infty)^\vee) + \sum_{l\in\Sigma_0} \lambda(\mathcal{H}_l(\mathbb{Q}_\infty, A_\psi)).$$

Moreover, by Proposition 3.9, for $l \in \Sigma_0$,

$$\lambda(\mathcal{H}_l(\mathbb{Q}_\infty, A_\varphi)) = \lambda(1-\tilde{\varphi}(l)l^{-1}(1+T)^{f_l}),$$
$$\lambda(\mathcal{H}_l(\mathbb{Q}_\infty, A_\psi)) = \lambda(1-\tilde{\psi}(l)l^{-1}(1+T)^{f_l}).$$

Thus, by the Mazur-Wiles theorem, we get

$$\lambda(\mathscr{L}_p^{\Sigma_0}(A_\varphi,T)) = \lambda(S_{A_\varphi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee), \qquad \lambda(\mathscr{L}_p^{\Sigma_0}(A_\psi,T)) = \lambda(S_{A_\psi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee).$$

Combining these results with Theorem 3.19, we obtain

$$\lambda_{f,\Sigma_0}^{\mathrm{anal}} = \lambda(\mathscr{L}_p(G,T)) = \lambda(\mathscr{L}_p^{\Sigma_0}(A_\varphi,T)) + \lambda(\mathscr{L}_p^{\Sigma_0}(A_\psi,T)) = \lambda_{\varphi,\Sigma_0} + \lambda_{\psi,\Sigma_0}.$$

Thus, by (3.4), $\lambda_{f,\Sigma_0}^{\mathrm{alg}} = \lambda_{f,\Sigma_0}^{\mathrm{anal}}$, which by Theorem 3.17 implies that $\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}$. We have completed the proof of Theorem 0.2.

Appendix A. Comparison theorem for torsion cohomology in the $\mathbf{GL_2}(\mathbb{Q})$ case

In this section, we retain the notation as before. Let p be an odd prime number and $N \geq 4$ a positive integer with $(p,N) = 1$. Let $C = X_1(N)$ be the modular curve over $\mathbb{Z}[1/N]$ parametrizing generarized elliptic curves with $\Gamma_1(N)$-structure. The cuspidal subscheme $Z = Z_N$ is étale over $\mathbb{Z}[1/N]$ and set $C^\circ = C - Z$. We write $\pi\colon \mathcal{E} \to C$ for the universal generalized elliptic curve with $\Gamma_1(N)$-structure. The map π is smooth away from Z and the fibers of π over Z are the standard Neron N'-gon, where N' divides N. Let $f\colon X \to C$ be the k-fold fiber product of $\mathcal{E}$ over C. If $k \geq 2$, X is singular and proper. Let $\tilde{f}\colon \tilde{X} \to C$ denote the desingularization of X constructed by Deligne [7], and explained by Scholl [31], [29], Ulmer [37], and Subsection A.2 in this paper. Put $X^\circ = \tilde{f}^*(C^\circ)$. Then X° is smooth and not proper. Note that $f^\circ\colon X^\circ \to C^\circ$ is smooth.

Let $G_k = (\mathbb{Z}/N\mathbb{Z} \rtimes \mu_2)^k \rtimes \mathfrak{S}_k$ and $H_k = \mu_2^k \rtimes \mathfrak{S}_k$, where $\mathfrak{S}_k$ is the symmetric group and the action of $\mu_2 = \{\pm 1\}$ on $\mathbb{Z}/N\mathbb{Z}$ is by the multiplication and the action of $\mathfrak{S}_k$ is by permutation. Moreover, $\mathbb{Z}/N\mathbb{Z}$ acts on X° by translation by points of order N, μ_2 acts on X° by inversion in the fibers and $\mathfrak{S}_k$ acts on X° by permuting the factors of the fiber product. Then both G_k and H_k act on X°. This action extends to X and $\tilde{X}$ by definition. Let $\varepsilon\colon G_k \to \{\pm 1\}$ be the homomorphism which is trivial on each factor $\mathbb{Z}/N\mathbb{Z}$, the identity on each factor μ_2 and the sign character sgn^k on $\mathfrak{S}_k$. Let $\Pi := \frac{1}{|G_k|}\sum_{g\in G_k}\varepsilon(g)g^{-1} \in \mathbb{Z}[\frac{1}{2N\cdot k!}][G_k]$ be the projector attached to ε. Also we denote by $\varepsilon_k = \varepsilon|_{H_k}$ the restriction of the character ε to the subgroup H_k, and $\Pi_k := \frac{1}{|H_k|}\sum_{g\in H_k}\varepsilon_k(g)g^{-1} \in \mathbb{Z}[\frac{1}{2N\cdot k!}][H_k]$ the projector associated to ε_k. If $p \geq 3$, $k < p$, and p is prime to N, then $\Pi \in \mathcal{O}[G_k]$ and $\Pi_k \in \mathcal{O}[H_k]$. We denote by $V(\varepsilon)$ the ε-eigenspace for any $\mathbb{Z}[\frac{1}{2N\cdot k!}][G_k]$-module V, and $W(\varepsilon_k)$ the ε_k-eigenspace for any $\mathbb{Z}[\frac{1}{2N\cdot k!}][H_k]$-module W. Note that $V(\varepsilon) = \mathrm{im}[\Pi\colon V \to V]$ for any $\mathbb{Z}[\frac{1}{2N\cdot k!}][G_k]$-module V and $W(\varepsilon_k) = \mathrm{im}[\Pi_k\colon W \to W]$ for any $\mathbb{Z}[\frac{1}{2N\cdot k!}][H_k]$-module W.

A.1. The Hecke correspondence and the Atkin correspondence. — We define the Hecke correspondence T_l and the Atkin correspondence U_l on the curves $X_1(N)$ and $Y_1(N)$ over $\mathbb{Z}[1/N]$.

First, we assume that l is prime to N. Let $Y_1(N,l)$ be the fine moduli scheme over $\mathbb{Z}[1/N]$ which represents the functor of triples (E,P,C), where $E \to S$ is an elliptic curves over a $\mathbb{Z}[1/N]$-scheme S, P a point of exact order N on E, and C a finite locally free subgroup scheme of order l in $E[l]$. The morphism $p_1\colon Y_1(N,l) \to Y_1(N)$ defined by

$$p_1\colon (E,P,C) \longmapsto (E,P)$$

is finite flat. Since $(l,N)=1$, we can define a morphism $p_2\colon Y_1(N,l) \to Y_1(N)$ of schemes over $\mathbb{Z}[1/N]$ by

$$p_2\colon (E,P,C) \longmapsto (E/C, P\,(\mathrm{mod}\,C)).$$

We define a morphism $\psi\colon Y_1(N,l) \to Y_1(N,l)$ of schemes over $\mathbb{Z}[1/N]$ by

$$\psi\colon (E,P,C) \longmapsto (E/C, P\,(\mathrm{mod}\,C), E[l]/C).$$

Since $\psi^2(E,P,C) = (E,lP,C)$, ψ is an automorphism of $Y_1(N,l)$. Hence $p_2 = p_1 \circ \psi$ implies that p_2 is also finite flat.

Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^k \arrow[d] & p_1^*\mathcal{E}^k \arrow[l, "\phi_1^k"'] \arrow[r, "\psi^k"] \arrow[d] \arrow[l, phantom, shift right=8, "\square"] & p_2^*\mathcal{E}^k \arrow[r, "\phi_2^k"] \arrow[d] \arrow[l, phantom, shift right=8, "\circlearrowleft"] \arrow[r, phantom, shift right=8, "\square"] & \mathcal{E}^k \arrow[d] \\
Y_1(N) & Y_1(N,l) \arrow[l, "p_1"] \arrow[r, equal] & Y_1(N,l) \arrow[r, "p_2"'] & Y_1(N),
\end{tikzcd}$$

where the first and third squares are cartesian. Thus we define the Hecke correspondence T_l on X° by scheme-theoretic image of the morphism

$$(\phi_1^k, \phi_2^k \circ \psi^k)\colon p_1^*\mathcal{E}^k \to \mathcal{E}^k \times \mathcal{E}^k,$$

which induces an endomorphism of $H^*_?(X^\circ)$ for $? =$ ét or dR (see §A.2 and §A.3). We also define the Hecke correspondence T'_l on $Y_1(N)$ by

$$(p_1, p_2)\colon Y_1(N, l) \to Y_1(N) \times Y_1(N).$$

Then T'_l and ψ^k induce an endomorphism of $H^*_?(Y_1(N))$ for $? =$ ét or dR (see §A.2 and §A.3). If (E, P) is a $\overline{\mathbb{Q}}$-valued point of $Y_1(N)$, then

$$T'_l(E, P) = \sum_{\varphi} (\varphi E, \varphi P),$$

where the sum runs over the l-isogenies φ with source E.

Similarly, we define the Hecke correspondence T'_l on $X_1(N)$ and it induces an endomorphism of compact support cohomologies $H^*_{?,\mathrm{c}}(Y_1(N))$ for $? =$ ét or dR (see §A.2 and §A.3).

Next we assume that l divides N. Let $X_1(N, l)$ be the fine moduli scheme over $\mathbb{Z}[1/N]$ which represents the functor of triples (E, P, C), where $E \to S$ is a generalized elliptic curves over a $\mathbb{Z}[1/N]$-scheme S, P a point of exact order N on E, and C a finite locally free subgroup scheme of order l in $E[l]$ which is not contained in the subgroup generated by P. The morphism $p_1\colon X_1(N, l) \to X_1(N)$ defined by

$$p_1\colon (E, P, C) \longmapsto (E, P)$$

is finite. Since C is not contained in the subgroup generated by P, we can define a finite morphism $p_2\colon X_1(N, l) \to X_1(N)$ of schemes over $\mathbb{Z}[1/N]$ by

$$p_2\colon (E, P, C) \longmapsto (E/C, P \,(\mathrm{mod}\, C)).$$

Then, we define the Atkin correspondence U_l on X by scheme-theoretic image of the map

$$(\phi_1^k, \phi_2^k \circ \psi^k)\colon p_1^*\mathcal{E}^k \to \mathcal{E}^k \times \mathcal{E}^k,$$

which induces an endomorphism of $H^{k+1}_?(X^\circ)$, and U'_l on $X_1(N)$ by

$$(p_1, p_2)\colon Y_1(N, l) \to Y_1(N) \times Y_1(N),$$

which induces an endomorphism of $H^*_?(Y_1(N))$ for $? =$ ét or dR (see §A.2 and §A.3). If (E, P) is a $\overline{\mathbb{Q}}$-valued point of $Y_1(N)$, then

$$U'_l(E, P) = \sum_{\varphi} (\varphi E, \varphi P),$$

where the sum runs over the l-isogenies φ with source E such that $\ker(\varphi)$ is not contained in the subgroup generated by P.

Similarly, we define the Hecke correspondence U'_l on $X_1(N)$ and it induces an endomorphism of compact support cohomologies $H^*_{?,\mathrm{c}}(Y_1(N))$ for $? =$ ét or dR (see §A.2 and §A.3).

We now define the Hecke correspondence $\tilde{T}_l$ on $\tilde{X}$ as the closure of T_l in $\tilde{X} \times \tilde{X}$ and the Atkin correspondence $\tilde{U}_l$ on $\tilde{X}$ as the closure of U_l in $\tilde{X} \times \tilde{X}$. These induce an endomorphism of $H^*_?(\tilde{X})$ for $? =$ ét or dR (see §A.2 and §A.3).

A.2. Comparison with p-adic étale cohomology. — In this subsection, we assume that $2 \leq k < p$. Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$ and $\varpi \in \mathcal{O}$ a uniformizer. Let $A = \mathcal{O}$ or $\mathcal{O}/\varpi^n$. For any scheme T over $\mathcal{O}$, we denote by $T_{\overline{\mathbb{Q}}_p} = T \times_{\mathcal{O}} \overline{\mathbb{Q}}_p$ its base change to $\mathrm{Spec}(\overline{\mathbb{Q}}_p)$. The aim of this subsection is to prove the following proposition which gives an isomorphism between p-adic torsion étale cohomology for the modular curve with non-constant coefficients and for the Kuga-Sato variety with constant coefficients.

Proposition A.1. — *Assume that $k < p$. Then there exists the canonical exact sequence*

$$
\begin{aligned}
\text{(A.1)}\quad & 0 \to H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \to H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to H^0_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, A)(-k-1) \\
& \to H^{k+2}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \to H^{k+2}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to 0
\end{aligned}
$$

and canonical isomorphisms

$$\text{(A.2)}\qquad H^1_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^k R^1\pi_* A) \simeq H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k),$$

$$\text{(A.3)}\qquad H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^k R^1\pi_* A) \simeq H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon),$$

$$\text{(A.4)}\qquad H^n_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0 \ \textit{if}\ n \neq k+1, k+2 \ \textit{and}\ k > 0,$$

as Hecke modules endowed with a continuous $\mathbb{Q}_p$-linear action of $G_{\mathbb{Q}_p}$. Here the parabolic cohomology group in p-adic theories was defined by Deligne as

$$
\begin{aligned}
& H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^k R^1\pi_* A) \\
& \qquad = \mathrm{im}\left(H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}_p}, j_! \mathrm{Sym}^k R^1\pi_* A) \to H^1_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^k R^1\pi_* A) \right),
\end{aligned}
$$

where j denotes the open immersion $j : C^\circ \hookrightarrow C$.

In order to prove this proposition, we strictly follow the arguments in [29].

First we construct the isomorphism (A.2). There exists the Leray spectral sequence for $f^\circ : X^\circ \to C^\circ$:

$$E_2^{i,j} = H^i_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, R^j f^\circ_* A) \Rightarrow H^{i+j}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A).$$

Since C° is affine, we have $E_2^{i,j} = 0$ for $i \geq 2$. By using the Künneth formula, we have

$$R^j f^\circ_* A \simeq \bigoplus_{r_1 + \cdots + r_k = j} R^{r_1}\pi_* A \otimes \cdots \otimes R^{r_k}\pi_* A.$$

Note that $-1 \in \mu_2$ acts as $(-1)^r$ on $R^r\pi_* A$. Hence, if $k < p$, then we have

$$R^r f^\circ_* A(\varepsilon_k) = \begin{cases} \mathrm{Sym}^k R^1\pi_* A & \text{if } r = k, \\ 0 & \text{if } r \neq k. \end{cases}$$

Therefore we get

$$H^i_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^k R^1\pi_* A) \simeq H^{i+k}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \tag{A.5}$$

for $i = 0, 1$. This proves (A.2).

Secondly we prove (A.1). We begin by considering the cuspidal fibers of $\tilde{X}$.

We remark that, for any smooth scheme of finite type S over $\overline{\mathbb{Q}}_p$, the Künneth map induces an isomorphism

$$H^*_{\text{ét}}(\mathbb{G}^k_m \times_{\bar{\mathbb{Q}}_p} S, A)(k) \simeq H^*_{\text{ét}}(S, A) \otimes \bigwedge(At_1 + \cdots + At_k), \tag{A.6}$$

where $t_i = \mathrm{pr}_i^*(t) \in H^1(\mathbb{G}_m, A)(1)$ and $\bigwedge(At_1+\cdots+At_k)$ is the exterior algebra on $At_1 + \cdots + At_k$.

The symmetric group $\mathfrak{S}_k$ acts on $\mathbb{G}^k_m$ by permuting the coordinates and μ_2^k acts on $\mathbb{G}^k_m$ by $(x_i) \mapsto (x_i^{a_i})$ for any $(a_i)_i \in \mu_2^k$. Then the group H_k acts on $\mathbb{G}^k_m$.

Proposition A.2. — *Cup product with $t_1 \cup \cdots \cup t_k$ defines isomorphisms*

$$H^*_{\text{ét}}(S, A)(-k) \simeq H^{*+k}_{\text{ét}}(\mathbb{G}^k_m \times_{\bar{\mathbb{Q}}_p} S, A)(\varepsilon_k)$$

for any smooth scheme of finite type S over $\overline{\mathbb{Q}}_p$.

Proof. — We denote by Π_k the projector associated to ε_k. Then, by (A.6), it is enough to show that $\Pi_k(t_1 \cup \cdots \cup t_r) = 0$ for each $r < k$. Since μ_2 acts on k-th component of $\mathbb{G}^k_m$ by $x_k \mapsto x_k^{a_k}$, it acts trivially on $t_1 \cup \cdots \cup t_r$ and $\varepsilon_k|_{\mu_2}$ is non-trivial. Hence we obtain the assertion as required. $\square$

Let $P_k = \operatorname{Proj} \mathcal{O}[x_1, y_1, \ldots, x_k, y_k]/(x_1y_1 = x_2y_2 = \cdots = x_ky_k)$ be the closed subscheme of the projective space $\mathbb{P}^{2k-1}_{\mathcal{O}}$ over $\mathcal{O}$ defined by the equations $x_1y_1 = x_2y_2 = \cdots = x_ky_k$. Note that H_k acts on P_k. We define a subscheme P^{reg}_k as

$$P^{\text{reg}}_k = \{(x_i, y_i) \in P_k | \text{there are no two pairs } (x_i, y_i) \text{ simultaneously vanish}\}.$$

As in the proof of [31, Proposition 2.4.1] or [29, Proposition 7.2.3.1], we obtain the following result.

Proposition A.3. — *Assume that $k < p$. Then $H^*_{\text{ét}}(P^{\text{reg}}_{k,\overline{\mathbb{Q}}_p} \times_{\bar{\mathbb{Q}}_p} S, A)(\varepsilon_k) = 0$ for any smooth scheme of finite type S over $\overline{\mathbb{Q}}_p$.*

Let X^{reg} be the regular locus of X and $X^* = X^\circ \cup (Z \times \mathbb{G}^k_m)$ the open variety whose fiber over $x \in C$ is the connected component of the Néron model of $X^\circ \to C^\circ$.

Proposition A.4. — (1) $H^j_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^j_{\text{ét}}(X^{\text{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon)$.

(2) $H^j_{\text{ét}}(X^{\text{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^j_{\text{ét}}(X^*_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k)$.

Proof. — (1). We define $V = X \times_C Z = X - X^\circ$ and a filtration of V by closed subschemes

$$V = V_k \supset V_{k-1} \supset \cdots \supset V_0 \supset V_{-1} = \emptyset,$$

where V_i is the set of $(x_1, x_2, \ldots, x_k)$ such that at least $(k-i)$ of the components x_i are singular points of corresponding Néron polygon. We define a desingularization $\tilde{X} = X\langle k-1\rangle$ of X and a filtration on $\tilde{V} = \tilde{X} \times_C Z$. We put $X\langle 0\rangle = X$ and $P\langle 0\rangle = V_0$. We define inductively $X\langle j\rangle$ and $P\langle j\rangle$ as follows: Let $\phi_j \colon X\langle j\rangle \to X\langle j-1\rangle$ be the blowing-up with center $P\langle j-1\rangle$ and let $P\langle j\rangle \subset X\langle j\rangle$ be the strict transform of V_j. We write $\tilde{X} = X\langle k-1\rangle$ and

$$\psi_j \colon \tilde{X} = X\langle k-1\rangle \xrightarrow{\phi_{k-1}} X\langle k-2\rangle \xrightarrow{\phi_{k-2}} \cdots \xrightarrow{\phi_{j+1}} X\langle j\rangle$$

for the composition $\phi_{j+1} \circ \cdots \circ \phi_{k-2} \circ \phi_{k-1}$. We will show that

$$H^*(\tilde{V}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0.$$

We define a filtration on $\tilde{V}$ by

$$\tilde{V} \supset W_0 \supset W_1 \supset \cdots \supset W_{k-2} \supset W_{k-1} = \emptyset$$

given by

$$W_j = \psi_j^{-1}(X\langle j\rangle^{\mathrm{sing}}).$$

Here $X\langle j\rangle^{\mathrm{sing}}$ is the singular locus of $X\langle j\rangle$.

We claim that

$$H^*((W_j - W_{j+1})_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0$$

for all $0 \leq j \leq k-2$. The proof of this claim is same as [31, Theorem 3.1.0,(ii)] using Proposition A.3 instead of [31, Proposition 2.4.1]. Thus we have, for $j \geq 0$,

$$H^*_{\text{ét}}(W_{j,\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(W_{j+1,\overline{\mathbb{Q}}_p}, A)(\varepsilon).$$

Since $\tilde{V} - W_0 \simeq V^{\mathrm{reg}}$,

$$H^*_{\text{ét}}(\tilde{V}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(W_{0,\overline{\mathbb{Q}}_p}, A)(\varepsilon).$$

Therefore we get

$$H^*_{\text{ét}}(\tilde{V}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(W_{0,\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq \cdots \simeq H^*_{\text{ét}}(W_{k-1,\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0.$$

Since $\tilde{X} - \tilde{V} \simeq X^{\mathrm{reg}}$, we obtain

$$H^*_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(X^{\mathrm{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon),$$

as required.

(2). Fix a cusp $x \in Z$. Then we have, on $f^{-1}(x)$,

$$\begin{aligned} V_{k-1} - V_{k-2} &= \{(x_1, \dots, x_k) \in V \mid \text{there exists one pair such that } x_i \text{ is singular}\} \\ &= \bigcup_{\sigma \in G_k} \sigma\{(x_1, \dots, x_k) \in V \mid x_1 \text{ is singular, } x_i \text{ is non-singular for any } i \neq 1\} \\ &= \coprod_{\sigma \in G_k / \mu_2 \times (\mu_2^{k-1} \rtimes \mathfrak{S}_{k-1})} \sigma T. \end{aligned}$$

Here T is the component

$$T := \{(x_1, \dots, x_k) \in V \mid x_1 \text{ is singular and } x_i \text{ is non-singular for any } i \neq 1\}$$

and $\mu_2 \times (\mu_2^{k-1} \rtimes \mathfrak{S}_{k-1})$ is the stabilizer of T under G_k. Note that the first factor μ_2 acts on T trivially. Therefore

$$H^*_{\text{ét}}((V_{k-1} - V_{k-2})_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = \operatorname{Ind}^{G_k}_{\mu_2 \times (\mu_2^{k-1} \rtimes \mathfrak{S}_{k-1})} H^*_{\text{ét}}(T_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0$$

by Frobenius reciprocity. Then by using the Gysin sequence for $X - V_{k-1} \hookrightarrow X^{\text{reg}} = X - V_{k-2}$, we have

$$H^*_{\text{ét}}(X^{\text{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}((X - V_{k-1})_{\overline{\mathbb{Q}}_p}, A)(\varepsilon).$$

Note that

$$(X - V_{k-1}) \times Z \simeq (\mathbb{G}_m \times \mathbb{Z}/N\mathbb{Z})^k \times Z \text{ and } X^* - (X - V) = X^* \times Z = \mathbb{G}_m^k \times Z.$$

Then by using the Gysin sequence for $X^\circ = X - V \hookrightarrow X^{\text{reg}}$ and $X^\circ = X - V \hookrightarrow X^*$, we see that

$$H^*_{\text{ét}}(X^{\text{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(X^*_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k). \qquad \square$$

By the Gysin sequence for $X^\circ \hookrightarrow X^*$, we have the exact sequence

(A.7)
$$\begin{aligned} \cdots \to H^{j-2}_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p} \times \mathbb{G}_m^k, A)(-1)(\varepsilon_k) &\to H^j_{\text{ét}}(X^*_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to H^j_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \\ \to H^{j-1}_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p} \times \mathbb{G}_m^k, A)(-1)(\varepsilon_k) &\to H^{j+1}_{\text{ét}}(X^*_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to \cdots . \end{aligned}$$

Therefore (A.1) follows from (A.7), Proposition A.2, and Proposition A.4. Also (A.4) follows from (A.7), Proposition A.2, (A.5), and $H^0_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^k R^1\pi_* A) = 0$ if $(p, N) = 1$ and $k > 0$ (by (2.4)).

Thirdly, we construct the isomorphism (A.3). Let j (resp. $\tilde{j}$) denote the open immersion $C^\circ \hookrightarrow C$ (resp. $X^\circ \hookrightarrow \tilde{X}$). There exists the Leray spectral sequence

$$E_2^{a,b} = H^a_{\text{ét}}(C_{\overline{\mathbb{Q}}_p}, R^b \tilde{f}_* \tilde{j}_! A) \Rightarrow H^{a+b}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, \tilde{j}_! A).$$

As before in the proof of (A.5), by using the proper base change theorem and the Künneth formula, we see that

$$R^b\tilde{f}_*\tilde{j}_!A(\varepsilon) \simeq j_!R^bf^\circ_*A(\varepsilon) \simeq \begin{cases} j_! \operatorname{Sym}^k R^1\pi_*A & \text{if } b = k, \\ 0 & \text{if } b \neq k. \end{cases}$$

Therefore we have a commutative diagram

$$\text{(A.8)}\qquad \begin{tikzcd} H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}_p}, j_! \operatorname{Sym}^k R^1\pi_*A) \arrow[r, "\simeq"] \arrow[dd] & H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, \tilde{j}_!A)(\varepsilon) \arrow[d] \\ \arrow[r, phantom, "\circlearrowleft"] & H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \arrow[d] \\ H^1_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^k R^1\pi_*A) \arrow[r, "\simeq"] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \end{tikzcd}$$

from the functoriality of the Leray spectral sequence and (A.2). By Poincaré duality, we see that

$$H^{k+1}_{\text{ét,c}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \times H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k^{-1}) \xrightarrow{\text{trace}} \mathbb{Q}/\mathbb{Z},$$
$$H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \times H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon^{-1}) \xrightarrow{\text{trace}} \mathbb{Q}/\mathbb{Z}$$

are perfect pairings. Note that $\varepsilon = \varepsilon^{-1}$ and

$$H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \to H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k)$$

is an injection by (A.1). Thus we see that

$$H^{k+1}_{\text{ét,c}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon)$$

is a surjection by duality. Therefore we have

$$H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^k R^1\pi_*A) \simeq H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon).$$

This proves (A.3).

We prove that the isomorphisms (A.2) and (A.3) are compatible with the Hecke operator and the Atkin operator. From the Leray spectral sequence and its functoriality for $A \to p_{1*}p_1{}^*A$, $A \to \psi_*\psi^*A$, and the trace map $\mathrm{tr}_{p_2} : p_{2*}p_2{}^*A \to A$, the diagram

$$\begin{tikzcd} \mathcal{E}^k \arrow[d] \arrow[dr, phantom, "\square"] & p_1^*\mathcal{E}^k \arrow[l, "\phi_1^k"'] \arrow[r, "\psi^k"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & p_2^*\mathcal{E}^k \arrow[r, "\phi_2^k"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}^k \arrow[d] \\ Y_1(N) & Y_1(N,l) \arrow[l, "p_1"] \arrow[r, equal] & Y_1(N,l) \arrow[r, "p_2"'] & Y_1(N) \end{tikzcd}$$

implies that the isomorphism (A.2) is an isomorphism of Hecke modules as desired.

In order to prove that the isomorphism (A.3) is compatible with the Hecke operator and the Atkin operator, from the diagram (A.8) it suffices to show that the commutative diagram

$$\begin{tikzcd}
\tilde{T}_l \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{X}\times\tilde{X} \arrow[r, "\mathrm{pr}_i"] \arrow[dr, phantom, "\square"] & \tilde{X} \\
T_l \arrow[u] \arrow[r] & X^\circ\times X^\circ \arrow[u, "\tilde{j}\times\tilde{j}"'] \arrow[r, "\mathrm{pr}_i"'] & X^\circ \arrow[u, "\tilde{j}"']
\end{tikzcd}$$

induces the following commutative diagram

$$\begin{tikzcd}
H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A) \arrow[r] \arrow[d, "\mathrm{pr}_1^*"] \arrow[dr, phantom, "\circlearrowleft"] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A) \arrow[d, "\mathrm{pr}_1^*"] \\
H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}\times\tilde{X}_{\overline{\mathbb{Q}}_p}, A) \arrow[r] \arrow[d, "\cup\mathrm{cl}(\tilde{T}_l)"] \arrow[dr, phantom, "\circlearrowleft"] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}\times X^\circ_{\overline{\mathbb{Q}}_p}, A) \arrow[d, "\cup\mathrm{cl}(T_l)"] \\
H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}\times\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(k+1) \arrow[r] \arrow[d, "\mathrm{pr}_{2*}"] \arrow[dr, phantom, "\circlearrowleft"] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}\times X^\circ_{\overline{\mathbb{Q}}_p}, A)(k+1) \arrow[d, "\mathrm{pr}_{2*}"] \\
H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A) \arrow[r] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A).
\end{tikzcd}$$

Here cl is the cycle map. The first square is compatible by the smooth base change theorem and the second square is compatible by the semi-purity theorem. The compatibility of third square follows from the fact that trace maps are compatible with base change. This completes the proof of Proposition A.1.

A.3. Comparison with algebraic de Rham cohomology. — The aim of this subsection is to prove Proposition A.8, which gives an isomorphism between mod p de Rham cohomology for the modular curve with non-constant coefficients and for the Kuga-Sato variety with constant coefficients. In order to do it, we use the terminology of logarithmic structures in Kato [19].

Let $\mathcal{Y}$ be a regular scheme and $\mathcal{D}$ a reduced divisor with normal crossings on $\mathcal{Y}$. Then the subsheaf L of monoids on $\mathcal{Y}_{\text{ét}}$ defind by

$$L(\mathcal{U})=\{g\in\mathcal{O}_{\mathcal{Y}}(\mathcal{U})|g \text{ is invertible outside } \mathcal{D}\times_{\mathcal{Y}}\mathcal{U}\} \tag{A.9}$$

for each étale $\mathcal{Y}$-scheme $\mathcal{U}$ is a fine log structure ([19, (2.5)]).

We fix an algebra $A_0 = \mathbb{Z}[1/N]$. We define a log scheme $C^\times$ over A_0 to be the scheme C over A_0 endowed with the log structure $L = \{g \in \mathcal{O}_C | g$ is invertible outside $Z\}$, and $\mathcal{E}^\times$ the scheme $\mathcal{E}$ over A_0 endowed with the log structure $M = \{g \in \mathcal{O}_{\mathcal{E}} | g$ is invertible outside $\pi^{-1}(Z)\}$. Then the morphism of log schemes $\mathcal{E}^\times \to C^\times$ over A_0 is log smooth ([19, Theorem 3.5]) and hence the $\mathcal{O}_C$-module $\Omega^i_{\mathcal{E}^\times/C^\times} = \Omega^i_{\mathcal{E}/C}(\log(M/L))$ is locally free of finite type ([19, Theorem 3.10]).

For any A_0-algebra A and A_0-scheme $\mathcal{Y}$, we denote by $\mathcal{Y}_A$ its base change to $\mathrm{Spec}(A)$. Moreover, for any A_0-algebra A and A_0-log scheme $\mathcal{Y}^\times = (\mathcal{Y}, L)$, we denote by $\mathcal{Y}^\times_A = (\mathcal{Y}, L)_A$ its base change to $(\mathrm{Spec}(A), \mathrm{triv})$ with the trivial log structure.

In this subsection, let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$ and κ the residue field of $\mathcal{O}$.

We define the de Rham cohomology sheaf on C_κ by

$$\mathcal{L}_\kappa = R^1\pi_*\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa}.$$

We have the invertible sheaf

$$\boldsymbol{\omega}_\kappa = \pi_*\Omega^1_{\mathcal{E}^\times_\kappa/C^\times_\kappa}$$

([8, II.1.6], [24, §10.13]). The exact sequence

$$0 \to \Omega^1_{\mathcal{E}^\times_\kappa/C^\times_\kappa}[-1] \to \Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa} \to \mathcal{O}_{\mathcal{E}^\times_\kappa} \to 0$$

induces an exact sequence

$$0 \to \boldsymbol{\omega}_\kappa \to \mathcal{L}_\kappa \to \boldsymbol{\omega}_\kappa^{-1} \to 0 \tag{A.10}$$

(cf. [23, A1.2.1, page 163]). This sequence (A.10) defines the Hodge filtration

$$\mathcal{L}_\kappa = F^0(\mathcal{L}_\kappa) \supset F^1(\mathcal{L}_\kappa) = \boldsymbol{\omega}_\kappa \supset F^2(\mathcal{L}_\kappa) = 0.$$

We have the canonical integrable Gauss-Manin connection

$$\nabla_\kappa \colon \mathcal{L}_\kappa \to \mathcal{L}_\kappa \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}.$$

For a non-negative integer k, we denote by $\mathcal{L}_{\kappa,k}$ the k-th symmetric tensor $\mathrm{Sym}^k\,\mathcal{L}_\kappa$ of $\mathcal{L}_\kappa$ and by $\nabla_{\kappa,k} \colon \mathcal{L}_{\kappa,k} \to \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}$ the k-th symmetric power of ∇_κ. Explicitly, it is given by

$$\nabla_{\kappa,k}(x_1, \ldots, x_k) = \sum_{r=1}^{k} x_1 \cdots x_{r-1}x_{r+1} \cdots x_k \nabla_\kappa(x_r). \tag{A.11}$$

We define a complex of sheaves $\Omega^\bullet(\mathcal{L}_{\kappa,k})$ by

$$\Omega^0(\mathcal{L}_{\kappa,k}) = \mathcal{L}_{\kappa,k},\ \Omega^1(\mathcal{L}_{\kappa,k}) = \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}$$

and the cohomology $H^m(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$ by

$$H^m(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) = H^m(C_\kappa, \Omega^\bullet(\mathcal{L}_{\kappa,k})).$$

Let R_κ denote the canonical residue map in the sense of Deligne, which gives an exact sequence

$$0 \to \Omega^1_{C_\kappa/\kappa} \to \Omega^1_{C_\kappa^\times/\kappa} \xrightarrow{\mathrm{R}_\kappa} \mathcal{O}_{Z_\kappa} \to 0.$$

Since $\mathrm{R}_\kappa(\nabla_{\kappa,k}(ax)) = \mathrm{R}_\kappa(a\nabla_{\kappa,k}(x)) + \mathrm{R}_\kappa(x \otimes \mathrm{d}a) = \mathrm{R}_\kappa(a\nabla_{\kappa,k}(x))$ on $\mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}$ for any $a \in \mathcal{O}_{C_\kappa}$ and $x \in \mathcal{L}_{\kappa,k}$, the morphism R_κ induces an $\mathcal{O}_{C_\kappa}$-linear morphism

$$\mathcal{L}_{\kappa,k} \xrightarrow{\nabla_{\kappa,k}} \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \xrightarrow{\mathrm{R}_\kappa} \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}.$$

We define a complex of sheaves $\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})$ by

$$\Omega^0_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) = \mathcal{L}_{\kappa,k},\ \Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) = \nabla_{\kappa,k}(\mathcal{L}_{\kappa,k}) + \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}$$

and the parabolic cohomology $H^m_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$ in the sense of Scholl [30] by

$$H^m_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) = H^m(C_\kappa, \Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})).$$

Proposition A.5. — *Assume that $k < p$. Then, the morphism R_κ induces an exact sequence*

$$0 \to H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \to H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \xrightarrow{\mathrm{R}_\kappa} H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes \mathcal{O}_{Z_\kappa}).$$

Proof. — Fix a cusp s. The level structure on $\mathrm{Tate}_{N'}(q) = \mathbb{G}_m/(q^{1/N'})^{\mathbb{Z}}$ defines a morphism

$$\psi\colon \mathrm{Spec} A_0[[q^{1/N'}]] \to C$$

identifying $A_{N'} = A_0[[q^{1/N'}]]$ with the formal completion of C along the cusp s, where $N'|N$. Then $\psi^*(\boldsymbol{\omega}_{A_0})$ has the nowhere vanishing section dt/t on the formal completion of C along the cusp s, where t is the parameter on $\mathbb{G}_m$ (cf. [8, VII,1.16.2], [23, A1.3.18]). Let ω be the canonical generator. Since $(p, N) = 1$, ∇_{A_0} induces

$$\nabla_{A_0} : \psi^*\mathcal{L}_{A_0} \to \psi^*\mathcal{L}_{A_0} \cdot d\log(q^{1/N'}) = \psi^*\mathcal{L}_{A_0} \cdot \frac{dq}{q},$$

and we have

$$\psi^*\mathcal{L}_{A_0} = A_{N'} \cdot \omega \oplus A_{N'} \cdot \xi,$$

where $\nabla_{A_0}(\omega) = \xi \cdot \frac{dq}{q}$ and $\nabla_{A_0}(\xi) = 0$ (cf. [23, A.1.3]). Then we get

$$\psi^*\mathcal{L}_{A_0,k} = \bigoplus_{r=0}^{k} A_{N'} \cdot \omega^{k-r}\xi^r \tag{A.12}$$

and, by (A.11),

$$\nabla_{A_0,k}(\omega^{k-r}\xi^r) = \sum_{i=1}^{k-r}\omega^{k-r-1}\xi^r\nabla_{A_0}(\omega) + \sum_{j=k-r+1}^{k}\omega^{k-r}\xi^{r-1}\nabla_{A_0}(\xi)$$
$$= \begin{cases} (k-r)\omega^{k-r-1}\xi^{r+1}\frac{dq}{q} & \text{if } r \neq k, \\ 0 & \text{if } r = k. \end{cases}$$

Since $k < p$, we obtain the exact sequence

$$0 \to \Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) \to \Omega^\bullet(\mathcal{L}_{\kappa,k}) \xrightarrow{\mathrm{R}_\kappa} \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}[-1] \to 0. \tag{A.13}$$

This proves the theorem. □

We denote by $X^\times$ the k-fold fiber product of $\mathcal{E}^\times$ over $C^\times$.

Proposition A.6. — *Assume that $k < p$. Then there exists a canonical isomorphism*

$$H^m(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \simeq H^{m+k}(X_\kappa, \Omega^\bullet_{X_\kappa^\times/\kappa})(\varepsilon) \textit{ for all } m.$$

Proof. — Similarly as in the proof of (A.2), by using the Künneth formula, we see that

$$R^j f_*\Omega^\bullet_{X_\kappa^\times/C_\kappa^\times}(\varepsilon) \simeq \begin{cases} \mathcal{L}_{\kappa,k} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Thus, the Leray spectral sequence ([22, Remark 3.3]) implies the assertion as required. □

We define $\tilde{X}^\times$ to be the scheme $\tilde{X}$ endowed with the log structure defined by the subsheaf of functions invertible on the cuspidal fibers as (A.9).

Proposition A.7. — *The morphism $g\colon \tilde{X}^\times \to X^\times$ induces isomorphisms*

$$Rg_*\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa} \simeq \Omega^\bullet_{X_\kappa^\times/\kappa} \quad \textit{and} \quad Rg_*\Omega^\bullet_{\tilde{X}_\kappa^\times/C_\kappa^\times} \simeq \Omega^\bullet_{X_\kappa^\times/C_\kappa^\times}.$$

Proof. — In virtue of [20, Theorem 11.3], $Rg_*\mathcal{O}_{\tilde{X}_\kappa} \simeq \mathcal{O}_{X_\kappa}$. Since g is log étale, applying [19, (3.12)] to $\tilde{X}_\kappa^\times \xrightarrow{g} X_\kappa^\times \to (\kappa, \mathrm{triv})$ (resp. $\tilde{X}_\kappa^\times \xrightarrow{g} X_\kappa^\times \to C_\kappa^\times$), we obtain

$$g^*\Omega^i_{X_\kappa^\times/\kappa} = \Omega^i_{\tilde{X}_\kappa^\times/\kappa} \text{ (resp. } g^*\Omega^i_{X_\kappa^\times/C_\kappa^\times} = \Omega^i_{\tilde{X}_\kappa^\times/C_\kappa^\times})$$

for all i. Hence the projection formula implies that

$$Rg_*\Omega^j_{\tilde{X}_\kappa^\times/\kappa} \simeq \Omega^j_{X_\kappa^\times/\kappa} \text{ and } Rg_*\Omega^j_{\tilde{X}_\kappa^\times/C_\kappa^\times} \simeq \Omega^j_{X_\kappa^\times/C_\kappa^\times}.$$ □

PROPOSITION A.8. — *Assume that $k < p$. Then there exist canonical isomorphisms*

$$H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \simeq H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon), \tag{A.14}$$

$$H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \simeq H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon), \tag{A.15}$$

as filtered Hecke modules. Here the filtration on $H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$ (resp. $H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$) is induced by the filtration (A.21) *(resp.* (A.23)*) and the filtration on $H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon)$ and $H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon)$ are defined by the Hodge filtration.*

Proof. — First, the isomorphism (A.14) is obtained by Proposition A.6 and Proposition A.7.

Secondly, we construct the isomorphism (A.15) by using the Leray spectral sequence [22]. In order to do it, we make a general observation on logarithmic differential. Let $\mathcal{Y}$ be a regular scheme and suppose that $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}+\mathcal{D}'$ are reduced divisors with normal crossings on $\mathcal{Y}$. Let M be the log structure associated to $\mathcal{D}$ as (A.9). Étale locally on $\mathcal{Y}$, we can write $\mathcal{D} = \sum_{i=1}^r \mathcal{C}_i$, where $\mathcal{C}_i$ is a regular closed subsheme of $\mathcal{Y}$ defined by $\pi_i = 0$ for a non-zero divisor $\pi_i \in \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ and M is isomorphic to the log structure associated to $(\mathbb{N}^r)_\mathcal{Y} \to \mathcal{O}_\mathcal{Y} : (n_i) \mapsto \Pi \pi_i^{n_i}$. The residue map Res from $\Omega^\bullet_\mathcal{Y}(\log(\mathcal{D}+\mathcal{D}'))$ to $\Omega^{\bullet-1}_{\mathcal{C}_i}(\log(\mathcal{C}_i \cap \mathcal{D}'))$ is defined by the formula

$$\mathrm{Res}(d\log(\pi_i) \wedge \omega) = \omega|_{\mathcal{C}_i}.$$

Summing over all components, we get the morphism

$$\mathrm{Res}\colon \Omega^\bullet_\mathcal{Y}(\log(\mathcal{D}+\mathcal{D}')) \to \alpha_*(\Omega^{\bullet-1}_{\tilde{\mathcal{D}}}(\log(\alpha^*\mathcal{D} \cap \mathcal{D}')))$$

for the normalization $\alpha : \tilde{\mathcal{D}} \to \mathcal{D}$ of $\mathcal{D}$.

We define D_j as the strict transform of the exceptional divisor $\phi_{j+1}^{-1}(P\langle j\rangle)$ in $\tilde{X}_\kappa$ for $j = 0, 1, \ldots, k-2$ and D_k as the strict transform in $\tilde{X}_\kappa$ of the cuspidal fibers of $f\colon X_\kappa \to C_\kappa$ over the cusps. We write $\tilde{D}_j$ for the normalization of D_j for all j. Put $D = D_0 + \cdots + D_{k-2} + D_k$ and $E_j = D_0 + \cdots + D_j$ for $0 \le j \le k-2$. Let $\tilde{D}_k^\times$ be the scheme $\tilde{D}_k$ endowed with the log structure associated to the normal crossing divisor $\sum_{j=0}^{k-2}(\tilde{D}_k \cap D_j)$.

Filtrations on $\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}$ and $\Omega^\bullet_{\tilde{D}_k^\times/\kappa}$ are defined by

$$\begin{aligned} &\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa} = F^0(\Omega^\bullet_{X_\kappa^\times/\kappa}) \supset F^1(\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}) \supset F^2(\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}) = 0 \text{ and} \\ &\Omega^\bullet_{\tilde{D}_k^\times/\kappa} = F^0(\Omega^\bullet_{\tilde{D}_k^\times/\kappa}) = F^1(\Omega^\bullet_{\tilde{D}_k^\times/\kappa}) \supset F^2(\Omega^\bullet_{\tilde{D}_k^\times/\kappa}) = 0, \end{aligned} \tag{A.16}$$

respectively, where

$$F^1(\Omega^q_{\tilde{X}_\kappa^\times/\kappa}) = \mathrm{im}\left[\Omega^{q-1}_{\tilde{X}_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \to \Omega^q_{\tilde{X}_\kappa^\times/\kappa}\right]$$

(see, for example, [22, (3.2)]). The residue map $\mathrm{Res}\colon \Omega^{\bullet}_{\tilde{X}^{\times}_{\kappa}/\kappa} \to \alpha_*\Omega^{\bullet-1}_{\tilde{D}^{\times}_{k}/\kappa}$ for the canonical morphism $\alpha : \tilde{D}^{\times}_{k} \to \tilde{X}^{\times}_{\kappa}$ and the exact sequence $0 \to \mathrm{Gr}^1_F \to F^0 \to \mathrm{Gr}^0_F \to 0$ induce a commutative diagram

$$\begin{tikzcd}
\mathcal{L}_{\kappa,k} \arrow[r, "\nabla_{\kappa,k}"] \arrow[d, "\mathrm{Res}"'] & \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^{\times}_{\kappa}/\kappa} \arrow[d, "\mathrm{Res}"] \\
0 \arrow[r, "d"] & R^k\tilde{f}_*\Omega^{\bullet}_{\tilde{D}^{\times}_{k}/\kappa}(\varepsilon)
\end{tikzcd} \tag{A.17}$$

obtained by the projection formula and cutting out by ε. Here we note that $R^k\tilde{f}_*\Omega^{\bullet}_{\tilde{X}^{\times}_{\kappa}/C^{\times}_{\kappa}} \simeq R^k f_*\Omega^{\bullet}_{X^{\times}_{\kappa}/C^{\times}_{\kappa}}$ by Proposition A.7 and hence $R^k\tilde{f}_*\Omega^{\bullet}_{\tilde{X}^{\times}_{\kappa}/C^{\times}_{\kappa}}(\varepsilon) \simeq R^k f_*\Omega^{\bullet}_{X^{\times}_{\kappa}/C^{\times}_{\kappa}}(\varepsilon) \simeq \mathcal{L}_{\kappa,k}$ by the proof of Proposition A.6. Thus the Leray spectral sequence [22, Remark 3.3] and its functoriality induce a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\ker(\mathrm{Res}) \arrow[r, "\simeq"] \arrow[d] & H^{k+1}(\tilde{X}_\kappa, \Omega^{\bullet}_{\tilde{X}_\kappa/\kappa})(\varepsilon) \arrow[d] \\
H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Res}"'] & H^{k+1}(\tilde{X}_\kappa, \Omega^{\bullet}_{\tilde{X}^{\times}_{\kappa}/\kappa})(\varepsilon) \arrow[d, "\mathrm{Res}"] \\
H^1(C_\kappa, 0 \xrightarrow{d} R^k\tilde{f}_*\Omega^{\bullet}_{\tilde{D}^{\times}_{k}/\kappa}(\varepsilon)) \arrow[r, "\simeq"] & H^k(\tilde{D}_k, \Omega^{\bullet}_{\tilde{D}^{\times}_{k}/\kappa})(\varepsilon).
\end{tikzcd}$$

Here, using

$$\begin{aligned}
H^{k+1}(\tilde{X}_\kappa, \Omega^{\bullet}_{\tilde{X}_\kappa/\kappa}(\log(E_{k-2})))(\varepsilon) &\simeq H^{k+1}(\tilde{X}_\kappa, \Omega^{\bullet}_{\tilde{X}_\kappa/\kappa}(\log(E_{k-3})))(\varepsilon) \\
&\simeq \cdots \simeq H^{k+1}(\tilde{X}_\kappa, \Omega^{\bullet}_{\tilde{X}_\kappa/\kappa})(\varepsilon),
\end{aligned}$$

obtained by an inductive argument with the help of the vanishing results [37, p.146] and $R^i\tilde{f}_*\Omega^{\bullet}_{\tilde{D}^{\times}_{k}/\kappa}(\varepsilon) = 0$ if $i \neq k$ by [37, p.145], we see that the second arrow in the right vertical sequence is an injection and the bottom horizontal morphism is an isomorphism. Since the image of $\Omega^{\bullet}_{\tilde{X}^{\times}_{\kappa}/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \tilde{f}^*\Omega^1_{C_\kappa/\kappa}[-1]$ under $\mathrm{Gr}^1_F \mathrm{Res} : \mathrm{Gr}^1_F \Omega^{\bullet}_{\tilde{X}^{\times}_{\kappa}/\kappa} = \Omega^{\bullet}_{\tilde{X}^{\times}_{\kappa}/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \tilde{f}^*\Omega^1_{C^{\times}_{\kappa}/\kappa}[-1] \to \mathrm{Gr}^1_F \Omega^{\bullet}_{\tilde{D}^{\times}_{k}/\kappa}[-1]$ is equal to 0, we have $\mathrm{Res}(\mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}) = 0$. Combining with (A.17), we get $\mathrm{Res}(\Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) = 0$. Thus, by the exact sequence (A.13), the map Res factors through $\Omega^1(\mathcal{L}_{\kappa,k})/\Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) \simeq \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}$. Then Res induces $\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa} \to R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon)$ and we have a commutative diagram

$$\text{(A.18)}\qquad \begin{tikzcd} \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \arrow[r, equal] \arrow[d, "\mathrm{R}_\kappa"] & \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \arrow[d, "\mathrm{Res}"] \\ \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa} \arrow[r] & R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon). \end{tikzcd}$$

Therefore, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r] \arrow[d] & \ker(\mathrm{Res}) \arrow[d] \\ H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r, equal] \arrow[d, "\mathrm{R}_\kappa"'] & H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[d, "\mathrm{Res}"] \\ H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}) \arrow[r] & H^1(C_\kappa, 0 \xrightarrow{d} R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon)). \end{tikzcd}$$

Here the left vertical sequence is exact by Proposition A.5.

We claim that $\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa} \to R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon)$ is an injective morphism. Recall that from (A.12),

$$\psi^*\mathcal{L}_{\kappa,k} = \bigoplus_{r=0}^{k} \kappa_{_{N'}}\omega^{k-r}\xi^r,$$

where $\kappa_{_{N'}} = A_{N'} \otimes_{A_0} \kappa$. Since the claim is local on Z_κ, it is enough to show that $\psi^*(\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}) \to \psi^*(R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon))$ is injective. Hereafter we drop the notation ψ^* and write $\mathcal{L}_{\kappa,k}$ for $\psi^*\mathcal{L}_{\kappa,k}$ and so on. For any $a = \sum_r b_r\omega^{k-r}\xi^r \in \mathcal{L}_{\kappa,k}$, $\mathrm{R}_\kappa(a d\log(q)) = \overline{b_0}\omega^k \otimes 1$, where $b_r \in \kappa_{_{N'}}$ and $\overline{b_0} \in \kappa_{_{N'}}/(q^{1/N'})$. Therefore, by (A.18), it suffices to show that $\mathrm{Res}(\omega^k \otimes d\log(q))$ is non-zero. Recall that Res is the composition of the following morphisms (A.19) and (A.20):

$$\text{(A.19)}\qquad \begin{aligned} \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa} \xrightarrow{\text{Künneth}} & R^k\tilde{f}_*\Omega^\bullet_{\tilde{X}_\kappa^\times/C_\kappa^\times} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \\ & \simeq R^k\tilde{f}_*\left(\Omega^\bullet_{\tilde{X}_\kappa^\times/C_\kappa^\times} \otimes_{\tilde{f}^*\mathcal{O}_{C_\kappa}} \tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa}\right); \end{aligned}$$

$$\text{(A.20)}\qquad R^k\tilde{f}_*\left(\Omega^\bullet_{\tilde{X}_\kappa^\times/C_\kappa^\times} \otimes_{\tilde{f}^*\mathcal{O}_{C_\kappa}} \tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa}\right) \xrightarrow{\mathrm{Res}} R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}.$$

Here the morphism (A.20) is induced by Res : $\Omega^\bullet_{\tilde{X}^\times_\kappa/C^\times_\kappa} \otimes_{\tilde{f}^*\mathcal{O}_{C_\kappa}} \tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \to \Omega^\bullet_{\tilde{D}^\times_k/\kappa}$.

We shall compute the image of $\omega^k \otimes d\log(q)$ under (A.19). We denote p_i by the i-th projection $X^\times \to \mathcal{E}^\times$ and $\tilde{p}_i : \tilde{X}^\times \to \mathcal{E}^\times$ by $p_i \circ g$ for any i. Note that the image of $\omega^k \otimes d\log(q)$ under the map $\mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa} \to (R^1\pi_*\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa})^{\otimes k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}$ is $\omega^{\otimes k} \otimes d\log(q)$ and the image of this element under the map $(R^1\pi_*\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa})^{\otimes k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa} \xrightarrow{\text{Künneth}} R^k f_*\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}$ is $p_1^*(\omega) \cup \cdots \cup p_k^*(\omega) \otimes d\log(q)$. The commutative diagram

$$\begin{array}{ccc} p_i^{-1}\Omega^1_{\mathcal{E}^\times_\kappa/C^\times_\kappa}[-1] & \longrightarrow & \sigma_{\geq 1}\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa} \\ \downarrow & & \downarrow \\ p_i^{-1}\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa} & \longrightarrow & \Omega^\bullet_{X^\times_\kappa/C^\times_\kappa} \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} \pi_*\Omega^1_{\mathcal{E}^\times_\kappa/C^\times_\kappa} & \xrightarrow{h_i} & R^1 f_*(\sigma_{\geq 1}\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}) \\ \downarrow & & \downarrow \\ R^1\pi_*\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa} & \xrightarrow{p_i^*} & R^1 f_*\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}. \end{array}$$

Here we denote by h_i the upper horizontal morphism. We have $h_i(\omega) \in R^1 f_*(\sigma_{\geq 1}\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}) = \ker[f_*\Omega^1_{X^\times_\kappa/C^\times_\kappa} \to f_*\Omega^2_{X^\times_\kappa/C^\times_\kappa}]$. Similarly, we have $g^*h_i(\omega) \in \ker[\tilde{f}_*\Omega^1_{\tilde{X}^\times_\kappa/C^\times_\kappa} \to \tilde{f}_*\Omega^2_{\tilde{X}^\times_\kappa/C^\times_\kappa}]$. By putting $\tilde{h}_i(\omega) = g^*h_i(\omega)$, we have $\tilde{h}_1(\omega) \wedge \cdots \wedge \tilde{h}_k(\omega) \in \tilde{f}_*\Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa} = R^k\tilde{f}_*(\sigma_{\geq k}\Omega^\bullet_{\tilde{X}^\times_\kappa/C^\times_\kappa})$. The image of $\omega^k \otimes d\log(q)$ under (A.19) is $\tilde{p}_1^*(\omega) \wedge \cdots \wedge \tilde{p}_k^*(\omega) \otimes d\log(q)$. Thus our claim follows from $\tilde{f}_*\Omega^k_{\tilde{D}^\times_k/\kappa}(\varepsilon) \simeq R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}^\times_k/\kappa}(\varepsilon)$ obtained by [37, p.145] and $\tilde{h}_1(\omega) \wedge \cdots \wedge \tilde{h}_k(\omega) = \mathrm{d}t_1/t_1 \wedge \cdots \wedge \mathrm{d}t_k/t_k \neq 0$ on the smooth locus for the parameter t_i on $\mathbb{G}_m$.

Next, we prove that the isomorphisms (A.14) and (A.15) are filtered isomorphisms.

Case (A.14). — We construct the filtration of $H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$. The Hodge filtration (A.10) on $\mathcal{L}_\kappa$ defines a decreasing filtration

$$F^r(\mathcal{L}_\kappa^{\otimes k}) = \sum_{\sigma \in \mathfrak{S}_k} \sigma \cdot [F^1(\mathcal{L}_\kappa)^{\otimes r} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes (k-r)}]$$

on $\mathcal{L}_\kappa^{\otimes k}$ and

$$F^r(\mathcal{L}_{\kappa,k}) = \mathrm{im}(F^r(\mathcal{L}_\kappa^{\otimes k}) \xrightarrow{\mathrm{pr}} \mathcal{L}_{\kappa,k})$$

on $\mathcal{L}_{\kappa,k}$, where $\mathrm{pr}\colon \mathcal{L}_\kappa^{\otimes k} \to \mathcal{L}_{\kappa,k}$ is the canonical projection map. We can define a filtration on the complex $\Omega^\bullet(\mathcal{L}_{\kappa,k})$ by

$$\begin{aligned} F^r(\Omega^0(\mathcal{L}_{\kappa,k})) &= F^r(\mathcal{L}_{\kappa,k}), \\ F^r(\Omega^1(\mathcal{L}_{\kappa,k})) &= \Omega^1(\mathcal{L}_{\kappa,k}) \cap \left(F^{r-1}(\mathcal{L}_{\kappa,k}) \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\right). \end{aligned} \tag{A.21}$$

In order to use the Hodge to de Rham spectral sequence

$$E_1^{i,j} = H^{i+j}(C_\kappa, \mathrm{Gr}^i(\Omega^\bullet(\mathcal{L}_{\kappa,k}))) \Rightarrow H^{i+j}(C_\kappa, \Omega^\bullet(\mathcal{L}_{\kappa,k})),$$

we compute the E_1-terms.

Proposition A.9. — *There is the canonical isomorphism*

$$\mathrm{Gr}^i(\mathcal{L}_{\kappa,k}) \simeq \boldsymbol{\omega}_\kappa^{2i-k} = \boldsymbol{\omega}_\kappa^i(\boldsymbol{\omega}_\kappa^{-1})^{k-i}.$$

Proof. — The canonical morphism

$$h\colon \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i)} \to \mathrm{Gr}^i(\mathcal{L}_{\kappa,k}) = F^i(\mathcal{L}_{\kappa,k})/F^{i+1}(\mathcal{L}_{\kappa,k})$$

is surjective, since any element $\sum_\sigma \sigma \cdot m_\sigma \in F^i(\mathcal{L}_{\kappa,k})$ with $\sigma \in \mathfrak{S}_k$ and $m_\sigma \in \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i)}$ is equal to $\sum_\sigma m_\sigma$ in $F^i(\mathcal{L}_{\kappa,k})/F^{i+1}(\mathcal{L}_{\kappa,k})$. The kernel of $\mathrm{pr}\colon \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i)} \to \boldsymbol{\omega}_\kappa^{2i-k}$ is equal to

$$K := \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \boldsymbol{\omega}_\kappa \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i-1)} + \cdots + \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i-1)} \otimes_{\mathcal{O}_{C_\kappa}} \boldsymbol{\omega}_\kappa.$$

Indeed, over some open subset, if we fix a splitting of the exact sequence (A.10) and write e_1 and e_2 for a basis of $\boldsymbol{\omega}_\kappa$ and $\boldsymbol{\omega}_\kappa^{-1}$ respectively, then we can identify $\{e_1, e_2\}$ with a basis of $\mathcal{L}_\kappa$ and we see that ${e_1}^{\otimes i} \otimes {e_2}^{\otimes(k-i)}$ is a basis of $\boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i)}/K$. Since $K \subset \ker(h)$, we obtain a surjective morphism

$$\boldsymbol{\omega}_\kappa^{2i-k} \to \mathrm{Gr}^i(\mathcal{L}_{\kappa,k}),$$

and hence it is an isomorphism since $\mathrm{Gr}^i(\mathcal{L}_{\kappa,k})$ is free of rank 1 by the definition of $F^r(\mathcal{L}_{\kappa,k})$. □

The Kodaira-Spencer map

$$\theta\colon \boldsymbol{\omega}_\kappa \hookrightarrow \mathcal{L}_\kappa \xrightarrow{\nabla_\kappa} \mathcal{L}_\kappa \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \to \boldsymbol{\omega}_\kappa^{-1} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa},$$

which is $\mathcal{O}_{C_\kappa}$-linear, induces an isomorphism

$$\boldsymbol{\omega}_\kappa^2 \simeq \Omega^1_{C_\kappa^\times/\kappa}$$

([23, A1.3.17], [8, VI §4.5], [24, Theorem 10.13.11]). Then θ induces

$$\begin{aligned} \mathrm{Gr}^0(\Omega^\bullet(\mathcal{L}_{\kappa,k})) &= [\boldsymbol{\omega}_\kappa^{-k} \to 0], \\ \mathrm{Gr}^r(\Omega^\bullet(\mathcal{L}_{\kappa,k})) &= \left[\boldsymbol{\omega}_\kappa^{(2r-k)} \xrightarrow{\theta\otimes\mathrm{id}} \boldsymbol{\omega}_\kappa^{(2r-k-2)} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\right] \ (\text{for } 1 \le r \le k), \\ \mathrm{Gr}^{k+1}(\Omega^\bullet(\mathcal{L}_{\kappa,k})) &= \left[0 \to \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\right]. \end{aligned}$$

We claim that $\mathrm{Gr}^r(\mathcal{L}_{\kappa,k}\otimes_{\mathcal{O}_{C_\kappa}}\Omega^\bullet_{C_\kappa^\times/\kappa})=0$ for $1\le r\le k$, that is, $\theta\otimes\mathrm{id}$ is an isomorphism. Over some open set, if we write ω and η for a basis of $\boldsymbol{\omega}$ and $\Omega^1_{C_\kappa^\times/\kappa}$, respectively, then there is a basis ξ of $\boldsymbol{\omega}^{-1}$ such that $\theta(\omega)=\xi\otimes\eta$. Let $\bar{\xi}\in\mathcal{L}_\kappa$ such that $\nabla_\kappa(\omega)=\bar{\xi}\otimes\eta$. Since $\nabla_{\kappa,k}(ax)=a\nabla_{\kappa,k}(x)$ in $\mathrm{Gr}^{r-1}(\mathcal{L}_{\kappa,k})\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa}$ for any $a\in\mathcal{O}_{C_\kappa}$ and $x\in\mathrm{Gr}^r(\mathcal{L}_{\kappa,k})$,

$$\nabla_{\kappa,k}(\omega^r\bar{\xi}^{k-r})=\sum_{j=1}^{r}\omega^{r-1}\bar{\xi}^{k-r+1}\otimes\eta+\sum_{j=r+1}^{k}\omega^r\bar{\xi}^{k-r-1}\nabla_\kappa(\bar{\xi}).$$

Since the second term of the right hand side belongs to $F^r(\mathcal{L}_{\kappa,k})$, we have $(\theta\otimes\mathrm{id})(\omega^r\bar{\xi}^{k-r})=r\omega^r\xi^{k-r+1}\otimes\eta$. Thus we have $E_\infty^{i,j}=0$, $E_\infty^{k+2,j}=0$ if $1\le i\le k$ and any $j\in\mathbb{Z}$. Therefore we obtain

$$\begin{aligned}H^1(C_\kappa,\mathcal{L}_{\kappa,k},\nabla_{\kappa,k})&=F^0_\kappa\supset F^1_\kappa=\cdots=F^{k+1}_\kappa\\&=H^0(C_\kappa,\boldsymbol{\omega}_\kappa^k\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa})\supset F^{k+2}_\kappa=0.\end{aligned}$$

Here $F^{k+1}_\kappa=H^0(C_\kappa,\boldsymbol{\omega}_\kappa^k\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa})$ follows from the following long exact sequence induced by the exact sequence $0\to F^1\to F^0\to\mathrm{Gr}^0_F\to 0$, the quasi-isomorphism $F^k\to\cdots\to F^1$, and $\mathrm{Gr}^{k+1}_F=F^k$:

$$\begin{aligned}&0\to H^0(C_\kappa,\mathcal{L}_{\kappa,k},\nabla_{\kappa,k})\to H^0(C_\kappa,\boldsymbol{\omega}_\kappa^{-k})\\&\to H^0(C_\kappa,\boldsymbol{\omega}_\kappa^k\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa})\to H^1(C_\kappa,\mathcal{L}_{\kappa,k},\nabla_{\kappa,k})\to\cdots.\end{aligned}$$

Hence it suffices to show that the isomorphism (A.14) induces an isomorphism

$$F^{k+1}_\kappa\simeq F^{k+1}_{\kappa,\mathrm{Hdg}}(\varepsilon),$$

where $F^{k+1}_{\kappa,\mathrm{Hdg}}=H^0(\tilde{X}_\kappa,\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa})$. Recall that the filtration $F^\bullet(\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})$ on $\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}$ is defined by (A.16).

We define the filtration $F^\bullet(\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])$ on $\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)]$ by

$$\begin{aligned}\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)]&=F^0(\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])\\&=F^1(\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])\supset F^2(\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])=0.\end{aligned}$$

Similarly as the construction of (A.17), the canonical map $\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)]\to\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}$ and the exact sequence $0\to\mathrm{Gr}^1_F\to F^0\to\mathrm{Gr}^0_F\to 0$ induce a commutative diagram

$$\begin{tikzcd}0 \arrow[r, "d"] \arrow[d] & R^{k+1}\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)](\varepsilon) \arrow[d]\\ \mathcal{L}_{\kappa,k} \arrow[r, "\nabla_{\kappa,k}"] & \mathcal{L}_{\kappa,k}\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa}.\end{tikzcd}$$

The Leray spectral sequence [22, Remark 3.3] and its functoriality induce a commutative diagram

$$\begin{tikzcd}
H^1(C_\kappa, 0 \xrightarrow{d} R^{k+1}\tilde{f}_*\Omega^{k+1}_{\tilde{X}^\times_\kappa/\kappa}[-(k+1)](\varepsilon)) \arrow[r, "\simeq"] \arrow[d] & H^{k+1}(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}^\times_\kappa/\kappa}[-(k+1)])(\varepsilon) \arrow[d] \\
H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r, "\simeq"] & H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}^\times_\kappa/\kappa})(\varepsilon).
\end{tikzcd}$$

Hence, in order to prove that (A.14) is a filtered isomorphism, it suffices to show that

$$\begin{tikzcd}
H^0(C_\kappa, \boldsymbol{\omega}^k_\kappa \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}) \arrow[r, "\text{Künneth}"] \arrow[dr] & H^0(C_\kappa, R^0\tilde{f}_*\Omega^{k+1}_{\tilde{X}^\times_\kappa/\kappa})(\varepsilon) \arrow[d] \\
 & H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})
\end{tikzcd}$$

is commutative and the horizontal morphism is an isomorphism. Since $\tilde{f} : \tilde{X} \to C$ is a log-smooth morphism, we have the exact sequence

$$0 \to \tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa} \to \Omega^{k+1}_{\tilde{X}^\times_\kappa/\kappa} \to \Omega^{k+1}_{\tilde{X}^\times_\kappa/C^\times_\kappa} \to 0 \tag{A.22}$$

([19, (3.12)]).

The filtration $F^\bullet(\tilde{f}^*\Omega^1_{C^\times_\kappa/C^\times_\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa}[-(k+1)])$ on $\tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa}[-(k+1)]$ is defined by

$$\begin{aligned}
\tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa}[-(k+1)] &= F^0(\tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa}[-(k+1)]) \\
&= F^1(\tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa}[-(k+1)]) \\
&\supset F^2(\tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa}[-(k+1)]) = 0.
\end{aligned}$$

Then, similarly as in the proof of (A.17), the canonical diagram

$$\begin{tikzcd}
\tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa}[-(k+1)] \arrow[r] \arrow[dr] & \Omega^{k+1}_{\tilde{X}^\times_\kappa/\kappa}[-(k+1)] \arrow[d] \\
 & \Omega^\bullet_{\tilde{X}^\times_\kappa/\kappa}
\end{tikzcd}$$

and the exact sequence $0 \to \mathrm{Gr}_F^1 \to F^0 \to \mathrm{Gr}_F^0 \to 0$ induce a commutative diagram

$$\begin{tikzcd}[column sep=small]
\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \arrow[r, "\text{Künneth}"] \arrow[drr] & R^0\tilde{f}_*\Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}(\varepsilon) \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \arrow[r] \arrow[dr] & R^0\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}(\varepsilon) \arrow[d] \\
 & & \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}.
\end{tikzcd}$$

It remains to check that the composition of the horizontal arrows $\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \to R^0\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}(\varepsilon)$ is an isomorphism.

Lemma A.10. — *There are canonical isomorphisms*

$$R^j f_*\Omega^i_{X_\kappa^\times/C_\kappa^\times}(\varepsilon) \simeq \begin{cases} \boldsymbol{\omega}_\kappa^{i-j} & \text{if } i+j=k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — We define complexes on $\mathcal{E}_\kappa^\times$ and $X_\kappa^\times$ as

$$\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times} = \Omega^0_{\mathcal{E}_\kappa^\times/C_\kappa^\times} \oplus \Omega^1_{\mathcal{E}_\kappa^\times/C_\kappa^\times}[-1] \text{ and } \mathcal{C}^\bullet_{X_\kappa^\times/C_\kappa^\times} = \bigoplus_{i=0}^k \Omega^i_{X_\kappa^\times/C_\kappa^\times}[-i],$$

respectively. Then we have

$$\mathcal{C}^\bullet_{X_\kappa^\times/C_\kappa^\times} \simeq \oplus_{j=1}^k p_j^*\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times}.$$

By using the Künneth formula, we obtain

$$\begin{aligned} R^n f_*\mathcal{C}^\bullet_{X_\kappa^\times/C_\kappa^\times} &\simeq \bigoplus_{n_1+\cdots+n_k=n} R^{n_1}\pi_*\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times} \otimes_{\mathcal{O}_{C_\kappa}} \cdots \otimes_{\mathcal{O}_{C_\kappa}} R^{n_k}\pi_*\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times} \\ &\simeq \bigoplus_{n_1+\cdots+n_k=n} (R^{n_1}\pi_*\mathcal{O}_{\mathcal{E}_\kappa} \oplus R^{n_1-1}\pi_*\Omega^1_{\mathcal{E}_\kappa^\times/C_\kappa^\times}) \\ &\qquad \otimes_{\mathcal{O}_{C_\kappa}} \cdots \otimes_{\mathcal{O}_{C_\kappa}} (R^{n_k}\pi_*\mathcal{O}_{\mathcal{E}_\kappa} \oplus R^{n_k-1}\pi_*\Omega^1_{\mathcal{E}_\kappa^\times/C_\kappa^\times}). \end{aligned}$$

As in the proof of (A.2) or Proposition A.6, we obtain

$$R^n f_*\mathcal{C}^\bullet_{X_\kappa^\times/C_\kappa^\times}(\varepsilon) \simeq \begin{cases} \mathrm{Sym}^k R^1\pi_*\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times} & \text{if } n=k, \\ 0 & \text{otherwise.} \end{cases}$$ □

By Lemma A.10, the long exact sequence of $R^\bullet\tilde{f}_*$ coming from (A.22), and $g_*\Omega^i_{\tilde{X}_\kappa^\times/C_\kappa^\times} = \Omega^i_{X_\kappa^\times/C_\kappa^\times}$ mentioned in the proof of Proposition A.7, we obtain an isomorphism

$$\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \simeq R^0\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}(\varepsilon).$$

Hence the isomorphism (A.14) is a filtered isomorphism.

Case (A.15). — We can define a filtration on the complex $\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})$ by

$$(\mathrm{A.23})\qquad \begin{aligned} F^r_{\mathrm{par}}(\Omega^0_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) &= F^r(\mathcal{L}_{\kappa,k}),\\ F^r_{\mathrm{par}}(\Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) &= \Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) \cap \Big(F^{r-1}(\mathcal{L}_{\kappa,k}) \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\Big). \end{aligned}$$

Then we have

$$\begin{aligned} \mathrm{Gr}^0(\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) &= [\boldsymbol{\omega}_\kappa^{-k} \to 0],\\ \mathrm{Gr}^r(\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) &= \Big[\boldsymbol{\omega}_\kappa^{(2r-k)} \xrightarrow{\theta\otimes\mathrm{id}} \boldsymbol{\omega}_\kappa^{(2r-k-2)} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\Big] \ (\text{for } 1\le r\le k),\\ \mathrm{Gr}^{k+1}(\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) &= \Big[0 \to \boldsymbol{\omega}_\kappa^{k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}\Big]. \end{aligned}$$

Thus we have $E_\infty^{i,j}=0$, $E_\infty^{k+2,j}=0$ if $1\le i\le k$ and any $j\in\mathbb{Z}$. Therefore we obtain

$$\begin{aligned} H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) &= F^0_{\kappa,\mathrm{par}} \supset F^1_{\kappa,\mathrm{par}} = \cdots = F^{k+1}_{\kappa,\mathrm{par}}\\ &= H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}) \supset F^{k+2}_{\kappa,\mathrm{par}} = 0. \end{aligned}$$

Here $F^{k+1}_{\kappa,\mathrm{par}} = H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa})$ follows from the same argument as (A.14). Moreover we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa/\kappa})(\varepsilon) \arrow[r] & H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon) \arrow[r, "\mathrm{Res}"] & H^0(\widetilde{D}_k, \Omega^k_{\tilde{D}_k^\times/\kappa})(\varepsilon) \\ & & H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}) \arrow[u, dash, "\simeq"'] \arrow[r, "\mathrm{R}_\kappa"] & H^0(Z_\kappa, \mathcal{O}_{Z_\kappa}) \arrow[u, dash, "\bigstar", "\simeq"'] \end{tikzcd}$$

(cf. [37, p.150]). Here the isomorphism ★ is obtained by [37, p.145]. Then $\ker(\mathrm{Res}) = H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa/\kappa})(\varepsilon) \simeq \ker(\mathrm{R}_\kappa) = H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa})$. Thus we obtain a commutative diagram

$$\begin{tikzcd} H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}) \arrow[r, "\simeq"] \arrow[d] & H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa/\kappa})(\varepsilon) \arrow[d] \\ H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}) \arrow[r, "\simeq"] \arrow[d] & H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon) \arrow[d] \\ H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r, "\simeq"] & H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon). \end{tikzcd}$$

In the same manner as in the proof of Proposition A.1, we see that the isomorphisms (A.14) and (A.15) are compatible with the Hecke operators and the Atkin operators. This completes the proof of Proposition A.8. □

In the next subsection, we will use the following lemma obtained by [37, Theorem 5.5] or [38, p.391].

LEMMA A.11. — *Assume that $k < p$. Then*

$$H^m(\tilde{X}_\kappa, \Omega^n_{\tilde{X}_\kappa/\kappa})(\varepsilon) \simeq 0 \text{ if } m+n \neq k+1.$$

A.4. Rank of parabolic cohomology. — We retain the notation as before. Let $\mathcal{O}$ be the ring of integers of a finite extension K over $\mathbb{Q}_p$, ϖ a uniformizer, and κ the residue field. We will use the results in §A.1, A.2, and A.3 by substituting $k-2$ for k.

Let $\Gamma = \Gamma_1(N)$, $k \geq 2$, $S_k = S_k(\Gamma, \mathcal{O})$, and $\bar{S_k} = H^0(C_\kappa, \boldsymbol{\omega}_\kappa^{k-2} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa})$. We denote by $f \in S_k$ a normalized Hecke eigenform with character ε, and by $\mathfrak{M}_f$ a maximal ideal of the Hecke algebra generated by ϖ, $T_l - a(l,f)$ (for $(l,N)=1$), $U_l - a(l,f)$ (for $l|N$), and $\langle d\rangle - \varepsilon(d)$ over $\mathcal{O}$. The goal of this subsection is to understand the eigenspaces of the complex conjugation acting on the $\mathfrak{M}_f$-part $H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*(\mathcal{O}/\varpi^n))[\mathfrak{M}_f]$. We will prove the following theorem in this subsection. The author would like to express his deep gratitude to Professor Takeshi Tsuji whose guidance was crucial in proving the following theorem.

THEOREM A.12. — *Assume that $2 \leq k \leq p-1$ and the residual Galois representation $\bar{\rho}_f : G_\mathbb{Q} \to \mathrm{GL}_2(\kappa)$ associated to f is reducible of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix}$$

satisfying that either ξ_1 or ξ_2 is unramified at p. Then, for any positive integer n and a parity $\alpha \in \{\pm 1\}$ as (A.27),

$$H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}}, \operatorname{Sym}^{k-2} R^1\pi_*(\mathcal{O}/\varpi^n))^\alpha[\mathfrak{M}_f] \simeq \mathcal{O}/\varpi^n,$$
$$H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O})^\alpha[\mathfrak{M}_f] \simeq \mathcal{O}.$$

In order to prove this theorem, we need the following proposition. For each n, we write

$$\tilde{V}(n) = H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}}, \operatorname{Sym}^{k-2} R^1\pi_*(\mathcal{O}/\varpi^n))[\mathfrak{M}_f],$$
$$\tilde{V}(\infty) = H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O})[\mathfrak{M}_f].$$

We put $\tilde{V} = \tilde{V}(1)$.

PROPOSITION A.13. — *All of the constituents of $\tilde{V}$ are isomorphic to $\kappa(\xi_1)$ or $\kappa(\xi_2)$.*

Proof. — We denote by ρ the Galois representation $\tilde{V}$ of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Put $d = \dim_{\kappa} \tilde{V}$. Fix a rational prime number l with $(l, pN) = 1$. The Eichler-Shimura relations impose the relation $\rho(\mathrm{Frob}_l)^2 - a(l,f)\rho(\mathrm{Frob}_l) + \varepsilon(l)l^{k-1} = 0$. We denote by $\alpha(l)$ and $\beta(l)$ the solutions of $X^2 - a(l,f)X + \varepsilon(l)l^{k-1} = 0$. Let

$$\tilde{V}^* = \mathrm{Hom}(\tilde{V}, \kappa(\varepsilon\omega^{k-1}))$$

and

$$W = \tilde{V} \oplus \tilde{V}^*$$

the direct sum of $\tilde{V}$ and $\tilde{V}^*$. We consider the characteristic polynomial of Frob_l acting on W. Let G denote a finite quotient of $G_{\mathbb{Q}}$ through which the actions on W, $\kappa(\xi_1)$, and $\kappa(\xi_2)$ factor. We denote by $N_{\alpha(l)}$ and $N_{\beta(l)}$ the generalized eigenspaces of $\rho(\mathrm{Frob}_l)$ with respect to $\alpha(l)$ and $\beta(l)$ respectively. Then $\tilde{V} = N_{\alpha(l)} \oplus N_{\beta(l)}$. Since the operation $\mathrm{Hom}(*, \kappa(\varepsilon\omega^{k-1}))$ interchanges the eigenvalues of the action of Frob_l, the characteristic polynomial of Frob_l acting on W is $(T-\alpha(l))^d(T-\beta(l))^d$. On the other hand, the characteristic polynomial of Frob_l acting on $\kappa(\xi_1)^{\oplus d} \oplus \kappa(\xi_2)^{\oplus d}$, which is regarded as a G-module, is also $(T-\alpha(l))^d(T-\beta(l))^d$. By the Chebotarev density theorem, any element of G is the image of some Frob_l with $l \nmid pN$. Thus, by the Brauer-Nesbitt theorem,

$$W^{\mathrm{ss}} \simeq \kappa(\xi_1)^{\oplus d} \oplus \kappa(\xi_2)^{\oplus d},$$

where W^{ss} is the semi-simplification of W. Hence there exists a Jordan-Hölder filtration

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_d = \tilde{V} \tag{A.24}$$

of $\tilde{V}$ satisfying

$$V_i/V_{i-1} \simeq \kappa(\alpha_i),$$

where α_i is equal to ξ_1 or ξ_2 for each i. □

Using integral p-adic Hodge theory, we shall prove Theorem $A.12$ by determining a character such that the number of constituents of $\tilde{V}$ isomorphic to it is equal to one. The key ingredients in our proof are to restrict the action of $G_{\mathbb{Q}}$ on $\tilde{V}$ to $G_{\mathbb{Q}_p}$ and to use that the Hodge-Tate weights of ξ_1 and ξ_2 are distinct.

First we will briefly review the fully faithful functor from the category of finitely generated filtered φ-module to the category of $\mathcal{O}$-representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of finite length, and state the comparison theorem between the parabolic étale cohomology and the parabolic log-crystalline cohomology, which we will use in this subsection.

For a non-negative integer r, let $\mathbf{MF}^r_{\mathcal{O}}$ denote the category whose objects are the following triples $(M, (\mathrm{Fil}^i M)_{i\in\mathbb{Z}}, (\varphi^i_M)_{i\in\mathbb{Z}})$:

(1) M is a finitely generated $\mathcal{O}$-module;

(2) $(\mathrm{Fil}^i M)_{i\in\mathbb{Z}}$ is a decreasing filtration on M by $\mathcal{O}$-submodules such that $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{r+1} M = 0$;
(3) $\varphi_M^i\colon \mathrm{Fil}^i M \to M$ is an $\mathcal{O}$-linear homomorphism such that $\varphi_M^i|_{\mathrm{Fil}^{i+1}M} = p\varphi_M^{i+1}$ and $\sum_{i=0}^r \varphi_M^i(\mathrm{Fil}^i M) = M$.

A morphism in $\mathbf{MF}_{\mathcal{O}}^r$ is a homomorphism of filtered $\mathcal{O}$-modules compatible with $\varphi^\bullet$. It is known that any morphism $\eta : M \to M'$ in $\mathbf{MF}_{\mathcal{O}}^r$ is strict with respect to the filtrations, that is, $\eta(\mathrm{Fil}^i M) = \mathrm{Fil}^i M' \cap \eta(M)$ for each $i \in \mathbb{Z}$ ([12, 1.10 (b)]). This implies that $\mathbf{MF}_{\mathcal{O}}^r$ is an abelian category as follows. Let $\eta : M \to M'$ be a morphism in $\mathbf{MF}_{\mathcal{O}}^r$, and let $\underline{\eta}$ denote η regarded as a homomorphism of underlying $\mathcal{O}$-modules. Then the $\mathcal{O}$-module $N := \ker(\underline{\eta})$ with $\mathrm{Fil}^i N$ and φ_N^i defined by $\mathrm{Fil}^i N = N \cap \mathrm{Fil}^i M$ and $\varphi_N^i = \varphi_M^i|_N$, respectively, belongs to $\mathbf{MF}_{\mathcal{O}}^r$ and gives the kernel of η in $\mathbf{MF}_{\mathcal{O}}^r$. Let N' denote $\mathrm{coker}(\underline{\eta})$. We define a filtration $\mathrm{Fil}^i N'$ and an $\mathcal{O}$-linear homomorphism $\varphi_{N'}^i$ by $\mathrm{Fil}^i N' = \mathrm{Fil}^i M'/\eta(\mathrm{Fil}^i M)$ and the homomorphism induced by φ_M^i and $\varphi_{M'}^i$, respectively. Note that $\mathrm{Fil}^i N' \to N'$ is injective because η is strict, and hence $\mathrm{Fil}^i N'$ may be regarded as an $\mathcal{O}$-submodule of N'. The triple $(N', (\mathrm{Fil}^i N')_{i\in\mathbb{Z}}, (\varphi_{N'}^i)_{i\in\mathbb{Z}})$ belongs to $\mathbf{MF}_{\mathcal{O}}^r$ and gives the cokernel of η in $\mathbf{MF}_{\mathcal{O}}^r$. The strictness of η further shows that we have $\mathrm{Fil}^i(\mathrm{im}(\eta)) = \eta(M) \cap \mathrm{Fil}^i M' = \eta(\mathrm{Fil}^i M) \simeq \mathrm{Fil}^i(\mathrm{coim}(\eta))$ and hence $\mathrm{im}(\eta) = \mathrm{coim}(\eta)$ in $\mathbf{MF}_{\mathcal{O}}^r$.

Let $\mathbf{MF}_\kappa^r$ denote the full subcategory of $\mathbf{MF}_{\mathcal{O}}^r$ consisting of objects M satisfying $\varpi M = 0$. Let $\mathbf{Rep}_{\mathcal{O}}(G_{\mathbb{Q}_p})$ denote the category of representations of $G_{\mathbb{Q}_p}$ on $\mathcal{O}$-modules of finite length. For an integer r such that $0 \le r \le p-2$, there exists a fully faithful functor

$$T_{\mathrm{cris}}\colon \mathbf{MF}_{\mathcal{O}}^r \to \mathbf{Rep}_{\mathcal{O}}(G_{\mathbb{Q}_p})$$

given by J.-M. Fontaine and G. Laffaille ([12], [6], [41]). Let $\mathbf{Rep}_{\mathcal{O},\mathrm{cris}}^r(G_{\mathbb{Q}_p})$ denote the essential image of $\mathbf{MF}_{\mathcal{O}}^r$ by T_{cris}. For an object T of $\mathbf{Rep}_{\mathcal{O},\mathrm{cris}}^r(G_{\mathbb{Q}_p})$, the Hodge-Tate weights of T mean $s \in \mathbb{Z}$ for which $\mathrm{Gr}^s M \neq 0$, where M is an object of $\mathbf{MF}_{\mathcal{O}}^r$ such that $T_{\mathrm{cris}}(M) \simeq T$.

By (A.15), we have a filtered isomorphism

$$H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k-2}, \nabla_{\kappa,k-2}) \simeq H^{k-1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon).$$

Here a filtration is given by

$$0 \subset \bar{S}_k = F^{k-1}_{\kappa,\mathrm{par}} = \cdots = F^1_{\kappa,\mathrm{par}} \subset F^0_{\kappa,\mathrm{par}} = H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k-2}, \nabla_{\kappa,k-2}).$$

THEOREM A.14. — *Assume that $k-1 \le p-2$. Then there is an isomorphism of Hecke modules*

$$\begin{aligned} T_{\mathrm{cris}}(H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k-2}, \nabla_{\kappa,k-2})) &\simeq T_{\mathrm{cris}}(H^{k-1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon)) \\ &\simeq H^{k-1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, \kappa)(\varepsilon) \simeq H^1_{\text{ét},\mathrm{par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^{k-2} R^1\pi_*\kappa). \end{aligned}$$

Proof. — The first and last isomorphisms follow from Proposition A.8 (A.15) and Proposition A.1 (A.3) respectively. The second isomorphism is obtained by the comparison theorem for proper smooth varieties with constant coefficients (proved by Fontaine-Messing ([13, III 6.4]) and Faltings ([9, Theorem 5.3]) and improved by Breuil-Tsuji ([5, Theorem 3.2.4.6]=[35, Theorem 5.1] and [5, Theorem 3.2.4.7])). It remains to check that these morphisms are Hecke equivariant. By the Hodge to de Rham spectral sequence and Lemma A.11, we have

$$H^k(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon) = 0, \tag{A.25}$$

$$H^{k-2}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon) = 0. \tag{A.26}$$

By the long exact sequence of cohomology for an exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi} \mathcal{O} \to \kappa \to 0$, (A.25), and Nakayama's lemma, we obtain

$$H^k(\tilde{X}_\mathcal{O}, \Omega^\bullet_{\tilde{X}_\mathcal{O}/\mathcal{O}})(\varepsilon) = 0.$$

Therefore, for any integer $n \geq 1$, by the long exact sequence of cohomology for an exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi^n} \mathcal{O} \to \mathcal{O}/\varpi^n \to 0$, we obtain

$$\begin{aligned} H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O})/\varpi^n H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O}) \\ \simeq H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*(\mathcal{O}/\varpi^n)). \end{aligned}$$

Moreover, (A.26) implies that $H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O})$ is torsion-free. Therefore the proof reduces to showing that the comparison isomorphism between $H^{k-1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ and $H^{k-1}(\tilde{X}_{\mathbb{Q}_p}, \Omega^\bullet_{\tilde{X}_{\mathbb{Q}_p}/\mathbb{Q}_p})$ is compatible with the Hecke correspondences and Atkin correspondences. This follows from the de Rham conjecture for proper smooth varieties with constant $\mathbb{Q}_p$-coefficients [36, Theorem A1]. □

Since $(p, N) = 1$, $\tilde{V}(n)$ is a crystalline representation of $G_{\mathbb{Q}_p}$.

Next we construct a filtration of $H^{k-1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon)[\mathfrak{M}_f]$ by using the filtration (A.24) of $\tilde{V}$. We put $\tilde{M} = H^{k-1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon)[\mathfrak{M}_f]$.

Case 1. — ξ_1 is unramified at p.

Then there exists $M(\xi_1) \in \mathbf{MF}^{k-1}_\kappa$ satisfying $T_{\text{cris}}(M(\xi_1)) = \kappa(\xi_1)$ and $F^1 = 0 \subset F^0 = M(\xi_1)$. Similarly, there is $M(\xi_2) \in \mathbf{MF}^{k-1}_\kappa$ satisfying $T_{\text{cris}}(M(\xi_2)) = \kappa(\xi_2)$ and $F^k = 0 \subset F^{k-1} = M(\xi_2)$.

Thus we obtain $M(\alpha_1) \in \mathbf{MF}^{k-1}_\kappa$ satisfying $T_{\text{cris}}(M(\alpha_1)) = V_1$. Since the length of module is preserved under T_{cris}, we have $\dim_\kappa M(\alpha_1) = 1$. Since T_{cris} is fully faithful, the image of $M(\alpha_1)$ in $\tilde{M}$ is non-trivial. We write $M_1 = \operatorname{im}(M(\alpha_1) \to \tilde{M})$.

Similarly, by replacing V_1 by V_2/V_1, there exists $M(\alpha_2) \in \mathbf{MF}_\kappa^{k-1}$ satisfying $T_{\mathrm{cris}}(M(\alpha_2)) = V_2/V_1$. Let $M^1 = \tilde{M}/M_1$, $\overline{M_2} = \mathrm{im}(M(\alpha_2) \to M^1)$, and

$$M_2 = \ker(\tilde{M} \to M^1/\overline{M_2}).$$

Then we have

$$T_{\mathrm{cris}}(M_2) = V_2.$$

Repeating this arguments, we obtain a Jordan-Hölder filtration

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_d = \tilde{M}$$

of $\tilde{M}$ satisfying

$$T_{\mathrm{cris}}(M_i/M_{i-1}) = V_i/V_{i-1} = \kappa(\alpha_i)$$

where α_i is equal to ξ_1 or ξ_2. By noting that, for any integer r,

$$\dim_\kappa \mathrm{Gr}_F^r(\tilde{M}) = \sum_{j=1}^{d} \dim_\kappa \mathrm{Gr}_F^r(M_j/M_{j-1}),$$

we have the following proposition.

Proposition A.15. — *We have*

$$\dim_\kappa \mathrm{Gr}_F^r(\tilde{M}) = \begin{cases} 0 & \text{if } r \neq 0, k-1, \\ \sharp\{j | \alpha_j = \xi_1\} & \text{if } r = 0, \\ \sharp\{j | \alpha_j = \xi_2\} & \text{if } r = k-1. \end{cases}$$

Case 2. — ξ_2 is unramified at p.

Similarly as in Case 1, we have the following proposition.

Proposition A.16. — *We have*

$$\dim_\kappa \mathrm{Gr}_F^r(\tilde{M}) = \begin{cases} 0 & \text{if } r \neq 0, k-1, \\ \sharp\{j | \alpha_j = \xi_2\} & \text{if } r = 0, \\ \sharp\{j | \alpha_j = \xi_1\} & \text{if } r = k-1. \end{cases}$$

Now we can prove Theorem A.12. By the q-expansion principle [23, § 1.6],

$$\mathrm{Gr}_F^{k-1}(\tilde{M}) = \bar{S}_k[\mathfrak{M}_f] \simeq \kappa.$$

Then, by the above propositions, we have $\sharp\{j | \alpha_j = \xi_2\} = 1$ in Case 1, and $\sharp\{j | \alpha_j = \xi_1\} = 1$ in Case 2. This proves Theorem A.12 in the case $n = 1, \infty$. In particular,

$$\alpha = \xi_2(-1) \text{ in Case 1 and } \alpha = \xi_1(-1) \text{ in Case 2.} \tag{A.27}$$

Next, we prove Theorem A.12 for any $n > 1$. As noted in the proof of Theorem A.14, $H^1_{\text{ét},\mathrm{par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^{k-2} R^1\pi_*\mathcal{O})$ is torsion free. Then the exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi^n} \mathcal{O} \to \mathcal{O}/\varpi^n \to 0$ on $\tilde{X}_{\overline{\mathbb{Q}}_p,\text{ét}}$ with Proposition A.1 (A.3) and (A.4) induces an exact sequence

$$0 \to \tilde{V}(\infty) \xrightarrow{\times\varpi^n} \tilde{V}(\infty) \to \tilde{V}(n).$$

Similarly, using the exact sequence $0 \to \varpi^{n-1}/\varpi^n \to \mathcal{O}/\varpi^n \to \mathcal{O}/\varpi^{n-1} \to 0$ on $\tilde{X}_{\overline{\mathbb{Q}}_p,\text{ét}}$, we obtain an exact sequence

$$0 \to \tilde{V}(1) \to \tilde{V}(n) \to \tilde{V}(n-1).$$

Thus an inductive argument proves Theorem A.12.

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SOMMES DE BIRKHOFF ITÉRÉES SUR DES EXTENSIONS FINIES D'ODOMÈTRES. CONSTRUCTION DE SOLUTIONS AUTO-SIMILAIRES À DES ÉQUATIONS DIFFÉRENTIELLES AVEC DÉLAI

PAR JEAN-FRANÇOIS BERTAZZON & VINCENT DELECROIX

RÉSUMÉ. — Nous estimons le comportement asymptotique *des sommes de Birkhoff itérées* pour des systèmes dynamiques qui sont le codage d'extensions finies d'odomètres.

Si u est une suite de réels, sa première somme itérée $S^{(1)}(u)$ est la suite des sommes cumulatives des premiers termes. Puis par récurrence, nous définissons la ℓ-ème somme itérée par $S^{(\ell)}(u) = S^{(1)}\big(S^{(\ell-1)}(u)\big)$.

Les systèmes dynamiques symboliques que nous étudions sont engendrés par des substitutions apériodiques et primitives particulières, que nous qualifierons de *fortement uniformes.*

Nous montrons que pour tout entier $\ell \geq 1$, il existe un polynôme d'approximation p_ℓ tel que la différence des suites $S^{(\ell)}(u) - \big(p_\ell(n)\big)_{n\in\mathbb{N}}$ soient bornées, où u est l'image de l'orbite d'un point par une fonction qui ne dépend que de la première lettre.

Nous nous restreignons ensuite à des alphabets sur deux lettres $\{a, b\}$ puis nous imposons une condition combinatoire sur la substitution permettant d'estimer la croissance de ces bornes. Nous restreignons également notre étude au point fixe d'une substitution, et non plus sur l'ensemble du système dynamique symbolique.

Ces sommes itérées correctement renormalisées convergent en un certain sens et permettent de construire une fonction qui est solution d'équations intégrales du type :

$$\int_0^{\lambda x} f(t)\,dt = \eta\big(f(x) - f(0)\big) \qquad \text{pour tout } x \geq 0$$

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pour des paramètres entiers $\lambda \geq 2$ et $\eta \in \mathbb{Z}^*$.

ABSTRACT (*Iterated Birkhoff sums on finite odometer extesnions. Construction of auto-similar solutions to differential equations with delay*). — We build bounded solutions to a linear integral equation. Our functions are built as limit of iterated Birkhoff sums over auto-similar dynamical systems.

1. Introduction

1.1. Principaux résultats. — Étant donnés un espace compact K et une application continue $T : K \to K$ ergodique pour une mesure de probabilité μ alors le théorème de Birkhoff donne une asymptotique au premier ordre des sommes d'une fonction le long des orbites. Plus précisément, soit $f \in L^1_\mu(K, \mathbb{R})$ alors :

$$\sum_{k=0}^{n-1} f\big(T^k(x)\big) - n \int_K f d\mu$$

est négligeable devant n lorsque n tend vers $+\infty$ pour μ-presque tout x. La somme apparaissant dans cette expression s'appelle *une somme de Birkhoff* et une question importante est d'en donner une estimation précise pour des classes particulières de systèmes dynamiques. Dans un cadre aléatoire apparaît un terme en $\sqrt{n}$ ou $\sqrt{n \log\log(n)}$ lié au théorème centrale limite et la loi du logarithme itéré. Dans le cadre des rotations, le théorème de Denjoy-Koksma ([12]) montre que ces expressions sont bornées pour certains temps n indépendamment de x. Plus récemment, ces questions ont été étudiées pour les systèmes dynamiques provenant de systèmes substitutifs [1] et d'échanges d'intervalles [18], [11] et [5]. Mentionnons que lorsque la dynamique est une translation sur un groupe abélien compact et que f et suffisamment régulière, cette quantité reste uniformément bornée en n. Nous pouvons alors étudier le comportement asymptotique de la somme des différences entre les sommes de Birkhoff et l'intégrale de la fonction f :

$$\sum_{n=1}^{N-1} \left(\sum_{k=0}^{n-1} f\big(T^k(x)\big) - n \int_K f d\mu \right).$$

Nous pouvons poursuivre ce processus et cela nous conduit à la notion de *sommes de Birkhoff itérées.*

DÉFINITION 1. — Soit $u = (u_i)_{i \in \mathbb{N}}$ une suite de réels, sa *somme de Birkhoff* $S^{(1)}(u)$ est la suite des sommes cumulatives définie par $S^{(1)}_0 = 0$ et :

$$S^{(1)}_n(u) = \sum_{i=0}^{n-1} u_i \qquad \text{pour } n \in \mathbb{N}^*.$$

La $(\ell+1)$-ème *somme de Birkhoff itérée* est définie par récurrence par $S^{(\ell+1)} = S^{(1)}\big(S^{(\ell)}(u)\big)$ soit encore $S_0^{(\ell+1)}(u) = 0$ et :

$$S_n^{(\ell+1)}(u) = S_n^1(S^{(\ell)}(u)) = \sum_{i=0}^{n-1} S_i^{(\ell)}(u) \qquad \text{pour tout entier } n \in \mathbb{N}^*.$$

Quitte à prolonger un vecteur $u = (u_i)_{0\leq i\leq n-1} \in \mathbb{R}^n$ en une suite en posant $u_i = 0$ pour $i \geq n$, nous prolongeons la notion de sommes de Birkhoff itérées à des vecteurs.

Étant donnés un système dynamique $T : K \to K$ et une fonction $f : K \to \mathbb{R}$, nous construisons ces sommes de Birkhoff en prenant la suite $u_n = f(T^n x)$. Dans le cadre des rotations du cercle, ces sommes itérées ont été étudiées dans [6].

Nous introduisons maintenant la classe de systèmes dynamiques que nous allons étudier qui sont des extensions finies d'odomètres. Nous renvoyons à la partie 2 pour plus de précisions. Une *substitution* σ sur un alphabet fini $\mathcal{A}$ est un endomorphisme du monoïde libre $\mathcal{A}^*$ formé de l'ensemble des mots finis. Une substitution est entièrement déterminée par les images des lettres de $\mathcal{A}$ et s'étend naturellement en un endomorphisme des mots infinis. À une substitution est associée sa *matrice d'incidence* pour laquelle l'entrée en position (α, β) est le nombre d'occurrences de la lettre α dans l'image $\sigma(\beta)$ que l'on notera aussi $|\sigma(\beta)|_\alpha$.

Nous ne considérons dans notre travail que des substitutions dites *positives* (c'est-à-dire telle que $|\sigma(\beta)|_\alpha > 0$ pour toutes paires de lettres (α, β)) et *apériodiques* (c'est-à-dire telles qu'il n'existe pas un mot infini périodique u tel que $\sigma(u) = u$). À une substitution positive et apériodique, nous pouvons associer un sous-ensemble compact K_σ de $\mathcal{A}^{\mathbb{N}}$ appelé *sous-décalage* induit par σ. Il est construit en prenant les adhérences des mots finis $\sigma^n(\alpha)$ et leurs décalages dans $\mathcal{A}^{\mathbb{N}}$. On parle parfois de *systèmes auto-induits* dans le sens où il admet un induit (sur un ouvert-fermé) isomorphe à lui-même.

NOTATION. —

1. Si φ est une fonction réelle définie sur l'alphabet $\mathcal{A}$, nous la prolongeons en une fonction définie sur $\mathcal{A}^*$ ou $\mathcal{A}^{\mathbb{N}}$, encore notée φ, en associant à tout mot (fini ou infini) u la valeur $\varphi(u) = \varphi(u_0)$ où u_0 est la première lettre de u. Par exemple *la fonction caractéristique de la lettre* $\alpha \in \mathcal{A}$ notée $\chi_\alpha : \mathcal{A} \to \mathbb{R}$ et définie par :
$$\chi_\alpha(\beta) = 0 \quad \text{si} \quad \beta \neq \alpha \quad \text{et} \quad \chi_\alpha(\alpha) = 1,$$
sera indifféremment considérée comme une fonction définie sur $\mathcal{A}$, $\mathcal{A}^*$ ou $\mathcal{A}^{\mathbb{N}}$.
2. Pour tout mot infini u et toute fonction $\varphi : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$, nous définissons la suite de réels :
$$\varphi * u = \big(\varphi(u_i u_{i+1} \cdots)\big)_{i\in\mathbb{N}} \qquad \text{où} \quad u = u_0 u_1 \cdots \in \mathcal{A}^{\mathbb{N}}.$$

Nous utiliserons la même notation si u est un mot de longueur n, auquel cas $\varphi * u \in \mathbb{R}^n$.

Définition 2. — Une substitution σ est *uniforme* de longueur λ si toutes les images des lettres par σ ont la même longueur λ. C'est-à-dire si la somme des éléments de chaque colonne vaut λ.

Elle est *fortement uniforme* si pour toute lettre $\beta \in \mathcal{A}$, le nombre d'occurrence de chacune des lettres $\alpha \in \mathcal{A}$ dans l'image $\sigma(\beta)$ ne dépend pas de la lettre β. Autrement dit, toutes les colonnes de la matrice d'incidence de σ sont les mêmes.

Une substitution positive, apériodique et uniforme σ de longueur λ engendre un système K_σ qui est une extension finie de l'odomètre $z \in \mathbb{Z}_\lambda \to z+1 \in \mathbb{Z}_\lambda$ où $\mathbb{Z}_\lambda$ est la limite inverse des groupes $(\mathbb{Z}/(\lambda^n\mathbb{Z}), +)$ (voir [7], [14], [3]). Nous revenons sur les odomètres dans la section 2.4. Nous y décrivons le comportement des sommes de Birkhoff itérées pour des fonctions $\varphi : \mathbb{Z}_\lambda \to \mathbb{R}$ de moyennes nulles et qui ne dépendent que d'un nombre fini de coordonnées.

Un exemple important de substitution fortement uniforme est la substitution $\sigma : a \mapsto ab, b \mapsto ba$ dite *de Prouet-Thue-Morse*. Les premières itérations de cette substitution sur la lettre a donnent :

$$\sigma^1(a) = ab, \qquad \sigma^2(a) = abba,$$
$$\sigma^3(a) = abbabaab, \qquad \sigma^4(a) = abbabaabbaababba, \ldots$$

Cette suite de préfixes emboîtés converge (pour la topologie produit) vers un mot infini :

$$abbabaabbaababbabaababbaabbabaabbaababbaabbabaababbabaabb\ldots$$

appelé mot *de Prouet-Thue-Morse*. C'est l'unique mot (infini) u commençant par a tel que $\sigma(u) = u$. Le sous-décalage K_σ associé est l'adhérence des translations de ce mot u.

Soit $\varphi : \{a, b\} \to \mathbb{R}$ la fonction $\varphi = \chi_a - \chi_b$. Dans l'article [2] du premier auteur du présent texte, il est démontré que d'une part les sommes de Birkhoff $S^{(\ell)}\big(\varphi * u\big)$ sont bornées et que d'autre part ces sommes correctement renormalisées en temps et en espace convergent vers une fonction limite f_u (voir la figure 1.1). Dans le cas de la substitution de Prouet-Thue-Morse, cette fonction limite est la fonction de Fabius [9] qui peut également être construite à partir de la loi d'une somme infinie de variables aléatoires indépendantes.

L'objectif de cet article est de généraliser la construction de [2] obtenue pour la substitution de Prouet-Thue-Morse. Nous montrons tout d'abord que le caractère borné des sommes itérées est une conséquence de l'hypothèse fortement uniforme des substitutions.

Théorème 3. — *Soient $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ une substitution positive, apériodique et fortement uniforme et $\varphi : \mathcal{A} \to \mathbb{R}$. Alors pour tout point $u \in K_\sigma$, il existe une*

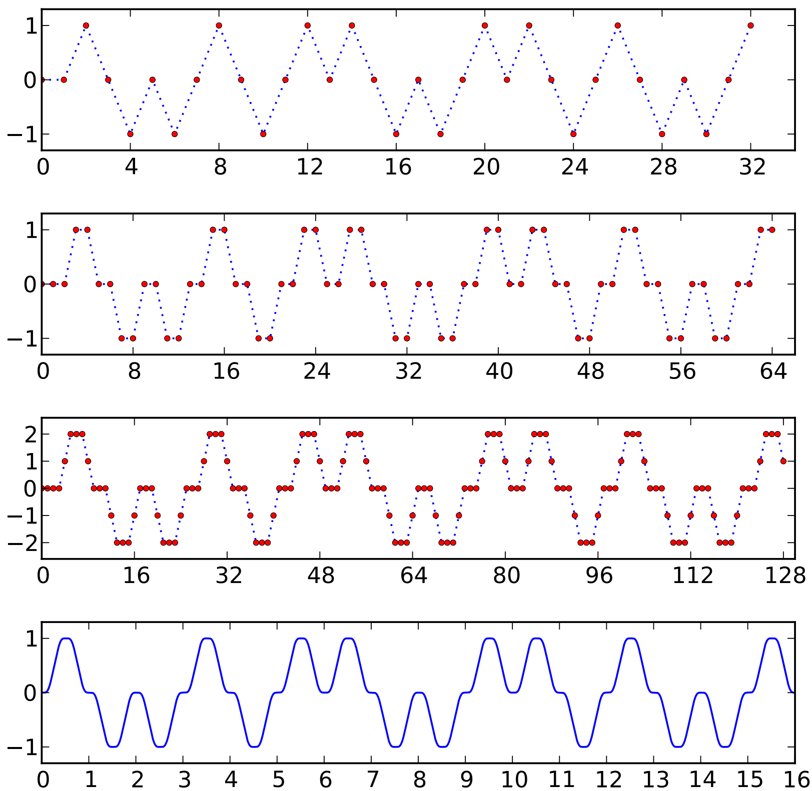

FIGURE 1.1. Les trois premières sommes $S^{(\ell)}(\varphi * u)$ et la fonction limite f_u avec u le mot de Prouet-Thue-Morse et $\varphi = \chi_a - \chi_b$.

unique suite de polynômes $(p_\ell)_{\ell\in\mathbb{N}}$ *vérifiant pour tout* $\ell \in \mathbb{N}$ *:*

$$\sup_{n\in\mathbb{N}} \left| S_n^{(\ell)}(\varphi * u) - p_\ell(n) \right| < \infty \quad \textit{où} \quad \varphi * u = \big(\varphi(u_i)\big)_{i\in\mathbb{N}}.$$

Sous la condition $p_{\ell+1}(n+1) - p_{\ell+1}(n) = p_\ell(n)$ *pour tous les entiers* $(n, \ell) \in \mathbb{N}^2$*, la suite de polynômes vérifiant cette relation est unique.*

Le cœur de l'article est l'étude asymptotique des quantités

$$\sup\{|S_n^{(\ell)}(\varphi * u) - p_\ell(n)|; n \in \mathbb{N}\}.$$

Notre approche nécessite plus de restrictions sur les substitutions. En plus d'être positives, apériodiques et fortement uniformes, nous imposons

1. que l'alphabet $\mathcal{A}$ soit constitué de deux lettres,
2. que les substitutions soient «non dégénérées» (voir la condition $\delta_2 \neq 0$ dans les deux énoncés ci-dessous).

Sous ces deux conditions, nous donnons une asymptotique précise et un théorème limite pour les sommes de Birkhoff itérées.

Si σ est une substitution uniforme sur $\{a, b\}$ de longueur λ alors pour toute fonction injective $\varphi : \{a, b\} \to \mathbb{R}$ et pour tout entier $\ell \geq 1$, nous définissons :

$$\delta_\ell = \frac{S_\lambda^{(\ell)}\big(\varphi * \sigma(a)\big) - S_\lambda^{(\ell)}\big(\varphi * \sigma(b)\big)}{\varphi(a) - \varphi(b)}.$$

Remarquons que ces expressions ne dépendent pas de la fonction injective φ choisie et que $\delta_1 = 0$ si et seulement si σ est fortement uniforme (voir lemme 10 section 2.2). Par exemple si $\varphi = \chi_a$, alors $\delta_1 = |\sigma(a)|_a - |\sigma(b)|_b$. Nous pouvons maintenant énoncer un résultat important de cet article :

Théorème 4. — *Soient σ une substitution fortement uniforme sur $\{a, b\}$ de longueur λ vérifiant $\delta_2 \neq 0$. Soient u un point fixe de σ et $\varphi : \{a, b\} \to \mathbb{R}$ une fonction injective. Soit $(p_\ell)_{\ell \in \mathbb{N}}$ la suite de polynômes du théorème 3. Alors la suite de terme général :*

$$\frac{1}{\delta_2^{\ell-1} \lambda^{(\ell-1)(\ell-2)/2}} \sup_{n \geq 0} \left| S_n^{(\ell)}\big(\varphi * u\big) - p_\ell(n) \right| \quad \textit{converge lorsque } \ell \textit{ tend vers } +\infty.$$

Afin de donner un sens précis à la limite des sommes de Birkhoff nous introduisons la concaténation de fonctions.

Définition 5. — Soient $\tau \in \mathbb{R}_+^*$, $f_a : [0, \tau[\to \mathbb{R}$ et $f_b : [0, \tau[\to \mathbb{R}$ deux fonctions et $u = u_0 u_1 \cdots u_{n-1}$ un mot de longueur n sur $\{a, b\}$, nous noterons $f_u : [0, n\tau[\to \mathbb{R}$ la fonction définie par :

$$f_u(k\tau + x) = f_{u_k}(x) \quad \text{pour } x \in [0, \tau[\text{ et } 0 \leq k \leq n - 1.$$

Nous dirons que la fonction f_u est la *concaténation des fonctions f_a et f_b le long du mot u*. Cette définition se prolonge aux mots infinis u et la fonction f_u est alors définie sur le domaine $\mathbb{R}_+$.

Notons que pour les mots de longueur 1, nous retrouvons bien les fonctions de départ f_a et f_b.

Théorème 6. — *Soient σ une substitution vérifiant les conditions du théorème 4 et u un point fixe de σ. Fixons une fonction injective $\varphi : \{a, b\} \to \mathbb{R}$ et $(p_\ell)_{\ell \in \mathbb{N}}$ la suite de polynômes du théorème 3. Alors il existe deux fonctions dérivables non-nulles f_a et f_b définies sur $[0, \lambda]$ telles que la concaténation f_u vérifie pour tout réel $x \geq 0$ de la forme $x = \frac{n}{\lambda^{m-1}}$:*

$$f_u(x) = f_u\left(\frac{n}{\lambda^{m-1}}\right) = \lim_{\ell \to +\infty} \frac{S_{n\lambda^\ell}^{(m+\ell)}\big(\varphi * u\big) - p_{m+\ell}(n\lambda^\ell)}{\delta_2^{\ell-1} \lambda^{(\ell-1)(\ell-2)/2}}.$$

De plus, pour tout $x \in \mathbb{R}_+$: $\quad f_u'(x) = \dfrac{\lambda}{\delta_2} f_u(\lambda x)$.

Cette dernière équation différentielle peut s'exprimer de manière équivalente sous la forme intégrale suivante :

$$\int_0^{\lambda x} f_u(t)\,dt = \delta_2\big(f_u(x) - f_u(0)\big) \qquad \text{pour tout } x \geq 0.$$

Pour $\tau \in \mathbb{R}^+$ et $\nu \in \mathbb{R}^*$, nous revenons dans l'annexe B sur les équations intégrales de la forme :

$$(E_{\tau,\nu}) \qquad \int_0^{\tau x} f(t)\,dt = \nu\big(f(x) - f(0)\big) \qquad \text{pour tout } x \geq 0.$$

Nous pouvons citer les travaux de T. Yoneda qui obtient pour chaque paramètre (τ, ν) une solution particulière de l'équation fonctionnelle (voir [16, 17]). Pour le cas $\tau = 2$ et $\nu = 1$, cette solution est la fonction de Fabius. Lorsque $\tau > 2$ est un entier, ces solutions sont distinctes de celles obtenues via le théorème 6.

1.2. Exemples. — Présentons quelques exemples de fonctions limites obtenues dans le théorème 6.

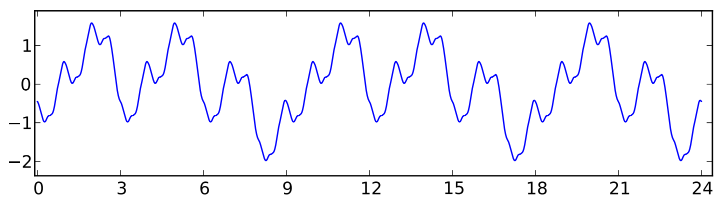

FIGURE 1.2. Graphe de la fonction f_u pour le point fixe commençant par a de la substitution $(a \mapsto aab, b \mapsto aba)$ obtenue avec notre construction pour la fonction $\varphi = \chi_a - 2\chi_b$.

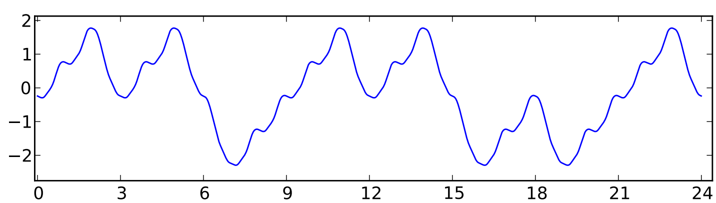

FIGURE 1.3. Graphe de la fonction f_u pour le point fixe commençant par a de la substitution $(a \mapsto aab, b \mapsto baa)$ obtenue avec notre construction avec la fonction $\varphi = \chi_a - 2\chi_b$.

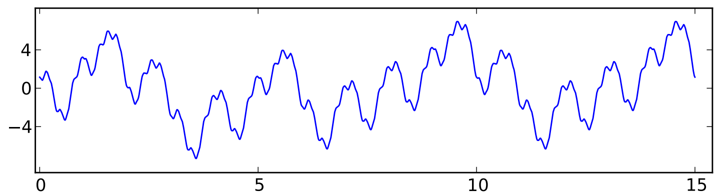

FIGURE 1.4. Graphe de la fonction f_u pour le point fixe commençant par a de la substitution $(a \mapsto abbaa, b \mapsto baaab)$ obtenue avec notre construction avec la fonction $\varphi = 2\chi_a - 3\chi_b$.

1.3. Organisation de l'article et notations. — Le premier résultat que nous obtenons sur l'approximation des sommes de Birkhoff itérées (théorème 3) est prouvé dans la section 2.5. Ces sommes sont particulièrement faciles à calculer pour des fonctions qui sont des cobords « infinis » (voir le lemme 11) et nous montrons que les fonctions qui ne dépendent que de la première coordonnée sont de cette forme.

Afin de prouver le théorème 4, nous recherchons une relation de récurrence reliant les sommes de Birkhoff itérées d'ordre $\ell + 1$ aux temps $n\lambda$ à celles d'ordre $j \leq \ell$ aux temps n. Pour cela, nous revenons sur la définition des quantités $(\delta_\ell)_{\ell \in \mathbb{N}^*}$ et en introduisons de nouvelles notées $(\gamma_\ell)_{\ell \in \mathbb{N}^*}$.

Si σ est une substitution *uniforme* de longueur λ sur $\{a, b\}$ alors pour toute fonction injective $\varphi : \{a, b\} \to \mathbb{R}$ et pour tout entier $\ell \geq 1$, nous définissons les quantités :

$$\delta_\ell = \frac{S_\lambda^{(\ell)}\big(\varphi * \sigma(a)\big) - S_\lambda^{(\ell)}\big(\varphi * \sigma(b)\big)}{\varphi(a) - \varphi(b)}$$
(1)
$$\text{et} \quad \gamma_\ell = \frac{\varphi(b) S_\lambda^{(\ell)}\big(\varphi * \sigma(a)\big) - \varphi(a) S_\lambda^{(\ell)}\big(\varphi * \sigma(b)\big)}{\varphi(b) - \varphi(a)}.$$

Si $\ell \geq \lambda + 1$, alors $\delta_\ell = 0$ et nous noterons :

$$\delta = \sum_{\ell=0}^{\lambda-1} \delta_{\ell+1} X^\ell = \sum_{\ell=0}^{+\infty} \delta_{\ell+1} X^\ell \quad \text{et} \quad \gamma = \sum_{\ell=0}^{\lambda-1} \gamma_{\ell+1} X^\ell = \sum_{\ell=0}^{+\infty} \gamma_{\ell+1} X^\ell.$$

Rappelons que les quantités $(\delta_\ell)_{\ell \in \mathbb{N}^*}$ ne dépendent que de la substitution σ et non pas de la fonction injective φ choisie (lemme 10 de la section 2.2).

Pour démontrer le théorème 4, nous commençons par introduire dans la section 3 un groupe $\mathcal{G}$ construit comme produit semi-direct de l'espace des séries formelles par $\mathbb{R}$. Dans ce groupe, pour toute suite u de réels, les sommes de Birkhoff $\big(S_n^{(\ell)}(u)\big)_{(n,\ell) \in \mathbb{N}^2}$ apparaissent naturellement comme des produits dans $\mathcal{G}$ (voir la section 3.2).

Étant données une substitution uniforme σ de longueur λ et une fonction injective $\varphi : \{a,b\} \to \mathbb{R}$, nous construisons dans la section 3.5 un morphisme π_φ de l'ensemble des mots finis sur $\{a,b\}$ (muni de la concaténation) à valeurs dans $\mathcal{G}$. Nous construisons ensuite dans la section 3.6 (proposition 25) un endomorphisme $\mathcal{L}$ de $\mathcal{G}$ qui généralise l'abélianisé d'une substitution et qui vérifie $\mathcal{L} \circ \pi_\varphi = \pi_\varphi \circ \sigma$. Si $\delta_1 = 0$, cet endomorphisme permet de relier les sommes de Birkhoff d'ordre $\ell+1$ et j pour $j \le \ell$ aux temps n et $n\lambda$ pour tout mot u selon la formule de récurrence :
(2)

$$S_{n\lambda}^{(\ell+1)}\big(\varphi * \sigma(u)\big) = \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i}(\lambda) S_n^{(j+1)}\big(\varphi * u\big)\delta_{i+1} + \sum_{j=0}^{\ell}\sum_{i=0}^{\ell-j} q_{j,\ell-i}(\lambda)\gamma_{i+1}\binom{n}{j+1}.$$

Dans la section 4.1, nous étudions les polynômes $(q_{j,\ell})_{(j,\ell)\in\mathbb{N}^2}$ (définition 26) intervenant dans cette formule : nous estimons leur comportement asymptotique (proposition 31) et en donnons un majorant (lemme 32).

L'endomorphisme $\mathcal{L}$ permet aussi de décrire précisément les polynômes d'approximation $(p_\ell)_{\ell\in\mathbb{N}}$ (voir section 4.2).

La condition de non-nullité de δ_2 nous permet de décrire la croissance des sommes itérées. Pour cela nous définissons les coefficients de renormalisation $(\rho_\ell)_{\ell\in\mathbb{N}}$ de la manière suivante :

Définition 7. — Pour tout $\ell \in \mathbb{N}$, nous définissons : $\rho_\ell = \delta_2^{\ell-1}\lambda^{(\ell-1)(\ell-2)/2}$.

Nous notons pour tout entier $\ell \in \mathbb{N}$, $c_\ell = p_\ell(0)$ et $\tilde{c}_\ell = c_\ell/\rho_\ell$. Une difficulté importante dans ce travail est d'étudier la croissance de cette suite $(c_\ell)_{\ell\in\mathbb{N}}$ et nous montrons (proposition 38) que la suite $(\tilde{c}_\ell)_{\ell\in\mathbb{N}}$ converge. Nous apportons alors la preuve du théorème 4 dans la section 4.4.

Nous montrons la proposition suivante dans la section 5 en étudiant le comportement asymptotique des puissances de $\mathcal{L}$ dans la proposition 40 :

Proposition 8. — *Sous les hypothèses du théorème 4, il existe une fonction* $\Psi : \mathbb{N}^2 \to \mathbb{R}$ *définie pour tous les entiers* m *et* n *par :*

$$\lim_{\ell\to+\infty} \tilde{S}_{n\lambda^\ell}^{(m+\ell)}\big(\varphi * u\big) = \Psi(m,n) \quad \text{où} \quad \tilde{S}_n^{(\ell)}\big(\varphi * u\big) = \frac{1}{\rho_\ell}\Big(S_n^{(\ell)}\big(\varphi * u\big) - p_\ell(u)\Big).$$

De plus, la suite $\big(\Psi(1,n)\big)_{n\in\mathbb{N}^*}$ *n'est pas identiquement nulle.*

La condition de non-nullité permet de s'assurer que la limite obtenue dans le théorème 4 n'est pas nulle.

Ce résultat nous permet dans la section 6 de définir correctement la fonction solution annoncée dans le théorème 6. En effet, la fonction solution f_u est définie par :

$$f_u\left(\frac{n}{\lambda^{m-1}}\right) = \Psi(m,n) \quad \text{pour tout couple d'entiers } (n,m) \in \mathbb{N}^2.$$

Nous vérifions également dans cette section les différentes propriétés vérifiées par cette fonction.

Nous revenons enfin dans l'annexe B sur les équations intégrales $(E_{\tau,\nu})$.

Remerciements. — Nous tenons à remercier X. Bressaud qui nous a aidé au lancement de ce projet ainsi que N. Bédaride pour ses nombreux conseils tout au long de l'écriture de cet article.

2. Quelques rappels de combinatoire et preuve du théorème 3

2.1. Mots finis, mots infinis et substitutions. — Soit $\mathcal{A}$ un ensemble fini que l'on appelle *alphabet*. Dans la suite, nous prendrons très souvent $\mathcal{A} = \{a, b\}$. Les éléments de $\mathcal{A}$ sont appelés des *lettres*. Les lettres forment par concaténation des mots (finis) dont *la longueur* est le nombre de lettres dont ils sont constitués. Nous notons $\mathcal{A}^*$ l'ensemble des *mots finis* sur $\mathcal{A}$. Il s'agit d'un monoïde pour la concaténation et ε désigne le mot vide. Un *facteur* d'un mot $u = u_0 u_1 \dots u_{n-1}$ est un mot v de la forme $v = u_i u_{i+1} \dots u_{i+k-1}$. Nous considérons également l'ensemble des *mots infinis* $\mathcal{A}^{\mathbb{N}}$ que nous munissons de la topologie produit, ce qui en fait un espace compact. Une *substitution* sur $\mathcal{A}$ est un endomorphisme de $\mathcal{A}^*$ vu comme monoïde. Une substitution est *positive* si chaque image de lettre contient toutes les lettres de l'alphabet. Par exemple la substitution de Prouet-Thue-Morse $a \mapsto ab, b \mapsto ba$ est positive. Le *langage* d'une substitution positive est le plus petit sous-ensemble de $\mathcal{A}^*$ contenant $\mathcal{A}$, stable par facteur et par σ. C'est aussi l'ensemble des facteurs des mots $\sigma^n(\alpha)$ où α est une lettre de l'alphabet et n un entier. Par exemple, le langage de la substitution de Prouet-Thue-Morse est :

$$\{\ \varepsilon,\ a,\ b,\ aa,\ ab,\ ba,\ bb,\ aab,\ aba,\ abb,\ baa,\ bab,\ bba,\ \dots\ \}.$$

À une substitution positive σ, nous associons également le *sous-décalage associé* noté K_σ, qui est un sous-ensemble de $\mathcal{A}^{\mathbb{N}}$ formé des mots infinis dont tous les facteurs appartiennent au langage de σ. C'est un ensemble compact et non-vide. La substitution σ agit continûment sur K_σ. Un mot est *substitutif* s'il est le point fixe d'une substitution, comme par exemple, le mot de Prouet-Thue-Morse commençant par :

$$abbabaabbaababbabaababbaabbabaabbaababbaabbabaababbabaabbaababbabaaba\dots$$

Le *décalage* sur $A^{\mathbb{N}}$ est l'application qui consiste à décaler l'origine $T : (u_i)_{i\in\mathbb{N}} \mapsto (u_{i+1})_{i\in\mathbb{N}}$. $A^{\mathbb{N}}$ est un ensemble compact et T est une application continue. L'*orbite* d'un point $u \in A^{\mathbb{N}}$ est l'ensemble de points $\{T^n(u); n \in \mathbb{N}\}$.

Si σ est une substitution positive, alors K_σ est invariant par T : c'est un *sous-décalage* de $\mathcal{A}^{\mathbb{N}}$. Il est *minimal* : pour tout mot $u \in K_\sigma$ son orbite est dense dans K_σ.

2.2. Sommes de Birkhoff itérées. — Soit $u = (u_n)_{n\in\mathbb{N}}$ une suite de nombres réels. Reprenant la définition 1, les sommes de Birkhoff itérées d'ordre $\ell \in \mathbb{N}^*$ au rang $n \in \mathbb{N}^*$ vérifient la relation :

$$S_{n+1}^{(\ell)}(u) = S_n^{(\ell)}(u) + S_n^{(\ell-1)}(u), \tag{3}$$

où par convention : $S_n^{(0)}(u) = u_n$ pour tout entier $n \geq 1$.

Autrement dit, la double suite $(S_n^{(\ell)}(u))_{(n,\ell)\in\mathbb{N}^2}$ est un triangle de Pascal généralisé, pour lequel la première colonne, constituée de 1 dans le triangle de Pascal, est remplacée par u :

$n \backslash \ell$	0	1	2	3	4	5
0	u_0	0	0	0	0	0
1	u_1	u_0	0	0	0	0
2	u_2	u_0+u_1	u_0	0	0	0
3	u_3	$u_0+u_1+u_2$	$2u_0+u_1$	u_0	0	0
4	u_4	$u_0+u_1+u_2+u_3$	$3u_0+2u_1+u_2$	$3u_0+u_1$	u_0	0
5	u_5	$u_0+u_1+u_2+u_3+u_4$	$4u_0+3u_1+2u_2+u_3$	$6u_0+3u_1+u_2$	$4u_0+u_1$	u_0

TABLE 2.1. Premières valeurs des sommes itérées $\big(S_n^{(\ell)}(u)\big)_{(n,\ell)\in\mathbb{N}^2}$ pour $0 \leq n, \ell \leq 5$.

On peut montrer aisément par récurrence que pour tous les entiers $0 \leq \ell \leq n$, $S_n^{(\ell)}(u)$ s'exprime avec les coefficients $u_0, \ldots, u_{n-\ell}$ selon la formule :

$$S_n^{(\ell)}(u) = \sum_{k=0}^{n-\ell} \binom{n-k-1}{\ell-1} u_k.$$

Soit σ une substitution positive, K_σ le sous-décalage associé et $\varphi : K_\sigma \to \mathbb{R}$ une application continue. Le théorème suivant est un résultat classique de théorie ergodique. Il est souvent écrit avec $\ell = 1$. L'estimation pour $\ell \geq 2$ découle directement du cas $\ell = 1$ en remarquant que les sommes au rang $\ell + 1$ sont les sommes de celles au rang ℓ. Nous renvoyons à [10] pour des détails.

PROPOSITION 9 (Unique ergodicité). — *Soient σ une substitution positive et (K_σ, T) le décalage associé. Alors il existe une unique mesure de probabilité μ sur K_σ invariante par T. De plus, pour toute fonction $\varphi \in C(K_\sigma)$, pour tout entier $\ell \geq 1$, on a uniformément en $u \in K_\sigma$ lorsque n tend vers $+\infty$:*

$$S_n^{(\ell)}(\varphi * u) - \binom{n}{\ell} \int_{K_\sigma} \varphi d\mu = o(n^\ell).$$

Ce résultat montre que la première échelle d'approximation de $S_n^{(\ell)}\big(\varphi * u\big)$ est $\binom{n}{\ell}\int_{K_\sigma}\varphi d\mu$. Dans cet article, pour certaines substitutions et certaines fonctions, nous montrons que les sommes de Birkhoff restent bornées quitte à soustraire un polynôme. Ce résultat est très spécifique et ne concerne pas toutes les substitutions. On trouvera une étude précise de la croissance des sommes de Birkhoff d'ordre 1 pour une substitution quelconque dans [1].

LEMME 10. — *Soient σ une substitution uniforme de longueur $\lambda \in \mathbb{N}^*$ et $\varphi : \{a,b\} \to \mathbb{R}$ une fonction injective. Nous associons les quantités $(\delta_\ell)_{1\leq \ell \leq \lambda}$ définies en* (1).

1. *Les quantités $(\delta_\ell)_{1\leq\ell\leq\lambda}$ ne dépendent pas de la fonction injective φ choisie.*
2. *La substitution (uniforme) est fortement uniforme si et seulement si $\delta_1 = 0$.*

Preuve du lemme 10. — 1. Fixons un entier $\ell \in \mathbb{N}^*$. D'après la définition 1 des sommes itérées, l'application $S^{(\ell)} : u \mapsto S^{(\ell)}(u)$ est un endomorphisme de l'ensemble des suites réelles $\mathbb{R}^{\mathbb{N}}$. Fixons maintenant une fonction φ, alors pour tous les réels x et y :

$$S_\lambda^{(\ell)}\big((x\varphi+y)*\sigma(a)\big)-S_\lambda^{(\ell)}\big((x\varphi+y)*\sigma(b)\big) = x\Big(S_\lambda^{(\ell)}\big(\varphi*\sigma(a)\big)-S_\lambda^{(\ell)}\big(\varphi*\sigma(b)\big)\Big)$$

Prenons maintenant une seconde fonction injective $\psi : \{a,b\} \to \mathbb{R}$. Alors il existe deux réels x_0 et y_0 tels que $\psi = x_0\varphi + y_0$. En divisant la relation précédente par $\psi(a)-\psi(b) = x_0\big(\varphi(a)-\varphi(b)\big)$ avec $x = x_0$ et $y = y_0$, nous trouvons :

$$\frac{S_\lambda^{(\ell)}\big(\psi*\sigma(a)\big)-S_\lambda^{(\ell)}\big(\psi*\sigma(b)\big)}{\psi(a)-\psi(b)} = \frac{S_\lambda^{(\ell)}\big(\varphi*\sigma(a)\big)-S_\lambda^{(\ell)}\big(\varphi*\sigma(b)\big)}{\varphi(a)-\varphi(b)}$$

La quantité δ_ℓ est donc bien indépendante de la fonction φ injective choisie.

2. Il suffit pour cela de calculer δ_1. Nous notons $n_{\alpha,\beta} = |\sigma(\alpha)|_\beta$ pour deux lettres $(\alpha,\beta) \in \mathcal{A}^2$. Nous commençons par calculer les expressions suivantes :

$$S_\lambda^{(1)}\big(\varphi*\sigma(a)\big) = n_{a,a}\varphi(a)+n_{a,b}\varphi(b) \quad \text{et} \quad S_\lambda^{(1)}\big(\varphi*\sigma(b)\big) = n_{b,a}\varphi(a)+n_{b,b}\varphi(b).$$

Nous utilisons les relations $n_{a,b} = \lambda - n_{a,a}$ et $n_{b,b} = \lambda - n_{b,a}$:

$$\begin{aligned} S_\lambda^{(1)}\big(\varphi*\sigma(a)\big)-S_\lambda^{(1)}\big(\varphi*\sigma(b)\big) &= n_{a,a}\varphi(a)+n_{a,b}\varphi(b)-n_{b,a}\varphi(a)-n_{b,b}\varphi(b)\\ &= \big(n_{a,a}-n_{b,a}\big)\cdot\big(\varphi(a)-\varphi(b)\big). \end{aligned}$$

En divisant par $\varphi(a)-\varphi(b)$, nous trouvons donc $\delta_1 = n_{a,a}-n_{b,a}$. Cette quantité est nulle si et seulement s'il y a le même nombre de a dans l'image de b et de a. Donc également le même nombre de b, ce qui est la définition d'une substitution fortement uniforme. □

2.3. Cobords infinis. — Nous fixons un décalage minimal $K \subset \mathcal{A}^{\mathbb{N}}$. Une fonction continue $\varphi : X \to \mathbb{R}$ est un *cobord continu* s'il existe une fonction continue $\psi : X \to \mathbb{R}$ telle que $\varphi = \psi \circ T - \psi$. Autrement dit, si la fonction φ est dans l'image de l'opérateur $U_T - I : \psi \mapsto \psi \circ T - \psi$. Si φ est un cobord, alors la fonction ψ telle que $\varphi = \psi \circ T - \psi$ est déterminée à une constante près. Si $\varphi = \psi \circ T - \psi$ est un cobord, alors sa n-ème somme de Birkhoff se réécrit simplement en fonction de ψ :

$$S_n^{(1)}\big(\varphi * u\big) = \psi(T^n u) - \psi(u).$$

En particulier, la suite $\big(S_n^{(1)}\big(\varphi * u\big)\big)_{n\in\mathbb{N}}$ est bornée. La réciproque est vraie, c'est *le théorème de Morse-Hedlund* : si K est minimal, une fonction continue $\varphi : K \to \mathbb{R}$ dont la somme de Birkhoff est bornée est un cobord continu.

Nous dirons que la fonction φ est *un cobord infini* s'il existe une constante ψ_0 et une suite de fonctions continues $(\psi_n)_{n\geq 1}$ telles que

$$\begin{cases} \varphi + \psi_0 = \psi_1 \circ T - \psi_1, \\ \psi_n = \psi_{n+1} \circ T - \psi_{n+1} \quad \text{pour tout entier } n \in \mathbb{N}^*. \end{cases}$$

Autrement dit, la fonction φ est à une constante près dans $\bigcap_{n\in\mathbb{N}} (U_T - I)^n C(K)$. La suite de fonctions $(\psi_n)_{n\in\mathbb{N}}$ est unique et pour tout entier $n \in \mathbb{N}^*$, la moyenne de ψ_n selon toute mesure T-invariante est nulle. Avec ces notations, si φ est un cobord infini, alors pour tout mot $u \in K$ et tout entier $n \in \mathbb{N}$, un calcul direct donne :

$$\begin{aligned} S_n^{(1)}\big(\varphi * u\big) &= \psi_1 \circ T^n(u) - \psi_1(u) - n\psi_0, \\ S_n^{(2)}\big(\varphi * u\big) &= \psi_2(T^n u) - \psi_2(u) - n\psi_1(u) - \tbinom{n}{2}\psi_0, \\ S_n^{(3)}\big(\varphi * u\big) &= \psi_3(T^n u) - \psi_3(u) - n\psi_2(u) - \tbinom{n}{2}\psi_1(u) - \tbinom{n}{3}\psi_0. \end{aligned}$$

Nous déduisons aisément par récurrence que pour tous les entiers $n \geq 0$ et $\ell \geq 0$,

$$S_n^{(\ell)}\big(\varphi * u\big) = \psi_\ell(T^n u) - \sum_{j=0}^{\ell} \binom{n}{\ell - j} \psi_j(u). \tag{4}$$

En particulier, en notant $p_{\ell,u}$ le polynôme prenant les valeurs $-\sum_{k=0}^{\ell} \binom{n}{\ell-k}\psi_k(u)$ aux points n entiers, alors $S_n^{(\ell)}\big(\varphi * u\big) - p_{\ell,u}(n) = \psi_\ell(T^n u)$ est une suite bornée. Nous venons donc de démontrer le résultat suivant :

Lemme 11. — *Soient K un sous-décalage minimal de $\mathcal{A}^{\mathbb{N}}$, $u \in K$ et $\varphi : K \to \mathbb{R}$ un cobord infini continu. Alors il existe une suite de polynômes $(p_{\ell,u})_{\ell\in\mathbb{N}}$ tels que pour tout entier ℓ, la suite $\big(S_n^{(\ell)}\big(\varphi * u\big) - p_{\ell,u}(n)\big)_{n\in\mathbb{N}}$ est bornée.*

Les propriétés élémentaires vérifiées par les coefficients binomiaux nous assurent que pour tous les entiers n et ℓ : $\quad p_{\ell+1,u}(n+1) - p_{\ell+1,u}(n) = p_{\ell,u}(n)$.

2.4. Odomètres. — Nous montrons dans cette section que les sommes de Birkhoff itérées sur des odomètres sont à distance bornées d'une suite de polynômes. Fixons un entier $\lambda \geq 2$ et commençons par rappeler la structure de l'odomètre $(\mathbb{Z}_\lambda, +1)$. L'anneau $\mathbb{Z}_\lambda$ est homéomorphe au produit $\{0, 1, \ldots, \lambda - 1\}^{\mathbb{N}}$. Autrement dit, chaque élément de $\mathbb{Z}_\lambda$ est une suite $(a_0, a_1, a_2, \ldots)$. Cette suite peut être vue comme une série $\sum a_i \lambda^i$ (qui est convergente dans $\mathbb{Z}_\lambda$). Les suites finies (*i.e.* se terminant par des zéros) correspondent exactement aux entiers écrits en base λ. L'anneau $\mathbb{Z}_\lambda$ admet des projections $\mathbb{Z}_\lambda \to \mathbb{Z}/\lambda^n\mathbb{Z}$ qui sont des homomorphismes d'anneaux. Une définition possible de $\mathbb{Z}_\lambda$ consiste à le voir comme la limite projective de ces groupes. Du point de vue des suites, ces projections correspondent à la troncation. Lorsque $\lambda = p$ est un nombre premier, il s' agit de l'anneau des entiers p-adiques habituel.

L'odomètre correspond à l'addition de 1 dans l'anneau $\mathbb{Z}_\lambda$ (voir [13] chapitre 4). Notons que l'odomètre est uniquement ergodique et l'unique mesure invariante est la mesure produit des mesures uniformes sur $\{0, 1, \ldots, \lambda - 1\}$.

Il est possible de voir l'anneau $\mathbb{Z}_\lambda$ comme une suite de partitions périodiques de $\mathbb{Z}$. Plus précisément, soit $a = (a_0, a_1, \ldots)$ un élément fixé de $\mathbb{Z}_\lambda$. On définit pour chaque entier $k \in \mathbb{N}$ et chaque entier $m \in \mathbb{Z}$ un intervalle d'entiers :

$$I_{k,m}(a) = a_0 + a_1\lambda + \ldots + a_{k-1}\lambda^{k-1} + [\![m\lambda^k, (m+1)\lambda^k - 1]\!].$$

Chaque intervalle $I_{k,m}(a)$ a pour longueur λ^k et pour chaque entier k, la suite $\{I_{k,m}(a)\}_{m\in\mathbb{Z}}$ forme une partition périodique de $\mathbb{Z}$. D'autre part, chaque bloc de niveau $k \geq 1$ est composé de λ blocs de niveau $k+1$ (voir figure 2.1) :

$$I_{k,m}(a) = \bigcup_{b=-a_k}^{\lambda - a_k - 1} I_{k+1, m+\lambda b}(a). \tag{5}$$

On peut numéroter ces λ blocs de niveau $k+1$ de 0 à $\lambda - 1$. Il est alors facile de voir à partir de (5) que l'élément $0 \in \mathbb{Z}$ est dans le a_k-ème bloc de génération k. Cette construction s'inverse et fournit une bijection entre $\mathbb{Z}_\lambda$ et ces systèmes de partitions emboîtées (voir la figure 2.1).

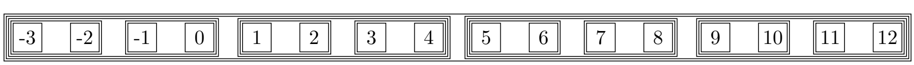

FIGURE 2.1. $\mathbb{Z}_2$ vu comme une structure de partitions emboîtées de $\mathbb{Z}$. Ici il s'agit d'un point $a = (1, 1, 0, \ldots)$. L'odomètre sur $\mathbb{Z}_2$ correspond alors au décalage de ces blocs.

L'odomètre correspond alors au décalage de ces blocs, en effet nous avons l'égalité des partitions :

$$\{I_{k,m}(a+1)\}_{m\in\mathbb{Z}} = \{I_{k,m}(a) + 1\}_{m\in\mathbb{Z}}.$$

Signalons que nous venons de décrire le *diagramme de Bratelli-Vershik* associé à l'odomètre. Nous renvoyons à [8] pour plus de généralités sur ces constructions et leurs liens avec les substitutions.

PROPOSITION 12. — *Soit $\varphi : \mathbb{Z}_\lambda \to \mathbb{R}$ une fonction de moyenne nulle pour l'unique mesure invariante pour l'odomètre. Nous supposons que φ ne dépend que des k premières coordonnées. Alors il existe une suite $(\psi_n)_{n\in\mathbb{N}}$ de fonctions de moyennes nulles telles que*

- $\psi_0 = \varphi$,
- *pour tout entier $n \geq 0$, la fonction ψ_n ne dépend que des k premières coordonnées,*
- *pour tout $a \in \mathbb{Z}_\lambda$, $\psi_n(a) = \psi_{n+1}(a+1) - \psi_{n+1}(a)$.*

En particulier, pour tout $a \in \mathbb{Z}_\lambda$ et tout entier ℓ, si nous notons $u = \big(\varphi(a), \varphi(a+1), \dots\big)$, alors la somme itérée $S^{(\ell)}(u)$ est la somme d'un polynôme de degré au plus ℓ et d'une suite périodique de période au plus λ^k.

Preuve de la proposition 12. — Comme φ ne dépend que de k coordonnées, on peut la voir comme une application de $\mathbb{Z}/\lambda^k\mathbb{Z} \to \mathbb{R}$. Soient $a = (a_0, a_1, \dots)$ un point de $\mathbb{Z}_\lambda$ et $u = \big(\varphi(a), \varphi(a+1), \varphi(a+2), \dots\big)$. La suite u est périodique de période (au plus) λ^k. Comme f est de moyenne nulle, pour tout entier i nous avons $\sum_{j=0}^{\lambda^k-1} u_{i+j} = 0$. Ainsi, $S^{(1)}(u)$ est également périodique de période au plus k.

On définit la suite $(v_n)_{n\in\mathbb{N}}$ par :

$$v_0 = -\frac{(k-1)u_0 + (k-2)u_1 + \ldots + u_{k-2}}{k} \quad \text{et pour } n \in \mathbb{N} : \quad v_{n+1} = v_n + u_n.$$

Cette suite vérifie par construction : $v_{n+1} - v_n = u_n$. La fonction $\psi_1 : \mathbb{Z}_\lambda \to \mathbb{R}$ définie par $\psi_1(n) = v_{n \mod \lambda^k}$ vérifie alors :

- pour tout $a \in \mathbb{Z}_\lambda$, $\varphi(a) = \psi_1(a+1) - \psi_1(a)$,
- ψ_1 est périodique de période au plus λ^k,
- ψ_1 est de moyenne nulle.

On peut itérer cette construction et définir la suite de fonctions $\psi_2, \psi_3, \dots$. Ceci démontre la première partie de la proposition. La fonction φ est alors un cobord infini et nous pouvons lui associer naturellement une unique suite de polynômes, selon le processus décrit dans la partie 2.3 précédente. Comme chaque fonction ψ_n ne dépend que des k premières coordonnées cela prouve la seconde partie. □

2.5. Preuve du théorème 3. — Nous montrons dans cette section que les sommes de Birkhoff itérées sur les sous-décalages de substitution fortement uniforme sont à distance bornées d'une suite de polynômes.

L'énoncé suivant est une reformulation du théorème 3 de l'introduction.

PROPOSITION 13. — *Soit $\sigma : \mathcal{A} \to \mathcal{A}$ une substitution positive, apériodique et fortement uniforme de longueur λ et $\varphi : \mathcal{A} \to \mathbb{R}$. Soit K_σ le sous-décalage associé à σ. Alors il existe une suite de fonctions $(\psi_n)_{n\in\mathbb{N}}$ telles que*

- $\psi_0 = \varphi$,
- ψ_n *ne dépend que des λ^n premières coordonnées,*
- $\psi_n = \psi_{n+1} \circ T - \psi_{n+1}$,

En particulier, pour tout mot infini $u \in K_\sigma$ et tout entier ℓ, la suite $S^{(\ell)}(\varphi * u)$ est la somme d'un polynôme de degré au plus ℓ et d'une suite périodique de période au plus λ^ℓ où nous rappelons que $\varphi * u = \big(\varphi(u), \varphi(Tu), \varphi(T^2u), \dots\big)$.

Preuve de la proposition 13. — La preuve de cette proposition repose sur une décomposition du sous-décalage K_σ (similaire à (5)). Cette décomposition n'est valide que sur le sous-décalage bilatère $\widetilde{K}_\sigma \subset \mathcal{A}^{\mathbb{Z}}$ associé à σ. Le théorème 3 ne souffre pas de cette restriction et afin de ne pas alourdir les notations, K_σ désignera encore le sous-décalage bilatère associé à σ tout au long de cette preuve.

Soit v_α le nombre d'occurrences de la lettre α dans une image $\sigma(\beta)$. Nous avons $\lambda = \sum_{\alpha\in\mathcal{A}} v_\alpha$. Pour une lettre α, nous notons pour $0 \le m < \lambda^k$:

$$\mathrm{Cyl}(k, m, \alpha) = T^m \sigma^k([\alpha]) \subset K_\sigma \subset \mathcal{A}^{\mathbb{Z}}.$$

Pour tout entier k, nous notons F^k l'ensemble des fonctions de $K_\sigma \subset \mathcal{A}^{\mathbb{Z}}$ dans $\mathbb{R}$ qui sont constantes sur chaque cylindre $\mathrm{Cyl}(k, m, \alpha)$ et F_0^k les fonctions de F^k de moyenne nulle. La proposition 13 découle alors des deux lemmes suivants.

LEMME 14. — *Pour tout entier k, tout entier $m \in \{0, \dots, \lambda^k\}$ et toute lettre $\alpha \in \mathcal{A}$,*

$$\mu\big(\mathrm{Cyl}(k, m, \alpha)\big) = \frac{v_\alpha}{\lambda^{k+1}}.$$

D'autre part, si $\varphi \in F^k$ alors pour toute lettre $\alpha \in \mathcal{A}$ et tout mot $u \in \mathrm{Cyl}(k+1, 0, \alpha)$:

$$\frac{1}{\lambda^{k+1}} \sum_{m=0}^{\lambda^{k+1}-1} \varphi(T^m u) = \int_{K_\sigma} \varphi d\mu,$$

où μ est l'unique mesure de probabilité sur $K_\sigma \subset \mathcal{A}^{\mathbb{Z}}$ invariante par T.

LEMME 15. — *Soient k un entier et $\varphi \in F_0^k$. Alors φ est un cobord et si ψ est telle que $\varphi = \psi \circ T - \psi$ alors ψ appartient à F^{k+1}. Autrement dit,*

$$\forall k \in \mathbb{N}, \quad F_0^k \subset (U_T - I)(F_0^{k+1}).$$

Nous montrons tout d'abord comment déduire le théorème de ces deux lemmes. Puisque $\varphi = \psi_0$ est définie sur $\mathcal{A}$, alors $\psi_0 \in F^0$ et le lemme 15 nous assure l'existence d'une fonction $\psi_1 \in F^1$ telle que $\psi_0 = \psi_1 \circ T - \psi_1$. Quitte à considérer $\psi_1 - \int_{K_\sigma} \psi_1 d\mu$, nous pouvons supposer que $\psi_1 \in F_0^1$. Supposons avoir obtenu des fonctions $(\psi_i)_{1\leq i\leq n}$ telles que

- pour tout $1 \leq i \leq n$, $\psi_i \in F_0^i$;
- pour $1 \leq i \leq n-1$, $\psi_{i+1} \circ T - \psi_{i+1} = \psi_i$.

Alors il ne nous reste plus qu'à utiliser le lemme 15 pour obtenir une fonction ψ_{n+1} de F^{n+1} telle que $\psi_{n+1} \circ T - \psi_{n+1} = \psi_n$. Le résultat suit en renommant ψ_{n+1} la fonction $\psi_{n+1} - \int_{K_\sigma} \psi_{n+1} d\mu$. □

Pour prouver les deux lemmes ci-dessus l'ingrédient essentiel et le théorème de Mossé [14] qui nous permet de décomposer les cylindres en cylindres d'ordre supérieur (de manière similaire à (5)). Pour chaque entier $k \geq 0$, chaque cylindre se décompose dans K_σ en une réunion disjointe de $v_\alpha \times |\mathcal{A}|$ cylindres de niveau supérieur la manière suivante :

$$\mathrm{Cyl}(k, m, \alpha) = \bigcup_{\sigma(\beta)=p\alpha s} \mathrm{Cyl}(k+1, \lambda m + |p|, \beta). \tag{6}$$

Mettons en avant que pour certaines substitutions σ ce fait n'est pas vrai sur le sous-décalage unilatère associé à σ.

Preuve du lemme 14. — Comme la mesure μ est invariante, $\mu(\mathrm{Cyl}(k, m, \alpha)) = \mu(\mathrm{Cyl}(0, m, \alpha))$. Notons μ_k la mesure définie pour tout ensemble mesurable Y par :

$$\mu_k(Y) = \mu\big(T^k \sigma(Y)\big).$$

Alors d'après la décomposition (6)

- les mesures μ_0, μ_1, ..., $\mu_{\lambda-1}$ sont à supports disjoints,
- $\mu_0 + \mu_1 + \ldots + \mu_{\lambda-1}$ est une mesure de probabilité invariante.

De l'unique ergodicité de K_σ on en déduit que $\mu = \mu_0 + \mu_1 + \ldots + \mu_{\lambda-1}$. Ainsi pour tout ensemble mesurable $Y \subset K_\sigma$:

$$\mu(\sigma(Y)) = \frac{\mu(Y)}{\lambda}.$$

Ceci prouve la première partie du lemme.

Maintenant prouvons la formule pour l'intégrale de φ. Soit $u \in \mathrm{Cyl}(k+1, 0, \alpha)$. Alors, pour toute lettre $\beta \in \mathcal{A}$ et tout $m \in \{0, 1, \ldots, \lambda^k - 1\}$, la suite $\{u, Tu, T^2u, \ldots, T^{\lambda^{k+1}-1}u\}$ passe exactement v_β fois par le cylindre $\mathrm{Cyl}(k, m, \beta)$. Maintenant la fonction φ appartient à F^k, alors elle est constante sur chaque

$\mathrm{Cyl}(k,m,\beta)$ et nous avons :

$$\begin{aligned}\sum_{m=0}^{\lambda^{k+1}-1} \varphi(T^m u) &= \sum_{m=0}^{\lambda^k-1}\sum_{\beta\in\mathcal{A}} \frac{v_\beta}{\mathrm{Cyl}(k,m,\beta)} \int_{\mathrm{Cyl}(k,m,\beta)} \varphi \, d\mu \\ &= \sum_{m=0}^{\lambda^k-1}\sum_{\beta\in\mathcal{A}} \lambda^{k+1} \int_{\mathrm{Cyl}(k,m,\beta)} \varphi \, d\mu \\ &= \int_{K_\sigma} \varphi d\mu.\end{aligned}$$

La deuxième égalité provient de la formule prouvée auparavant pour les mesures des cylindres $\mathrm{Cyl}(k,m,\beta)$. □

Preuve du lemme 15. — Fixons une fonction $\varphi \in F_0^k$. Prenons un mot $u \in \mathrm{Cyl}(k+1,0,\alpha)$ et posons pour tout entier $0 \leq m < \lambda^{k+1}$, $g(T^m u) = S_m^{(1)}(\varphi,u)$. La fonction g s'étend de manière unique en une fonction de F^{k+1}. D'après la seconde partie du lemme 14, pour tout $u \in \mathrm{Cyl}(k+1,0,\alpha)$ et pour tout $m \geq 0$, $g(T^m u) = S_m^{(1)}(\varphi,u)$. Et donc pour tout $u \in K_\sigma$ et tout entier m, $S_m^{(1)}(\varphi,u) = g(T^m u) - g(u)$. □

Le théorème 3 ne se généralise pas simplement à d'autres substitutions. Les fonctions qui ne dépendent que de la première lettre et qui sont des cobords se lisent sur la matrice d'incidence de la substitution (voir [1]). Par exemple, pour les mots *sturmiens* u sur $\{a,b\}$ (qui sont des codages de rotations et dont certains sont substitutifs), il est bien connu que les fonctions $\chi_a - \mu([a])$ et $\chi_b - \mu([b])$ sont des cobords : il existe un réel α tel que :

$$\left\{ \left| S_n^{(1)}(\chi_a * u) - n\alpha \right| ; n \in \mathbb{N}^* \right\} \quad \text{est bornée.}$$

Cependant, il est montré dans [15] que pour tout paramètre $\beta \in \mathbb{R}$,

$$\left\{ \left| S_n^{(2)}(\chi_a * u) - \alpha \binom{n}{2} - \beta n \right| ; n \in \mathbb{N}^* \right\} \quad \text{n'est par bornée.}$$

3. Construction d'un groupe $\mathcal{G}$ propice aux calculs des sommes de Birkhoff itérées

3.1. Définitions. — Soit $V = \mathbb{R}[[X]]$ l'algèbre des séries formelles. Afin de définir le groupe $\mathcal{G}$, nous sommes amenés à étudier la multiplication par $A = 1 + X$ sur V :

$$A \cdot s = \sum_{\ell=0}^{+\infty} (s_\ell + s_{\ell-1}) X^\ell \quad \text{pour} \quad s = \sum_{\ell=0}^{+\infty} s_\ell X^\ell \in V,$$

où par convention $s_{-1} = 0$. Les puissances entières de A sont simplement données par le binôme de Newton. Afin de définir une action de $\mathbb{R}$ sur $\mathcal{G}$, nous nous intéressons également aux puissances réelles de A. Pour tout réel x, nous définissons la série entière A^x par :

$$A^x = \sum_{\ell=0}^{+\infty} \binom{x}{\ell} X^\ell \tag{7}$$

où les coefficients binomiaux sont définis par $\binom{x}{0} = 1$ et pour tout entier $\ell \geq 1$:

$$\binom{x}{\ell} = \frac{1}{\ell!}\, x(x-1)\cdots(x-\ell+1).$$

Pour tout réel x, nous pouvons écrire en coordonnées :

$$A^x \cdot s = \sum_{\ell=0}^{+\infty} \left(\sum_{i=0}^{\ell} \binom{x}{i} s_{\ell-i} \right) X^\ell \quad \text{pour} \quad s = \sum_{\ell=0}^{+\infty} s_\ell X^\ell \in V.$$

Il est clair que pour tous les entiers $(n,m) \in \mathbb{Z}^2$, $A^{n+m} = A^n A^m$. De plus l'application $x \to A^x$ est polynomiale (dans le sens où il existe des polynômes $P_0, P_1, \ldots \in \mathbb{R}[X]$ tels que $A^x = P_0(x) + P_1(x)X + \cdots$). Donc :

$$\text{pour tous les réels } (x,y) \in \mathbb{Z}^2 : A^{x+y} = A^x A^y \text{ et en particulier } A^x A^{-x} = 1. \tag{8}$$

Nous reviendrons sur ce genre de raisonnement par la suite.

Définition 16. — Nous définissons le groupe $\mathcal{G}$ comme le produit semi-direct $\mathbb{R} \ltimes V$ avec la multiplication

$$\langle x, s\rangle \cdot \langle y, t\rangle = \langle x+y,\ A^y \cdot s + t\rangle \quad \text{où } (x,y) \in \mathbb{R}^2 \text{ et } (s,t) \in V^2.$$

Le *commutateur* de deux éléments g et h de $\mathcal{G}$ est $[g,h] = g^{-1} \cdot h^{-1} \cdot g \cdot h$.

Soulevons immédiatement quelques points de notation délicats avec ces objets. L'élément neutre du groupe $\mathcal{G}$ est $1_{\mathcal{G}} = \langle 0, 0\rangle$. Nous verrons également plus loin que pour tout élément $g \in \mathcal{G}$, l'application $x \to g^x$ (définie au lemme 21) de $\mathbb{R}$ dans $\mathcal{G}$ est polynomiale en x.

Lemme 17. — *Soient $\langle x, s\rangle$ et $\langle y, t\rangle$ deux éléments de $\mathcal{G}$. Alors :*

$$\langle x, s\rangle^{-1} = \langle -x, -A^{-x} \cdot s\rangle \qquad \text{et} \qquad [\langle x, s\rangle, \langle y, t\rangle] = \langle 0, (A^y - 1) \cdot s - (A^x - 1) \cdot t\rangle.$$

En particulier $\langle x, s\rangle$ et $\langle y, t\rangle$ commutent si et seulement si $(A^y - 1) \cdot s = (A^x - 1) \cdot t$.

Preuve du lemme 17. — Soient $\langle x, s\rangle$ et $\langle y, t\rangle$ deux éléments de $\mathcal{G}$. Alors d'après (8) :

$$\langle x, s\rangle \cdot \langle -x, -A^{-x} \cdot s\rangle = \langle 0, A^{-x} \cdot s - A^{-x} \cdot s\rangle = \langle 0, 0\rangle = 1_{\mathcal{G}}.$$

Donc $\langle x, s\rangle^{-1}$ existe et vaut $\langle -x, -A^{-x} \cdot s\rangle$. Un calcul direct nous donne pour le commutateur :

$$\begin{aligned}[\langle x, s\rangle, \langle y, t\rangle] &= \langle -x-y, -A^{-x-y} \cdot s - A^{-y} \cdot t\rangle \cdot \langle x+y, A^y \cdot s + t\rangle \\ &= \langle 0, A^y \cdot s + t - s - A^x \cdot t\rangle. \end{aligned}$$ □

Pour tout entier $\ell \in \mathbb{N}$, nous notons $V_\ell = X^\ell \cdot V$. Les groupes $(V_\ell, +)$ s'injectent naturellement dans le groupe $(\mathcal{G}, \cdot)$ via inj : $s \mapsto \langle 0, s\rangle$. Nous noterons $\mathcal{V}_\ell = \{\langle 0, s\rangle; s \in V_\ell\}$ l'image dans $\mathcal{G}$ de V_ℓ.

Remarquons que pour tout réel x : $A^x - 1 = \binom{x}{1}X + \binom{x}{2}X^2 + \ldots$. Ainsi $(A^x - 1) \cdot V_\ell \subset V_{\ell+1}$ et donc $[\mathcal{G}, \mathcal{G}] \subset \mathcal{V}_0$ et $[\mathcal{G}, \mathcal{V}_\ell] \subset \mathcal{V}_{\ell+1}$. D'autre part, d'après le lemme 17 : $[\langle 1, s\rangle, \langle 0, -t\rangle] = \langle 0, X \cdot t\rangle$ pour tout $(s,t) \in V^2$. Nous venons de montrer que $[\mathcal{G}, \mathcal{G}] = \mathcal{V}_0 = \mathcal{V}$ et pour tout entier $\ell \geq 0$, $[\mathcal{G}, \mathcal{V}_\ell] = \mathcal{V}_{\ell+1}$: les groupes $(\mathcal{V}_\ell)_{\ell \in \mathbb{N}^*}$ sont donc les éléments de *la série centrale descendante* de $\mathcal{G}$.

Définition 18. — Récapitulons les différents groupes introduits. Nous notons pour tout $\ell \in \mathbb{N}$, $V_\ell = X^\ell \cdot V$ ainsi que $\mathcal{V}_\ell = \{\langle 0, s\rangle; s \in V_\ell\}$ les sous-groupes de la série centrale descendante de $\mathcal{G}$. Le sous-groupe $\mathcal{V}_0$ est parfois simplement noté $\mathcal{V}$. Pour $\ell \in \mathbb{N}^*$, nous notons $G_\ell = V/V_\ell$ et $\mathcal{G}_\ell = \mathcal{G}/\mathcal{V}_\ell$. G_ℓ s'identifie avec l'ensemble des polynômes réels de degré au plus $\ell - 1$.

Soit $\ell \in \mathbb{N}^*$ un entier. Comme le groupe $\mathcal{G}_\ell$ est un quotient de $\mathcal{G}$ par un élément de sa série centrale descendante, le groupe $\mathcal{G}_\ell$ est *nilpotent*. Il s'identifie bien sûr au groupe $\mathbb{R} \ltimes \mathbb{R}^\ell$. Pour $\ell = 2$, il s'agit du groupe de Heisenberg. Le groupe $\mathcal{G}$ est la limite projective des groupes $(\mathcal{G}_\ell)_{\ell \in \mathbb{N}^*}$.

Nous munissons V et V_ℓ de la topologie produit. Pour cette topologie, la multiplication par X de V dans V est continue. Ainsi la topologie produit sur $\mathcal{G}$ en fait un groupe topologique (vu comme limite projective de groupes topologiques). L'injection $(V, +) \to (\mathcal{V}, \cdot)$ et les projections $G \to G_\ell$ et $\mathcal{G} \to \mathcal{G}_\ell$ sont alors des homomorphismes continus.

Nous dirons qu'une application de $f : V \to V$ définie par $f(s) = f_0(s) + f_1(s)X + \cdots + f_n(s)X^n + \cdots$ est *polynomiale* si les coordonnées $(f_0, f_1, \ldots)$ de cette application sont polynomiales. C'est-à-dire que chaque composante est un polynôme ne faisant intervenir qu'un nombre fini de coordonnées. Par exemple X et A^x sont des applications polynomiales. Par extension, une application à valeurs dans $\mathcal{G}$ est polynomiale si ses coordonnées sont des polynômes. C'est le cas de l'opération de multiplication par un élément donné. Une application polynomiale est continue.

3.2. Le groupe $\mathcal{G}$ et les sommes de Birkhoff itérées. — Soit $u = (u_k)_{k \in \mathbb{N}}$ une suite de nombres réels. Nous définissons pour tout entier $n \in \mathbb{N}$, la série formelle :

$$S_n(u) = \sum_{\ell=0}^{+\infty} S_n^{(\ell+1)}(u) X^\ell.$$

Reprenant la définition 1 des sommes itérées, un calcul direct donne la relation :

$$S_{n+1}(u) = (1+X)\cdot S_n(u) + u_n = A\cdot S_n(u) + u_n.$$

L'ensemble des réels s'injecte dans le groupe $\mathcal{G}$ via l'application $i(x) = \langle 1, x\rangle$ pour $x \in \mathbb{R}$, qui n'est pas un morphisme de groupes. Alors en reprenant la relation précédente et par définition 16 du produit dans le groupe $\mathcal{G}$, nous avons pour toute suite de réels $u = (u_k)_{k\in\mathbb{N}}$ et pour tout entier $n \in \mathbb{N}^*$:

$$i(u_0)\cdot i(u_1)\cdot\dots\cdot i(u_{n-1}) = \langle n, S_n(u)\rangle = \left\langle n\ ,\ \sum_{\ell=0}^{+\infty} S_n^{(\ell+1)}(u)\ X^\ell \right\rangle.$$

Soient $\mathcal{A}$ un alphabet, K un sous-ensemble de $\mathcal{A}^{\mathbb{N}}$ stable par le décalage T et $\varphi : K \to \mathbb{R}$ une application. Nous définissons une application $\hat{\varphi} : K \to \mathcal{G}$ pour $u \in K$ par :

$$\hat{\varphi}(u) = i\circ\varphi(u) = \langle 1, \varphi(u)\rangle. \tag{9}$$

Alors pour tout mot $u \in K$, en considérant la suite $\varphi * u$ nous avons pour tout entier $n \in \mathbb{N}^*$:

(10)

$$\hat{\varphi}(u)\cdot\hat{\varphi}(Tu)\cdot\dots\cdot\hat{\varphi}\big(T^{n-1}(u)\big) = \big\langle n, S_n\big(\varphi*u\big)\big\rangle = \left\langle n\ ,\ \sum_{\ell=0}^{+\infty} S_n^{(\ell+1)}\big(\varphi*u\big)\ X^\ell \right\rangle.$$

Cette relation permettra de définir la projection de l'ensemble des mots finis dans le groupe $\mathcal{G}$ dans la section 3.5 et de relier ces différents objets avec les sommes de Birkhoff itérées.

3.3. Puissances dans $\mathcal{G}$. — Nous allons définir la puissance r-ème d'un élément de $\mathcal{G}$ pour tout nombre réel r. Notons déjà que pour toute puissance entière $n \in \mathbb{N}$:

(11)

$$\langle 1, s\rangle^n = \big\langle n, (1 + A + \dots + A^{n-1})\cdot s\big\rangle \quad \text{pour toute série formelle } s \in V.$$

DÉFINITION 19. — Cela nous amène comme pour les puissances de A (relation (7)), à définir une série formelle pour tout nombre réel r par :

$$B(r) = \sum_{\ell=0}^{+\infty} \binom{r}{\ell+1} X^\ell,$$

où nous rappellons pour tout réel $r \in \mathbb{R}$ et tout entier $\ell \in \mathbb{N}$, $\binom{r}{\ell+1} = \frac{1}{(\ell+1)!} r(r-1)\cdots(r-\ell)$.

Pour tout réel r non nul, un calcul élémentaire assure que les séries A^r et $B(r)$ sont liées via la relation :

$$A^r = 1 + XB(r), \quad \text{que nous écrirons parfois :} \quad B(r) = \frac{A^r - 1}{X}.$$

La fonction $r \to B(r)$ est une fonction polynomiale, donc continue. Rappelons qu'une série formelle est inversible si et seulement le terme constant est non nul. Celui de $B(r)$ vaut r et donc $B(r)$ est inversible si et seulement si $r \neq 0$.

Définition 20. — Ceci nous permet de définir la puissance r-ème d'un élément $\langle x, s\rangle$ de $\mathcal{G}$ pour tout réel r de la manière suivante :

$$\langle x, s\rangle^r = \begin{cases} \left\langle rx, B(rx)\, B(x)^{-1} \cdot s\right\rangle = \left\langle rx, \dfrac{A^{rx}-1}{A^x-1} \cdot s\right\rangle & \text{si } x \neq 0, \\ \langle 0, r \cdot s\rangle & \text{si } x = 0. \end{cases}$$

Cette définition coïncide avec la définition donnée lorsque la puissance r est entière.

Lemme 21. —
1. *Pour tout élément $g \in \mathcal{G}$ et tout couple de réels (r, r') : $g^{r+r'} = g^r \cdot g^{r'}$.*
2. *Soit $g \in \mathcal{G} \setminus \mathcal{V}$, alors l'ensemble des éléments de $\mathcal{G}$ qui commutent avec g sont les éléments de la forme g^r avec $r \in \mathbb{R}$.*
3. *L'application de $\mathcal{G} \times \mathbb{R} \to \mathcal{G}$ définie par $(g, r) \to g^r$ est polynomiale donc continue.*

Nous donnons la preuve de ce lemme dans l'annexe A.1.

3.4. Relations dans $\mathcal{G}$. —

Définition 22. — Nous définissons des éléments d et $(f_i)_{i\in\mathbb{N}}$ de $\mathcal{G}$ par :

$$d = \langle 1, 0\rangle, \quad f = f_0 = \langle 0, 1\rangle \quad \text{et pour } i \in \mathbb{N}^*, \quad f_i = \langle 0, X^i\rangle.$$

Rappelons que A est inversible et que $A = 1 + X$. D'après le lemme 17, nous avons :

$$[f_i, d] = \langle 0, (A-1) \cdot X^i\rangle = f_{i+1} \quad \text{pour tout entier } i \in \mathbb{N}. \tag{12}$$

De plus, pour tous les entiers i et j, les éléments f_i et f_j commutent car ils appartiennent au groupe $\mathcal{V}$.

Lemme 23. — *Tout élément $\langle x, s\rangle$ de $\mathcal{G}$ s'écrit sous la forme canonique :*

$$\langle x, s\rangle = d^x \cdot f_0^{s_0} \cdot f_1^{s_1} \cdot f_2^{s_2} \cdots f_\ell^{s_\ell} \cdots \quad \textit{pour } x \in \mathbb{R} \textit{ et } s = \sum_{\ell=0}^{+\infty} s_\ell X^\ell \in V.$$

C'est-à-dire, il existe des réels $d, f_0, f_1, \ldots$ tels que pour tout entier ℓ, la projection de $\langle x, s\rangle$ dans le groupe $\mathcal{G}_\ell$ est $d^x \cdot f_0^{s_0} \cdot f_1^{s_1} \cdot f_2^{s_2} \cdots f_\ell^{s_\ell}$.

Preuve du lemme 23. — Il suffit de reprendre le calcul des puissances (définition 20) et la définition 16 du produit dans $\mathcal{G}$. □

Pour tout entier $\ell \in \mathbb{N}^*$, le groupe $\mathcal{G}_\ell$ est donc engendré par d et $f = f_0$. Pour $\ell = 2$, on obtient assez simplement la liste des relations :

$$\big[f, [f, d]\big] = 1_{\mathcal{G}} \quad \text{et} \quad \big[d, [f, d]\big] = 1_{\mathcal{G}}$$

qui sont les relations standards pour le groupe d'Heisenberg à coefficients entiers.

3.5. Projection du monoïde libre dans $\mathcal{G}$. — Soit $\varphi : \{a, b\} \to \mathbb{R}$ une fonction injective. Rappelons que nous pouvons lui associer une fonction $\hat{\varphi} : \{a, b\} \to \mathcal{G}$ selon la relation (9). À un mot fini $u = u_0 \dots u_{n-1}$ de longueur n du monoïde libre généré par a et b, nous pouvons lui associer ses sommes de Birkhoff itérées (voir la section 3.2). Nous définissons la projection π_φ d'un tel mot dans $\mathcal{G}$ par :

$$\pi_\varphi(u) = \big\langle 1, \varphi(u_0)\big\rangle \cdot \big\langle 1, \varphi(u_1)\big\rangle \dots \big\langle 1, \varphi(u_{n-1})\big\rangle = \hat{\varphi}(u_0) \cdot \hat{\varphi}(u_1) \cdots \hat{\varphi}(u_{n-1}).$$

L'application $\pi_\varphi : \{a, b\}^* \to \mathcal{G}$ est un morphisme : pour tous les mots finis u et v sur l'alphabet $\{a, b\}$, $\pi_\varphi(uv) = \pi_\varphi(u) \cdot \pi_\varphi(v)$.

En reprenant le travail effectué dans la section 3.2 (relation (10)), $\pi_\varphi(u)$ peut s'exprimer à l'aide des sommes de Birkhoff itérées selon la formule :

$$\pi_\varphi(u) = \big\langle n, S_n\big(\varphi * u\big)\big\rangle. \tag{13}$$

Si u est un mot fini de longueur n, $S_n\big(\varphi * u\big)$ est un polynôme de degré au plus $n - 1$ et $\pi_\varphi(u)$ peut être vu comme un élément de $\mathcal{G}_n$.

$bbbb$	$\langle 4, (0,0,0,0)\rangle$
$bbba$	$\langle 4, (1,0,0,0)\rangle$
$bbab$	$\langle 4, (1,1,0,0)\rangle$
$babb$	$\langle 4, (1,2,1,0)\rangle$
$abbb$	$\langle 4, (1,3,3,1)\rangle$

$bbaa$	$\langle 4, (2,1,0,0)\rangle$
$baba$	$\langle 4, (2,2,1,0)\rangle$
$baab$	$\langle 4, (2,3,1,0)\rangle$
$abba$	$\langle 4, (2,3,3,1)\rangle$
$abab$	$\langle 4, (2,4,3,1)\rangle$
$aabb$	$\langle 4, (2,5,4,1)\rangle$

$baaa$	$\langle 4, (3,3,1,0)\rangle$
$abaa$	$\langle 4, (3,4,3,1)\rangle$
$aaba$	$\langle 4, (3,5,4,1)\rangle$
$aaab$	$\langle 4, (3,6,4,1)\rangle$
$aaaa$	$\langle 4, (4,6,4,1)\rangle$

TABLE 3.1. Projection π_φ des mots de longueur 4 dans le groupe $\mathcal{G}$ avec $\varphi = \chi_a$ la fonction caractéristique de la lettre a. Nous avons identifié le polynôme $a_0 + a_1X + a_2X^2 + a_3X^3$ avec le quadruplet (a_0, a_1, a_2, a_3).

3.6. Endomorphismes de $\mathcal{G}$ associés aux substitutions uniformes

LEMME 24. — *Soit $\mathcal{L}$ un endomorphisme de $\mathcal{G}$. Alors $\mathcal{L}$ est continu et pour tout nombre réel r et tout élément g de $\mathcal{G}$, $\mathcal{L}(g^r) = \mathcal{L}(g)^r$.*

Preuve du lemme 24. — Pour tout entier ℓ, les endomorphismes du groupe $\mathcal{G}_\ell \cong \mathbb{R} \rtimes \mathbb{R}^\ell$ sont continus (voir définition 18). D'autre part, comme les sous-groupes $\mathcal{V}_\ell = \{\langle 0, s\rangle; s \in V_\ell\}$ pour $\ell \in \mathbb{N}^*$ sont les groupes de la série centrale descendante, ils sont *caractéristiques* (c'est-à-dire préservés par tout endomorphisme). Tout endomorphisme de $\mathcal{G}$ passe donc au quotient en un endomorphisme de $\mathcal{G}_\ell$. Ainsi, comme $\mathcal{G}$ est la limite projective des groupes $(\mathcal{G}_\ell)_{\ell\in\mathbb{N}^*}$, les endomorphismes de $\mathcal{G}$ sont également tous continus.

Intéressons nous maintenant aux images des puissances r-ème de $g \in \mathcal{G}$. La relation est claire si la puissance r est un nombre entier (relatif). Elle est donc aussi vérifiée pour les puissances $r \in \mathbb{Q}$. Comme l'application puissance est continue dans $\mathcal{G}$ (lemme 21), la relation s'étend aux nombres réels $r \in \mathbb{R}$ par continuité de $\mathcal{L}$. □

Reprenons la forme canonique d'un élément de $\mathcal{G}$ donnée dans le lemme 23 :

$$\langle x, s\rangle = d^x \cdot f_0^{s_0} \cdot f_1^{s_1} \cdot f_2^{s_2} \cdots f_\ell^{s_\ell} \cdots \quad \text{pour } x \in \mathbb{R} \text{ et } s = \sum_{\ell=0}^{+\infty} s_\ell X^\ell \in V.$$

La donnée des valeurs prises par un endomorphisme sur $\{d, f_0, f_1, \ldots\}$ détermine donc cet endomorphisme sur $\mathcal{G}$ d'après le lemme 24 précédent. Cependant un endomorphisme doit préserver les relations du groupe. La première famille de relations est que pour tout entier $i \in \mathbb{N}$, $f_{i+1} = [f_i, d]$. Donc les valeurs en $\{f_i; i \in \mathbb{N}^*\}$ sont contraintes par celles en d et $f_0 = f$ et les données de $\mathcal{L}$ en d et f déterminent $\mathcal{L}$ sur $\mathcal{G}$ de manière unique. Si on souhaite définir un endomorphisme sur $\mathcal{G}$ à partir de la donnée des images de d et f, il faut alors vérifier que la deuxième famille de relations soit vérifiée : $[f_i, f_j] = 1_{\mathcal{G}}$ pour tout couple $(i, j) \in \mathbb{N}^2$.

Proposition 25. — *Soient σ une substitution uniforme de longueur $\lambda \in \mathbb{N}^*$ sur $\{a, b\}$ et $\varphi : \{a, b\} \to \mathbb{R}$ une fonction injective. Alors les assertions suivantes sont vérifiées.*

1. *Il existe un unique endomorphisme $\mathcal{L}$ de $\mathcal{G}$ tel que $\mathcal{L} \circ \pi_\varphi = \pi_\varphi \circ \sigma$.*
2. *Les polynômes δ et γ (voir (1)) et les éléments d et f (définition 22) sont reliés selon les relations : $\mathcal{L}(d) = \langle \lambda, \gamma\rangle$ et $\mathcal{L}(f) = \langle 0, \delta\rangle$.*
3. *De plus cet endomorphisme préserve $\mathcal{V}$ qui est engendré par $\{f_j; j \in \mathbb{N}\}$ et plus précisément pour tout $j \in \mathbb{N}$, $\mathcal{L}(f_j) = \big\langle 0, (A^\lambda - 1)^j \cdot \delta\big\rangle$.*
4. *Il existe un endomorphisme L de V, ne dépendant que de σ et pas de la fonction φ injective choisie, tel qu'en notant* inj : $V \to \mathcal{V}$ *l'injection définie pour $s \in V$ par* $\mathrm{inj}(s) = \langle 0, s\rangle$: $\mathrm{inj} \circ L = \mathcal{L} \circ \mathrm{inj}$.
5. *Pour tout élément $\langle x, s\rangle \in G$ avec $s = s_0 + s_1 X + \cdots \in V$:*

$$\mathcal{L}\big(\langle x, s\rangle\big) = \left\langle \lambda x, \left(\sum_{j=0}^{\infty} s_j (A^\lambda - 1)^j\right) \cdot \delta + B(\lambda x) B(\lambda)^{-1} \cdot \gamma \right\rangle.$$

Nous prouvons ce résultat dans l'annexe A.2.

3.7. Propriétés et forme explicite des endomorphismes associés aux substitutions uniformes. — Les preuves des résultats 27 et 28 de cette section étant essentiellement techniques, nous les apportons dans l'annexe A.3.

DÉFINITION 26. — Nous définissons pour des couples d'entiers $(j,k) \in \mathbb{N}^2$ des polynômes $(q_{j,k})_{(j,k)\in\mathbb{N}^2}$,

$$\text{pour tout réel } y \text{ par :} \quad q_{j,k}(y) = \sum_{\substack{k_1+\ldots+k_j=k \\ k_1\geq 1,\ldots,k_j\geq 1}} \binom{y}{k_1}\cdots\binom{y}{k_j}.$$

Nous rappelons que par convention, $\binom{y}{0} = 1$ et pour $i \geq 1$, $\binom{y}{i} = \frac{1}{i!}y\cdots(y-i+1)$. Nous posons $q_{0,0} = 1$, $q_{0,k} = 0$ si $k > 0$ et $q_{j,0} = 0$ si $j > 0$. Pour $j > k$ deux entiers, il n'existe pas d'entiers $(k_1,\ldots,k_j) \in (\mathbb{N}^*)^j$ tels que $k_1 + \cdots + k_j = k$ et nous définissons donc $q_{j,k} = 0$.

Ce sont les uniques polynômes satisfaisant l'égalité de séries formelles suivantes pour tout $y \in \mathbb{R}$:

$$\begin{aligned}(A^y-1)^j &= \left(\sum_{k\geq 1}\binom{y}{k}X^k\right)^j = \sum_{k\geq 1}\left(\sum_{\substack{k_1+\ldots+k_j=k \\ k_1\geq 1,\ldots,k_j\geq 1}}\binom{y}{k_1}\cdots\binom{y}{k_j}\right)X^k \\ &= \sum_{k=0}^{\infty} q_{j,k}(y)X^k.\end{aligned} \tag{14}$$

$j \backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	y	$\binom{y}{2}$	$\binom{y}{3}$	$\binom{y}{4}$	$\binom{y}{5}$
2	0	0	y^2	y^3-y^2	$\frac{7y^4-18y^3+11y^2}{12}$	$\frac{3y^5-14y^4+21y^3-10y^2}{12}$
3	0	0	0	y^3	$\frac{3}{2}(y^4-y^3)$	$\frac{5y^5-12y^4+7y^3}{4}$
4	0	0	0	0	y^4	$2(y^5-y^4)$
5	0	0	0	0	0	y^5

TABLE 3.2. Les premiers polynômes $q_{j,k}(y)$ pour $0 \leq j,k \leq 5$ avec $y \in \mathbb{R}$.

Il est clair que pour tout $y \in \mathbb{R}$:

(15)
$$\forall k \in \mathbb{N}^*, \quad q_{1,k}(y) = \binom{y}{k}, \qquad q_{k,k}(y) = y^k \quad \text{et} \quad q_{k,k+1}(y) = \frac{y-1}{2} k y^k.$$

Les deux premières relations sont évidentes. Pour prouver la dernière, il faut remarquer que si des entiers strictement positifs $a_1, \ldots, a_k$ vérifient $a_1 + \cdots + a_k = k+1$, alors l'un des $(a_1, \ldots, a_k)$ vaut 2 et tous les autres valent 1. Le produit des coefficients binomiaux associés est $\binom{y}{2}\binom{y}{1}\ldots\binom{y}{1} = \frac{1}{2}y(y-1)y^{k-1}$. Il suffit de compter le nombre de k-uplets vérifiant cette relation : il y en a k.

Proposition 27. — *Soient x et y deux réels. Pour tout entier $k \geq 0$, on a :*

$$\binom{xy}{k} = \sum_{j=0}^{k} q_{j,k}(y) \binom{x}{j}. \tag{16}$$

Ou, de manière équivalente :

$$B(xy)B(y)^{-1} = \sum_{j=1}^{\infty} \binom{x}{j} (A^y - 1)^{j-1}.$$

Cette proposition permet d'obtenir une forme plus explicite pour l'endomorphisme $\mathcal{L}$:

Corollaire 28. — *Soient σ une substitution uniforme de longueur λ sur $\{a,b\}$ et $\varphi : \{a,b\} \to \mathbb{R}$ injective. Soient $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ l'endomorphisme associé obtenu dans la proposition 25 et $\langle x, s\rangle$ un élément de $\mathcal{G}$ avec $s = s_0 + s_1 X + \cdots \in V$. Alors,*

$$\mathcal{L}\big(\langle x, s\rangle\big) = \left\langle \lambda x, \sum_{j=0}^{\infty} (A^\lambda - 1)^j \ \left(s_j \delta + \binom{x}{j+1}\gamma\right)\right\rangle.$$

Ainsi, en écrivant $\mathcal{L}(\langle x, s\rangle) = \langle \lambda x, t\rangle$ avec $t = t_0 + t_1 X + \cdots \in V$, nous avons pour tout $\ell \geq 2$ l'expression :

$$t_\ell = \lambda^\ell s_\ell \delta_1 + \lambda^\ell \gamma_1 \binom{x}{\ell+1} + \sum_{j=0}^{\ell-1}\sum_{i=0}^{\ell-j} q_{j,\ell-i}(\lambda) \left(s_j \delta_{i+1} + \gamma_{i+1}\binom{x}{j+1}\right). \tag{17}$$

Supposons que σ est fortement uniforme de longueur λ et que u est un mot (fini ou infini) sur $\{a,b\}$. Pour tout entier $n \in \mathbb{N}$, les sommes itérées vérifient la relation de récurrence suivante, donnée en introduction :

(2)
$$S_{n\lambda}^{(\ell+1)}\big(\varphi * \sigma(u)\big) = \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i}(\lambda) S_n^{(j+1)}\big(\varphi * u\big)\delta_{i+1} + \sum_{j=0}^{\ell}\sum_{i=0}^{\ell-j} q_{j,\ell-i}(\lambda)\gamma_{i+1}\binom{n}{j+1}.$$

Le corollaire suivant est immédiat :

Corollaire 29. — *Soient σ une substitution fortement uniforme de longueur λ sur $\{a,b\}$ et $\varphi : \{a,b\} \to \mathbb{R}$ injective. Soient $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ l'endomorphisme associé obtenu dans la proposition 25 et $g = \langle x, s\rangle$ un élément de $\mathcal{G}$ avec $s = s_0 + s_1X + \cdots \in V$. Fixons deux entiers j et n.*

Alors si on écrit $\mathcal{L}^n(g) = \langle \lambda^n x, t\rangle$ avec $t = t_0 + t_1X + \cdots \in V$, alors la j-ème coordonnée t_j de t ne dépend que de $(x, s_1, \ldots, s_{j-n})$.

Si $j < n$, cette condition signifie que cette coordonnée ne dépend que de x, et est nulle si $x = 0$.

4. Preuve du théorème 4

4.1. Majoration et étude asymptotique des polynômes $(q_{j,k})_{(j,k)\in\mathbb{N}^2}$. — Il n'est pas raisonnable d'exhiber une formule explicite analogue à la relation (15) pour tous les polynômes $(q_{j,k})_{(j,k)\in\mathbb{N}^2}$. Nous en donnerons le comportement asymptotique dans la proposition 31 et nous les majorerons dans le lemme 32.

Le principal outil de cette section va être la récurrence suivante, elle sera prouvée dans l'annexe A.4.

Lemme 30. — *Pour tous les entiers $k \in \mathbb{N}^*$ et $j \in \mathbb{N}^*$ et tout réel y, les polynômes $(q_{j,k})_{(j,k)\in\mathbb{N}^2}$ vérifient la relation :*

$$q_{j,k+1}(y) = \frac{jy}{k+1}\, q_{j-1,k}(y) + \frac{jy-k}{k+1}\, q_{j,k}(y).$$

Nous en déduisons le résultat suivant, dont la preuve est également apportée dans l'annexe A.4.

Proposition 31. — *Pour tout entier $i \in \mathbb{N}^*$ et tout réel $y > 1$, on a lorsque k tend vers $+\infty$:*

$$q_{k,k+i}(y) \underset{k}{\sim} \frac{(y-1)^i}{2^i i!}\, k^i y^k. \tag{18}$$

Lemme 32. — *Soient $n \geq 1$ un entier et $y > 1$ un réel. Alors,*

$$\sup\{|q_{j,k}(y)| \ ;\ 1 \leq j \leq k \leq n\} \leq (2y-1)^n.$$

Preuve du lemme 32. — Fixons $y > 1$ et notons $A_n = \sup\{|q_{j,k}(y)|; 1 \leq j \leq k \leq n\}$. Nous montrons ce résultat par récurrence. Pour $n = 1$, $A_1 = |q_{1,1}(y)| = y \leq 2y - 1$. Fixons un entier $n \geq 2$ et $j \in \{2, \ldots, n\}$, alors d'après le lemme 30 :

$$\begin{aligned}|q_{j,n+1}(y)| = \left|\frac{jy}{n+1} q_{j-1,n}(y) + \frac{yj-n}{n+1} q_{j,n}(y)\right| &\leq \frac{ny}{n+1} A_n + n\frac{\lambda-1}{n+1} A_n \\ &\leq \frac{n}{n+1}(2y-1)A_n = (2y-1)(2y-1)^n = (2y-1)^{n+1}.\end{aligned}$$

D'autre part, le résultat est aussi vrai pour $q_{n+1,n+1}(y) = y^{n+1}$ ce qui termine la preuve. □

4.2. Approximation polynomiale des sommes itérées. — Soient σ une substitution uniforme de longueur λ sur $\{a,b\}$ et $\varphi : \{a,b\} \to \mathbb{R}$ injective. Soit $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ l'endomorphisme associé obtenu dans la proposition 25.

Définition 33. — Afin d'étudier les puissances de l'endomorphisme $\mathcal{L}$, nous introduisons des polynômes $\big(R_{m,\ell}\big)_{(m,\ell)\in\mathbb{N}\times\mathbb{N}^*}$ en posant :

$$\mathcal{L}^m\big(\langle x,0\rangle\big) = \left\langle \lambda^m x, \sum_{\ell=0}^{+\infty} R_{m,\ell+1}(x)X^\ell \right\rangle \quad \text{pour tout } m\in\mathbb{N} \text{ et tout } x\in\mathbb{R}.$$

Rappelons que la fonction puissance est polynomiale dans le groupe $\mathcal{G}$ (lemme 21 au sens de la relation (25)). Les fonctions $\big(R_{m,\ell}\big)_{(m,\ell)\in\mathbb{N}\times\mathbb{N}^*}$ sont bien des polynômes (dépendants des paramètres γ, δ, φ et λ) d'après la définition 20 pour les puissances, le lemme 24 et le fait que $\langle x,0\rangle = \langle 1,0\rangle^x = d^x$ et donc $\mathcal{L}^i\big(\langle x,0\rangle\big) = \big(\mathcal{L}^i(d)\big)^x$.

Définition 34. — Nous définissons pour tout $\ell\in\mathbb{N}$, $c_\ell = R_{\ell+1,\ell+1}\left(\dfrac{1}{\lambda^{\ell+1}}\right)$.

Proposition 35. — *Pour tout entier $\ell\in\mathbb{N}^*$, tout réel x et tout entier $m\in\mathbb{N}$ on a,*

1. $R_{\ell+m,\ell}(x) = R_{\ell,\ell}(\lambda^m x)$,
2. $R_{\ell,\ell}\left(\dfrac{x}{\lambda^\ell}\right) = \displaystyle\sum_{k=0}^{\ell-1}\binom{x}{\ell-k}c_k$,
3. $\mathcal{L}(\langle 1,c\rangle) = \langle 1,c\rangle^\lambda$ *en notant* $c = c_0 + c_1X + \cdots \in V$.

Nous apportons la preuve de la proposition 35 dans l'annexe A.5.

Le dernier item permet de calculer les valeurs de la suite $(c_\ell)_{\ell\in\mathbb{N}}$. Par exemple, dans le groupe $\mathcal{G}_1$, c_0 est l'unique réel vérifiant :

$$\mathcal{L}\big(\langle 1,c_0\rangle\big) = \langle 1,c_0\rangle^\lambda = \langle \lambda, \lambda c_0\rangle.$$

Le morphisme π_φ induit une application de $\{a,b\}^*$ dans le groupe $\mathcal{G}_1$ (définition 18). Si nous notons encore π_φ cette application, pour une lettre $\alpha\in\{a,b\}$ fixée :

$$\begin{aligned}\mathcal{L}\big(\langle 1,c_0\rangle\big) = \mathcal{L}\big(\langle 1,\varphi(\alpha)\rangle\big) = \pi_\varphi\big(\sigma(\alpha)\big) &= \langle \lambda, S_\lambda^{(1)}(\varphi * \sigma(\alpha))\rangle \\ &= \langle \lambda, \varphi(a)|\sigma(a)|_a + \varphi(b)|\sigma(a)|_b\rangle.\end{aligned}$$

Nous trouvons donc :

$$c_0 = \frac{1}{\lambda}\Big(\varphi(a)|\sigma(a)|_a + \varphi(b)|\sigma(b)|_a\Big). \tag{19}$$

	c_0	c_1	c_2	c_3	c_4	c_5
$a \mapsto ab, b \mapsto ba$	1/2	0	0	0	0	0
$a \mapsto aab, b \mapsto aba$	2/3	1/3	10/9	11	8567/27	718435/27
$a \mapsto aab, b \mapsto baa$	1/3	1/3	23/9	440/9	74431/27	455949

TABLE 4.1. Valeurs des premiers coefficients $(c_\ell)_{\ell\in\mathbb{N}}$ pour différentes substitutions σ associées à la fonction $\varphi = \chi_a$.

DÉFINITION 36. — Nous définissons pour tout entier $\ell \in \mathbb{N}^*$ un polynôme p_ℓ pour tout réel x par :

$$p_\ell(x) = \sum_{i=0}^{\ell} c_i \binom{x}{\ell - i} = R_{\ell,\ell}\left(\frac{x}{\lambda^\ell}\right) + c_\ell.$$

Nous posons de plus $p_0(x) = c_0$.

PROPOSITION 37. — *Soient σ une substitution fortement uniforme de longueur $\lambda \in \mathbb{N}$ sur $\{a,b\}$ et $\varphi : \{a,b\} \to \mathbb{R}$ une fonction injective. Alors, pour tout entier $\ell \in \mathbb{N}^*$ et tout mot $u \in \sigma^\ell(\{a,b\}^{\mathbb{N}})$ (c'est-à-dire de la forme $\sigma^\ell(v)$), les assertions suivantes sont vérifiées.*

1. *Pour tout réel x,* $\quad p_\ell(x+1) - p_\ell(x) = p_{\ell-1}(x)$.
2. *Pour tout entier n,*

$$\big(S^{(\ell)}_{n+1}\big(\varphi * u\big) - p_\ell(n+1)\big) - \big(S^{(\ell)}_{n}\big(\varphi * u\big) - p_\ell(n)\big) = S^{(\ell-1)}_{(n)}\big(\varphi * u\big) - p_{\ell-1}(n).$$

3. *Pour tout entier n, $S^{(\ell)}_{n\cdot\lambda^\ell}\big(\varphi * u\big) = p_\ell(n\lambda^\ell) - c_\ell = R_{\ell,\ell}(n)$.*
4. *La suite $\big(S^{(\ell)}_{n}\big(\varphi * u\big) - p_\ell(n)\big)_{n\geq 0}$ est bornée.*

Nous retrouvons bien la relation entre les coefficients $(c_\ell)_{\ell\in\mathbb{N}}$ et les polynômes $(p_\ell)_{\ell\in\mathbb{N}}$ annoncée dans l'introduction, à savoir $p_\ell(0) = c_\ell$. Rappelons également que la suite des polynômes $(p_\ell)_{\ell\in\mathbb{N}}$ qui approchent les sommes itérées et qui vérifient $p_\ell(x+1) - p_\ell(x) = p_{\ell-1}(x)$ est unique. Les polynômes de la section 2 partagent ces mêmes propriétés, donc si u est un mot infini point fixe de σ, les objets de ces différentes sections sont reliées pour tous les entiers $n \in \mathbb{N}$ et $\ell \in \mathbb{N}$ via :

$$\psi_\ell(T^n u) = S^{(\ell)}_{n}\big(\varphi * u\big) - p_\ell(n) \quad \text{et} \quad c_\ell = -\psi_\ell(u).$$

Preuve de la proposition 37. — Le premier point est immédiat par la définition des polynômes $(p_\ell)_{\ell\in\mathbb{N}}$. En effet, pour tout nombre réel x et tout entier $\ell \geq 1$:

$$p_\ell(x+1) - p_\ell(x) = \sum_{i=0}^{\ell} c_i \binom{x+1}{\ell - i} - \sum_{i=0}^{\ell} c_i \binom{x}{\ell - i} = \sum_{i=0}^{\ell-1} c_i \binom{x}{\ell - 1 - i} = p_{\ell-1}(x).$$

Le deuxième point en découle immédiatement car les sommes itérées vérifient les mêmes relations de récurrence (voir la relation (3)).

Notons $u_0u_1\ldots = \sigma^\ell(v_0v_1\ldots)$ un mot de $\sigma^\ell(\{a,b\}^{\mathbb{N}})$. Nous considérons l'endomorphisme $\mathcal{L}$ associé (proposition 25). Fixons un entier $n \in \mathbb{N}$. Puisque $\mathcal{L} \circ \pi_\varphi = \pi_\varphi \circ \sigma$, nous avons :

$$\pi_\varphi(u_0\cdots u_{n\lambda^\ell-1}) = \pi_\varphi \circ \sigma^\ell(v_0\cdots v_{n-1}) = \mathcal{L}^\ell \circ \pi_\varphi(v_0\cdots v_{n-1}).$$

Donc dans le groupe $\mathcal{G}_\ell$:

$$\begin{aligned}\pi_\varphi(u_0\cdots u_{\lambda^\ell n-1}) &= \left\langle n\lambda^\ell, R_{\ell,1}(n) + \cdots + R_{\ell,\ell}(n)X^{\ell-1}\right\rangle \text{ par le corollaire 29}\\ &= \left\langle n\lambda^\ell, S^{(1)}_{n\lambda^\ell}(\varphi * u) + \cdots + S^{(\ell)}_{n\lambda^\ell}(\varphi * u)X^{\ell-1}\right\rangle\\ &\qquad \text{par la relation (13).}\end{aligned}$$

Nous obtenons donc $S^{(\ell)}_{n\lambda^\ell}(\varphi * u) = R_{\ell,\ell}(n) = p_\ell(n\lambda^\ell) - c_\ell$, ce qui prouve la relation de l'item 3.

Le dernier point se démontre par récurrence sur ℓ. Pour $\ell = 1$, le résultat est clair car la suite est constante aux temps $\lambda\mathbb{N}$. De plus les incréments sont en nombre fini et liés aux images de φ, donc bornés.

Supposons que la suite $\big(S_n^{(\ell-1)}(\varphi * u) - p_{\ell-1}(n)\big)_{n\geq 0}$ soit bornée pour $\ell > 1$. Alors comme dans le cas $\ell = 1$, la suite $\big(S_n^{(\ell)}(\varphi * u) - p_\ell(n)\big)_{n\geq 0}$ est constante aux temps $n\lambda^\ell$ par l'item 2. De plus les incréments sont en nombre fini (majoré par λ) et valent $\big(S_n^{(\ell-1)}(\varphi * u) - p_{\ell-1}(n)\big)_{n\geq 0}$ par l'item 2, qui est bornée. □

4.3. Convergence de la suite renormalisée $(\tilde{c}_\ell)_{\ell\in\mathbb{N}}$. — Soient σ une substitution fortement uniforme de longueur $\lambda \in \mathbb{N}$ sur $\{a,b\}$ et $\varphi : \{a,b\} \to \mathbb{R}$ une fonction injective. Nous supposons que $\delta_2 \neq 0$. Notons pour tout entier $\ell \in \mathbb{N}$, $\rho_\ell = \delta_2^{\ell-1}\lambda^{(\ell-1)(\ell-2)/2}$ (définition 7) et $\tilde{c}_\ell = c_\ell/\rho_\ell$.

PROPOSITION 38. — *Sous ces hypothèses, la suite $(\tilde{c}_\ell)_{\ell\in\mathbb{N}}$ converge. De plus il existe un réel $\theta \in]0,1[$ et un réel $M > 0$ tel qu'à partir d'un certain rang, $|\lim_{j\to+\infty} \tilde{c}_j - c_\ell| \leq M\cdot\theta^\ell$.*

Preuve de la proposition 38. — Nous notons $q_{j,k}$ pour $q_{j,k}(\lambda)$. Fixons un entier $\ell > \lambda$. Commençons par rappeler la relation du troisième item de la proposition 35 :

$$\mathcal{L}(\langle 1,c\rangle) = \langle 1,c\rangle^\lambda \quad \text{avec} \quad c = c_0 + c_1X + \cdots \in V.$$

D'après la forme explicite de l'endomorphisme $\mathcal{L}$ (relation (17) du corollaire 28), puisque $\binom{1}{j+1} = 0$ pour tout entier $j \geq 0$ et d'après la définition 20 des puissances dans $\mathcal{G}$:

$$\lambda c_\ell + \binom{\lambda}{2} c_{\ell-1} + \cdots + \binom{\lambda}{\ell+1} c_0 = \sum_{j=0}^{\ell-1} \sum_{i=0}^{\ell-j} q_{j,\ell-i} c_j \delta_{i+1}$$

(20)

$$\iff \qquad \lambda c_\ell + \binom{\lambda}{2} c_{\ell-1} + \cdots + \binom{\lambda}{\ell+1} c_0 = \lambda^{\ell-1} \delta_2 c_{\ell-1} + \sum_{j=0}^{\ell-2} \sum_{i=1}^{\ell-j} q_{j,\ell-i} \delta_{i+1} c_j.$$

Nous noterons $||\delta|| = \max\{|\delta_i|; 1 \leq i \leq \lambda\} = \sup\{|\delta_i|; 1 \leq i\}$ car $\delta_i = 0$ pour tout entier $i \geq \lambda + 1$. La double somme se majore alors avec le lemme 32 :

$$\begin{aligned} \left| \sum_{j=0}^{\ell-2} \sum_{i=1}^{\ell-j} q_{j,\ell-i} \delta_i c_j \right| &\leq ||\delta|| \cdot \sup_{0 \leq j \leq \ell-2} |c_j| \cdot \left| \sum_{j=1}^{\ell-1} \sum_{i=2}^{\ell-j+2} q_{j-1,\ell+1-i} \right| \\ &\leq ||\delta|| \cdot \sup_{0 \leq j \leq \ell-2} |c_j| \cdot \sum_{j=1}^{\ell-1} \sum_{i=2}^{\ell-j+2} |q_{j-1,\ell+1-i}| \\ &\leq ||\delta|| \cdot \sup_{0 \leq j \leq \ell-2} |c_j| \cdot \ell^2 \cdot (2\lambda - 1)^{\ell-1}. \end{aligned}$$

Nous utilisons également la majoration,

$$\left| \binom{\lambda}{2} c_{\ell-1} + \cdots + \binom{\lambda}{\ell+1} c_0 \right| \leq 2^\lambda \sup_{0 \leq j \leq \ell-1} |c_j|.$$

Les coefficients $(c_0, \ldots, c_\ell)$ vérifient donc :

$$|\lambda c_\ell - \lambda^{\ell-1} \delta_2 c_{\ell-1}| \leq ||\delta|| \cdot \ell^2 \cdot (2\lambda - 1)^{\ell-1} \sup_{0 \leq j \leq \ell-2} |c_j| + 2^\lambda \sup_{0 \leq j \leq \ell-1} |c_j|.$$

Pour $0 \leq j \leq \ell - 2$, $\binom{\ell-1}{2} - \binom{j-1}{2} \geq 2\ell - 1$ et de plus $\binom{\ell-1}{2} - \binom{\ell-2}{2} = \ell - 1$. Donc

(21)

$$\text{pour } 0 \leq j \leq \ell - 2,\ |\rho_\ell| \geq \lambda^{2\ell-1} |\rho_j| \text{ et } \rho_\ell = \delta_2 \lambda^{\ell-1} \text{ et donc } |\rho_\ell| \geq \lambda^{\ell-1} |\rho_{\ell-1}|.$$

En divisant la relation précédente par ρ_ℓ, nous obtenons alors :

$$|\tilde{c}_\ell - \tilde{c}_{\ell-1}| \leq \|\delta\| \binom{\ell+1}{2} \frac{(2\lambda-1)^{\ell-1}}{\lambda^{2\ell-1}} \sup_{0 \leq j \leq \ell-2} |\tilde{c}_j| + \frac{2^\lambda}{\lambda^{\ell-1}} \sup_{0 \leq j \leq \ell-1} |\tilde{c}_j|.$$

Maintenant, si θ est tel que $\frac{2\lambda-1}{\lambda^2} < \theta < 1$ alors pour ℓ assez grand :

$$|\tilde{c}_\ell - \tilde{c}_{\ell-1}| \leq \theta^\ell \sup_{1 \leq j \leq \ell-1} |\tilde{c}_j|.$$

Nous pouvons alors conclure en utilisant le lemme 39 suivant. $\square$

LEMME 39. — *Soit* $(u_n)_{n\in\mathbb{N}}$ *une suite de réels telle qu'il existe* $\theta \in [0,1[$ *et une suite* $v = (v_n)_{n\in\mathbb{N}^*}$ *de termes positifs dont la série converge. Supposons qu'à partir d'un certain rang,*

$$|u_n - u_{n-1}| \leq v_n + \theta^n \max_{0\leq k\leq n-1} |u_k|.$$

Alors la suite $(u_n)_{n\in\mathbb{N}}$ *converge. Notons* L *cette limite et prenons pour* v *la suite nulle, alors il existe une constante* M *telle qu'à partir d'un certain rang,* $|L - u_n| \leq M \cdot \theta^n$.

Preuve du lemme 39. — Si nous montrons que $(u_n)_{n\in\mathbb{N}}$ est bornée alors $|u_n - u_{n-1}|$ est le terme général d'une série convergente, ce qui montrera que $(u_n)_{n\in\mathbb{N}}$ est une suite de Cauchy. Nous supposons que la majoration est vérifiée pour tout entier $n \in \mathbb{N}^*$. Pour obtenir une borne sur $(u_n)_{n\in\mathbb{N}}$, il suffit de remarquer que pour $n \geq 2$:

$$|u_n| \leq |u_0| \prod_{k=0}^{n}(1+\theta^k) + v_n + \sum_{k=1}^{n-1} v_k \prod_{i=1}^{k-1}(1+\theta^{n-i}) \leq \left(\prod_{k=0}^{+\infty}(1+\theta^k)\right) \cdot \left(|u_0| + \sum_{k=1}^{+\infty} v_k\right).$$

Notons U un majorant de $\{u_n; n \in \mathbb{N}\}$ et supposons que v soit la suite nulle. Alors la relation de récurrence se réécrit :

$$|u_{n+1} - u_n| \leq \theta^n \cdot U \quad \text{et pour tout entier } p : |u_{n+p} - u_n| \leq \sum_{k=0}^{p-1} \theta^{n+k} \cdot U.$$

Par passage à la limite, il existe une constante M, $|L - u_n| \leq M \cdot \theta^n$. □

4.4. Preuve du théorème 4. — Nous nous plaçons sous les hypothèses du théorème 4. Fixons un entier $\ell \geq \lambda+1$ et notons $q_{j,k} = q_{j,k}(\lambda)$ et $\|\delta\| = \max\{\delta_i, 1 \leq i \leq \lambda\} = \sup\{\delta_i; i \geq 1\}$. Pour tous les entiers $(i,j) \in \mathbb{N}^2$, nous noterons $S_i^{(\ell)}$ pour $S_i^{(\ell)}(\varphi * u)$, $\bar{S}_i^{(\ell)} = S_i^{(\ell)} - p_\ell(i)$ et $\tilde{S}_i^{(\ell)} = \bar{S}_i^{(\ell)}/\rho_\ell$ où les coefficients $(\rho_\ell)_{\ell\in\mathbb{N}}$ sont définis en 7. Nous noterons également pour tout entier $\ell \in \mathbb{N}$:

$$\theta_\ell = \sup_{n\in\mathbb{N}} \frac{1}{\rho_\ell}\left|S_n^{(\ell)} - p_\ell(n)\right| = \sup_{n\in\mathbb{N}} \frac{1}{\rho_\ell}\left|\bar{S}_n^{(\ell)}\right| = \sup_{n\in\mathbb{N}} \left|\tilde{S}_n^{(\ell)}\right|.$$

Le but est de montrer que la suite $(\theta_\ell)_{\ell\in\mathbb{N}}$ converge.

Rappelons la formule (2) (corollaire 28), pour tout entier n, puisque $\sigma(u) = u$:

$$(2) \qquad S_{n\lambda}^{(\ell+1)} = \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i} S_n^{(j+1)} \delta_{i+1} + \sum_{j=0}^{\ell}\sum_{i=0}^{\ell-j} q_{j,\ell-i}\gamma_{i+1}\binom{n}{j+1}.$$

Rappelons également la formule de l'item 3 de la proposition 37 : pour tous les entiers i et m, $S_{i\cdot\lambda^m}^{(m)} = p_m(i\lambda^m) - c_m$. Nous en déduisons que pour tous les entiers n de la forme $i\lambda^\ell$, la relation suivante est vérifiée :

$$p_{\ell+1}(n\lambda)-c_{\ell+1} = \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i}\big(p_{j+1}(n)-c_{j+1}\big)\delta_{i+1}+\sum_{j=0}^{\ell}\sum_{i=0}^{\ell-j} q_{j,\ell-i}\gamma_{i+1}\binom{n}{j+1}.$$

Et puisque cette équation est polynomiale, elle est donc vérifiée pour tous les entiers n. Nous trouvons donc :

$$\bar{S}_{n\lambda}^{(\ell+1)} + c_{\ell+1} = \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i}\bar{S}_n^{(j+1)}\delta_{i+1} + \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i}c_{j+1}\delta_{i+1}.$$

Divisons alors cette relation par $\rho_{\ell+1}$, nous obtenons :

$$\tilde{S}_{n\lambda}^{(\ell+1)} + \tilde{c}_{\ell+1} = \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i}\frac{\rho_{j+1}}{\rho_{\ell+1}}\tilde{S}_n^{(j+1)}\delta_{i+1} + \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i}\frac{\rho_{j+1}}{\rho_{\ell+1}}\tilde{c}_{j+1}\delta_{i+1}.$$

En notant :

$$\varepsilon_{\ell+1} = \sum_{j=0}^{\ell-1}\sum_{i=1}^{\ell-j} q_{j,\ell-i}\frac{\rho_{j+1}}{\rho_{\ell+1}}\tilde{c}_{j+1}\delta_{i+1} - \tilde{c}_{\ell+1},$$

et en remarquant que $\rho_\ell \cdot \delta_2 \cdot \lambda^{\ell-1} = \rho_{\ell+1}$, nous trouvons donc :

$$\tilde{S}_{n\lambda}^{(\ell+1)} - \tilde{S}_n^{(\ell)} = \sum_{j=0}^{\ell-2}\sum_{i=1}^{\ell-j} q_{j,\ell-i}\frac{\rho_{j+1}}{\rho_{\ell+1}}\tilde{S}_n^{(j+1)}\delta_{i+1} + \varepsilon_{\ell+1}.$$

Avec les relations $q_{j,\ell-i} \leq (2\lambda-1)^{\ell-2} \leq (2\lambda-1)^{\ell-1}$ (lemme 32), $|\rho_{j+1}|\lambda^{2\ell-1} \leq \rho_{\ell+1}$ (relation (21)) et $|\tilde{S}_n^{(j+1)}| \leq \theta_{j+1}$, nous trouvons :

$$\left|\tilde{S}_{n\lambda}^{(\ell+1)} - \tilde{S}_n^{(\ell)}\right| \leq \|\delta\| \cdot \ell^2 \cdot \lambda \cdot \left(\frac{2\lambda-1}{\lambda^2}\right)^{\ell-1} \cdot \sup_{0\leq j\leq \ell-1} \theta_j + |\varepsilon_{\ell+1}|.$$

Pour tout entier n :

$$\big|\tilde{S}_{n\lambda}^{(\ell+1)}\big| \leq \left|\tilde{S}_{n\lambda}^{(\ell+1)} - \tilde{S}_n^{(\ell)}\right| + \big|\tilde{S}_n^{(\ell)}\big|.$$

Ces suites sont bornées et en prenant la borne supérieur sur n, pour un réel $K \in](2\lambda-1)/\lambda^2, 1[$ et pour ℓ assez grand, nous avons :

$$\sup_{n\in\mathbb{N}} \big|\tilde{S}_{n\lambda}^{(\ell+1)}\big| \leq \theta_\ell + K^\ell \cdot \sup_{0\leq j\leq \ell-1} \theta_j + \beta_{\ell+1}.$$

Soit N un entier, il s'écrit de manière unique sous la forme $N = n\lambda+k$ avec $0 \leq k < \lambda$. Alors par construction (item 2 de la proposition 37) puis en divisant par $\rho_{\ell+1}$:

$$
\begin{aligned}
\bar{S}_N^{(\ell+1)} &= \sum_{i=0}^{k-1} \bar{S}_{n\lambda+i}^{(\ell)} + \bar{S}_{n\lambda}^{(\ell+1)} \\
\text{donc}\quad |\tilde{S}_N^{(\ell+1)}| &\leq \sum_{i=0}^{k-1} \frac{1}{\lambda^{\ell-1}} \cdot |\tilde{S}_{n\lambda+i}^{(\ell)}| + |\tilde{S}_{n\lambda}^{(\ell+1)}| \leq \frac{1}{\lambda^{\ell-1}}\theta_\ell + |\tilde{S}_{n\lambda}^{(\ell+1)}|.
\end{aligned}
$$

Donc quitte à redéfinir K (avec toujours $K < 1$), à partir d'un certain rang, nous avons la relation :

$$\theta_{\ell+1} \leq \theta_\ell + K^\ell \cdot \sup_{0 \leq j \leq \ell} \theta_j + \varepsilon_{\ell+1}.$$

Afin de conclure par le lemme 39, il nous reste à vérifier que la série de terme général $|\varepsilon_\ell|$ converge.

Rappelons que d'après la proposition 38, à partir d'un certain rang, il existe deux réels θ et M tels que pour tout entier j, $|\tilde{c}_j - \lim_{m\to+\infty} \tilde{c}_m| \leq M \cdot \theta^j$. On note $C = \sup\{\tilde{c}_j ; j \in \mathbb{N}\}$. On utilise encore le lemme 32 et la relation (21) :

$$
\begin{aligned}
|\varepsilon_{\ell+1}| &\leq \left| \sum_{j=0}^{\ell-1} \sum_{i=1}^{\ell-j} q_{j,\ell-i} \frac{\rho_{j+1}}{\rho_{\ell+1}} \tilde{c}_{j+1} \delta_{i+1} - \tilde{c}_{\ell+1} \right| \\
&\leq \left| \sum_{j=0}^{\ell-2} \sum_{i=1}^{\ell-j} q_{j,\ell-i} \frac{\rho_{j+1}}{\rho_{\ell+1}} \tilde{c}_{j+1} \delta_{i+1} \right| + |\tilde{c}_\ell - \tilde{c}_{\ell+1}| \\
&\leq \ell^2 \|\delta\| \frac{1}{\lambda^{\ell-1}} C + |\tilde{c}_\ell - \lim_{m\to+\infty} \tilde{c}_m| + |\tilde{c}_{\ell+1} - \lim_{m\to+\infty} \tilde{c}_m| \\
&\leq \ell^2 \|\delta\| \frac{1}{\lambda^{\ell-1}} C + M\theta^\ell + m\theta^{\ell+1}.
\end{aligned}
$$

Ce qui est le terme général d'une série convergente.

5. Preuve de la proposition 8

Rappelons que la proposition 8 nous permettra par la suite de démontrer le théorème 6.

5.1. Comportement asymptotique diagonal. — Soit σ une substitution fortement uniforme de longueur $\lambda \in \mathbb{N}^*$ sur $\{a, b\}$. Soit $\varphi : \{a, b\} \to \mathbb{R}$ une fonction injective. Nous notons $\mathcal{L}$ et L les endomorphismes respectivement de $\mathcal{G}$ et V obtenus dans la proposition 25. Nous supposons que $\delta = \delta_2 X + \cdots \in V_1 \backslash V_2$, c'est-à-dire que $\delta_1 = 0$ et $\delta_2 \neq 0$. Nous noterons $q_{j,k}$ à la place de $q_{j,k}(\lambda)$.

NOTATION. — Nous avons supposé que δ_1 était nul. Donc d'après le corollaire 29, si $s = s_0 + s_1 X + \cdots \in V$ alors pour tout entier $m \in \mathbb{N}$, $L^m(s)$ est de la forme :

$$L^m(s) = s' \quad \text{avec} \quad s' = s'_0 X^m + s'_1 X^{m+1} + \cdots \in V.$$

Nous noterons pour tout entier m, $s^{(m)} = s_0^{(m)} + s_1^{(m)} X + \cdots \in V$ la suite telle que $L^m(s) = X^m s^{(m)}$ soit encore $L^m(s) = X^m s_0^{(m)} + s_1^{(m)} X^{m+1} + \cdots$.

$$\begin{tikzcd}
s \arrow[d] & = & s_0^{(0)} \arrow[drr] & + & s_1^{(0)}X \arrow[drr] & + & \cdots & & & & \\
L(S) \arrow[d, dashed] & = & & & s_0^{(1)}X \arrow[drr, dashed] & + & s_1^{(1)}X^2 \arrow[drr, dashed] & + & \cdots & & \\
L^m(s) & = & & & & & s_0^{(m)}X^m & + & s_1^{(m)}X^{m+1} & + & \cdots
\end{tikzcd}$$

PROPOSITION 40. — *Avec les notations de l'introduction de cette section, pour tout entier $\ell \geq 0$, il existe une fonction $\Phi_\ell : V \to \mathbb{R}$ linéaire et qui ne dépend que des ℓ premières coordonnées telle que pour tout $s \in V$ on ait :*

$$s_\ell^{(m)} = \Phi_\ell(s)\ \delta_2^m\ \lambda^{\frac{(m+\ell)(m+\ell-1)}{2}} + \underset{m\to+\infty}{o}\left(\delta_2^m \lambda^{\frac{(m+\ell)(m+\ell-1)}{2}}\right).$$

De plus, $\Phi_\ell(X^\ell) = 1$.

Nous pouvons obtenir des expressions explicites de $s_0^{(m)}$ et $s_1^{(m)}$ pour tout entier m :

$$\begin{cases} s_0^{(m)} = s_0\ \delta_2^m\ \lambda^{m(m-1)/2}, \\ s_1^{(m)} = \left(s_1 + \dfrac{s_0}{\delta_2}\displaystyle\sum_{i=0}^{m-1}\frac{1}{\lambda^{i+1}}\left(\frac{i(\lambda-1)}{2}\delta_2 + \delta_3\right)\right)\cdot\ \delta_2^{m+1}\ \lambda^{m(m+1)/2}. \end{cases}$$

En particulier $\quad \Phi_0(s) = s_0 \quad$ *et* $\quad \Phi_1(s) = s_1 + \dfrac{1}{\lambda-1}\dfrac{1}{\delta_2}\left(\dfrac{\delta_2}{2} + \delta_3\right) s_0.$

Nous apportons la preuve de ce résultat dans l'annexe A.6.

5.2. Asymptotiques des sommes de Birkhoff et des polynômes d'approximation

COROLLAIRE 41. — *Soit σ une substitution fortement uniforme de longueur $\lambda \in \mathbb{N}^*$ sur $\{a, b\}$. Soit $\varphi : \{a, b\} \to \mathbb{R}$ une fonction injective. Nous supposons que $\delta_2 \neq 0$. Soient $(R_{m,\ell})_{(m,\ell)\in\mathbb{N}\times\mathbb{N}^*}$ les polynômes d'approximation définis* *en (33) de la section 4.2. Alors pour tout mot fini $u = u_0 u_1 \dots u_{n-1}$ de longueur n et tout entier $m \geq 0$, avec la définition 7 des coefficients de renormalisation :*

$$\lim_{\ell\to\infty} \frac{S^{(\ell+m)}_{n\lambda^\ell}\big(\varphi * \sigma^m(u)\big) - R_{\ell,\ell+m}(n)}{\delta_2^\ell\ \lambda^{(\ell+m-1)(\ell+m-2)/2}} = \delta_2^{m-1} \lim_{\ell\to\infty} \frac{S^{(\ell+m)}_{n\lambda^\ell}\big(\varphi * \sigma^\ell(u)\big) - R_{\ell,\ell+m}(n)}{\rho_{m+\ell}}$$
$$= \Phi_m\Big(S_{n\lambda^\ell}(\varphi * \sigma^\ell(u)) - (R_{m,1}(n) + R_{m,2}(n)X + \cdots)\Big).$$

Pour tout entier m et tout réel x,

$$\lim_{\ell\to\infty} \frac{R_{\ell+m,\ell+m}(x) - R_{\ell,\ell+m}(\lambda^m x)}{\delta_2^\ell\ \lambda^{(\ell+m-1)(\ell+m-2)/2}} = \delta_2^{m-1} \lim_{\ell\to\infty} \frac{R_{\ell+m,\ell+m}(x) - R_{\ell,\ell+m}(\lambda^m x)}{\rho_{m+\ell}}$$
$$= \Phi_m\big(R_{m,1}(x) + R_{m,2}(x)X + \cdots\big).$$

Preuve du corollaire 41. — Nous notons $\mathcal{L}$ l'endomorphisme associé obtenu dans la proposition 25. Rappelons que c'est l'unique endomorphisme vérifiant $\mathcal{L} \circ \pi_\varphi = \pi_\varphi \circ \sigma$ avec π_φ définie en (13).

Par construction, on a pour tout entier n :

$$\begin{cases} \mathcal{L}^m \circ \pi_\varphi(u) = \pi_\varphi\big(\sigma^m(u)\big) = \Big\langle n\lambda^m, S^{(\ell)}_{n\lambda^\ell}\big(\varphi * \sigma^m(u)\big)\Big\rangle, \\ \mathcal{L}^m\big(\langle n, 0\rangle\big) = \big\langle n\lambda^m, R_{m,1}(n) + R_{m,2}(n)X + \cdots\big\rangle. \end{cases}$$

Maintenant, remarquons que dans $\mathcal{G}$: $\langle x, s\rangle^{-1} \cdot \langle x, t\rangle = \langle 0, t-s\rangle$ pour $x \in \mathbb{R}$ et $(s,t) \in V^2$. En particulier

$$\mathcal{L}^m\big(\langle n, 0\rangle^{-1}\pi_\varphi(u)\big) = \big\langle n\lambda^m, R_{m,1}(n) + R_{m,2}(n)X + \cdots\big\rangle^{-1} \cdot \big\langle n\lambda^\ell, S_{n\lambda^\ell}\big(\varphi * \sigma^\ell(u)\big)\big\rangle$$
$$= \big\langle 0, S_{n\lambda^m}(\varphi * \sigma^m(u)) - (R_{m,1}(n) + R_{m,2}(n)X + \cdots)\big\rangle.$$

Il suffit alors, d'appliquer la proposition 40 avec la série $S_{n\lambda^m}(\varphi * \sigma^m(u)) - (R_{m,1}(n) + R_{m,2}(n)X + \cdots)$.

Nous montrons la seconde partie de manière similaire. Soient x un réel et (ℓ, m) deux entiers :

$$\mathcal{L}^\ell \circ \mathcal{L}^m\big(\langle x, 0\rangle\big) = \big\langle \lambda^{\ell+m}x, R_{\ell+m,1}(x) + \cdots + R_{\ell+m,j}(x)X^{j-1} + \cdots\big\rangle$$
$$\mathcal{L}^\ell\big(\langle \lambda^m x, 0\rangle\big) = \big\langle \lambda^{\ell+m}x, R_{\ell,1}(\lambda^m x) + \cdots + R_{\ell,j}(\lambda^m x)X^{j-1} + \cdots\big\rangle.$$

On peut alors écrire :

$$\mathcal{L}^\ell\big(\langle 0, R_{m,1}(x) + R_{m,2}(x)X + \cdots\rangle\big) = \mathcal{L}^\ell \circ \mathcal{L}^m\big(\langle x, 0\rangle \cdot \langle \lambda^m x, 0\rangle^{-1}\big)$$
$$= \Big\langle 0, \big(R_{\ell+m,1}(x) - R_{\ell,1}(\lambda^m x)\big) + \big(R_{\ell+m,2}(x) - R_{\ell,2}(\lambda^m x)\big)X + \cdots\Big\rangle.$$

Il suffit d'appliquer encore la proposition 40 pour conclure. □

5.3. Preuve de la proposition 8. — On se donne une substitution σ fortement uniforme de longueur λ telle que $\delta_2 \neq 0$ et admettant un point fixe $u = u_0 u_1 \cdots$. Soit $\varphi : \{a, b\} \to \mathbb{R}$ une fonction non constante. Pour tous les entiers $(i, j) \in \mathbb{N}^2$, nous noterons $S_i^{(\ell)}$ pour $S_i^{(\ell)}(\varphi * u)$ et $\tilde{S}_i^{(\ell)} = \big(S_i^{(\ell)} - p_\ell(i)\big)/\rho_\ell$.

Pour tous les entiers n, m et ℓ :

$$\begin{aligned} S_{n\lambda^\ell}^{(m+\ell)} - p_{m+\ell}(n\lambda^\ell) &= S_{n\lambda^\ell}^{(m+\ell)} - R_{m+\ell,m+\ell}\left(\frac{n}{\lambda^m}\right) - c_{m+\ell} \text{ d'après la définition 36} \\ &= \left(S_{n\lambda^\ell}^{(m+\ell)} - R_{\ell,m+\ell}(n)\right) \\ &\quad + \left(R_{\ell,m+\ell}(n)) - R_{m+\ell,m+\ell}\left(\frac{n}{\lambda^m}\right)\right) - c_{m+\ell}. \end{aligned}$$

Il ne reste alors plus qu'à diviser par $\rho_{m+\ell}$ (définition 7) pour obtenir $\tilde{S}_{n\lambda^\ell}^{(m+\ell)}$. Les trois termes convergent lorsque ℓ tend vers $+\infty$ d'après le corollaire 41 et la proposition 38. Nous venons de définir une fonction $\Psi : \mathbb{N}^2 \to \mathbb{R}$ par :

$$\Psi(m, n) = \lim_{\ell \to +\infty} \tilde{S}_{n\lambda^\ell}^{(m+\ell)} \quad \text{pour tout } (m, n) \in \mathbb{N}^2.$$

Rappelant que pour tous les entiers ℓ et n, d'après l'item 3 de la proposition 37, $S_{n\lambda^\ell}^{(\ell)} = p_\ell(n\lambda^\ell) - c_\ell$ donc en renormalisant par ρ_ℓ, nous obtenons $\tilde{S}_{n\lambda^\ell}^{(\ell)} = -\tilde{c}_\ell$. Nous venons donc de remarquer que pour tout entier n,

$$\Psi(0, n) = \lim_{\ell \to +\infty} \tilde{S}_{n\lambda^\ell}^{(\ell)} = -\lim_{\ell \to +\infty} \tilde{c}_\ell. \tag{22}$$

Le reste de la preuve consiste à vérifier que la suite $\big(\Psi(1, n)\big)_{n \in \mathbb{N}^*}$ n'est pas nulle. Fixons deux entiers n et ℓ. Le groupe $\mathcal{G}_{\ell+1}$ est caractéristique (voir définition 18), donc $\mathcal{L}$ induit un endomorphisme sur ce groupe encore noté $\mathcal{L}$ et on a :

$$\mathcal{L}^\ell\big(\pi_\varphi(u_0 \cdots u_{n-1})\big) = \langle n\lambda^\ell, S_{n\lambda^\ell}^{(1)} + \cdots + S_{n\lambda^\ell}^{(\ell)} X^{\ell-1} + S_{n\lambda^\ell}^{(\ell+1)} X^\ell \rangle.$$

Mais d'après le corollaire 29, dans le groupe le groupe $\mathcal{G}_{\ell+1}$, l'action de $\mathcal{L}$ sur un élément $\langle x, s_0 + \cdots + s_\ell X^\ell \rangle$ ne dépend que de x et s_0. Puisque :

$$\pi_\varphi(u_0 \cdots u_{n-1}) = \langle n, S_n^{(1)} + \cdots + S_n^{(\ell+1)} X^\ell \rangle,$$

alors $\mathcal{L}^\ell\big(\pi_\varphi(u_0, \ldots, u_{n-1})\big)$ ne dépend que des coordonnées n et $S_n^{(1)}$ et

$$\mathcal{L}^\ell\big(\langle n, S_n^{(1)} \rangle\big) = \langle n\lambda^\ell, S_{n\lambda^\ell}^{(1)} + \cdots + S_{n\lambda^\ell}^{(\ell)} X^{\ell-1} + S_{n\lambda^\ell}^{(\ell+1)} X^\ell \rangle.$$

Nous utilisons alors la formule du produit (définition 17), des puissances (définition 20) et le calcul explicite dans la proposition 40 (calcul de $s_0^{(m)}$) et nous obtenons d'une part :

$$\begin{aligned} \mathcal{L}^\ell\big(\langle n, S_n^{(1)} \rangle\big) &= \mathcal{L}^\ell\big(\langle n, 0 \rangle\big) \cdot \mathcal{L}^\ell\big(\langle 0, S_n^{(1)} \rangle\big) \\ &= \mathcal{L}^\ell\big(\langle n, 0 \rangle\big) \cdot \big\langle 0, \delta_2^\ell \lambda^{\ell(\ell-1)/2} S_n^{(1)} X^\ell \big\rangle. \end{aligned}$$

Et d'autre part, puisque $\mathcal{L}\langle n/\lambda, 0\rangle = \langle n, R_{1,1}(n/\lambda) + \cdots + R_{1,\ell+1}(n/\lambda)X^\ell\rangle$ et que $\mathcal{L}^\ell$ ne dépend que de n et $R_{1,1}(n/\lambda) = p_1(n) + c_1$:

$$\begin{aligned}\mathcal{L}^\ell\big(\langle n,0\rangle\big) =\ & \mathcal{L}^\ell\big(\mathcal{L}(\langle n/\lambda,0\rangle)\cdot\langle 0,-p_1(n)-c_1\rangle\big)\\ &= \langle n/\lambda, R_{\ell+1,1}(n/\lambda)+\cdots+R_{\ell+1,\ell+1}(n/\lambda)\rangle)\\ &\quad\cdot\langle 0,-\delta_2^\ell\lambda^{\ell(\ell-1)/2}(p_1(n)+c_1)X^\ell\rangle\big).\end{aligned}$$

Après avoir effectué le produit et en identifiant les coefficients devant X^ℓ, nous trouvons :

$$S^{(\ell+1)}_{n\lambda^\ell} = p_{\ell+1}(n\lambda^\ell) + \delta_2^\ell\lambda^{\ell(\ell-1)/2}\Big(S^{(1)}_n - p_1(n) - c_1\Big).$$

Puis en renormalisant, nous trouvons :

$$\Psi(1,n) = \lim_{\ell\to+\infty}\tilde{S}^{(\ell+1)}_{n\lambda^\ell} = S^{(1)}_n - p_1(n) - c_1 = S^{(1)}_n - c_0 n.$$

Reprenant la valeur de c_0 donnée en (19) et rappelant que $|\sigma(a)|_a + |\sigma(a)|_b = |\sigma(a)| = \lambda$:

$$\Psi(1,n+1) - \Psi(n) = \varphi(u_n) - c_0 = \begin{cases}\frac{|\sigma(a)|_b}{\lambda}\big(\varphi(a)-\varphi(b)\big) & \text{si } u_n = a\\ \frac{|\sigma(a)|_a}{\lambda}\big(\varphi(a)-\varphi(b)\big) & \text{si } u_n = b.\end{cases}\tag{23}$$

La suite $\big(\Psi(1,n)\big)_{n\in\mathbb{N}}$ n'est donc pas nulle.

6. Preuve du théorème 6

Nous nous plaçons sous les hypothèses et notations du théorème 6. On fixe une fonction $\varphi : \{a,b\} \to \mathbb{R}$ injective. Un réel positif x est dit λ-*adique* s'il existe un entier n tel que $x\lambda^n \in \mathbb{N}$. Pour tous les entiers $(i,j) \in \mathbb{N}^2$, nous noterons $S^{(\ell)}_i$ pour $S^{(\ell)}_i(\varphi * u)$, $\bar{S}^{(\ell)}_i = S^{(\ell)}_i - p_\ell(i)$ et $\tilde{S}^{(\ell)}_i = \bar{S}^{(\ell)}_i/\rho_\ell$.

– *Étape 1 : quelques résultats préliminaires.* Soient n, n' et ℓ trois entiers. D'après l'item 2 de la proposition 37 :

$$\bar{S}^{(\ell+1)}_{n+n'} - \bar{S}^{(\ell+1)}_n = \sum_{k=0}^{n'-1}\bar{S}^{(\ell)}_{n_1+k}.$$

Donc en divisant par $\rho_{\ell+1} = \delta_2\lambda^{\ell-1}\rho_\ell$ (définition 7), alors les sommes normalisées $\tilde{S}^{(\ell+1)}_n$ vérifient :

$$\tilde{S}^{(\ell+1)}_{n+n'} - \tilde{S}^{(\ell+1)}_n = \frac{1}{\delta_2\lambda^{\ell-1}}\sum_{k=0}^{n'-1}\bar{S}^{(\ell)}_{n_1+k}.\tag{24}$$

Rappelons que d'après le théorème 4, les sommes normalisées $|\tilde{S}^{(\ell)}_n|$ sont uniformément bornées. Nous noterons Θ la borne supérieure.

– *Étape 2 : définition de la fonction pour les réels λ-adiques.* On définit une fonction sur l'ensemble des points λ-adiques en posant (proposition 8) :

$$f\left(\frac{n}{\lambda^{m-1}}\right) = \lim_{\ell\to+\infty} \tilde{S}^{(m+\ell)}_{n\lambda^\ell} = \Psi(m,n) \quad \text{pour tout couple d'entiers } (m,n)\in\mathbb{N}^2.$$

La fonction f est bien définie car :

$$\Psi(m,n) = \lim_{\ell\to+\infty} \tilde{S}^{(m+\ell)}_{n\lambda^\ell} = \lim_{\ell\to+\infty} \tilde{S}^{(m+\ell+1)}_{n\lambda^{\ell+1}} = \Psi(m+1,\lambda n).$$

D'après le théorème 4, la fonction f est bornée et nous notons $\|f\| = \sup\{|f(x)|; x \text{ est un point } \lambda\text{-}\}$. On a $\|f\| \leq \Theta$. Puisque Ψ n'est pas identiquement nulle, f ne l'est pas non plus. Plus précisément cette fonction n'est pas identiquement nulle sur l'ensemble $\mathbb{N}$ car $\Psi(1,n)$ ne l'est pas (proposition 8).

– *Étape 3 : continuité uniforme et prolongement à $\mathbb{R}_+$.* On vérifie que f est uniformément continue.

$$\begin{aligned} f\left(\frac{n+1}{\lambda^{m-1}}\right) - f\left(\frac{n}{\lambda^{m-1}}\right) &= \lim_{\ell\to+\infty} \tilde{S}^{(m+\ell)}_{(n+1)\lambda^\ell} - \tilde{S}^{(m+\ell)}_{n\lambda^\ell} \\ &= \lim_{\ell\to+\infty} \frac{1}{\delta_2\lambda^{m+\ell-2}} \sum_{k=0}^{\lambda^\ell-1} \tilde{S}^{(m+\ell-1)}_{n\lambda^\ell+k} \quad \text{relation (24)} \end{aligned}$$

Or pour tout entier ℓ,

$$\frac{1}{\lambda^\ell}\sum_{k=0}^{\lambda^\ell-1} \left|\tilde{S}^{(m+\ell-1)}_{n\lambda^\ell+k}\right| \leq \Theta \quad \text{et donc} \quad \lambda^{m-1}\left|f\left(\frac{n+1}{\lambda^{m-1}}\right) - f\left(\frac{n}{\lambda^{m-1}}\right)\right| \leq \frac{\lambda}{\delta_2}\Theta.$$

Soient $x = n_x\lambda^{1-m}$ et $y = n_y\lambda^{1-m}$ deux points λ-adiques avec $y < x$. Alors d'après la relation précédente :

$$\begin{aligned} |f(x)-f(y)| = \left|f\left(\frac{n_x}{\lambda^{m-1}}\right) - f\left(\frac{n_y}{\lambda^{m-1}}\right)\right| &\leq \sum_{n=n_y}^{n_x-1} \left|f\left(\frac{n+1}{\lambda^{m-1}}\right) - f\left(\frac{n}{\lambda^{m-1}}\right)\right| \\ &\leq \sum_{n=n_y}^{n_x-1} \frac{\lambda}{\delta_2}\frac{1}{\lambda^{m-1}}\Theta = \frac{\lambda}{\delta_2}\frac{n_x-n_y}{\lambda^{m-1}}\Theta = \frac{\lambda}{\delta_2}\Theta\cdot(x-y). \end{aligned}$$

La fonction f est donc uniformément continue et on la prolonge en une fonction uniformément continue, définie sur $\mathbb{R}_+$ encore notée f.

– *Étape 4. Équation différentielle.* Fixons quatre entiers (n_1,n_2,m_1,m_2) tel que

$$\frac{n_1}{\lambda^{m_1-1}} > 0 \quad \text{et} \quad \frac{n_1}{\lambda^{m_1-1}} + \frac{n_2}{\lambda^{m_2-1}} > 0.$$

Alors,

$$f\left(\frac{n_1}{\lambda^{m_1-1}} + \frac{n_2}{\lambda^{m_2-1}}\right) = f\left(\frac{n_1\lambda^{m_2}+n_2\lambda^{m_1}}{\lambda^{m_1+m_2-1}}\right) = \lim_{\ell\to+\infty} \tilde{S}^{(m_1+m_2+\ell)}_{(n_1\lambda^{m_2}+n_2\lambda^{m_1})\lambda^\ell}.$$

Pour tout entier ℓ :

$$\bar{S}^{(m_1+m_2+\ell)}_{(n_1\lambda^{m_2}+n_2\lambda^{m_1})\lambda^\ell} - \bar{S}^{(m_1+m_2+\ell)}_{n_1\lambda^{m_2+\ell}} = \sum_{k=0}^{n_2\lambda^{m_1+\ell}} \bar{S}^{(m_1+m_2+\ell-1)}_{n_1\lambda^{m_2+\ell}+k}.$$

Et maintenant pour k fixé :

$$\begin{aligned}\left|\bar{S}^{(m_1+m_2+\ell-1)}_{n_1\lambda^{m_2+\ell}+k} - \bar{S}^{(m_1+m_2+\ell-1)}_{n_1\lambda^{m_2+\ell}}\right| &= \left|\sum_{j=0}^{k-1} \bar{S}^{(m_1+m_2+\ell-2)}_{n_1\lambda^{m_2+\ell}+j}\right| \\ &\leq k\cdot\sup_{j\in\mathbb{N}}\left|\bar{S}^{(m_1+m_2+\ell-2)}_{j}\right|.\end{aligned}$$

Maintenant, nous normalisons en divisant ces expressions respectivement par $\rho_{m_1+m_2+\ell}$ et $\rho_{m_1+m_2+\ell-1}$. Les deux expressions ci-dessus s'écrivent alors :

$$\begin{cases}\tilde{S}^{(m_1+m_2+\ell)}_{(n_1\lambda^{m_2}+n_2\lambda^{m_1})\lambda^\ell} - \tilde{S}^{(m_1+m_2+\ell)}_{n_1\lambda^{m_2+\ell}} = \dfrac{1}{\delta_2\lambda^{m_1+m_2+\ell-2}} \displaystyle\sum_{k=0}^{n_2\lambda^{m_1+\ell}} \tilde{S}^{(m_1+m_2+\ell-1)}_{n_1\lambda^{m_2+\ell}+k}, \\ \left|\tilde{S}^{(m_1+m_2+\ell-1)}_{n_1\lambda^{m_2+\ell}+k} - \tilde{S}^{(m_1+m_2+\ell-1)}_{n_1\lambda^{m_2+\ell}}\right| \leq k\dfrac{\lambda}{|\delta_2|}\dfrac{1}{\lambda^{m_1+m_2+\ell-1}}\cdot\Theta.\end{cases}$$

En rassemblant ces relations, on a :

$$\begin{aligned}&\left|\tilde{S}^{(m_1+m_2+\ell)}_{(n_1\lambda^{m_2}+n_2\lambda^{m_1})\lambda^\ell} - \tilde{S}^{(m_1+m_2+\ell)}_{n_1\lambda^{m_2+\ell}} - \frac{\lambda}{\delta_2}\frac{n_2}{\lambda^{m_2-1}}\tilde{S}^{(m_1+m_2+\ell-1)}_{n_1\lambda^{m_2+\ell}}\right| \\ &\leq \frac{\lambda}{|\delta_2|}\frac{1}{\lambda^{m_1+m_2+\ell-1}}\sum_{k=0}^{n_2\lambda^{m_1+\ell}} k\frac{\lambda}{|\delta_2|}\frac{1}{\lambda^{m_1+m_2+\ell-1}}\cdot\Theta \\ &\leq \frac{\lambda^2}{\delta_2^2}\frac{1}{\lambda^{2m_1+2m_2+2\ell-2}}n_2^2\lambda^{2m_1+2\ell}\cdot\Theta \leq \frac{\lambda^2}{\delta_2^2}\left(\frac{n_2}{\lambda^{m_2-1}}\right)^2\Theta.\end{aligned}$$

Finalement, en prenant la limite sur ℓ, on obtient :

$$\left|f\left(\frac{n_1}{\lambda^{m_1-1}}+\frac{n_2}{\lambda^{m_2-1}}\right) - f\left(\frac{n_1}{\lambda^{m_1-1}}\right) - \frac{\lambda}{\delta_2}\frac{n_2}{\lambda^{m_2-1}}f\left(\frac{n_1}{\lambda^{m_1-2}}\right)\right| \leq \frac{\lambda^2}{\delta_2^2}\left(\frac{n_2}{\lambda^{m_2-1}}\right)^2\Theta.$$

Par densité des points λ-adiques, on récupère pour tout couple de réels (x,h) tels que $x>0$ et $x+h>0$:

$$\left|f(x+h) - f(x) - h\frac{\lambda}{\delta_2}f(\lambda x)\right| \leq \frac{\lambda^2}{\delta_2^2}h^2\Theta.$$

Ceci prouve que f est dérivable en tout point x, et que sa dérivée vérifie $f'(x) = \frac{\lambda}{\delta_2}f(\lambda x)$.

– *Étape 5. Concaténation.* Nous allons prouver par récurrence sur $\ell\in\mathbb{N}$ que pour tout entier $k\in\{0,\dots,\lambda^\ell-1\}$, la suite $\left(\bar{S}^{(\ell)}_{n\lambda^\ell+k}\right)_{n\in\mathbb{N}}$ ne prend que deux valeurs. Plus précisément, pour tout entier $k\in\{0,\dots,\lambda^\ell-1\}$ fixé, pour tout entier $n \in \mathbb{N}$, $\bar{S}^{(\ell)}_{n\lambda^\ell+k}$ ne dépend que de u_n. Le résultat suivra alors par passage à la limite.

Ce résultat est immédiat pour $\ell = 0$ car $\bar{S}^{(0)}_n = u_n - c_0$ ne dépend que de u_n. Supposons le résultat vrai au rang ℓ. Soit $k \in \{0, \dots, \lambda^{\ell+1} - 1\}$ fixé. Alors d'après les items 2 et 3 de la proposition 37 :

$$\bar{S}^{(\ell+1)}_{n\lambda^{\ell+1}+k} = \bar{S}^{(\ell+1)}_{n\lambda^{\ell+1}} + \sum_{i=0}^{k-1} \bar{S}^{(\ell)}_{n\lambda^{\ell+1}+i} = -c_{\ell+1} + \sum_{i=0}^{k-1} \bar{S}^{(\ell)}_{n\lambda^{\ell+1}+i}.$$

Le résultat est alors prouvé car d'après l'hypothèse de récurrence, chaque $\bar{S}^{(\ell)}_{n\lambda^{\ell+1}+i}$ ne dépend que de $u_{n\lambda} \cdots u_{(n+1)\lambda-1}$ (et de i).

Or le mot $u_{n\lambda} \cdots u_{(n+1)\lambda-1}$ est l'image par σ de u_n.

Ce résultat pourrait également être prouvé directement par le travail effectué dans la partie 2.5. En effet nous avons vu dans cette partie que les fonctions ψ_i sont constantes sur chaque ensemble $\mathrm{Cyl}(i, m, \alpha)$. □

Il est *a priori* très difficile de calculer les valeurs des fonctions limites f_u du théorème 6. La valeur de la fonction aux points $\lambda\mathbb{N}$ est donnée en (22) et nous ne pouvons pas la relier «simplement» à la substitution σ. Les différences de valeurs prises par la fonction entre deux points entiers sont cependant assez accessibles et données par la formule (23). On peut voir ce fait sur les figures 6.1, 6.2 et 6.3.

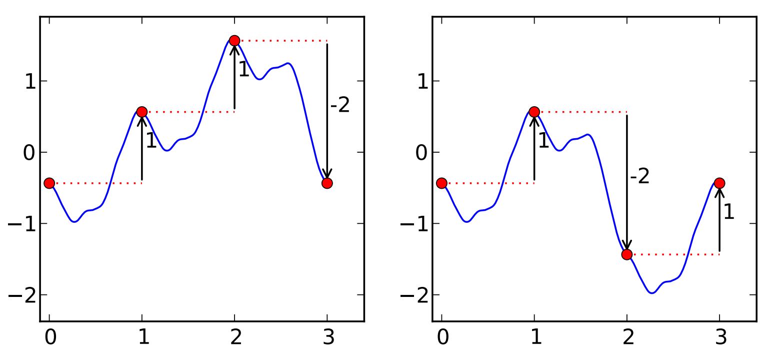

FIGURE 6.1. Les fonctions f_a et f_b pour $a \mapsto aab, b \mapsto aba$ et pour $\varphi = \chi_a - 2\chi_b$.

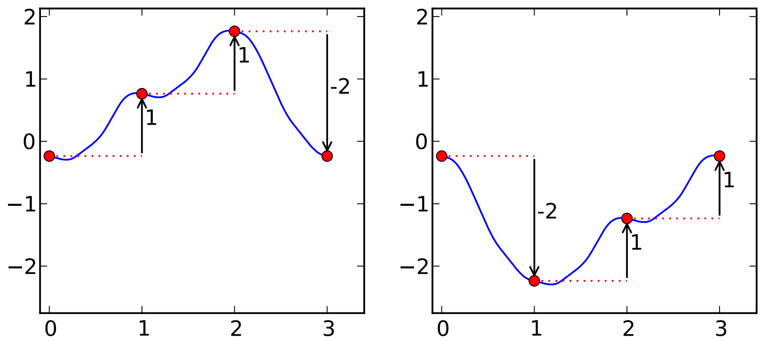

FIGURE 6.2. Les fonctions f_a et f_b pour $a \mapsto aab, b \mapsto baa$ et pour $\varphi = \chi_a - 2\chi_b$.

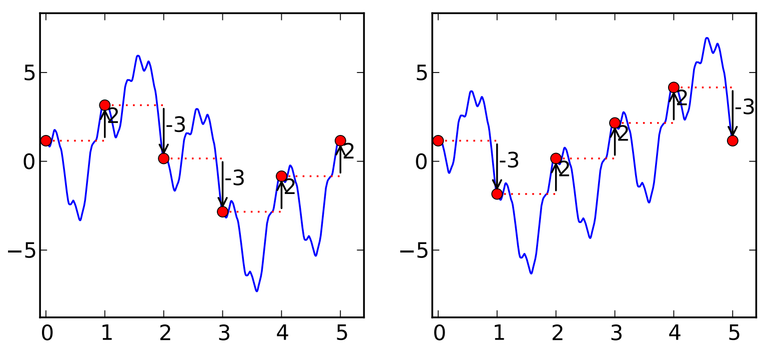

FIGURE 6.3. Les fonctions f_a et f_b pour $a \mapsto abbaa, b \mapsto baaab$ et pour $\varphi = 2\chi_a - 3\chi_b$.

Annexe A. Preuves des résultats

A.1. Preuve du lemme 21

1. Soient $(r, r') \in \mathbb{R}^2$ et $\langle x, s\rangle \in \mathcal{G}$. Nous avons d'une part :

$$\langle x, s\rangle^{r+r'} = \left\langle (r+r')x, B\bigl(rx + r'x\bigr)B(x)^{-1} \cdot s \right\rangle,$$

et d'autre part,

$$\begin{aligned}\langle x, s\rangle^{r} \cdot \langle x, s\rangle^{r'} &= \langle rx, B(rx)B(x)^{-1} \cdot u\rangle \, \langle r'x, B(r'x)B(x)^{-1} \cdot s\rangle \\ &= \Bigl\langle (r+r')x, \Bigl(A^{r'x}B(rx) + B(r'x)\Bigr)\, B(x)^{-1} \cdot s \Bigr\rangle.\end{aligned}$$

Il nous reste alors à prouver que pour tout couple de réels $(y, y') \in \mathbb{R}^2$:

$$B(y+y') = B(y) + A^y B(y') = B(y') + A^{y'} B(y).$$

Un calcul direct donne :

$$\begin{aligned} A^{y+y'} &= A^y A^{y'} = (1 + XB(y)) \cdot (1 + XB(y')) \\ &= 1 + X\Big(B(y) + B(y') + XB(y)B(y')\Big) \\ &= 1 + X\Big(B(y) + \big(I + XB(y)\big)B(y')\Big) \\ &= 1 + X\Big(B(y) + A^y B(y')\Big). \end{aligned}$$

2. D'après le lemme 17, deux éléments $\langle x, s\rangle$ et $\langle y, t\rangle$ de $\mathcal{G}$ commutent si et seulement si $(A^y - 1) \cdot s = (A^x - 1) \cdot t$. Cette relation peut également s'écrire $B(y) \cdot s = B(x) \cdot t$ et si $x \neq 0$: $t = B(y)B(x)^{-1} \cdot s$. Autrement dit $\langle x, s\rangle^{y/x} = \langle y, t\rangle$.
3. Fixons un élément $g = \langle x, s\rangle$ de $\mathcal{G}$ avec x non nul. Pour tout nombre réel r :

$$\begin{cases} B(rx) &= rx + \frac{rx(rx-1)}{2} X + \cdots \\ B(x) &= x + \frac{x(x-1)}{2} X + \cdots \\ B(x)^{-1} &= \frac{1}{x} - \frac{1}{x}\frac{x-1}{2} X + \cdots \end{cases}$$

Si c_n désigne le n-ème coefficient de $B(x)$, alors ceux de $B(x)^{-1}$ que l'on note $(c'_n)_n$ se trouvent en résolvant le système triangulaire :

$$c_0 c'_0 = 1, \quad c_1 c'_0 + c_0 c'_1 = 0, \quad \ldots \quad c_n c'_0 + \cdots + c_k c'_{n-k} + \cdots c_0 c'_n = 0, \quad \ldots$$

Donc $B(rx)B(x)^{-1}$ est une série formelle de la forme :

(25)

$$B(rx)\, B(x)^{-1} = P_0(x, r) + \cdots + P_\ell(x, r)X^\ell + \cdots \quad \text{où } \forall \ell \in \mathbb{N},\ P_\ell \in \mathbb{R}_2[X, Y].$$

Ainsi l'application $(g, r) \to g^r$ est continue (car polynomiale dans le sens de la relation (25) ci-dessus).

Il ne reste qu'à vérifier qu'elle se prolonge par la formule donnée lorsque x tend vers 0. D'après la relation (11) sur les puissances entières dans $\mathcal{G}$, pour tout entier n et tout réel x :

$$P_0(x, n) + P_1(x, n)X + \cdots + P_\ell(x, n)X^\ell + \cdots \underset{x \to 0}{\longrightarrow} n.$$

Donc,

$$\forall \ell \in \mathbb{N}^*,\ \forall n \in \mathbb{N},\ P_\ell(0, n) = 0 \quad \text{et} \quad \forall k \in \mathbb{N},\ P_0(0, n) = n.$$

Puisque pour tout entier ℓ, $y \to P_\ell(0, y)$ est un polynôme en une variable, alors :

$$\forall \ell \in \mathbb{N}^*,\ \forall y \in \mathbb{R},\ P_\ell(0, y) = 0 \quad \text{et} \quad \forall y \in \mathbb{R},\ P_0(0, y) = y.$$

En reprenant la relation (25) avec la topologie des séries formelles :

$$B(rx)B(x)^{-1} = P_0(x,r) + \cdots + P_\ell(x,r)X^\ell + \cdots \xrightarrow[x\to 0]{} r.$$

L'application puissance ainsi définie est donc continue sur $\mathcal{G} \times \mathbb{R}$. Et même polynomiale au sens de (25).

A.2. Preuve de la proposition 25. — Si un tel endomorphisme existe, alors $\mathcal{L} \circ \pi_\varphi = \pi_\varphi \circ \sigma$ et l'image des générateurs $\{a, b\}$ est :

$$\mathcal{L}\big(\pi_\varphi(a)\big) = \pi_\varphi\big(\sigma(a)\big) \quad \text{et} \quad \mathcal{L}\big(\pi_\varphi(b)\big) = \pi_\varphi\big(\sigma(b)\big).$$

Rappelons que d'après la relation (13), nous avons :

$$\begin{cases} \pi_\varphi(a) = \hat{\varphi}(a) = \big\langle 1, \varphi(a)\big\rangle, \\ \pi_\varphi(b) = \hat{\varphi}(b) = \big\langle 1, \varphi(b)\big\rangle, \end{cases} \quad \text{et} \quad \begin{cases} \pi_\varphi\big(\sigma(a)\big) = \big\langle \lambda, S_\lambda(\varphi * \sigma(a))\big\rangle, \\ \pi_\varphi\big(\sigma(b)\big) = \big\langle \lambda, S_\lambda(\varphi * \sigma(b))\big\rangle. \end{cases}$$

Donc le morphisme $\mathcal{L}$ recherché doit vérifier :

$$\mathcal{L}\Big(\big\langle 1, \varphi(a)\big\rangle\Big) = \big\langle \lambda, S_\lambda(\varphi * \sigma(a))\big\rangle \quad \text{et} \quad \mathcal{L}\Big(\big\langle 1, \varphi(b)\big\rangle\Big) = \big\langle \lambda, S_\lambda(\varphi * \sigma(b))\big\rangle. \tag{26}$$

Nous avons vu en introduction de la section 3.6 que les endomorphismes étaient caractérisés par leur image de d et f. Nous allons utiliser cette relation (26) pour obtenir ces images.

- Un premier calcul donne :

$$\big\langle 1, \varphi(a)\big\rangle^{-1} \cdot \big\langle 1, \varphi(b)\big\rangle = \big\langle 0, \varphi(b) - \varphi(a)\big\rangle = f^{\varphi(b)-\varphi(a)}$$
$$\iff f = \Big(\big\langle 1, \varphi(a)\big\rangle^{-1} \cdot \big\langle 1, \varphi(b)\big\rangle\Big)^{1/(\varphi(b)-\varphi(a))}.$$

  Ce qui nous permet d'obtenir ce que doit être l'image par $\mathcal{L}$ de f par le lemme 24 :

$$\begin{aligned} \mathcal{L}(f) &= \Big(\mathcal{L}\big(\langle 1, \varphi(a)\rangle\big)^{-1} \cdot \mathcal{L}\big(\langle 1, \varphi(b)\rangle\big)\Big)^{1/(\varphi(b)-\varphi(a))} \\ &= \Big(\big\langle \lambda, S_\lambda(\varphi * \sigma(a))\big\rangle^{-1} \cdot \big\langle \lambda, S_\lambda(\varphi * \sigma(b))\big\rangle\Big)^{1/(\varphi(b)-\varphi(a))} \qquad \text{d'après (26)} \\ &= \Big(\big\langle -\lambda, -A^\lambda \cdot S_\lambda(\varphi * \sigma(a))\big\rangle \cdot \big\langle \lambda, S_\lambda(\varphi * \sigma(b))\big\rangle\Big)^{1/(\varphi(b)-\varphi(a))} \\ &= \big\langle 0, S_\lambda(\varphi * \sigma(b)) - S_\lambda(\varphi * \sigma(a))\big\rangle^{1/(\varphi(b)-\varphi(a))} \\ &= \left\langle 0, \frac{S_\lambda(\varphi * \sigma(b)) - S_\lambda(\varphi * \sigma(a))}{\varphi(b) - \varphi(a)} \right\rangle \end{aligned}$$

  d'après la définition 20 des puissances.

- Pour trouver l'image de d par $\mathcal{L}$, nous procédons avec la même méthode en exprimant $d = \langle 1, 0\rangle$ à partir de $\langle 1, \varphi(a)\rangle$ en utilisant la relation :

$$d = \langle 1, \varphi(a)\rangle \cdot \langle 0, 1\rangle^{-\varphi(a)} = \langle 1, \varphi(a)\rangle \cdot f^{-\varphi(a)}.$$

Et donc toujours par le lemme 24 :

$$\begin{aligned}
\mathcal{L}(d) &= \mathcal{L}\big(\langle 1, \varphi(a)\rangle\big) \cdot \mathcal{L}\big(f\big)^{-\varphi(a)} \\
&= \big\langle \lambda, S_\lambda(\varphi * \sigma(a))\big\rangle \cdot \langle 0, \delta\rangle^{-\varphi(a)} \quad \text{d'après (26)} \\
&= \big\langle \lambda, S_\lambda(\varphi * \sigma(a))\big\rangle \cdot \langle 0, -\varphi(a)\delta\rangle = \big\langle \lambda, S_\lambda(\varphi * \sigma(a)) - \varphi(a)\delta\big\rangle \\
&= \left\langle \lambda, S_\lambda(\varphi * \sigma(a)) - \varphi(a)\frac{S_\lambda(\varphi * \sigma(b)) - S(\varphi * \sigma(a))}{\varphi(b) - \varphi(a)}\right\rangle \\
&= \left\langle \lambda, \frac{\varphi(b)S_\lambda(\varphi * \sigma(a)) - \varphi(a)S_\lambda(\varphi * \sigma(b))}{\varphi(b) - \varphi(a)}\right\rangle.
\end{aligned}$$

L'application $\mathcal{L}$ est correctement définie si et seulement si les relations vérifiées par les générateurs $(d, f_0, f_1, \ldots)$ sont préservées par $\mathcal{L}$. C'est-à-dire si pour tout couple $(i,j) \in \mathbb{N}^2$, $[f_i, f_j] = 1_{\mathcal{G}}$. Nous sommes donc amener à calculer l'image de f_j pour $j \geq 1$. En utilisant la formule obtenue dans le lemme 17 pour le commutateur de deux éléments de $\mathcal{G}$, nous trouvons $\mathcal{L}(f_1) = \big\langle 0, (A^\lambda - 1) \cdot \delta\big\rangle$. Nous retrouvons alors la formule donnée au troisième item par récurrence.

Il est clair que les images de f_j commutent pour $j \geq 0$ car elles appartiennent à $\mathcal{V} = \{\langle 0, s\rangle ; s \in V\}$. Donc $\mathcal{L}$ définit donc correctement un endomorphisme de $\mathcal{G}$. Nous venons de prouver que les trois premiers items de la proposition sont vérifiés.

Le quatrième item est évident car l'action de $\mathcal{L}$ sur $\mathcal{V}$ ne dépend pas de la fonction φ choisie car δ ne dépend pas de φ (lemme 10).

Il ne nous reste plus qu'à prouver la dernière partie de la proposition. En écrivant les éléments de $\langle x, s\rangle$ de $\mathcal{G}$ avec $x \in \mathbb{R}$ et $s = s_0 + s_1 X + \cdots \in V$ sous leur forme canonique (lemme 23), nous obtenons l'expression suivante :

$$\begin{aligned}
\mathcal{L}\big(\langle x, s\rangle\big) &= \mathcal{L}(d^x f_0^{s_0} \cdots f_j^{s_j} \cdots) = \mathcal{L}(d)^x \mathcal{L}(f_0)^{s_0} \cdots \mathcal{L}(f_j)^{s_j} \cdots \\
&= \langle \lambda, \beta\rangle^z \prod_{j=0}^{\infty} \langle 0, (A^\lambda - 1)^j \delta\rangle^{s_j} \quad \text{d'après les items 2 et 3} \\
&= \langle \lambda z, B(\lambda z)B(\lambda)^{-1}\beta\rangle \prod_{j=0}^{\infty} \langle 0, s_j(A^\lambda - 1)^j \cdot \delta\rangle \\
&\qquad \text{(définition 20 des puissances)} \\
&= \langle \lambda z, B(\lambda z)B(\lambda)^{-1}\beta\rangle \cdot \left\langle 0, \sum_{j=0}^{\infty} s_j(A^\lambda - 1)^j \cdot \delta\right\rangle \\
&= \left\langle \lambda z, \left(\sum_{j=0}^{\infty} s_j (A^\lambda - 1)^j\right) \cdot \delta + B(\lambda z)B(\lambda)^{-1} \cdot \beta\right\rangle.
\end{aligned}$$

A.3. Preuves des résultats de la partie 3.7

Preuve de la proposition 27. — Nous fixons un entier m. Rappelons que nous avons d'une part :

$$A^{my} = \sum_{k=0}^{+\infty} \binom{my}{k} X^k.$$

Et d'autre part, en écrivant $(A^y)^m$, nous avons :

$$A^{my} = \sum_{k=0}^{\infty} \left(\sum_{\substack{k_1+\ldots+k_m=k \\ k_1\geq 0,\ldots,k_m\geq 0}} \binom{y}{k_1} \cdots \binom{y}{k_m} \right) X^k = \sum_{k=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} q_{j,k}(y) X^k.$$

Pour obtenir cette formule, il suffit de dénombrer pour $j \in \{0, \ldots, m\}$ le nombre de m-uplets de la forme $(k_1, \ldots, k_m)$ dont j coordonnées sont nulles. Or $q_{j,k}$ est nul pour $j > k$. Donc,

$$\sum_{j=0}^{m} \binom{m}{j} q_{j,k}(y) = \sum_{j=0}^{\min(m,k)} \binom{m}{j} q_{j,k}(y).$$

Maintenant si $m < k$, alors pour tout entier $j > m$, $\binom{m}{j}$ s'annule et nous pouvons donc écrire :

$$\binom{my}{k} = \sum_{j=0}^{k} \binom{m}{j} q_{j,k}(y).$$

Donc l'équation polynomiale en x :

$$\binom{xy}{k} = \sum_{j=0}^{k} \binom{x}{j} q_{j,k}(y),$$

est vérifiée pour tous les entiers : l'égalité est vérifiée pour tous les réels.

Démontrons maintenant la relation sur l'expression $B(y)^{-1}B(xy)$. Pour cela, rappelons la définition 19 de $B(xy)$ pour tout couple de réels (x, y) :

$$\begin{aligned}
B(xy) &= \sum_{k=0}^{+\infty} \binom{xy}{k+1} X^k = \sum_{k=1}^{+\infty} \binom{xy}{k} X^{k-1} \\
&= \sum_{k=1}^{\infty} \left(\sum_{j=0}^{k} q_{j,k}(y) \binom{x}{j} \right) X^{k-1} \quad \text{d'après la relation (16)} \\
&= \sum_{j=0}^{\infty} \left(\sum_{k=1}^{\infty} q_{j,k}(y) X^{k-1} \right) \binom{x}{j}, \\
&= \sum_{j=0}^{\infty} \frac{(A^y - 1)^j}{X} \binom{x}{j} \quad \text{par (14) et car } A^y - 1 \text{ est divisible par } X.
\end{aligned}$$

Et donc nous trouvons :

$$B(y)^{-1}B(xy) = \frac{X}{A^y - 1}\sum_{j=0}^{\infty}\binom{x}{j}\frac{(A^y-1)^j}{X} = \sum_{j=1}^{\infty}\binom{x}{j}(A^y-1)^{j-1}. \qquad \square$$

Preuve du corollaire 28. — La première formule est claire avec la proposition 27 et la forme de l'endomorphisme décrite à l'item 5 de la proposition 25. Nous notons $q_{j,k}$ pour $q_{j,k}(\lambda)$. Nous allons expliciter le calcul pour obtenir la relation (17) :

$$\begin{aligned}\sum_{j=0}^{\infty}(A^\lambda - 1)^j\left(s_j\cdot\delta + \binom{x}{j+1}\gamma\right) &= \sum_{j=0}^{\infty}\sum_{k=0}^{+\infty}q_{j,k}\sum_{i=0}^{+\infty}\left(s_j\cdot\delta_{i+1} + \binom{x}{j+1}\gamma_{i+1}\right)X^{k+i}\\ &= \sum_{k=0}^{+\infty}\sum_{i=0}^{+\infty}\sum_{j=0}^{\infty}q_{j,k}\left(s_j\cdot\delta_{i+1} + \binom{x}{j+1}\gamma_{i+1}\right)X^{k+i}.\end{aligned}$$

Fixons un entier $\ell \in \mathbb{N}^*$ et intéressons nous au coefficient de X_ℓ que nous notons t_ℓ. En rappelant que $q_{j,k}$ est nul si $j > k$:

$$\begin{aligned}t_\ell &= \sum_{i=0}^{\ell}\sum_{j=0}^{\infty}q_{j,\ell-i}\left(s_j\cdot\delta_{i+1} + \binom{x}{j+1}\gamma_{i+1}\right)\\ &= \sum_{i=0}^{\ell}\sum_{j=0}^{\ell-i}q_{j,\ell-i}\left(s_j\cdot\delta_{i+1} + \binom{x}{j+1}\gamma_{i+1}\right)\\ &= \sum_{j=0}^{\ell}\sum_{i=0}^{\ell-j}q_{j,\ell-i}\left(s_j\cdot\delta_{i+1} + \binom{x}{j+1}\gamma_{i+1}\right).\end{aligned}$$

Nous retrouvons donc la relation (17) en isolant le cas où $j = \ell$. La relation suivante (2) s'en déduit immédiatement en simplifiant avec $\delta_1 = 0$. $\square$

A.4. Preuves des résultats de la section 4.1

Preuve du lemme 30. — Dérivons en X l'égalité (14) de séries formelles pour $y \in \mathbb{R}$ et $j \in \mathbb{N}^*$ fixés :

$$\sum_{k=0}^{+\infty}q_{j,k}(y)X^k = (A^y-1)^j = \left(\sum_{m=0}^{+\infty}\binom{y}{m}X^m\right)^j.$$

Nous obtenons :

$$\begin{aligned}\sum_{k=1}^{+\infty}kq_{j,k}(y)X^{k-1} &= j\left(\sum_{m=1}^{+\infty}m\binom{y}{m}X^m\right)(A^y-1)^{j-1}\\ &= j\left(\sum_{m=1}^{+\infty}m\binom{y}{m}X^m\right)\left(\sum_{n=0}^{\infty}q_{j-1,n}(y)X^n\right).\end{aligned}$$

En identifiant les coefficients de X^{k-1}, nous obtenons pour tout entier $k \geq 1$:

$$kq_{j,k}(y) = j \sum_{m=1}^{k-1} m \binom{y}{m} q_{j-1,k-m}(y). \tag{27}$$

Fixons un entier $k \in \mathbb{N}^*$ et notons $q_{j,k}$ pour $q_{k,k}(y)$. En écrivant pour tout entier m non nul, $\binom{y}{m+1} = \frac{y-m+1}{m}\binom{y}{m}$:

$$\begin{aligned}
kq_{j,k} &= j \sum_{m=1}^{k-1} m \binom{y}{m} q_{j-1,k-m} = yjq_{j-1,k-1} + j \sum_{m=2}^{k-1} m \binom{y}{m} q_{j-1,k-m} \\
&= yjq_{j-1,k-1} + j \sum_{m=1}^{k-2} (m+1) \binom{y}{m+1} q_{j-1,k-m-1} \\
&= yjq_{j-1,k-1} + j \sum_{m=1}^{k-2} (y-m) \binom{y}{m} q_{j-1,k-m-1} \\
&= yjq_{j-1,k-1} + jy \sum_{m=1}^{k-2} \binom{y}{m} q_{j-1,k-m-1} - j \sum_{m=1}^{k-2} m \binom{y}{m} q_{j-1,k-m-1} \\
&= yjq_{j-1,k-1} + jy \sum_{m=1}^{k-2} \binom{y}{m} q_{j-1,k-m-1} - (k-1)q_{j-1,k} \\
&\quad \text{en utilisant à nouveau (27).}
\end{aligned}$$

Il ne nous reste plus qu'à calculer la somme suivante :

$$\begin{aligned}
\sum_{m=1}^{k-2} \binom{y}{m} q_{j-1,k-m-1} &= \sum_{m=1}^{k-2} \binom{y}{m} \sum_{\substack{n_1+\ldots+n_{j-1}=k-m-1 \\ n_1\geq 1,\ldots,n_{j-1}\geq 1}} \binom{y}{n_1} \cdots \binom{y}{n_{j-1}} \\
&= \sum_{\substack{n_1+\ldots+n_{j-1}+m=k-m-1 \\ n_1\geq 1,\ldots,k_{j-1}\geq 1,m\geq 1}} \binom{y}{n_1} \cdots \binom{y}{n_{j-1}} \binom{y}{m} \\
&= q_{j,k-1}.
\end{aligned}$$

Finalement, nous trouvons pour tous les entiers $k \geq 1$ et $j \geq 1$:

$$kq_{j,k} = yjq_{j-1,k-1} + jyq_{j,k-1} - (k-1)q_{j-1,k}. \qquad \square$$

Preuve de la proposition 31. — Fixons un réel y. Pour tout couple d'entiers (i,k), nous posons :

$$Q_{k,k+i}(y) = \frac{k(k+1)\ldots(k+i-1)}{y^k}\, q_{k,k+i}(y).$$

Nous noterons $q_{j,k}$ pour $q_{j,k}(y)$ et $Q_{j,k}$ pour $Q_{j,k}(y)$. Alors, $Q_{k,k} = q_{k,k} = y^n$ et la récurrence du lemme 30 nous donne :

$$Q_{k,k+i+1} = Q_{k-1,k+i} + \big(k(y-1) - i\big)Q_{k,k+i}.$$

Nous en déduisons la formule :

$$Q_{k,k+i+1} = \sum_{m=0}^{k} (m(y-1) - i)Q_{m,m+i}. \tag{28}$$

Nous démontrons alors par récurrence que :

$$Q_{k,k+i} \underset{k}{\sim} \frac{(y-1)^i k^{2i}}{2^i\ i!}.$$

Cette formule est vraie pour $i = 0$, supposons-la vraie jusqu'au rang i. En utilisant la formule (28), nous trouvons :

$$Q_{k,k+i+1} \underset{k}{\sim} (y-1)\sum_{m=0}^{k} mQ_{m,m+i} \underset{k}{\sim} \frac{(y-1)^{i+1}}{2^i\ i!}\sum_{m=0}^{k} m^{2i+1} \underset{k}{\sim} \frac{(y-1)^{i+1}}{2^i\ i!}\ \frac{k^{2i+2}}{2i+2}.$$

Le résultat suit en revenant aux polynômes $q_{k,k+i}$. □

A.5. Preuve de la proposition 35. — Nous fixons un entier ℓ et nous effectuons les calculs dans $\mathcal{G}_\ell = \mathcal{G}/\mathcal{V}_\ell \cong \mathbb{R} \rtimes \mathbb{R}^\ell$ où $\mathcal{V}_\ell = \{\langle 0, s\rangle; s \in V_\ell\}$ (voir définition 18). Nous avons vu que le sous-groupe $\mathcal{V}_\ell$ était caractéristique, donc $\mathcal{L}$ induit un endomorphisme de $\mathcal{G}_\ell$ encore noté $\mathcal{L}$.

Pour tout élément $\langle y, s\rangle$ de $\mathcal{G}_\ell$ où y est un réel et $s = s_0 + \cdots + s_{\ell-1}X^\ell \in G/V_\ell$, comme $\delta_1 = 0$, d'après le corollaire 29 :

$$\mathcal{L}^\ell\big(\langle y, s\rangle\big) = \mathcal{L}^\ell\big(\langle y, 0\rangle\big) = \Big\langle \lambda^\ell y, R_{\ell,1}(y) + R_{\ell,2}(y)X + \cdots + R_{\ell,\ell}(y)X^{\ell-1}\Big\rangle.$$

Fixons un réel x. Pour tout entier m, dans le groupe $\mathcal{G}_\ell$, $\mathcal{L}^\ell\big(\langle \lambda^m x, 0\rangle\big) = \mathcal{L}^\ell \circ \mathcal{L}^m\big(\langle x, 0\rangle\big)$ et donc :

$$\begin{aligned}\Big\langle \lambda^{\ell+m}x, R_{\ell,1}(\lambda^m x) + \cdots + R_{\ell,\ell}(\lambda^m x)X^{\ell-1}\Big\rangle \\ = \Big\langle \lambda^{\ell+m}x, R_{\ell+m,1}(x) + \cdots + R_{\ell+m,\ell}(x)X^{\ell-1}\Big\rangle.\end{aligned} \tag{29}$$

Ce qui démontre le premier item.

Pour tout entier n, on a dans le groupe $\mathcal{G}_\ell$ d'une part

$$\mathcal{L}^\ell\big(\langle n, 0\rangle\big) = \Big\langle n\lambda^\ell, R_{\ell,1}(n) + \cdots + R_{\ell,\ell}(n)X^{\ell-1}\Big\rangle.$$

Mais d'après la définition 20 des puissances, on a $\langle n, 0\rangle = \langle 1/\lambda^\ell, 0\rangle^{n\lambda^\ell}$ et donc d'autre part :

$$\begin{aligned}\mathcal{L}^\ell\big(\langle n,0\rangle\big) &= \mathcal{L}\left(\langle 1/\lambda^\ell, 0\rangle^{n\lambda^\ell}\right)\\ &= \big(\mathcal{L}^\ell\big(\langle 1/\lambda^\ell,0\rangle\big)\big)^{n\lambda^\ell} \quad \text{d'après le lemme 24}\\ &= \left\langle 1, R_{\ell,1}(1/\lambda^\ell) + \cdots + R_{\ell,\ell}(1/\lambda^\ell)X^{\ell-1}\right\rangle^{n\lambda^\ell}.\end{aligned}$$

Toujours en utilisant la définition 20 pour les puissances, $\mathcal{L}^\ell\big(\langle n,0\rangle\big) = \langle n\lambda^\ell, s\rangle$ où :

$$\begin{aligned}s &= B(n\lambda^i)B(1)^{-1}\cdot\Big(R_{\ell,1}(1/\lambda^\ell)+\cdots+R_{\ell,\ell}(1/\lambda^\ell)X^{\ell-1}\Big)\\ &= \left(\sum_{k=0}^{+\infty}\binom{n\lambda^\ell}{k+1}X^k\right)\cdot\Big(R_{\ell,1}(1/\lambda^\ell)+\cdots+R_{\ell,\ell}(1/\lambda^\ell)X^{\ell-1}\Big)\\ &\qquad\text{par la définition 19 et car } B(1)=1\\ &= \left(\sum_{k=0}^{\ell-1}\binom{n\lambda^\ell}{k+1}X^k\right)\cdot\Big(R_{\ell,1}(1/\lambda^\ell)+\cdots+R_{\ell,\ell}(1/\lambda^\ell)X^{\ell-1}\Big) \quad \text{dans } \mathcal{G}_\ell.\end{aligned}$$

Donc pour tout $1 \le j \le \ell$:

$$R_{\ell,j}(n) = \sum_{k=1}^{j}\binom{n\lambda^\ell}{j-k+1}R_{\ell,k}\left(\frac{1}{\lambda^\ell}\right).$$

En utilisant le premier item de la proposition, nous pouvons remplacer $R_{\ell,k}(1/\lambda^\ell)$ par $R_{k,k}(1/\lambda^k)$. Le second item est alors prouvé en prenant $j=\ell$ pour tous les réels x de la forme $n\lambda^\ell$. Puisque l'équation :

$$R_{\ell,\ell}(x/\lambda^\ell) = \sum_{k=0}^{\ell-1}\binom{x}{\ell-k}c_k$$

est polynomiale, elle est donc vérifiée pour tout réel x.

Pour démontrer la dernière partie, remarquons tout d'abord que pour tout entier ℓ :

$$\begin{aligned}\mathcal{L}^\ell\left(\left\langle\frac{1}{\lambda^\ell},0\right\rangle\right) &= \left\langle 1, R_{\ell,1}\left(\frac{1}{\lambda^\ell}\right)+\cdots+R_{\ell,\ell}\left(\frac{1}{\lambda^\ell}\right)X^{\ell-1}\right\rangle\\ &= \left\langle 1, R_{1,1}\left(\frac{1}{\lambda}\right)+R_{2,2}\left(\frac{1}{\lambda^2}\right)X+\cdots+R_{\ell,\ell}\left(\frac{1}{\lambda^\ell}\right)X^{\ell-1}\right\rangle\\ &= \langle 1, c_0 + c_1X+\cdots+c_{\ell-1}X^{\ell-1}\rangle.\end{aligned}$$

En rappelant que nous effectuons les calculs dans le groupe $\mathcal{G}_\ell$, nous obtenons :

$$\begin{aligned}\mathcal{L}\big(\langle 1, c_0 + c_1X + \cdots + c_{\ell-1}X^{\ell-1}\rangle\big) &= \mathcal{L}^{\ell+1}\left(\left\langle \frac{1}{\lambda^\ell}, 0\right\rangle\right)\\ &= \left(\mathcal{L}^{\ell+1}\left(\left\langle \frac{1}{\lambda^{\ell+1}}, 0\right\rangle\right)\right)^\lambda \quad \text{car } \left\langle \frac{1}{\lambda^\ell}, 0\right\rangle = \left\langle \frac{1}{\lambda^{\ell+1}}, 0\right\rangle^\lambda\\ &= \langle 1, c_0 + c_1X + \cdots + c_{\ell-1}X^{\ell-1}\rangle^\lambda.\end{aligned}$$

A.6. Preuve de la proposition 40. — Nous prouvons le résultat général par récurrence sur ℓ. Nous commençons par étudier les suites $\big(s_0^{(m)}\big)_{m\in\mathbb{N}}$ et $\big(s_1^{(m)}\big)_{m\in\mathbb{N}}$ pour lesquelles tout est explicite. Nous notons $q_{j,k}$ pour $q_{j,k}(\lambda)$ pour tous les entiers $(j,k) \in \mathbb{N}^2$. Reprenons la forme de l'endomorphisme L donnée au corolaire 28 en simplifiant l'expression avec $\delta_1 = 0$ car la substitution est fortement uniforme. Nous trouvons pour toute série $r = r_0 + r_1X + \cdots \in V$, en notant $L(r) = t = t_0 + t_1X + \cdots \in V$, pour tout entier $i \in \mathbb{N}^*$:

$$t_i = q_{i-1,i-1}\delta_2 r_{i-1} + \left(\sum_{j=1}^{2} q_{i-2,i-j}\delta_{j+1}\right) r_{i-2} + \cdots + \left(\sum_{j=1}^{\ell} q_{i-i,i-j}\delta_{j+1}\right) r_{i-i}$$

$$\text{(30)}\qquad \iff \quad t_i = \lambda^{i-1}\delta_2 r_{i-1} + \left(\sum_{j=1}^{2} q_{i-2,i-j}\delta_{j+1}\right) r_{i-2} + \cdots + \left(\sum_{j=1}^{i} q_{0,i-j}\delta_{j+1}\right) r_0.$$

- *Cas* $\ell = 0$ *et* $\ell = 1$. Reprenons la relation (30) précédente. Sous les hypothèses de la proposition 40 :

  $$\begin{aligned}L(s) &= s_0^{(1)}X + s_1^{(1)}X^2 + s_2^{(1)}X^3 + s_3^{(1)}X^4 + \cdots\\ &= \delta_2 s_0 X + \big(\delta_3 s_0 + \lambda\delta_2 s_1\big)X^2 + s_2^{(1)}X^3 + s_3^{(1)}X^4 + \cdots.\end{aligned}$$

  Donc $s_0^{(1)} = \delta_2 s_0$ et $s_1^{(1)} = \delta_3 s_0 + \lambda\delta_2 s_1$. En appliquant à nouveau L, nous avons :

  $$L^2(s) = \lambda\delta_2 s_0^{(1)}X^2 + \left(\left(\binom{\lambda}{2}\delta_2 + \lambda\delta_3\right) s_0^{(1)} + \lambda^2\delta_2 s_1^{(1)}\right) X^3 + s_2^{(2)}X^4 + s_3^{(2)}X^5 + \cdots$$

  Nous trouvons donc $s_0^{(2)} = \lambda\delta_2^2 s_0$ et $s_1^{(2)} = \left(\binom{\lambda}{2}\delta_2 + \lambda\delta_3\right) s_0^{(1)} + \lambda^2\delta_2 s_1^{(1)}$. On obtient ainsi pour tout entier m, par applications successives de L, la relation de récurrence :

  $$\begin{cases} s_0^{(m+1)} = \delta_2\, q_{m,m}\, s_0^{(m)},\\ s_1^{(m+1)} = (q_{m,m+1}\,\delta_2 + q_{m,m}\,\delta_3)s_0^{(m)} + q_{m+1,m+1}\,\delta_2\, s_1^{(m)}.\end{cases}$$

D'après (15), pour tout entier k non nul, $q_{k,k} = q_{k,k}(\lambda) = \lambda^k$ et $q_{k,k+1} = \frac{k}{2}(\lambda - 1)\lambda^k$.

$$\begin{cases} s_0^{(m+1)} = \delta_2 \lambda^m s_0^{(m)}, \\ s_1^{(m+1)} = \left(\frac{m}{2}(\lambda-1)\lambda^m \delta_2 + \lambda^m\, \delta_3\right) s_0^{(m)} + \lambda^{m+1}\, \delta_2\, s_1^{(m)}. \end{cases}$$

Ce qui nous permet d'obtenir l'expression exacte de $s_0^{(m)}$ pour tout entier m :

$$s_0^{(m)} = s_0\, \delta_2^m\, \lambda^{\frac{m(m-1)}{2}}.$$

Pour l'étude de $\left(s_1^{(m)}\right)_m$, posons pour tout entier m :

$$\tilde{s}_1^{(m)} = \frac{s_1^{(m)}}{\delta_2^m \lambda^{m(m+1)/2}}.$$

En divisant la relation précédente par $\delta_2^{m+1} \lambda^{(m+1)(m+2)/2} = \rho_{m+3}/\delta_2$ où nous rappelons que $\rho_i = \delta_2^{i-1} \lambda^{(i-1)(i-2)/2}$ (voir définition 7), nous obtenons :

$$\begin{aligned} \tilde{s}_1^{(m+1)} &= \tilde{s}_1^{(m)} + \frac{1}{\delta_2^{m+1} \lambda^{(m+1)(m+2)/2}} \left(\frac{m}{2}(\lambda-1)\lambda^m \delta_2 + \lambda^m\, \delta_3\right) s_0^{(m)} \\ &= \tilde{s}_1^{(m)} + \frac{1}{\delta_2 \lambda^{2m+1}} \left(\frac{m}{2}(\lambda-1)\lambda^m \delta_2 + \lambda^m\, \delta_3\right) s_0 \\ &= \tilde{s}_1^{(m)} + \frac{1}{\lambda^{m+1}} \frac{1}{\delta_2} \left(\frac{m}{2}\delta(\lambda-1) + \delta_3\right) s_0. \end{aligned}$$

La série de terme général $\dfrac{1}{\lambda^{i+1}} \left(\dfrac{i}{2}\delta_2(\lambda-1) + \delta_3\right)$ est sommable et donc la suite $\left(\tilde{s}_1^{(m)}\right)_{m\in\mathbb{N}}$ converge vers

$$\Phi_1(s) = s_1 + s_0 \frac{1}{\delta_2} \sum_{i=0}^{\infty} \lambda^{-i-1} \left(\frac{i}{2}\delta_2(\lambda-1) + \delta_3\right) = s_1 + \frac{1}{\lambda-1} \frac{1}{\delta_2} \left(\frac{1}{2}\delta_2 + \delta_3\right) s_0.$$

- *Passage de $\ell - 1$ à ℓ.* Pour le cas général, on établit d'abord une relation de récurrence qui exprime $s_\ell^{(m)}$ sous la forme d'une somme de termes qui dépendent de $s_0, \ldots, s_{\ell-1}, s_\ell$. Nous utilisons ensuite les estimations asymptotiques des polynômes $q_{m,m+i}$ obtenues dans la proposition 31 pour conclure qu'elle converge. Rappelons que $s_j^{(i)}$ est le coefficient devant X^{i+j} de la série $L^i(s)$. D'après la forme explicite de L rappelée dans la relation (30) plus haut, pour l'indice $i = m + \ell + 1$, nous trouvons pour tout entier m :

(31)
$$s_\ell^{(m+1)} = \sum_{j=m+1}^{\ell+m-1} \sum_{i=0}^{\ell+m-j} q_{j,\ell+m-i}\ s_{j-m-1}^{(m)} \delta_{i+1} \sum_{j=1}^{\ell} \sum_{i=1}^{\ell-j+2} q_{j+m-1,m+\ell+1-i} \delta_i\ s_j^{(m)}.$$

Nous divisons la relation précédente par le coefficient $\rho_{m+\ell+2}$. Nous notons $\tilde{s}_i^{(j)} = s_i^{(j)}/\rho_{i+j+1}$.

$$\tilde{s}_\ell^{(m+1)} = \lambda^{m+\ell}\delta_2 \frac{\rho_{m+\ell+1}}{\rho_{m+\ell+2}} \tilde{s}_\ell^{(m)} + \frac{\rho_{m+\ell}}{\rho_{m+\ell+2}} \left(\sum_{j=1}^{2} q_{m+\ell-2,m+\ell-j}\delta_{j+1} \right) \tilde{s}_{\ell-1}^{(m)} + \cdots + \frac{\rho_{m+1}}{\rho_{m+\ell+2}} \left(\sum_{j=1}^{\ell} q_{m-1,m+\ell-j}\delta_{j+1} \right) \tilde{s}_0^{(m)}.$$

Nous simplifions immédiatement la relation en remarquant que $\rho_{i+n} = \delta_2^n \lambda^{n(n+2i-3)/2}\rho_i$ pour trouver :

$$\tilde{s}_\ell^{(m+1)} = \tilde{s}_\ell^{(m)} + \frac{1}{\delta_2^2 \lambda^{2(m+\ell)+1}} \left(\sum_{j=1}^{2} q_{m+\ell-2,m+\ell-j}\delta_{j+1} \right) \tilde{s}_{\ell-1}^{(m)} + \cdots + \frac{1}{\delta_2^{\ell+1} \lambda^{(\ell+1)(\ell+2m+2)/2}} \left(\sum_{j=1}^{\ell} q_{m-1,m+\ell-j}\delta_{j+1} \right) \tilde{s}_0^{(m)}.$$

Nous voulons montrer que la série de terme général $\tilde{s}_\ell^{(m+1)} - \tilde{s}_\ell^{(m)}$ converge. Cette série est elle-même une somme d'un nombre fini de termes. Nous allons étudier chacun de ces termes et montrer qu'ils sont sommables. Fixons $i \in \{1, \dots, \ell\}$, $j \in \{1, \dots, i\}$ et étudions le comportement asymptotique de la suite

$$z_m^{(i,j)} = \frac{1}{\delta_2^{i+1} \lambda^{(i+1)(2m+2\ell-i-2)/2}} q_{m+\ell-i-1,m+\ell-j}\delta_{j+1}\tilde{s}_{\ell-i}^{(m)}.$$

D'après la proposition 31 qui donne une estimation du comportement asymptotique de la suite $q_{m+\ell-i-1,m+\ell-j}$, nous trouvons :

$$q_{m+\ell-i-1,m+\ell-j} \underset{m\to+\infty}{\sim} \frac{(\lambda-1)^{i+1-j}}{2^{i+1-j} \times (i+j-1)!} \lambda^{m+\ell-i-1}(m+\ell-i-1)^{i+j-1}.$$

Donc lorsque m tend vers l'infini :

$$q_{m+\ell-i-1,m+\ell-j} \underset{m\to+\infty}{=} \mathcal{O}\big(\lambda^m \cdot m^{i+j-1}\big) \underset{m\to+\infty}{=} \mathcal{O}\big(\lambda^m \cdot m^{i+j-1}\big).$$

D'autre part, on a l'estimation :

$$\frac{1}{\delta_2^{i+1} \lambda^{(i+1)(2m+2\ell-i-2)/2}} \underset{m\to+\infty}{=} \mathcal{O}\left(\frac{1}{\lambda^{(i+1)m}} \right).$$

Enfin, par récurrence,

$$\tilde{s}_{\ell-i}^{(m)} \underset{m\to+\infty}{=} \mathcal{O}\left(\frac{1}{\delta_2^m \lambda^{(m+\ell-i)(m+\ell-i-1)/2}} \right).$$

Ces trois dernières relations permettent de donner une estimation du comportement asymptotique de $z_m^{(i,j)}$:

$$z_m^{(i,j)} \underset{m\to+\infty}{=} \mathcal{O}\left(\frac{m^{i+j-1}}{\lambda^{im}} \frac{1}{\lambda^{(m+\ell-i)(m+\ell-i-1)/2}}\right) \underset{m\to+\infty}{=} o\left(\frac{1}{\lambda^m}\right).$$

Donc la série de terme général $z_m^{(i,j)}$ converge et la suite $\tilde{s}_\ell^{(m)}$ converge donc vers un réel $\Phi_\ell(s)$ qui est la somme de $s_\ell^{(0)}$ et d'une combinaison linéaire de $\Phi_j(s)$. Donc la fonction $\Phi_\ell(s)$ est bien linéaire en $s_1,\dots,s_\ell$ et le coefficient de s_ℓ est 1. Ce qui finit la preuve du résultat.

Annexe B. Généralités sur les solutions de l'équation intégrale

B.1. Contexte général. — Reprenons les équations intégrales définies dans l'introduction. Pour $\tau \in \mathbb{R}^+$ et $\nu \in \mathbb{R}^*$, nous nous intéressons aux équations intégrales suivantes :

$$(E_{\tau,\nu}) \qquad \int_0^{\tau x} f(t)\,dt = \nu\big(f(x) - f(0)\big) \qquad \text{pour tout } x \geq 0.$$

Remarquons tout d'abord que le paramètre ν est accessoire : si f est solution de $(E_{\tau,\nu})$ alors $g(x) = f(|\nu| x)$ est solution de $(E_{\tau,1})$ ou $(E_{\tau,-1})$. Cependant, il s'avère naturel de considérer des paramètres τ et ν entiers dans notre construction. Les fonctions continues f solutions de $(E_{\tau,\nu})$ sont de classe C^∞ sur $\mathbb{R}_+$ et nous préférerons parfois la formulation équivalente :

$$(E'_{\tau,\nu}) \qquad f'(x) = \frac{\tau}{\nu} f(\tau x) \qquad \text{pour tout } x \geq 0.$$

Si f est solution de $(E_{\tau,\nu})$, les dérivés successives de f sont reliées aux valeurs de f via :

(32)
$$f^{(n)}(x) = \frac{\tau^{n(n+1)/2}}{\nu^n} f(\tau^n x) \qquad \text{pour tout réel } x \geq 0 \text{ et tout entier } n \in \mathbb{N}.$$

L'équation $(E_{\tau,\nu})$ ci-dessus est un cas particulier de *l'équation du pantographe* dont la forme générale est :

$$f'(x) = af(\tau x) + bf(x) \quad \text{avec } (a,b) \in \mathbb{R}^2 \text{ et } \tau \in \mathbb{R}_+ \text{ pour } x \geq 0.$$

Nous renvoyons à l'introduction de [4] pour une bibliographie récente sur le sujet.

B.2. Cas $0 < \tau < 1$. — Si $0 < \tau < 1$, alors les solutions f de $(E_{\tau,\nu})$ sont développables en séries entières car d'après (32) :

$|f^{(n)}(x)| \leq M \times K$ pour tout réel $r > 0$, tout entier n et tout réel $x \in [-r, r]$, où $M = \sup\{|f(x)|; x \in [-r, r]\}$ et $K = \sup\{\nu^{-n}\tau^{n(n+1)/2}; n \in \mathbb{N}\}$. Ceci implique que f est développable en série entière en 0. Il est clair que le rayon de convergence est infini, donc l'ensemble des solutions est un espace vectoriel de dimension 1 et les solutions sont de la forme :

$$x \mapsto \beta \sum_{n\geq 0} \frac{1}{n!}\frac{1}{\nu^n}\tau^{n(n+1)/2}x^n.$$

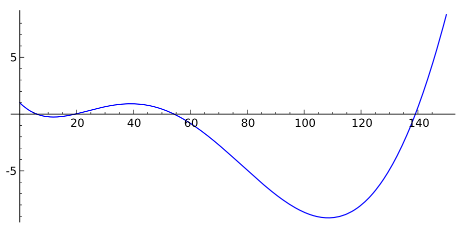

FIGURE B.1. Graphe de la solution f de $(E_{1/2,-1})$ telle que $f(0) = 1$.

B.3. Cas $\boldsymbol{\tau > 1}$. — Une conséquence de la proposition suivante est que l'espace des solutions de l'équation $(E_{\tau,\nu})$ est de dimension infinie.

PROPOSITION 42. — *Soient $\tau \in]1, +\infty[$ et $f \in C^\infty([1, \tau])$ telle que pour tout entier n, on ait $f^{(n)}(1) = f^{(n)}(\tau) = 0$. Alors pour tout $\nu \in \mathbb{R}^*$, f se prolonge de manière unique en une fonction de classe C^∞ sur $\mathbb{R}_+$, non nécessairement bornée, solution de $(E_{\tau,\nu})$.*

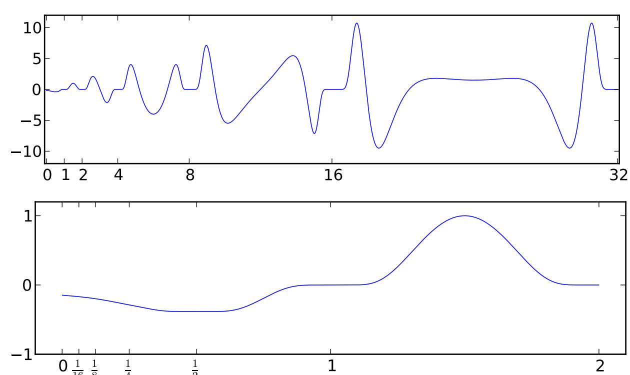

FIGURE B.2. Graphe du prolongement de la fonction $x \to e^4 \exp(-\frac{1}{(x-1)(2-x)})$ définie sur $]1, 2[$ pour $\tau = 2$ et $\nu = 1$.

Nous montrons que les solutions ne peuvent pas être périodiques et s'annulent nécessairement infiniment souvent si $\tau > 1$ et $\nu > 0$.

PROPOSITION 43. — *Soit f une solution de l'équation $(E_{\tau,\nu})$ avec $\tau \in]1,+\infty[$ et $\nu \in \mathbb{R}^*$.*

1. *Si f est périodique, alors f est la fonction nulle.*
2. *Si $\nu > 0$, alors pour tout $t > \dfrac{\nu}{\tau(\tau-1)}$ la fonction f s'annule au moins une fois dans l'intervalle $\left[\dfrac{t}{\nu}, \dfrac{t}{\nu}\tau^3\right]$.*

Si $\tau > 1$ et $\nu \neq 0$, les solutions non-nulles de $(E_{\tau,\nu})$ ne peuvent donc pas être périodiques. Nous pouvons nous demander qu'elles sont les fonctions les «plus simples» qui sont solutions. Pour cela, nous avons défini dans l'introduction (définition 5) la notion de *concaténation de fonctions.*

LEMME 44. — *Soient $f_a, f_b : [0,\tau] \to \mathbb{R}$ deux fonctions et u un mot fini ou infini sur $\{a,b\}$.*

1. *Si f_a et f_b sont continues et que $f_a(0) = f_a(\tau) = f_b(0) = f_b(\tau)$ alors f_u est continue.*
2. *Si f_u est continue et que trois des quatre mots aa, ab, ba et bb sont facteurs de u alors f_a et f_b sont continues et $f_a(0) = f_a(\tau) = f_b(0) = f_b(\tau)$. Cette condition signifie que u n'est pas de la forme $a^m b^n$, $a^m bb\cdots$, $b^m a^n$ ou bien $b^m aa\cdots$.*

Preuve du lemme 44. — Soient $f_a, f_b : [0,\tau] \to \mathbb{R}$ deux fonctions, $u = u_0 u_1 \ldots$ un mot (fini ou infini) sur l'alphabet $\{a,b\}$ et f_u la concaténation de f_a et f_b le long de u. Les conditions énoncées sur f_a et f_b sont clairement suffisantes pour la continuité de f_u.

Maintenant supposons que f_u soit continue. Si a apparaît dans u alors f_a est continue et de même pour b. De plus, pour tout $i < |u| - 1$ on a $f_{u_i}(\tau) = f_{u_{i+1}}(0)$. Ainsi si trois des quatre mots de longueur deux aa, ab, ba et bb apparaissent dans u, on en déduit que $f_a(0) = f_a(\tau) = f_b(0) = f_b(\tau)$. □

PROPOSITION 45. — *Soient $f_a, f_b : [0,\lambda] \to \mathbb{R}$ et $u \in \{a,b\}^{\mathbb{N}}$ tels que $f = f_u$ soit solution de $(E_{\lambda,\eta})$ avec $\eta \in \mathbb{Z}^*$ et $\lambda \geq 2$ un entier. Si f n'est pas identiquement nulle alors il existe une unique substitution uniforme σ de longueur λ telle que $\sigma(u) = u$. De plus, pour tout $x \in [0,\lambda[$:*

$$\frac{\eta}{\lambda} f_a'(x) = f_{\sigma_a}(\lambda x) \qquad \text{et} \qquad \frac{\eta}{\lambda} f_b'(x) = f_{\sigma_b}(\lambda x).$$

B.4. Preuves des propositions 42, 43 et 45

Preuve de la proposition 42. — Soit f une fonction qui vérifie les hypothèses de l'énoncé. En utilisant la relation $(E'_{\tau,\nu})$, nous pouvons prolonger f sur $[\tau,\tau^2]$, puis $[\tau^2,\tau^3]$ et ainsi construire de proche en proche une fonction $f : [1,+\infty[\to \mathbb{R}$.

La condition imposée sur les dérivées garantit que la fonction est bien définie aux points $\{\tau^n ; n \in \mathbb{N}\}$. Remarquons que la fonction ainsi prolongée n'est pas nécessairement bornée. Pour prolonger la fonction sur $[1/\tau, 1]$, nous utilisons la relation :

$$f(x) = f(1) - \frac{1}{\nu} \int_{\tau x}^{\tau} f(s)ds \quad \text{pour } x \in [1/\tau, 1].$$

Par construction, f est de classe C^∞ sur $[1/\tau, \tau]$. Plus généralement, si f est définie sur $[\tau^{-n}, \tau]$, elle se prolonge sur $[\tau^{-n-1}, \tau^{-n}]$ via la formule :

$$f(x) = f(\tau^{-n}) - \frac{1}{\nu} \int_{\tau x}^{\tau^{-n+1}} f(s)ds.$$

Aux points $\{\tau^{-n} ; n \in \mathbb{N}\}$, la fonction ainsi prolongée est bien lisse (les formules qui définissent f coïncident à gauche et à droite de ces points). Par contre, il n'est pas immédiat de voir que la fonction ainsi construite se prolonge en 0. Introduisons pour $n \geq 0$ les quantités :

$$u_n = \max_{x \in [\tau^{-n}, \tau^{-n+1}]} |f(x)| \quad \text{et} \quad v_n = \max_{x,y \in [\tau^{-n}, \tau^{-n+1}]} |f(x) - f(y)|.$$

Nous allons montrer que v_n est le terme général d'une série convergente. Par définition et l'équation fonctionnelle $(E_{\tau,\nu})$, ces suites vérifient pour $n \geq 1$:

$$\begin{aligned} v_n &\leq \frac{1}{|\nu|} \int_{\tau^{-n+1}}^{\tau^{-n+2}} |f(s)|ds \leq \frac{1}{|\nu|} \tau^{-n+1}(\tau - 1)u_{n-1} \leq \frac{1}{|\nu|} \tau^{-n+2} u_{n-1}, \\ u_n &\leq u_{n-1} + v_n, \\ u_{n-1} &\leq u_n + v_{n-1}. \end{aligned}$$

En particulier

$$|u_n - u_{n-1}| \leq |\nu|^{-1} \tau^{-n+2} \max(u_{n-1}, u_{n-2}).$$

Nous pouvons alors conclure en utilisant le résultat du lemme 39. Remarquons enfin que cette construction définie de manière unique f sur chaque intervalle $[\tau^k, \tau^{k+1}]$ pour $k \in \mathbb{Z}$. □

Preuve de la proposition 43. — Nous fixons pour toute cette preuve $\tau \in]1, +\infty[$.

Soient f une solution périodique et T sa plus petite période :

$$T = \inf\{t > 0 \ ; \ \forall x > 0, \, f(x) = f(x + t)\}.$$

Alors pour tout entier n :

$$n \int_0^T f(t)dt = \int_0^{nT} f(t)dt = \nu \left(f\left(\frac{nT}{\tau}\right) - f(0) \right).$$

Comme f est bornée car périodique, en passant à la limite sur n, on obtient que $\int_0^T f(t)dt = 0$.

Maintenant, si x est un réel positif, on a :

$$\begin{aligned} f(x) &= \frac{1}{\nu}\int_0^{\tau x} f(t)dt + f(0) = \frac{1}{\nu}\int_T^{\tau x+T} f(t)dt + \frac{1}{\nu}\int_0^T f(s)ds + f(0) \\ &= \frac{1}{\nu}\int_0^{\tau x+T} f(s)ds + f(0) = f\left(x+\frac{T}{\tau}\right). \end{aligned}$$

Comme $\tau > 1$, T/τ est une période inférieure à T, ce qui n'est possible que si $T = 0$.

Nous démontrons maintenant la deuxième partie de la proposition, à savoir l'annulation de f. Supposons donc que ν est strictement positif, et quitte à remplacer $f(x)$ par $f(\nu x)$, nous fixerons même sa valeur égale à 1.

L'équation $(E_{\tau,1})$ permet d'écrire pour tout couple de réels (x, y) :

$$f(y) - f(x) = \int_{\tau x}^{\tau y} f(t)dt.$$

Soit $t \geq 1/(\tau(\tau-1))$ tel que $f(t) \neq 0$. Quitte à changer f par $-f$, nous considérerons que f est strictement positive sur $]t, +\infty[$. Supposons que f ne s'annule pas sur $[t, (\tau^3+\varepsilon)t]$ pour $\varepsilon > 0$. L'équation $(E'_{\tau,1})$ montre que f est strictement croissante sur $[t, (\tau^2+\varepsilon/\tau)t]$. Nous obtenons alors que pour tout réel $x \in [\tau t, (\tau+\varepsilon/\tau^2)t]$:

$$\begin{aligned} f(x) &> f(x) - f(t) && \text{car } f(t) > 0 \\ &= \int_{\tau t}^{\tau x} f(s)ds && \text{par l'équation fonctionnelle} \\ &\geq \int_x^{\tau x} f(s)ds && \text{car } f \geq 0 \text{ sur } [\tau t, x] \subset [t, (\tau^2+\varepsilon/\tau)t] \\ &> (\tau x - x)f(x) && \text{car } f \text{ croissante sur } [x, \tau x] \subset [t, (\tau^3+\varepsilon)t]. \end{aligned}$$

Ce qui amène à une contradiction car $x \geq \tau t \geq 1/(\tau-1)$. Ainsi f s'annule en au moins un point de l'intervalle $[t, (\tau^3+\varepsilon)t]$ pour tout $\varepsilon > 0$. Par continuité, il existe un point de $[t, \tau^3 t]$ en lequel f s'annule. □

Preuve de la proposition 45. — Nous avons vu dans la proposition 43 que la solution f ne peut pas être périodique. Ainsi $f_a \neq f_b$. De plus, par le lemme 44 et le fait que dans un mot non périodique au moins 3 des facteurs de longueur deux apparaissent, on obtient que $f_a(0) = f_a(\lambda) = f_b(0) = f_b(\lambda)$. Comme $f_a(0) = f_b(0)$ et $f_a \neq f_b$, nous avons également $f'_a \neq f'_b$.

D'après la relation $(E'_{\tau,\nu})$, $f'(t) = \frac{\lambda}{\eta} f(\lambda t)$. Pour tout $k \geq 0$ entier et tout $x \in [0, \lambda]$, nous avons :

$$f(\lambda^2 k + \lambda x) = \frac{\eta}{\lambda} f'(\lambda k + x) = \begin{cases} \frac{\eta}{\lambda} f'_a(x) \text{ si } u_k = a \\ \frac{\eta}{\lambda} f'_b(x) \text{ si } u_k = b. \end{cases}$$

Ainsi pour tout entier $k \geq 0$, les valeurs de f sur le segment $[\lambda^2 k, \lambda^2 k + \lambda^2]$ ne dépendent que de u_k. Or f est la concaténation de $f_{u_{\lambda k}} f_{u_{\lambda k+1}} \cdots f_{u_{\lambda(k+1)-1}}$ sur $[\lambda^2 k, \lambda^2 k + \lambda^2]$. Donc pour tout entier k, u_k détermine le mot $u_{\lambda k} u_{\lambda k+1} \cdots u_{\lambda(k+1)-1}$.

Comme f_a est distinct de f_b, et en considérant des positions k_a et k_b telles que $u_{k_a} = a$ et $u_{k_b} = b$, on déduit qu'il existe deux mots finis de longueur λ :

$$\sigma_a = u_{\lambda k_a} \cdots u_{\lambda(k_a+1)-1} \quad \text{et} \quad \sigma_b = u_{\lambda k_b} \cdots u_{\lambda(k_b+1)-1},$$

uniquement déterminés et tels que l'on retrouve bien la relation annoncée :

$$\frac{\eta}{\lambda} f_a'(x) = f_{\sigma_a}(\lambda x) \qquad \text{et} \qquad \frac{\eta}{\lambda} f_b'(x) = f_{\sigma_b}(\lambda x).$$

Le mot u est donc point fixe de la substitution uniforme $a \mapsto \sigma_a$ et $b \mapsto \sigma_b$. □

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A NOTE ON CRYSTALLINE LIFTINGS IN THE $\mathbb{Q}_p$ CASE

BY HUI GAO

ABSTRACT. — Let $p > 2$ be a prime. Let ρ be a crystalline representation of $G_{\mathbb{Q}_p}$ with distinct Hodge-Tate weights in $[0, p]$, such that its reduction $\overline{\rho}$ is upper triangular. Under certain conditions, we prove that $\overline{\rho}$ has an upper triangular crystalline lift ρ' such that $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$. The method is based on the author's previous work, combined with an inspiration from the work of Breuil-Herzig.

RÉSUMÉ (*Note sur les élévations cristallines dans le cas* $\mathbb{Q}_p$). — Soit $p > 2$ un premier. Soit ρ une représentation cristalline de $G_{\mathbb{Q}_p}$ avec des poids distincts de Hodge-Tate dans $[0, p]$, de telle sorte que sa réduction $\overline{\rho}$ soit triangulaire supérieure. Dans certaines conditions, nous prouvons que $\overline{\rho}$ a une élévation cristalline triangulaire supérieure ρ' telle que $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$. La méthode est basée sur le travail antérieur de l'auteur, combiné avec une inspiration de l'oeuvre de Breuil-Herzig.

1. Introduction

1.1. Overview. — Given (a lattice in) a crystalline representation, it is natural to study its reduction. Conversely, given a representation over an $\overline{\mathbb{F}}_p$-vector space, it is natural to consider its crystalline lifts. We are particularly interested with crystalline representations, because they will have applications to

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weight part of Serre's conjectures (see e.g., [6, 7, 3]). In general, both these questions are notoriously difficult. For example, given an $\overline{\mathbb{F}}_p$-representation, we do not even know if it has any crystalline lift. However, for applications to weight part of Serre's conjectures, we can *assume* at the beginning that certain $\overline{\mathbb{F}}_p$-representation already have at least one crystalline lift; the key point then is to show that it has some other *nicer* crystalline lift. And this is what we do in this paper.

To state our main result, we introduce some notations first. Let $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the Galois group of $\mathbb{Q}_p$. Let $E/\mathbb{Q}_p$ be a finite extension, $\mathcal{O}_E$ the ring of integers, ω_E a fixed uniformizer, and $k_E = \mathcal{O}_E/\omega_E\mathcal{O}_E$ the residue field. We will use the following notations often, (CRYS):

- Let $p > 2$ be an odd prime. Let V be a crystalline representation of $G_{\mathbb{Q}_p}$ of E-dimension d, such that the Hodge-Tate weights $\mathrm{HT}(V) = \{0 = r_1 < \ldots < r_d \leq p\}$.
- Let $\rho = T$ be a $G_{\mathbb{Q}_p}$-stable $\mathcal{O}_E$-lattice in V, and $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi,\hat{G}}$ the $(\varphi,\hat{G})$-module (with $\mathcal{O}_E$-coefficient) attached to T. Let $\overline{\rho} := T/\omega_E T$ be the reduction. Let $\overline{\hat{\mathfrak{M}}}$ be the reduction of $\hat{\mathfrak{M}}$, and $\overline{\mathfrak{M}}$ the reduction of $\mathfrak{M}$.

1.1.1. Theorem. — *With notations in (*CRYS*). Suppose that $\overline{\rho}$ is upper triangular, i.e., $\overline{\rho}$ is a successive extension of d characters: $\overline{\chi}_1, \ldots, \overline{\chi}_d$. Suppose $\overline{\chi}_i\overline{\chi}_j^{-1} \neq \overline{\varepsilon}_p, \forall i \neq j$, where $\overline{\varepsilon}_p$ is the reduction of the cyclotomic character. Then there exists an upper triangular crystalline representation ρ' such that $\overline{\rho}' \cong \overline{\rho}$, and* $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$ *as sets.*

Theorem 1.1.1 strengthens [3, Cor. 0.2(1)] in the $\mathbb{Q}_p$-case, and of course have direct application to weight part of Serre's conjectures as in *loc. cit.*. In our Theorem 1.1.1,

- we do not require the Condition (C-1) of [3, §3], and
- we only require a weaker version of Condition (C-2A) of [3, §6].
- Note that Condition (C-2B) of [3, §6] in general will never be satisfied in our current paper.

Let us also remark that Condition (C-1) seems to be the most difficult condition to remove in [3].

The proof of our theorem still uses results in [3] to study the possible shape of upper triangular reductions of crystalline representations. The difference in the current paper is a different crystalline lifting technique, which is inspired by some group theory developed in [1]. Roughly speaking, we can use the group theory to conjugate our upper triangular $\overline{\rho}$ to another upper triangular form, which can be lifted to an *ordinary* (in particular, upper triangular) crystalline representation via the result of [5]. The lifting process via *loc. cit.* is in some sense easier than those used in [3] (which is generalization of methods in [6, 7]). However, we can only apply this technique in the $\mathbb{Q}_p$-case, because it seems that we cannot apply the group theory in [1] to deal with general $K/\mathbb{Q}_p$ case for our problem. Let us remark that our current paper shows a much refined structure for upper triangular reductions of crystalline representations. It is also worth pointing out that our result gives a very *natural* example (see (4.1.2)) for some of the group theories in [1].

The paper is organized as follows. In Section 2, we review the theory of Kisin modules and $(\varphi, \hat{G})$-modules with $\mathcal{O}_E$-coefficients. In Section 3, we review the group theory in [1]. In Section 4, we study the shape of upper triangular torsion $(\varphi, \hat{G})$-modules, using results in [3], as well as techniques inspired by the group theory in Section 3. Finally in Section 5, we prove our crystalline lifting theorem.

1.2. Notations. — The notations in the following are taken directly from [3]. In particular, they are valid for any finite extension $K/\mathbb{Q}_p$ (and we use K_0 to denote the maximal unramified sub-extension of K, and k the residue field of K). See *loc. cit.* for any unfamiliar terms and more details.

In this paper, we sometimes use boldface letters (e.g., $\boldsymbol{e}$) to mean a sequence of objects (e.g., $\boldsymbol{e} = (e_1, \ldots, e_d)$ a basis of some module). We use Mat(?) to mean the set of matrices with elements in ?. We use notations like $[u^{r_1}, \ldots, u^{r_d}]$ to mean a diagonal matrix with the diagonal elements in the bracket. We use Id to mean the identity matrix. For a matrix A, we use diag$A$ to mean the diagonal matrix formed by the diagonal of A.

In this paper, *upper triangular* always means successive extension of rank-1 objects. We use notations like $\mathcal{E}(m_d, \ldots, m_1)$ (note the order of objects) to mean the set of all upper triangular extensions of rank-1 objects in certain categories. That is, m is in $\mathcal{E}(m_d, \ldots, m_1)$ if there is an increasing filtration $0 = \mathrm{Fil}^0 m \subset \mathrm{Fil}^1 m \subset \ldots \subset \mathrm{Fil}^d m = m$ such that $\mathrm{Fil}^i m / \mathrm{Fil}^{i-1} m = m_i, \forall 1 \leq i \leq d$.

We normalize the Hodge-Tate weights so that $\mathrm{HT}_\kappa(\varepsilon_p) = 1$ for any $\kappa : K \to \overline{\mathbb{Q}_p}$, where ε_p is the p-adic cyclotomic character.

We fix a system of elements $\{\pi_n\}_{n=0}^{\infty}$ in $\overline{K}$, where $\pi_0 = \pi$ is a uniformizer of K, and $\pi_{n+1}^p = \pi_n, \forall n$. Let $K_n = K(\pi_n), K_\infty = \bigcup_{n=0}^{\infty} K(\pi_n)$, and $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$. We fix a system of elements $\{\mu_{p^n}\}_{n=0}^{\infty}$ in $\overline{K}$, where $\mu_1 = 1$, μ_p is a primitive p-th root of unity, and $\mu_{p^{n+1}}^p = \mu_{p^n}, \forall n$. Let $K_{p^\infty} = \bigcup_{n=0}^{\infty} K(\mu_{p^n})$, and $\hat{K} = K_{\infty, p^\infty} = \bigcup_{n=0}^{\infty} K(\pi_n, \mu_{p^n})$. Note that $\hat{K}$ is the Galois closure of K_∞, and let $\hat{G} = \mathrm{Gal}(\hat{K}/K)$, $H_K = \mathrm{Gal}(\hat{K}/K_\infty)$, and $G_{p^\infty} = \mathrm{Gal}(\hat{K}/K_{p^\infty})$. When $p > 2$, then $\hat{G} \simeq G_{p^\infty} \rtimes H_K$ and $G_{p^\infty} \simeq \mathbb{Z}_p(1)$, and so we can (and do) fix a topological generator τ of G_{p^∞}. And we can furthermore assume that $\mu_{p^n} = \frac{\tau(\pi_n)}{\pi_n}$ for all n.

Let $C = \hat{\overline{K}}$ be the completion of $\overline{K}$, with ring of integers $\mathcal{O}_C$. Let $R := \varprojlim \mathcal{O}_C/p$ where the transition maps are p-th power map. R is a valuation ring with residue field $\bar{k}$ ($\bar{k}$ is the residue field of C). R is a perfect ring of characteristic p. Let $W(R)$ be the ring of Witt vectors. Let $\underline{\epsilon} := (\mu_{p^n})_{n=0}^{\infty} \in R$, $\underline{\pi} = (\pi_n)_{n=0}^{\infty} \in R$, and let $[\underline{\epsilon}], [\underline{\pi}]$ be their Teichmüller representatives respectively in $W(R)$. We normalize the valuation on R so that $v_R(\underline{\pi}) = \frac{1}{e}$, where e is the ramification index of $K/\mathbb{Q}_p$.

There is a map $\theta : W(R) \to \mathcal{O}_C$ which is the unique universal lift of the map $R \to \mathcal{O}_C/p$ (projection of R onto the its first factor), and $\operatorname{Ker}\theta$ is a principle ideal generated by $\xi = [\overline{\omega}] + p$, where $\overline{\omega} \in R$ with $\omega^{(0)} = -p$, and $[\overline{\omega}] \in W(R)$ its Teichmüller representative. Let $B_{\mathrm{dR}}^+ := \varprojlim_n W(R)[\frac{1}{p}]/(\xi)^n$, and $B_{\mathrm{dR}} := B_{\mathrm{dR}}^+[\frac{1}{\xi}]$. Let $t := \log([\underline{\epsilon}])$, which is an element in B_{dR}^+. Let A_{cris} denote the p-adic completion of the divided power envelope of $W(R)$ with respect to $\operatorname{Ker}(\theta)$. Let $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$ and $B_{\mathrm{cris}} := B_{\mathrm{cris}}^+[\frac{1}{t}]$. The projection from R to $\bar{k}$ induces a projection $\nu : W(R) \to W(\bar{k})$, since $\nu(\operatorname{Ker}\theta) = pW(\bar{k})$, the projection extends to $\nu : A_{\mathrm{cris}} \to W(\bar{k})$, and also $\nu : B_{\mathrm{cris}}^+ \to W(\bar{k})[\frac{1}{p}]$. Write $I_+B_{\mathrm{cris}}^+ := \operatorname{Ker}(\nu : B_{\mathrm{cris}}^+ \to W(\bar{k})[\frac{1}{p}])$, and for any subring $A \subseteq B_{\mathrm{cris}}^+$, write $I_+A = A \cap \operatorname{Ker}(\nu)$.

Let $\mathfrak{S} := W(k)[\![u]\!]$, $E(u) \in W(k)[u]$ the minimal polynomial of π over $W(k)$, and S the p-adic completion of the PD-envelope of $\mathfrak{S}$ with respect to the ideal $(E(u))$. We can embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ by mapping u to $[\underline{\pi}]$. The embedding extends to the embeddings $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$.

2. Kisin modules and $(\varphi, \hat{G})$-modules

In this section, we briefly review some facts in the theory of Kisin modules and $(\varphi, \hat{G})$-modules with $\mathcal{O}_E$-coefficients. The materials in this section are based on works of [8, 10, 2, 6, 9] etc.. But here we only cite them in the form as in [3, §1], where the readers can find more detailed attributions.

2.1. Kisin modules and $(\varphi, \hat{G})$-modules with coefficients. — In this subsection, all the definitions and results are valid for any finite extension $K/\mathbb{Q}_p$.

Recall that $\mathfrak{S} = W(k)[\![u]\!]$ with the Frobenius endomorphism $\varphi_{\mathfrak{S}} : \mathfrak{S} \to \mathfrak{S}$ which acts on $W(k)$ via arithmetic Frobenius and sends u to u^p. Denote $\mathfrak{S}_{\mathcal{O}_E} := \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ and $\mathfrak{S}_{k_E} := \mathfrak{S} \otimes_{\mathbb{Z}_p} k_E = k[\![u]\!] \otimes_{\mathbb{F}_p} k_E$. We can extend $\varphi_{\mathfrak{S}}$ to $\mathfrak{S}_{\mathcal{O}_E}$ (resp. $\mathfrak{S}_{k_E}$) by acting on $\mathcal{O}_E$ (resp. k_E) trivially. Let r be any nonnegative integer.

- Let $'\mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ (called the category of Kisin modules of height r with $\mathcal{O}_E$-coefficients) be the category whose objects are $\mathfrak{S}_{\mathcal{O}_E}$-modules $\mathfrak{M}$, equipped with $\varphi : \mathfrak{M} \to \mathfrak{M}$ which is a $\varphi_{\mathfrak{S}_{\mathcal{O}_E}}$-semi-linear morphism such that the span of $\operatorname{Im}(\varphi)$ contains $E(u)^r\mathfrak{M}$. The morphisms in the category are $\mathfrak{S}_{\mathcal{O}_E}$-linear maps that commute with φ.

- Let $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ be the full subcategory of $'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ with $\mathfrak{M} \simeq \oplus_{i\in I}\mathfrak{S}_{\mathcal{O}_E}$ where I is a finite set. Let $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{k_E}}$ be the full subcategory of $'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ with $\mathfrak{M} \simeq \oplus_{i\in I}\mathfrak{S}_{k_E}$ where I is a finite set.

For any integer $n \geq 0$, write $n = (p-1)q(n) + r(n)$ with $q(n)$ and $r(n)$ the quotient and residue of n divided by $p-1$. Let $t^{\{n\}} = (p^{q(n)} \cdot q(n)!)^{-1} \cdot t^n$, we have $t^{\{n\}} \in A_{\mathrm{cris}}$.

We define a subring of B^+_{cris}, $\mathcal{R}_{K_0} := \left\{\sum_{i=0}^{\infty} f_i t^{\{i\}}, f_i \in S_{K_0}, f_i \to 0 \text{ as } i \to \infty\right\}$. Define $\hat{\mathcal{R}} := \mathcal{R}_{K_0} \cap W(R)$. Then $\hat{\mathcal{R}}$ is a φ-stable subring of $W(R)$, which is also G_K-stable, and the G_K-action factors through $\hat{G}$. Denote $\hat{\mathcal{R}}_{\mathcal{O}_E} := \hat{\mathcal{R}} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, $W(R)_{\mathcal{O}_E} := W(R) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$, and extend the G_K-action and φ-action on them by acting on $\mathcal{O}_E$ trivially. Note that $\mathfrak{S}_{\mathcal{O}_E} \subset \hat{R}_{\mathcal{O}_E}$, and let $\varphi : \mathfrak{S}_{\mathcal{O}_E} \to \hat{R}_{\mathcal{O}_E}$ be the composite of $\varphi_{\mathfrak{S}_{\mathcal{O}_E}} : \mathfrak{S}_{\mathcal{O}_E} \to \mathfrak{S}_{\mathcal{O}_E}$ and the embedding $\mathfrak{S}_{\mathcal{O}_E} \to \hat{R}_{\mathcal{O}_E}$.

2.1.1. Definition. — Let $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ be the category (called the category of $(\varphi,\hat{G})$-modules of height r with $\mathcal{O}_E$-coefficients) consisting of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where,

1. $(\mathfrak{M}, \varphi_{\mathfrak{M}}) \in {}'\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ is a Kisin module of height r;
2. $\hat{G}$ is a $\hat{\mathcal{R}}_{\mathcal{O}_E}$-semi-linear $\hat{G}$-action on $\hat{\mathfrak{M}} := \hat{\mathcal{R}}_{\mathcal{O}_E} \otimes_{\varphi,\mathfrak{S}_{\mathcal{O}_E}} \mathfrak{M}$;
3. $\hat{G}$ commutes with $\varphi_{\hat{\mathfrak{M}}} := \varphi_{\hat{\mathcal{R}}_{\mathcal{O}_E}} \otimes \varphi_{\mathfrak{M}}$;
4. Regarding $\mathfrak{M}$ as a $\varphi(\mathfrak{S}_{\mathcal{O}_E})$-submodule of $\hat{\mathfrak{M}}$, then $\mathfrak{M} \subseteq \hat{\mathfrak{M}}^{H_K}$;
5. $\hat{G}$ acts on the $\hat{\mathfrak{M}}/(I_+\hat{R})\hat{\mathfrak{M}}$ trivially.

A morphism between two $(\varphi,\hat{G})$-modules is a morphism in $\mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$ which commutes with $\hat{G}$-actions.

We denote $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ to be the full subcategory of $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ where $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}_{\mathcal{O}_E}}$; and we denote $\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{k_E}}$ for the full subcategory of $'\mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ where $\mathfrak{M} \in \mathrm{Mod}^{\varphi}_{\mathfrak{S}_{k_E}}$.

We can associate representations to $(\varphi,\hat{G})$-modules.

2.1.2. Theorem ([3, Thm. 1.2, Thm. 1.4]). — 1. *Suppose* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{\mathcal{O}_E}}$ *where* $\mathfrak{M}$ *is of* $\mathfrak{S}_{\mathcal{O}_E}$*-rank* d*, then*

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\hat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R))$$

is a finite free $\mathcal{O}_E$*-representation of* G_K *of rank* d*.*

2. *Suppose* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{\varphi,\hat{G}}_{\mathfrak{S}_{k_E}}$ *where* $\mathfrak{M}$ *is of* $\mathfrak{S}_{k_E}$*-rank* d*, then*

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\hat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)$$

is a finite free k_E*-representation of* G_K *of dimension* d*.*

3. *For* $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi,\hat{G}}$, *we have* $\hat{T}(\hat{\mathfrak{M}}/\omega_E\hat{\mathfrak{M}}) \simeq \hat{T}(\hat{\mathfrak{M}})/\omega_E\hat{T}(\hat{\mathfrak{M}})$.

When $p > 2$, the theory of $(\varphi, \hat{G})$-modules becomes simpler.

2.1.3. Lemma ([3, Lem. 1.6]). — *Suppose* $p > 2$. *Let* $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi,\hat{G}}$. *Then* $\hat{\mathfrak{M}}$ *is uniquely determined up to isomorphism by the following information:*

1. *A matrix* $A_\varphi \in \mathrm{Mat}(\mathfrak{S}_{\mathcal{O}_E})$ *for the Frobenius* $\varphi : \mathfrak{M} \to \mathfrak{M}$, *such that there exist* $B \in \mathrm{Mat}(\mathfrak{S}_{\mathcal{O}_E})$ *with* $A_\varphi B = E(u)^r Id$.
2. *A matrix* $A_\tau \in \mathrm{Mat}(\hat{R}_{\mathcal{O}_E})$ *(for the* τ*-action* $\tau : \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}$*) such that*
   - $A_\tau - Id \in \mathrm{Mat}(I_+\hat{\mathcal{R}}_{\mathcal{O}_E})$,
   - $A_\tau \tau(\varphi(A_\varphi)) = \varphi(A_\varphi)\varphi(A_\tau)$.
   - $g(A_\tau) = \prod_{k=0}^{\varepsilon_p(g)-1} \tau^k(A_\tau)$ *for all* $g \in G_\infty$ *such that* $\varepsilon_p(g) \in \mathbb{Z}^{\geq 0}$.

For $\overline{\hat{\mathfrak{M}}} \in \mathrm{Mod}_{\mathfrak{S}_{k_E}}^{\varphi,\hat{G}}$, *it is also uniquely determined up to isomorphism by its matrix* A_φ *and* A_τ *satisfying similar conditions as above.*

2.2. Rank 1 Kisin modules and $(\varphi, \hat{G})$-modules.

— We only recall the following definitions and results in the $\mathbb{Q}_p$ case.

2.2.1. Definition. —

1. Suppose t is a non-negative integer, $a \in k_E^\times$. Let $\overline{\mathfrak{M}}(t; a)$ be the rank-1 module in $\mathrm{Mod}_{\mathfrak{S}_{k_E}}^{\varphi}$ such that $\overline{\mathfrak{M}}(t; a)$ is generated by some basis e, and $\varphi(e) = au^t e$.
2. Suppose t is a non-negative integer, $\hat{a} \in \mathcal{O}_E^\times$. Let $\mathfrak{M}(t; \hat{a})$ be the rank-1 module in $\mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi}$ such that $\mathfrak{M}(t; \hat{a})$ is generated by some basis $\tilde{e}$, and $\varphi(\tilde{e}) = \hat{a}(u - p)^t \tilde{e}$.

2.2.2. Lemma ([3, Lem. 1.11]). —

1. *Any rank* 1 *module in* $\mathrm{Mod}_{\mathfrak{S}_{k_E}}^{\varphi}$ *is of the form* $\overline{\mathfrak{M}}(t; a)$ *for some* t *and* a.
2. *When* $\hat{a}$ *is a lift of* a, $\mathfrak{M}(t; \hat{a})/\omega_E\mathfrak{M}(t; \hat{a}) \simeq \overline{\mathfrak{M}}(t; a)$.
3. *There is a unique* $\hat{\mathfrak{M}}(t; \hat{a}) \in \mathrm{Mod}_{\mathfrak{S}_{\mathcal{O}_E}}^{\varphi,\hat{G}}$ *such that the ambient Kisin module of* $\hat{\mathfrak{M}}(t; \hat{a})$ *is* $\mathfrak{M}(t; \hat{a})$, *and* $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a}))$ *is a crystalline character. In fact,* $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a})) = \lambda_{\hat{a}}\psi^t$, *where* ψ *is a certain crystalline character such that* $\mathrm{HT}(\psi) = 1$, *and* $\lambda_{\hat{a}}$ *is the unramified character of* $G_{\mathbb{Q}_p}$ *which sends the arithmetic Frobenius to* $\hat{a}$.
4. *There is a unique* $\overline{\hat{\mathfrak{M}}}(t; a) \in \mathrm{Mod}_{\mathfrak{S}_{k_E}}^{\varphi,\hat{G}}$ *such that the ambient Kisin module is* $\overline{\mathfrak{M}}(t; a)$. *Furthermore,* $\hat{T}(\overline{\hat{\mathfrak{M}}}(t; a))$ *is the reduction of* $\hat{T}(\hat{\mathfrak{M}}(t; \hat{a}))$ *for any lift* $\hat{a} \in \mathcal{O}_E$ *of* a.

3. Some group theory

We recall some group theory, which will be useful for our work. All the materials in this section are developed in [1, §2.3], for general split connected reductive groups. But we will only need it for GL_d, which we recall.

Let H be the algebraic group GL_d, T the torus consisting of diagonal matrices, B the Borel consisting of upper triangular matrices, and U the unipotent radical consisting of unipotent matrices.

We have $X(T) := \mathrm{Hom}_{\mathrm{alg}}(T, \mathbb{G}_m) = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_d$, where ϵ_i is the character sending the diagonal matrix $[x_1, \ldots, x_d]$ to x_i. Let $S = \{\epsilon_i - \epsilon_{i+1} : 1 \leq i \leq d-1\}$ be the simple roots, and let $R^+ = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq d\}$ be the positive roots. Denote W the Weyl group of H, which is isomorphic to the permutation group S_d. If $\alpha = \epsilon_i - \epsilon_j \in R^+$, let $U_\alpha \subset H$ be the root subgroup, which corresponds to the unipotent upper triangular matrices where the only nonzero element above the diagonal is at the (i, j)-position.

3.0.1. Definition. — A subset $C \subseteq R^+$ is called *closed* if the following condition is satisfied: if $\alpha \in C, \beta \in C$ and $\alpha + \beta \in R^+$, then $\alpha + \beta \in C$.

For a closed subset $C \subseteq R^+$, let $U_C \subseteq U$ be the Zariski closed subgroup of B generated by the subgroups U_α for all $\alpha \in C$. Let $B_C = TU_C \subseteq B$. If $C = \{\epsilon_{i_1} - \epsilon_{j_1}, \ldots, \epsilon_{i_m} - \epsilon_{j_m}\}$ is a closed subset of R^+, then it is easy to see that B_C corresponds to the matrices where the only nonzero elements above the diagonal are at the positions (i_ℓ, j_ℓ) for all $1 \leq \ell \leq m$.

Recall that if we let $N_H(T)$ be the normalizer of T in H, then $N_H(T)/T$ is isomorphic to W. For each $\sigma \in W$ which is a permutation sending $(1, \ldots, d)$ to $(\sigma(1), \ldots, \sigma(d))$, we fix a representative of σ in H to be the $d \times d$ matrix $w_\sigma := (\delta_{i,\sigma(j)})_{1 \leq i,j \leq d} = (\delta_{\sigma^{-1}(i),j})$ where the notation is $\delta_{x,y} = 0$ if $x \neq y$, and $\delta_{x,y} = 1$ if $x = y$. Note that if we have another $d \times d$ matrix $A = (a_{k,l})$, we have the matrix multiplication:

$$(\delta_{i,\sigma(j)})(a_{k,l}) = (a_{\sigma^{-1}(k),l}), \quad (a_{k,l})(\delta_{i,\sigma(j)}) = (a_{k,\sigma(l)}),$$

and so in particular $\sigma^{-1}(a_{i,j})\sigma = (a_{\sigma(i),\sigma(j)})$.

Let $C \subseteq R^+$ closed, we define the following subset of W:

$$W_C := \{\sigma \in W : \sigma^{-1}(C) \subseteq R^+\}.$$

3.0.2. Lemma ([1, Lem. 2.3.6]). — *With notations as above, we have*

$$W_C = \{\sigma \in W : w_\sigma^{-1} B_C w_\sigma \subseteq B\}.$$

The above lemma says that conjugations of a matrix in B_C by permutations in W_C will stay upper triangular. In the following, we will sometimes simply use σ to mean the matrix w_σ. If ? is a ring, we will use $B_C(?)$ to mean the subring of $\mathrm{Mat}_d(?)$ corresponding to the algebraic group B_C. That is, if $C = \{\epsilon_{i_1} - \epsilon_{j_1}, \ldots, \epsilon_{i_m} - \epsilon_{j_m}\}$ is closed in R^+, then we let

$$B_C(?) := \{A = (a_{i,j}) \in \mathrm{Mat}_d(?) : \\ A \text{ is upper triangular, and } a_{i,j} = 0 \text{ if } \epsilon_i - \epsilon_j \notin C\}.$$

It is clear that for any $A \in B_C(?)$ and any $\sigma \in W_C$, $\sigma^{-1}A\sigma$ is an upper triangular matrix.

4. Shape of upper triangular $(\varphi, \hat{G})$-modules with k_E-coefficients

In this section, we study the shape of upper triangular torsion $(\varphi, \hat{G})$-modules, using results in [3], as well as ideas in Section 3.

4.1. Shape of φ

4.1.1. PROPOSITION. — *With notations from (*CRYS*). Suppose that $\overline{\rho}$ is upper triangular. Then $\overline{\mathfrak{M}} \in \mathcal{E}(\overline{\mathfrak{N}}_d, \ldots, \overline{\mathfrak{N}}_1)$, where $\overline{\mathfrak{N}}_i = \overline{\mathfrak{M}}(t_i; a_i)$ for some $a_i \in k_E^\times$, and $\{t_1, \ldots, t_d\} = \{r_1, \ldots, r_d\}$ as sets.*

Furthermore, there exists a basis $\boldsymbol{e}$ of $\overline{\mathfrak{M}}$, such that the matrix A_φ of φ with respect to this basis can be decomposed as $A_\varphi = \widetilde{A_\varphi} + u^p N$ where

1. *$\widetilde{A_\varphi}$ is upper triangular, with diagonal equal to $[a_1 u^{t_1}, \ldots, a_d u^{t_d}]$, and $(\widetilde{A_\varphi})_{i,j} = u^{t_i} y_{i,j}$ for $i < j$ (here $(\widetilde{A_\varphi})_{i,j}$ is the element of $\widetilde{A_\varphi}$ in the (i,j)-position), where*
   - *$y_{i,j} = 0$ if $t_j < t_i$.*
   - *$y_{i,j} \in k_E$ if $t_j > t_i$.*
2. *$N \in \mathrm{Mat}_d(k_E[u])$ is strictly upper triangular (i.e., the diagonal is 0).*

Proof. — This is a slight generalization of [3, Prop. 4.1] (using, in particular, [3, Prop. 2.2, Prop. 2.3]). The novelty here is that we can allow the existence of nonzero morphisms $\overline{\mathfrak{N}}_j \to \overline{\mathfrak{N}}_i$ for some $j > i$, i.e., the situation in Statement (3) of [3, Prop. 2.2] is allowed.

Step 1. — First of all, the existence and shape of $\overline{\mathfrak{N}}_i$ is proved in [3, Prop. 2.3]. To construct the basis $\boldsymbol{e}$ and the upper triangular matrix A_φ, we will apply [3, Prop. 2.2]. For the convenience of the reader, let us give some more explanation of *loc. cit.*, in the $\mathbb{Q}_p$-case (the general unramified case is similar).

The statement of [3, Prop. 2.2] is correct. A minor imperfection is that in the proof of *loc. cit.*, we cited [6, Prop. 7.4]. Indeed, to be more precise, we should have cited [7, Prop. 5.1.3] instead (although as mentioned in [7, Prop. 5.1.3], their proof are almost identical). The difference between [6, Prop. 7.4] and [7, Prop. 5.1.3] is that in the latter situation, we can allow all the Hodge-Tate numbers to be nonzero (we thank one of the referees for pointing this out).

The construction of the basis $\boldsymbol{e}$ (in [3, Prop. 2.2]) when $d > 2$ is really an easy inductive process from that of [7, Prop. 5.1.3] (where $d = 2$). Let us only sketch the case when $d = 3$. That is, suppose now we have $\overline{\mathfrak{M}} \in \mathcal{E}(\overline{\mathfrak{N}}_3, \overline{\mathfrak{N}}_2, \overline{\mathfrak{N}}_1)$. So we have a basis $\{f_1, f_2, f_3\}$ such that

$$\varphi(f_1, f_2, f_3) = (f_1, f_2, f_3)\begin{pmatrix} a_1u^{t_1} & x & z \\ 0 & a_2u^{t_2} & y \\ 0 & 0 & a_3u^{t_3} \end{pmatrix}$$

The key point then is to make change of bases so that x, y, z will satisfy the conditions in [3, Prop. 2.2]. By [7, Prop. 5.1.3] (the $d = 2$ case), we can and do assume that x already satisfies all the conditions in [3, Prop. 2.2]. That is: x is a polynomial in $k_E[u]$ of degree less than t_2, unless if there exists nonzero morphism $\overline{\mathfrak{N}}_2 \to \overline{\mathfrak{N}}_1$, then x can have an extra term of degree $t_2 + \frac{t_2 - t_1}{p-1}$.

The next step is to alter f_3 in order to make y, z satisfy [3, Prop. 2.2]. We can first change f_3 to $f_3' = f_3 + \alpha f_2$ as in the proof of [7, Prop. 5.1.3] to make y to some y' that satisfy [3, Prop. 2.2]. Note that this process will not have any effect on x, but it will alter z. So now we are in the situation

$$\varphi(f_1, f_2, f_3') = (f_1, f_2, f_3')\begin{pmatrix} a_1u^{t_1} & x & z' \\ 0 & a_2u^{t_2} & y' \\ 0 & 0 & 0a_3u^{t_3} \end{pmatrix}$$

where both x, y' satisfy [3, Prop. 2.2]. Now we only need to change f_3' to some $f_3'' = f_3' + \beta f_1$ in order to make z' satisfy [3, Prop. 2.2]. Note that there is no extension between $\overline{\mathfrak{N}}_3$ and $\overline{\mathfrak{N}}_1$, so we can not directly apply [7, Prop. 5.1.3] to get f_3''. However, the "f_2-parts" of $\varphi(f_3'') = \varphi(f_3') + \varphi(\beta)a_1u^{t_1}f_1$ and $\varphi(f_3')$ are the same. So we can "forget" about f_2 and pretend that there is an extension between $\overline{\mathfrak{N}}_3$ and $\overline{\mathfrak{N}}_1$. The same process as in [7, Prop. 5.1.3] will in the end produce our desired basis $\boldsymbol{e}$.

Step 2. — Now let us discuss about the "extra terms". Recall that in [3, Prop. 2.2], when there exists nonzero morphisms $\overline{\mathfrak{N}}_j \to \overline{\mathfrak{N}}_i$ for some $j > i$, then A_φ can have *extra* terms as described in Statement (3) of *loc. cit.*, and this extra term has degree $t_j + \frac{t_j - t_i}{p-1}$. Note that in order to have $\overline{\mathfrak{N}}_j \to \overline{\mathfrak{N}}_i$ for $j > i$, the only possibility is to have $t_j - t_i = p - 1$ and $a_i = a_j$ (easy by [3, Lem. 1.13] since we are in the $\mathbb{Q}_p$ situation). So the extra terms are always of degree p or $p + 1$, i.e., the extra terms are always divisible by u^p. (In fact, clearly we can only have at most two extra terms). Decompose A_φ as $\widetilde{A_\varphi} + u^pN$ where u^pN are the extra terms.

Step 3. — Finally, we only need to prove the properties regarding $y_{i,j}$. We argue similarly as in [3, Prop. 2.3], let $\boldsymbol{e}'$ be another basis of $\overline{\mathfrak{M}}$ such that $\varphi(\boldsymbol{e}') = \boldsymbol{e}'X[u^{r_1}, \ldots, u^{r_d}]$ where $X \in \mathrm{Mat}_d(k_E[\![u]\!])$ as in [3, Thm. 2.1]. Let $\boldsymbol{e}' = \boldsymbol{e}T$ for some matrix $T \in \mathrm{GL}_d(k_E[\![u]\!])$, then $A_\varphi = TX[u^{r_1}, \ldots, u^{r_d}]\varphi(T^{-1})$. Similarly as in [3, Prop. 4.1], let $\varphi(T) = P + u^pQ$ for some $P \in \mathrm{GL}_d(k_E)$,

$Q \in \mathrm{Mat}_d(k_E[\![u]\!])$, and let $R \in \mathrm{GL}_d(k_E)$ such that $R^{-1}[u^{r_1}, \ldots, u^{r_d}]R = [u^{t_1}, \ldots, u^{t_d}]$, then we have

$$(\widetilde{A_\varphi} + u^p N)(P + u^p Q)R = TXR[u^{t_1}, \ldots, u^{t_d}].$$

So we have $u^{t_i} \mid \mathrm{col}_i(\widetilde{A_\varphi} PR)$. Now we can again apply [3, Lem. 4.3] to conclude (note that $\widetilde{A_\varphi}$ satisfies property (DEG) of *loc. cit.*, since we removed the extra terms $u^p N$ from A_φ). □

With notations in Proposition 4.1.1, we can define the following subset C of R^+:

$$C := \{\epsilon_i - \epsilon_j : i < j, t_i < t_j\}. \tag{4.1.2}$$

It is easy to see that C is closed in R^+, and $\widetilde{A_\varphi}$ is a matrix in the subring $B_C(k_E[u])$. But in fact, we also have $A_\varphi \in B_C(k_E[u])$, because the extra terms in $u^p N$ only show up in positions (i, j) where $t_i < t_j$.

4.1.3. PROPOSITION. — *There exists a unique $\sigma \in W_C$ such that $\sigma^{-1} A_\varphi \sigma$ is still upper triangular, and* $\mathrm{diag}(\sigma^{-1} A_\varphi \sigma) = [a_{\sigma(1)} u^{r_1}, \ldots, a_{\sigma(d)} u^{r_d}]$.

Proof. — The uniqueness of σ is determined since we have $t_{\sigma(i)} = r_i, \forall i$, that is,

$$t_{\sigma(1)} < \ldots < t_{\sigma(d)}. \tag{4.1.4}$$

It suffices to show that $\sigma \in W_C$ ($\Leftrightarrow \sigma^{-1}(W_C) \subseteq R^+$), i.e., if $\epsilon_i - \epsilon_j \in C$, then $\sigma^{-1}(i) < \sigma^{-1}(j)$. Let $x = \sigma^{-1}(i)$ and $\sigma^{-1}(j) = y$. Then $t_i = t_{\sigma(x)} < t_{\sigma(j)} = t_j$. So by (4.1.4), we must have $x < y$. □

4.1.5. REMARK. — The following remark is suggested by one of the referees. Since we have already shown that $(A_\varphi)_{i,j} = 0$ when $t_j < t_i$ (where $j > i$), we could use an elementary "swapping" process to obtain the above proposition. Namely, suppose for example $t_{i+1} < t_i$, we could simply change the basis $(e_1, \ldots, e_i, e_{i+1}, \ldots, e_d)$ to $(e_1, \ldots, e_{i+1}, e_i, \ldots, e_d)$; the matrix for φ will remain upper triangular. After all these possible two by two swappings, the u-power on the diagonal will become eventually increasing.

As the readers can see, this elementary swapping process is precisely the key idea in the Breuil-Herzig group theory that we reviewed in Section 3. Indeed, the "ordinary part" of the p-adic Langlands in [1] is precisely built out from GL_2! It is also interesting to point out in the paper [4], a similar Weyl group element played a similar useful role in determining the locally algebraic vectors in the "ordinary part" of [1] (see the remarks following [4, Thm 1.2]).

We have chosen to keep the Breuil-Herzig theory in our paper (instead of the more elementary swapping process), because it does make the argument cleaner. Also, as we mentioned in the Introduction, this indeed provides a natural example of the Breuil-Herzig group theory.

4.2. Shape of $\boldsymbol{\tau}$. — Our following lemma (Lemma 4.2.1) is valid for any $K/\mathbb{Q}_p$. So we use notations introduced in Section 1. Recall that $u = (\pi_n)_{n=0}^{\infty} \in R$, and we normalize the valuation on R so that $v_R(u) = \frac{1}{e}$ where e is the ramification degree of $K/\mathbb{Q}_p$. For $\zeta \in R \otimes_{\mathbb{F}_p} k_E$, write it as $\zeta = \sum_{i=1}^{m} y_i \otimes a_i$ where $y_i \in R$, and $a_i \in k_E$ are independent over $\mathbb{F}_p$. Let

$$v_R(\zeta) := \min\{v_R(y_i)\}.$$

Then by [3, Lem. 5.6], v_R is a well-defined valuation on $R \otimes_{\mathbb{F}_p} k_E$ (so in particular, it does not depend on the sum representing ζ). In particular, $v_R(\varphi(\zeta)) = p v_R(\zeta)$. We also use the convention that $v_R(0) = +\infty$.

4.2.1. Lemma. — *Let $\zeta \in R \otimes_{\mathbb{F}_p} k_E$ with $v_R(\zeta) > 0$, such that*

$$\zeta \tau(\varphi(u^b)) = \varphi(u^a)\varphi(\zeta)$$

for some $a > b \geq 0$, then $\zeta = 0$.

Proof. — Note that $\tau(u) = u\underline{\epsilon}$, where $\underline{\epsilon} = (\mu_{p^n})_{n=0}^{\infty} \in R$. Consider the valuation on both side of the equation, then $v_R(\zeta) + \frac{pb}{e} = \frac{pa}{e} + p v_R(\zeta)$. The only possibility is when $v_R(\zeta) = +\infty$. □

Now we return to the $\mathbb{Q}_p$ case.

4.2.2. Proposition. — *With notations as in Proposition 4.1.1, let $A_\tau \in \mathrm{Mat}(R \otimes_{\mathbb{F}_p} k_E)$ be the matrix of τ with respect to the basis $1 \otimes_\varphi \boldsymbol{e}$. Then A_τ is in the subring $B_C(R \otimes_{\mathbb{F}_p} k_E)$ defined by (4.1.2), i.e., if $i < j$ and $t_i > t_j$, then $(A_\tau)_{i,j} = 0$.*

Proof. — This is easy consequence of the following lemma. Note that for any $i < j$, $v_R((A_\tau)_{i,j}) > 0$ by [3, Lem. 5.7]. □

4.2.3. Lemma. — *Let $F = (f_{i,j}) \in \mathrm{Mat}_d(k_E[\![u]\!])$, $M = (m_{i,j}) \in \mathrm{Mat}_d(R \otimes_{\mathbb{F}_p} k_E)$ two upper triangular matrices. Suppose $\mathrm{diag}(F) = [a_1 u^{t_1}, \ldots, a_d u^{t_d}]$ where $a_i \in k_E^\times$ and t_i are* distinct *non-negative integers. Suppose that*

- *If $i < j$ and $t_i > t_j$, then $f_{i,j} = 0$,*
- *$v_R(m_{i,j}) > 0, \forall i < j$, and*
- *$M\tau(\varphi(F)) = \varphi(F)\varphi(M)$.*

Then $M \in B_C(R \otimes_{\mathbb{F}_p} k_E)$, where $C := \{\epsilon_i - \epsilon_j : i < j, t_i < t_j\}$ the closed subset of R^+.

Proof. — We prove by induction on the dimension d. When $d = 1$, there is nothing to prove. Suppose the lemma is true for dimension less than d, and consider it for d.

We can apply the induction hypothesis to $F_{1,1}$ and $M_{1,1}$ (resp. $F_{d,d}$ and $M_{d,d}$), where $F_{1,1}$ is the co-matrix of F by deleting the 1st row and 1st column (and similarly for $M_{1,1}, F_{d,d}$ and $M_{d,d}$). So we only need to deal with the element on the most upper right corner. That is, we only need to prove that if $t_1 > t_d$, then $m_{1,d} = 0$.

For any $2 \leq i \leq d$, we have

- (Case 1) If $t_i > t_1 > t_d$, then $f_{i,d} = 0$ (property of F), and $m_{i,d} = 0$ (induction hypothesis).
- (Case 2) If $t_1 > t_i$, then $f_{1,i} = 0$ (property of F), and $m_{1,i} = 0$ (induction hypothesis).

By the condition that $M\tau(\varphi(F)) = \varphi(F)\varphi(M)$, we must have

$$\sum_{i=1}^{d} m_{1,i}\tau(\varphi(f_{i,d})) = \sum_{i=1}^{d} \varphi(f_{1,i})\varphi(m_{i,d}).$$

So we will always have $m_{1,d}\tau(\varphi(u^{t_d})) = \varphi(u^{t_1})\varphi(m_{1,d})$, because all the other terms vanish. Now we can conclude $m_{1,d} = 0$ by Lemma 4.2.1. $\square$

5. Crystalline lifting theorem

5.0.1. Theorem. — *With notations in (*CRYS*), and suppose that $\overline{\rho}$ is upper triangular. Suppose $\overline{\rho} \in \mathcal{E}(\overline{\chi}_1, \ldots, \overline{\chi}_d)$ such that $\overline{\chi}_i\overline{\chi}_j^{-1} \neq \overline{\varepsilon}_p, \forall i \neq j$. Then there exists an upper triangular crystalline representation ρ' such that $\overline{\rho}' \cong \overline{\rho}$, and* $\mathrm{HT}(\rho') = \mathrm{HT}(\rho)$ *as sets.*

Proof. — Recall that $\boldsymbol{e} = (e_1, \ldots, e_d)$ is the basis of $\overline{\mathfrak{M}}$ in Proposition 4.1.1. Let $\sigma \in W_C$ be the unique element as in Proposition 4.1.3, and denote $\boldsymbol{e}^\sigma := (e_{\sigma(1)}, \ldots, e_{\sigma(d)})$. By *loc. cit.*, the matrix of φ for $\overline{\mathfrak{M}}$ with respect to the basis $\boldsymbol{e}^\sigma$ (which is $\sigma^{-1}A_\varphi\sigma$) is still upper triangular. By Proposition 4.2.2 and Lemma 3.0.2, the matrix of τ for $\overline{\hat{\mathfrak{M}}}$ with respect to the basis $1 \otimes_\varphi \boldsymbol{e}^\sigma$ (which is $\sigma^{-1}A_\tau\sigma$) is also upper triangular. That is to say (by Lemma 2.1.3), $\overline{\hat{\mathfrak{M}}} \in \mathcal{E}(\overline{\hat{\mathfrak{N}}}_{\sigma(d)}, \ldots, \overline{\hat{\mathfrak{N}}}_{\sigma(1)})$, where $\overline{\hat{\mathfrak{N}}}_{\sigma(i)} := \overline{\hat{\mathfrak{M}}}(r_i; a_{\sigma(i)})$. And so $\overline{\rho} = \hat{T}(\overline{\hat{\mathfrak{M}}}) \in \mathcal{E}(\overline{\chi}_{\sigma(1)}, \ldots, \overline{\chi}_{\sigma(d)})$.

By Lemma 2.2.2(3), each $\overline{\chi}_{\sigma(i)}$ has a crystalline lift $\chi_{\sigma(i)} =: \hat{T}(\hat{\mathfrak{M}}(r_i; \hat{a}_{\sigma(i)}))$, where $\hat{a}_{\sigma(i)} \in \mathcal{O}_E^\times$ is any lift of $a_{\sigma(i)}$. Since $r_1 < \ldots < r_d$, by [5, Lem. 3.1.5] (note that our convention of Hodge-Tate weights is the opposite of *loc. cit.*), $\overline{\rho}$ has an upper triangular crystalline lift ρ' such that $\rho' \in \mathcal{E}(\chi_{\sigma(1)}, \ldots, \chi_{\sigma(d)})$. Let us remark here that ρ' is in fact *ordinary* in the sense of [5, Def. 3.1.3]. $\square$

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GLOBAL EXISTENCE AND ASYMPTOTICS FOR QUASI-LINEAR ONE-DIMENSIONAL KLEIN-GORDON EQUATIONS WITH MILDLY DECAYING CAUCHY DATA

by Annalaura Stingo

Abstract. — Let u be a solution to a quasi-linear Klein-Gordon equation in one-space dimension, $\Box u + u = P(u, \partial_t u, \partial_x u;\, \partial_t\partial_x u, \partial_x^2 u)$, where P is a homogeneous polynomial of degree three, and with smooth Cauchy data of size $\varepsilon \to 0$. It is known that, under a suitable condition on the nonlinearity, the solution is global-in-time for compactly supported Cauchy data. We prove in this paper that the result holds even when data are not compactly supported but just decaying as $\langle x\rangle^{-1}$ at infinity, combining the method of Klainerman vector fields with a semiclassical normal forms method introduced by Delort. Moreover, we get a one term asymptotic expansion for u when $t \to +\infty$.

Résumé (*Existence globale et comportement asymptotique de petites solutions pour des équation de Klein-Gordon critiques 1D*). — Soit u une solution d'une équation de Klein-Gordon quasi-linéaire en dim. 1 d'espace, $\Box u + u = P(u, \partial_t u, \partial_x u;\, \partial_t\partial_x u, \partial_x^2 u)$, où P est un polynôme homogène de degré trois, avec données initiales régulières de taille $\varepsilon \to 0$. Il est connu que, sous certaines conditions sur la non-linéarité, la solution est globale en temps pour des données initiales à support compact. Nous montrons que ce résultat est aussi vrai quand les données ne sont pas à support compact mais seulement décroissantes à l'infini comme $\langle x\rangle^{-1}$, en combinant la méthode des champs de vecteurs de Klainerman avec une méthode de formes normales semi-classiques introduite par Delort. De plus, nous obtenons un développement asymptotique à un terme pour u lorsque $t \to +\infty$, prouvant ainsi un résultat de scattering modifié.

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Introduction

The goal of this paper is to prove the global existence and to study the asymptotic behavior of the solution u of the one-dimensional nonlinear Klein-Gordon equation, when initial data are small, smooth and slightly decaying at infinity. We will consider the case of a quasi-linear cubic nonlinearity, namely a homogeneous polynomial P of degree 3 in $(u, \partial_t u, \partial_x u;\, \partial_t\partial_x u, \partial_x^2 u)$, affine in $(\partial_t\partial_x u, \partial_x^2 u)$, so that the initial valued problem is written as

$$(1)\qquad \begin{cases} \Box u + u = P(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u) \\ u(1,x) = \varepsilon u_0(x) \\ \partial_t u(1,x) = \varepsilon u_1(x) \end{cases} \qquad t \geq 1, x \in \mathbb{R}, \varepsilon \in \,]0,1[.$$

Our main concern is to obtain results for data which have only mild decay at infinity (i.e., which are $O(|x|^{-1})$, $x \to +\infty$), while most known results for quasi-linear Klein-Gordon equations in dimension 1 are proved for compactly supported data. In order to do so, we have to develop a new approach, that relies on semiclassical analysis, and that allows to obtain for Klein-Gordon equations results of global existence making use of Klainerman vector fields and usual energy estimates, instead of L^2 estimates on the hyperbolic foliation of the interior of the light cone, as done for instance in an early work of Klainerman [23] and more recently in the paper of LeFloch, Ma [26].

We recall first the state of the art of the problem. In general, the problem in dimension 1 is critical, contrary to the problem in higher dimension which is subcritical. In fact, in space dimension d, the best time decay one can expect for the solution is $\|u(t,\cdot)\|_{L^\infty} = O(t^{-\frac{d}{2}})$: therefore, in dimension 1 the decay rate is $t^{-\frac{1}{2}}$, and for a cubic nonlinearity, depending for example only on u, one has $\|P(u)\|_{L^2} \leq Ct^{-1}\|u(t,\cdot)\|_{L^2}$, with a time factor t^{-1} just at limit of integrability. In space dimension $d \geq 3$, it is well known from works of Klainerman [23] and Shatah [33] that the analogous problem has global-in-time solutions if ε is sufficiently small. In [23], Klainerman proved it for smooth, compactly supported initial data, with nonlinearities at least quadratic, using the Lorentz invariant properties of $\Box + 1$ to derive uniform decay estimates and generalized energy estimates for solutions u to linear inhomogeneous Klein-Gordon equations. Simultaneously, in [33] Shatah proved this result for smooth and integrable initial data, extending Poincaré's theory of normal forms for ordinary differential equations to the case of nonlinear Klein-Gordon equations. For space dimension $d = 2$, in [16] Hörmander refined Klainerman's techniques to obtain new time decay estimates of solutions to linear inhomogeneous Klein-Gordon equations. He showed that, for quadratic nonlinearities, the solution exists over $[-T_\varepsilon, T_\varepsilon]$ with an existence time T_ε such that $\lim_{\varepsilon\to 0} \varepsilon \log T_\varepsilon = \infty$ (while $\lim_{\varepsilon\to 0} \varepsilon^2 T_\varepsilon = \infty$ for $d = 1$). In addition, he conjectured that $T_\varepsilon = \infty$ (while for $d = 1$, $\liminf_{\varepsilon\to 0} \varepsilon^2 \log T_\varepsilon > 0$). The first conjecture has been proved by Ozawa, Tsutaya and Tsutsumi in [30] in the semi-linear case, after partial results by Georgiev, Popivanov in [10], and Kosecki in [25] (for nonlinearities verifying some "suitable null conditions"). Later, in [31] Ozawa, Tsutaya and Tsutsumi announced the extension of their proof to the quasi-linear case and studied scattering of solutions. In space dimension 1, Moriyama, Tonegawa and Tsutsumi [29] have shown that the solution exists on a time interval of length longer or equal to e^{c/ε^2}, where ε is the Cauchy data's size, with a nonlinearity vanishing at least at order three at zero, or semi-linear. They also proved that the corresponding solution asymptotically approaches the free solution of the Cauchy problem for the linear Klein-Gordon equation. The fact that in general the solution does not exist globally in time was proved by Yordanov in [35], and independently by Keel and Tao [21]. However, there exist examples of nonlinearities for which the corresponding solution is global-in-time: on one hand, if P depends only on u and not on its derivatives; on the other hand, for seven special nonlinearities considered by Moriyama in [28]. A natural question is then posed by Hörmander, in [15, 16]: can we formulate a structure condition for the nonlinearity, analogous to the null condition introduced by Christodoulou [3] and Klainerman [24] for the wave equation, which implies global existence? In [5, 6] Delort proved that, when initial data are compactly supported, one can find a *null condition*, under which global existence is ensured. This condition is likely optimal, in the sense that when the structure hypothesis is violated, he constructed in [4] approximate solutions blowing up at e^{A/ε^2}, for an explicit constant A. This suggests that also the exact solution of the problem blows up in time at e^{A/ε^2}, but this remains still unproven.

Once global existence is ensured, a natural question that arises concerns the long time behavior of the solutions. While for $d \geq 2$ it is known that the global solution behaves like a free solution, in space dimension one, only few results were known, including for the simpler equation

$$\Box u + u = \alpha u^2 + \beta u^3 + \text{order}\, 4.$$

For this equation, Georgiev and Yordanov [11] proved that, when $\alpha = 0$, the distance between the solution u and linear solutions cannot tend to 0 when $t \to \infty$, but they do not obtain an asymptotic description of the solution (except for the particular case of sine-Gordon $\Box u + \sin u = 0$, for which they use methods of "nonlinear scattering"). In [27], Lindblad and Soffer studied the scattering problem for long range nonlinearities, proving that for all prescribed asymptotic solutions there is a solution of the equation with such behavior, for some choice of initial data, and finding the complete asymptotic expansion of the solutions. In [14], a sharp asymptotic behavior of small solutions in the quadratic, semilinear case is proved by Hayashi and Naumkin, without the condition of compact support on initial data, using the method of normal forms of Shatah. The only other cases in dimension one for which the asymptotic behavior is known concern nonlinearities studied by Moriyama in [28], where he showed that solutions have a free asymptotic behavior, assuming the initial data to be sufficiently small and decaying at infinity.

Some results about global existence and long time behavior are also known for solutions to systems of coupled Klein-Gordon equations. In dimension $d = 3$, we cite the work of Germain [12], and of Ionescu, Pausader [18], for a system of coupled Klein-Gordon equations with different speeds, with a quadratic nonlinearity, respectively in the semilinear case for the former, and in the quasi-linear one for the latter. For data small, smooth and localized, they prove that a global solution exists and scatters. In dimension $d = 2$, Delort, Fang and Xue proved in [8] the global existence of solutions for a quasi-linear system of two Klein-Gordon equations, with masses m_1, m_2, $m_1 \neq 2m_2$ and $m_2 \neq 2m_1$, for small, smooth, compactly supported Cauchy data, extending the result proved by Sunagawa in [34] in the semilinear case. Moreover, they proved that the global existence holds true also in the resonant case, e.g., when $m_1 = 2m_2$, and a convenient null condition is satisfied by nonlinearities. The same result in the resonant case is also proved by Katayama, Ozawa [19], and by Kawahara, Sunagawa [20], in which the structural condition imposed on nonlinearities includes the Yukawa type interaction, which was excluded from the *null condition* in the sense of [8]. We should cite also the paper [32] by Schottdorf, where he proved global well-posedness and scattering result in the semilinear case, in dimension 2 and higher, for small H^s data, using the contraction mapping technique in U^2/V^2 based spaces. There are some results also in dimension 1. In [22], Kim shows that the solution to a system of semilinear cubic Klein-Gordon equations, verifying a suitable structure condition, and with small, non compactly supported initial data in some appropriate Sobolev space, is global-in-time and has the optimal decay $t^{-1/2}$, as t tends to infinity. We should also cite the work of Guo, Han and Zhang [13] on the global existence and the long time behavior of the solution to the one dimensional Euler-Poisson system, under weak conditions on the initial data, and of Candy and Lindblad [2], on the one dimensional cubic Dirac equation.

In most of above mentioned papers dealing with the one dimensional scalar problem, two key tools are used: normal forms methods and/or Klainerman vector fields Z. In particular, the latter are useful since they have good properties of commutation with the linear part of the equation, and their action on the nonlinearity $ZP(u)$ may be expressed from u, Zu using Leibniz rule. This allows one to prove easily energy estimates for $Z^k u$, and then to deduce from them L^∞ bounds for u, through Klainerman-Sobolev type inequalities. However, in these papers the global existence is proved assuming small, *compactly supported* initial data. This is related to the fact that the aforementioned authors use in an essential way a change of variable in hyperbolic coordinates, that does not allow for non compactly supported Cauchy data. Our aim is to extend the result of global existence for cubic quasi-linear nonlinearities in the case of small compactly supported Cauchy data of [5, 6], to the more general framework of data with mild polynomial decay. To do that, we will combine the Klainerman vector fields' method with the one introduced by Delort in [7].

In [7], Delort develops a semiclassical normal form method to study global existence for nonlinear hyperbolic equations with small, smooth, decaying Cauchy data, in the critical regime and when the problem does not admit Klainerman vector fields. The strategy employed is to construct, through semiclassical analysis, some *pseudo-differential* operators which commute with the linear part of the considered equation, and which can replace vector fields when combined with a microlocal normal form method. Our aim here is to show that one may combine these ideas together with the use of Klainerman vector fields to obtain, in one dimension, and for nonlinearities satisfying the null condition, global existence and modified scattering.

In our paper, we prove the global existence of the solution u by a *boostrap* argument, namely by showing that we can propagate some suitable *a priori* estimates made on u. We propagate two types of estimates: some energy estimates on u, Zu, and some uniform bounds on u. To prove the propagation of energy estimates is the simplest task. We essentially write an energy inequality for a solution u of the Klein-Gordon equation in the quasi-linear case (the main reference is the book of Hörmander [16], Chapter 7), and then we use the commutation property of the Klainerman vector fields Z with the linear part of the equation to derive an inequality also for Zu. Moreover, Z acts like a derivation on the nonlinearity, so the Leibniz rule holds and we can estimate ZP in term of u, Zu. Injecting *a priori* estimates in energy inequalities and choosing properly all involved constants allow us to obtain the result.

The main difficulty is to prove that the uniform estimates hold and can be propagated. Actually, as mentioned above, the one dimensional Klein-Gordon equation is critical, in the sense that the expected decay for $\|u(t,\cdot)\|^2_{L^\infty}$ is in t^{-1}, so is not integrable. A drawback of that is that one cannot prove energy estimates that would be uniform as time tends to infinity. Consequently, a Klainerman-Sobolev inequality, that would control $\|u(t,\cdot)\|_{L^\infty}$ by $t^{-1/2}$ times the L^2 norms of u, Zu, would not give the expected optimal L^∞-decay of the solution, but only a bound in $t^{-\frac{1}{2}+\sigma}$ for some positive σ, which is useless to close the bootstrap argument. The idea to overcome this difficulty is, following the approach of Delort in [7], to rewrite (1) in semiclassical coordinates, for some new unknown function v. The goal is then to deduce from the PDE satisfied by v an ODE from which one will be able to get a uniform L^∞ bound for v (which is equivalent to the optimal $t^{-1/2}$ L^∞-decay of u). Let us describe our approach for a simple model of Klein-Gordon equation. Denoting by D_t, D_x respectively $\frac{1}{i}\partial_t$, $\frac{1}{i}\partial_x$, we consider the following:

$$(D_t - \sqrt{1+D_x^2})u = \alpha u^3 + \beta|u|^2u + \gamma|u|^2\bar{u} + \delta\bar{u}^3, \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ are constants, β being *real* (this last assumption reflecting the null condition on that example). Performing a semiclassical change of variables and unknowns $u(t,x) = \frac{1}{\sqrt{t}} v(t, \frac{x}{t})$, we rewrite this equation as

$$[D_t - \mathrm{Op}_h^w(\lambda_h(x,\xi))]v = h(\alpha v^3 + \beta|v|^2 v + \gamma|v|^2\bar{v} + \delta\bar{v}^3), \tag{3}$$

where $\lambda_h(x,\xi) = x\xi + \sqrt{1+\xi^2}$, the semiclassical parameter h is defined as $h := 1/t$, and the Weyl quantization of a symbol a is given by

$$\mathrm{Op}_h^w(a)v = \frac{1}{2\pi h}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{i}{h}(x-y)\xi} a\Big(\frac{x+y}{2}, \xi\Big) v(y)\, dy d\xi.$$

One introduces the manifold $\Lambda = \{(x,\xi) \,|\, x + \frac{\xi}{\sqrt{1+\xi^2}} = 0\}$ as in Figure 0.1, which is the graph of the smooth function $d\varphi(x)$, where $\varphi :]-1,1[\to \mathbb{R}$ is $\varphi(x) = \sqrt{1-x^2}$.

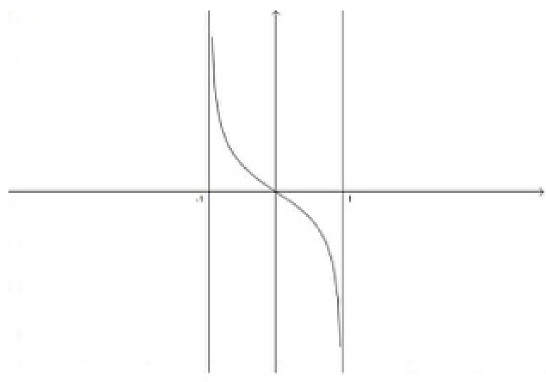

FIGURE 0.1. Λ for the Klein Gordon equation.

One can deduce an ODE from (3), developing the symbol $\lambda_h(x,\xi)$ on Λ, i.e., on $\xi = d\varphi(x)$. One obtains a first term $a(x)$ independent of ξ and a remainder, which turns out to be integrable in time as may be shown using some ideas of Ifrim-Tataru [17] and the L^2 estimates verified by v and by the action of the Klainerman vector field on v. In this way, one proves that v is solution of the equation

$$D_t v = a(x)v + h\beta|v|^2 v + \text{ non characteristic terms} \\ + \text{ remainder of higher order in } h. \tag{4}$$

Then the idea is to eliminate *non characteristic* terms by a normal forms argument, introducing a new function f which will be finally solution of an ordinary differential equation

$$D_t f = a(x)f + h\beta|f|^2 f + \text{remainder of higher order in } h. \tag{5}$$

From this equation, one easily derives an uniform control L^∞ on f, and then on the starting solution u. The analysis of the above ODE provides as well a one term asymptotic expansion of the solution of Equation (2) (or, more generally of the solution (1)), as proved in the last section of this paper. This expansion shows that, in general, scattering does not hold, and that one has only modified scattering. This is in contrast with higher dimensional problems for the Klein-Gordon equation where, as we already said, global solutions have at infinity the same behavior as free solutions.

We end this introduction with few words about the case of quadratic nonlinearities, in one space dimension. In [5], Delort proves global existence and modified scattering for an equation of the form (1), where the nonlinearity may have a quadratic component, i.e., for the equation

$$\begin{cases} \Box u + u = F(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u) \\ u(1,x) = \varepsilon u_0(x) \\ \partial_t u(1,x) = \varepsilon u_1(x) \end{cases} \qquad t \geq 1, x \in \mathbb{R}, \varepsilon \in]0,1[. \tag{6}$$

where

$$F(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u) \\ = Q(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u) + P(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u)$$

with Q (resp. P) homogeneous polynomial of degree 2 (resp. 3), and where one assumes a convenient null condition, that generalizes the one we impose here on the sole cubic terms. We believe that our method could be extended to that framework, providing global existence and modified scattering for (6), with small, mildly decaying initial data (instead of the compactly supported ones considered in [5]). Actually, it is well known that one may always perform a Shatah's normal form argument in order to reduce a Klein-Gordon equation with quadratic nonlinearities to a cubic one, when solutions are small. For quasi-linear equations, one should be cautious in order not to increase the number of derivatives in the nonlinearity, but this technical difficulty may be overcome using paradifferential calculus. Consequently, the case of quadratic nonlinearities can be reduced, at least in principle, to the cubic one, if one accepts to replace local cubic nonlinearities by nonlocal ones. We decided here to limit ourselves to the purely cubic case, in order to avoid the technicalities that are inherent to such reductions and keep the paper reasonably long.

1. Statement of the main results

The Cauchy problem we are considering is

$$\begin{cases} \Box u + u = P(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u) \\ u(1,x) = \varepsilon u_0(x) \\ \partial_t u(1,x) = \varepsilon u_1(x) \end{cases} \qquad t \geq 1, x \in \mathbb{R} \tag{1.1}$$

where $\square := \partial_t^2 - \partial_x^2$ is the D'Alembert operator, $\varepsilon \in]0,1[$, u_0, u_1 are smooth enough functions. P denotes a homogeneous polynomial of degree three, with real constant coefficients, affine in $(\partial_t\partial_x u, \partial_x^2 u)$. We can highlight this particular dependence on second derivatives following the approach of [5] and decomposing P as

$$P(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u) = P'(u\,; \partial_t u, \partial_x u) + P''(u, \partial_t\partial_x u, \partial_x^2 u\,; \partial_t u, \partial_x u), \tag{1.2}$$

where P', P'' are homogeneous polynomials of degree 3, P'' linear in $(\partial_t\partial_x u, \partial_x^2 u)$. Moreover

$$\begin{aligned} P'(X_1;\, Y_1, Y_2) &= \sum_{k=0}^{3} i^k P'_k(X_1; -iY_1, -iY_2) \\ P''(X_1, X_2, X_3;\, Y_1, Y_2) &= \sum_{k=0}^{2} i^k P''_k(X_1, -X_2, -X_3; -iY_1, -iY_2) \end{aligned} \tag{1.3}$$

where P'_k is homogeneous of degree k in (Y_1, Y_2) and of degree $3-k$ in X_1, while P''_k is homogeneous of degree 1 in (X_2, X_3) and of degree k in (Y_1, Y_2). We denote $P_k = P'_k + P''_k$. For $x \in]-1,1[$, define

$$\begin{aligned} \omega_0(x) &:= \frac{1}{\sqrt{1-x^2}}, \\ \omega_1(x) &:= \frac{-x}{\sqrt{1-x^2}}, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \Phi(x) := P'_1(1; \omega_0(x), \omega_1(x)) + P''_1(1, \omega_0(x)\omega_1(x), \omega_1^2(x); \omega_0(x), \omega_1(x)) \\ + 3P'_3(1; \omega_0(x), \omega_1(x)). \end{aligned} \tag{1.5}$$

Definition 1.1. — We say that the nonlinearity P satisfies the *null condition* if and only if $\Phi \equiv 0$.

Our goal is to prove that there is a global solution of (1.1) when ε is sufficiently small, u_0, u_1 decay rapidly enough at infinity, and when the cubic nonlinearity satisfies the *null condition*. We state the main theorem below.

Theorem 1.2 (Main Theorem). — *Suppose that the nonlinearity P satisfies the null condition. Then there exists an integer s sufficiently large, a positive small number σ, an $\varepsilon_0 \in]0,1[$ such that, for any real valued $(u_0, u_1) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$ satisfying*

$$\|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|xu_0\|_{H^2} + \|xu_1\|_{H^1} \leq 1, \tag{1.6}$$

for any $0 < \varepsilon < \varepsilon_0$, the problem (1.1) has an unique solution $u \in C^0([1, +\infty[; H^{s+1}) \cap C^1([1, +\infty[; H^s)$. Moreover, there exists a 1-parameter family of continuous function $a_\varepsilon : \mathbb{R} \to \mathbb{C}$, uniformly bounded and supported in $[-1, 1]$, a *function* $(t,x) \to r(t,x)$ *with values in* $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *bounded in* $t \geq 1$, *such that, for any* $\varepsilon \in]0,\varepsilon_0]$, *the global solution* u *of* (1.1) *has the asymptotic expansion*

(1.7)

$$u(t,x) = \Re\left[\frac{\varepsilon}{\sqrt{t}} a_\varepsilon\left(\frac{x}{t}\right) \exp\left[it\varphi\left(\frac{x}{t}\right) + i\varepsilon^2 \left|a_\varepsilon\left(\frac{x}{t}\right)\right|^2 \Phi_1\left(\frac{x}{t}\right)\log t\right]\right] + \frac{\varepsilon}{t^{\frac{1}{2}+\sigma}} r(t,x),$$

where $\varphi(x) = \sqrt{1-x^2}$, *and*

$$\begin{aligned}\Phi_1(x) = \frac{1}{8}\langle\omega_0(x)\rangle^{-4}\big[&3P_0(1,\omega_0(x)\omega_1(x),\omega_1(x)^2;\,\omega_0(x),\omega_1(x)) \\ &+P_2(1,\omega_0(x)\omega_1(x),\omega_1(x)^2;\,\omega_0(x),\omega_1(x))\big],\end{aligned} \tag{1.8}$$

with $\langle x\rangle = \sqrt{1+x^2}$.

We denote by Z the Klainerman vector field for the Klein-Gordon equation, that is $Z := t\partial_x + x\partial_t$, and by Γ a generic vector field in the set $\mathcal{Z} = \{Z,\partial_t,\partial_x\}$. The most remarkable properties of these vector fields are the commutation with the linear part of the equation in (1.1), namely

$$[\Box + 1, \Gamma] = 0, \tag{1.9}$$

and the fact that they act like a derivation on the cubic nonlinearity. We also denote by $W^{t,\rho,\infty}$ a modified Sobolev space, made by functions $t \to \psi(t,\cdot)$ defined on an interval, such that $\langle D_x\rangle^{\rho-i}D_t^i u \in L^\infty$, for $i \leq 2$, with the norm

$$\|\psi(t,\cdot)\|_{W^{t,\rho,\infty}(\mathbb{R})} := \sum_{i=0}^{2} \|\langle D_x\rangle^{\rho-i}D_t^i\psi(t,\cdot)\|_{L^\infty(\mathbb{R})}. \tag{1.10}$$

The proof of the main theorem is based on a *bootstrap* argument. In other words, we shall prove that we are able to propagate some *a priori* estimates made on a solution u of (1.1) on some interval $[1,T]$, for some $T>1$ fixed, as stated in the following theorem.

THEOREM 1.3 (Bootstrap Theorem). — *There exist two integers* s,ρ *large enough,* $s \gg \rho$, *an* $\varepsilon_0 \in]0,1[$ *sufficiently small, and two constants* $A,B>0$ *sufficiently large such that, for any* $0<\varepsilon<\varepsilon_0$, *if* u *is a solution of* (1.1) *on some interval* $[1,T]$, *for* $T>1$ *fixed, and satisfies*

$$\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \leq A\varepsilon t^{-\frac{1}{2}} \tag{1.11a}$$

$$\|Zu(t,\cdot)\|_{H^1} \leq B\varepsilon t^\sigma, \qquad \|\partial_t Zu(t,\cdot)\|_{L^2} \leq B\varepsilon t^\sigma \tag{1.11b}$$

$$\|u(t,\cdot)\|_{H^s} \leq B\varepsilon t^\sigma, \qquad \|\partial_t u(t,\cdot)\|_{H^{s-1}} \leq B\varepsilon t^\sigma, \tag{1.11c}$$

for every $t \in [1, T]$, for some $\sigma \geq 0$ small, then it verifies also

$$\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \leq \frac{A}{2}\varepsilon t^{-\frac{1}{2}} \tag{1.12a}$$

$$\|Zu(t,\cdot)\|_{H^1} \leq \frac{B}{2}\varepsilon t^{\sigma}, \qquad \|\partial_t Zu(t,\cdot)\|_{L^2} \leq \frac{B}{2}\varepsilon t^{\sigma} \tag{1.12b}$$

$$\|u(t,\cdot)\|_{H^s} \leq \frac{B}{2}\varepsilon t^{\sigma}, \qquad \|\partial_t u(t,\cdot)\|_{H^{s-1}} \leq \frac{B}{2}\varepsilon t^{\sigma}. \tag{1.12c}$$

In Section 2 we show that energy bounds (1.11b), (1.11c) can be propagated, simply recalling an energy inequality obtained by Hörmander in [16] for a solution u of a quasi-linear Klein-Gordon equation, and applying it to $\partial_x^{s-1}u$ and Zu. Sections from 3 to 5 concern instead the proof of the uniform estimate's propagation. Furthermore, in Section 5 we derive also the asymptotic behavior of the solution u.

To conclude, we can mention that we will mainly focus on not very high frequencies, for it is easier to control what happens for very large frequencies which correspond to points on Λ in Figure 0.1 close to vertical asymptotic lines. This is justified by the fact that contributions of frequencies of the solution larger than $h^{-\beta}$, for a small positive β, have L^2 norms of order $O(h^N)$ if $s\beta \gg N$, assuming small H^s estimates on v. In this way, most of the analysis is reduced to frequencies lower than $h^{-\beta}$.

2. Generalized energy estimates

With notations introduced in the previous section, we define

$$E_0(t,u) = \left(\|\partial_t u(t,\cdot)\|_{L^2}^2 + \|\partial_x u(t,\cdot)\|_{L^2}^2 + \|u(t,\cdot)\|_{L^2}^2\right)^{1/2} \tag{2.1}$$

as the square root of the energy associated to the solution u of (1.1) at time t, and $E_N^\Gamma(t,u) = \sum_{k=0}^{N} \left(E_0(t,\Gamma^k u)^2\right)^{1/2}$, for a fixed Γ. The goal of this section is to obtain an energy inequality involving $E_N^\Gamma(t,u)$. In particular, since the aim is to propagate *a priori* energy bounds on u, i.e., $\|u(t,\cdot)\|_{H^s}$, $\|\partial_t u(t,\cdot)\|_{H^{s-1}}$, $\|Zu(t,\cdot)\|_{H^1}$ and $\|\partial_t Zu(t,\cdot)\|_{L^2}$, we will consider on one hand $E_{s-1}^{\partial_x}(t,u)$ where all Γ are equal to ∂_x, and on the other $E_1^Z(t,u)$ where $\Gamma = Z$. Often in what follows we will denote partial derivatives with respect to t and x respectively by ∂_0 and ∂_1.

We will use the following result, which concerns the specific energy inequality for the Klein-Gordon equation in the quasi-linear case, and which is presented here without proof (see Lemma 7.4.1 in [16] for further details).

Lemma 2.1. — *Let u be a solution of*

$$\square u + u + \gamma^{01}\partial_0\partial_1 u + \gamma^{11}\partial_1^2 u + \gamma^0\partial_0 u + \gamma^1\partial_1 u = f, \tag{2.2}$$

where functions $\gamma^{ij}=\gamma^{ij}(t,x)$, $\gamma^j=\gamma^j(t,x)$ *are smooth, such that* $\sum_{i,j=0}^{1}|\gamma^{ij}|+|\gamma^j|\leq\frac{1}{2}$. *Then,*

$$E_0(t,u)\leq C\big[E_0(1,u)+\int_1^t\big(\|f(\tau,\cdot)\|_{L^2}+C(\tau)E_0(\tau,u)\big)d\tau\big], \tag{2.3}$$

where $C(\tau):=\sum_{i,j,h=0}^{1}\sup_x\big(|\partial_h\gamma^{ij}(\tau,x)|+|\partial_h\gamma^j(\tau,x)|\big)$.

We can rewrite the equation in (1.1) in the same form as in Lemma 2.1, especially highlighting the linear dependence on second derivatives,

$$\Box u+u+\gamma^{01}\partial_0\partial_1u+\gamma^{11}\partial_1^2u+\gamma^0\partial_0u+\gamma^1\partial_1u=0, \tag{2.4}$$

where coefficients γ^{ij},γ^j are homogeneous polynomials of degree two in $(u,\partial_0u,\partial_1u)$. Let us apply $\partial_1^{s'}$, $s':=s-1$, to this equation. If u is a solution of (2.4), then $\partial_1^{s'}u$ satisfies

$$\Box\partial_1^{s'}u+\partial_1^{s'}u+\partial_1^{s'}\big(\gamma^{01}\partial_0\partial_1u+\gamma^{11}\partial_1^2u+\gamma^0\partial_0u+\gamma^1\partial_1u\big)=0, \tag{2.5}$$

and applying the Leibniz rule, we obtain that $\partial_1^{s'}u$ is solution of the equation

$$\Box\partial_1^{s'}u+\partial_1^{s'}u+\gamma^{01}\partial_0\partial_1(\partial_1^{s'}u)+\gamma^{11}\partial_1^2(\partial_1^{s'}u)+\gamma^0\partial_0(\partial_1^{s'}u)+\gamma^1\partial_1(\partial_1^{s'}u)=f^{s'}, \tag{2.6}$$

where $f^{s'}$ is a linear combination of terms of the form

$$\begin{aligned}&(\partial_1^{s_1'}\partial_i^{\alpha_1}u)\,(\partial_1^{s_2'}\partial_j^{\alpha_2}u)\,(\partial_1^{s_3'}\partial_{ij}^2u),\\&(\partial_1^{s_1'}\partial_i^{\alpha_1}u)\,(\partial_1^{s_2'}\partial_j^{\alpha_2}u)\,(\partial_1^{s_3'}\partial_hu),\end{aligned} \tag{2.7}$$

for $i,j,h,\alpha_1,\alpha_2=0,1$, $s_1'+s_2'+s_3'=s'$, $s_3'<s'$. So taking the L^2 norm and observing that at most one index s_j' can be larger than $s'/2$, we have

$$\|f^{s'}(t,\cdot)\|_{L^2}\leq\Big(\sum_{\substack{i+j=0\\j\leq2}}^{[\frac{s'}{2}]+2}\|\partial_x^i\partial_t^ju(t,\cdot)\|_{L^\infty}^2\Big)E_{s'}^{\partial_1}(t,u)\leq\|u(t,\cdot)\|_{W^{t,\rho,\infty}}^2E_{s'}^{\partial_1}(t,u), \tag{2.8}$$

for any finite $\rho\geq[\frac{s'}{2}]+3$. Rewriting inequality (2.3) for $\partial_1^{s'}u$, where $s'=s-1$ and $C(\tau)\leq\|u(\tau,\cdot)\|_{W^{t,2,\infty}}^2$, we obtain

$$E_{s-1}^{\partial_1}(t,u)\leq C\left[E_{s-1}^{\partial_1}(1,u)+\int_1^t\|u(\tau,\cdot)\|_{W^{t,\rho,\infty}}^2E_{s-1}^{\partial_1}(\tau,u)d\tau\right]. \tag{2.9}$$

On the other hand, we want to obtain an analogous of (2.9) for $E_1^Z(t,u)$. Applying Z to (2.4), Leibniz rule and commutations, we derive that Zu is solution of the equation

$$\square Zu + Zu + \gamma^{01}\partial_0\partial_1 Zu + \gamma^{11}\partial_1^2 Zu + \gamma^0\partial_0 Zu + \gamma^1\partial_1 Zu = f^Z, \tag{2.10}$$

where f^Z is linear combination of $[\gamma^{ij}\partial^2_{ij}, Z]u$ and $[\gamma^h\partial_h, Z]u$. We calculate for instance the term $[\gamma^{01}\partial^2_{01}, Z]u$ and we find that it is equal to $-(Z\gamma^{01})\partial^2_{01}u - \gamma^{01}[\partial^2_{01}, Z]u$, that is a linear combination of

$$\begin{aligned} &(\partial_i^{\alpha_1}u)(\partial_j^{\alpha_2}Zu)(\partial^2_{01}u),\\ &(\partial_i^{\alpha_1}u)(\partial_j^{\alpha_2}u)(\partial^2_{hk}u), \end{aligned} \tag{2.11}$$

for $i, j, h, k, \alpha_1, \alpha_2 = 0, 1$. Therefore, the L^2 norm of f^Z can be estimated as follows

(2.12)

$$\|f^Z(t,\cdot)\|_{L^2} \leq \Big(\sum_{i+j=0}^{2} \|\partial_x^i\partial_t^j u(t,\cdot)\|^2_{L^\infty}\Big) E_1^Z(t,u) \leq \|u(t,\cdot)\|^2_{W^{t,3,\infty}} E_1^Z(t,u),$$

and applying Lemma 2.1 for Zu, we derive

$$E_1^Z(t,u) \leq C\left[E_1^Z(1,u) + \int_1^t \|u(\tau,\cdot)\|^2_{W^{t,3,\infty}} E_1^Z(\tau,\cdot)\,d\tau\right]. \tag{2.13}$$

Remark. — To make the above proof fully correct, one should check as well that the energy of Zu is actually finite at every fixed positive time. One may do that either using that the vector field Z is the infinitesimal generator of the action on the equation of a one parameter group, along the lines of Appendix A.2 in [1]. Alternatively, one may instead exploit finite propagation speed, remarking that if the data are cut off on a compact set, the solution remains compactly supported at every fixed time, so that the energy of Zu is actually finite, and that the bounds we get are uniform in terms of the cut off.

Proposition 2.2 (Propagation of Energy Estimates). — *There exist an integer s large enough, a $\rho \geq [\frac{s-1}{2}] + 3$, $\rho \ll s$, an $\varepsilon_0 \in]0,1[$ sufficiently small, a small $\sigma \geq 0$, and two constants $A, B > 0$ sufficiently large such that, for any $0 < \varepsilon < \varepsilon_0$, if u is a solution of* (1.1) *on some interval $[1,T]$, for $T > 1$ fixed, and satisfies*

$$\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \leq A\varepsilon t^{-\frac{1}{2}}, \tag{2.14a}$$

$$E^{\partial_1}_{s-1}(t,u) \leq B\varepsilon t^\sigma, \tag{2.14b}$$

$$E_1^Z(t,u) \leq B\varepsilon t^\sigma, \tag{2.14c}$$

for every $t \in [1,T]$, then it verifies also

$$E^{\partial_1}_{s-1}(t,u) \leq \frac{B}{2}\varepsilon t^\sigma, \tag{2.15a}$$

$$E_1^Z(t,u) \leq \frac{B}{2}\varepsilon t^\sigma. \tag{2.15b}$$

Proof. — Both estimates (2.14b) and (2.14c) can be propagated injecting *a priori* estimates (2.14) in energy inequalities (2.9) and (2.13) derived before, obtaining

$$E^{\partial_1}_{s-1}(t,u) \leq C\Big[E^{\partial_1}_{s-1}(1,u) + A^2B\varepsilon^3\int_1^t \tau^{-1+\sigma}d\tau\Big] \leq CE^{\partial_1}_{s-1}(1,u) + \frac{A^2BC\varepsilon^3}{\sigma}t^\sigma,$$
$$E^{Z}_{1}(t,u) \leq C\Big[E^{Z}_{1}(1,u) + A^2B\varepsilon^3\int_1^t \tau^{-1+\sigma}d\tau\Big] \leq CE^{Z}_{1}(1,u) + \frac{A^2BC\varepsilon^3}{\sigma}t^\sigma.$$

Then we can choose $B>0$ sufficiently large such that $CE^{\partial_1}_{s-1}(1,u) + CE^Z_1(1,u) \leq \frac{B}{4}\varepsilon$, and $\varepsilon_0 > 0$ sufficiently small such that $\frac{A^2C\varepsilon^2}{\sigma} \leq \frac{1}{4}$, to obtain (2.15a), (2.15b).

3. Semiclassical pseudo-differential operators

As told in the introduction, in order to prove an L^∞ estimate on u and on its derivatives we need to reformulate the starting problem (1.1) in term of an ODE satisfied by a new function v obtained from u, and this will strongly use the semiclassical pseudo-differential calculus. In the following two subsections, we introduce this semiclassical environment, defining classes of symbols and operators we shall use and several useful properties, some of which are stated without proof. More details can be found in [9] and [36].

3.1. Definitions and Composition Formula

Definition 3.1. — An order function on $\mathbb{R}\times\mathbb{R}$ is a smooth map from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}_+$: $(x,\xi) \rightarrow M(x,\xi)$ such that there exist $N_0 \in \mathbb{N}$, $C>0$ and for any $(x,\xi),(y,\eta) \in \mathbb{R}\times\mathbb{R}$

$$M(y,\eta) \leq C\langle x-y\rangle^{N_0}\langle \xi-\eta\rangle^{N_0}M(x,\xi), \tag{3.1}$$

where $\langle x\rangle = \sqrt{1+x^2}$.

Examples of order functions are $\langle x\rangle$, $\langle \xi\rangle$, $\langle x\rangle\langle \xi\rangle$.

Definition 3.2. — Let M be an order function on $\mathbb{R}\times\mathbb{R}$, $\beta\geq 0$, $\delta \geq 0$. One denotes by $S_{\delta,\beta}(M)$ the space of smooth functions

$$\begin{gathered}(x,\xi,h) \rightarrow a(x,\xi,h)\\ \mathbb{R}\times\mathbb{R}\times\,]0,1] \rightarrow \mathbb{C}\end{gathered}$$

satisfying for any $\alpha_1,\alpha_2,k,N \in \mathbb{N}$ bounds

$$|\partial^{\alpha_1}_x\partial^{\alpha_2}_\xi(h\partial_h)^k a(x,\xi,h)| \leq CM(x,\xi)\,h^{-\delta(\alpha_1+\alpha_2)}(1+\beta h^\beta|\xi|)^{-N}. \tag{3.2}$$

A key role in this paper will be played by symbols a verifying (3.2) with $M(x,\xi) = \langle\frac{x+f(\xi)}{\sqrt{h}}\rangle^{-N}$, for $N\in\mathbb{N}$ and a certain smooth function $f(\xi)$. This function M is no longer an order function because of the term $h^{-\frac{1}{2}}$ but nevertheless we continue to keep the notation $a \in S_{\delta,\beta}(\langle \frac{x+f(\xi)}{\sqrt{h}} \rangle^{-N})$.

DEFINITION 3.3. — We will say that $a(x,\xi)$ is a symbol of order r if $a \in S_{\delta,\beta}(\langle \xi \rangle^r)$, for some $\delta \geq 0$, $\beta \geq 0$.

Let us observe that when $\beta > 0$, the symbol decays rapidly in $h^\beta |\xi|$, which implies the following inclusion for $r \geq 0$

$$S_{\delta,\beta}(\langle \xi \rangle^r) \subset h^{-\beta r} S_{\delta,\beta}(1), \tag{3.3}$$

which will be often use in all the paper. This means that, up to a small loss in h, this type of symbols can be always considered as symbols of order zero. In the rest of the paper we will not indicate explicitly the dependence of symbols on h, referring to $a(x,\xi,h)$ simply as $a(x,\xi)$.

DEFINITION 3.4. — Let $a \in S_{\delta,\beta}(M)$ for some order function M, some $\delta \geq 0$, $\beta \geq 0$.

(i) We can define the *Weyl quantization* of a to be the operator $\mathrm{Op}_h^w(a) = a^w(x, hD)$ acting on $u \in \mathcal{S}(\mathbb{R})$ by the formula:

$$\mathrm{Op}_h^w(a(x,\xi))u(x) = \frac{1}{2\pi h} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\frac{i}{h}(x-y)\xi} a(\frac{x+y}{2}, \xi)\, u(y)\, dy d\xi\,; \tag{3.4}$$

(ii) We define also the *standard quantization*:

$$\mathrm{Op}_h(a(x,\xi))u(x) = \frac{1}{2\pi h} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\frac{i}{h}(x-y)\xi} a(x,\xi)\, u(y)\, dy d\xi. \tag{3.5}$$

It is clear from the definition that the two quantizations coincide when the symbol does not depend on x.

We introduce also a semiclassical version of Sobolev spaces, on which is more natural to consider the action of above operators.

DEFINITION 3.5. — (i) Let $\rho \in \mathbb{N}$. We define the semiclassical Sobolev space $W_h^{\rho,\infty}(\mathbb{R})$ as the space of families $(v_h)_{h\in]0,1]}$ of tempered distributions, such that $\langle hD \rangle^\rho v_h := \mathrm{Op}_h(\langle \xi \rangle^\rho) v_h$ is a bounded family of L^∞, i.e.,

$$W_h^{\rho,\infty}(\mathbb{R}) := \left\{ v_h \in \mathcal{S}'(\mathbb{R}) \,\Big|\, \sup_{h\in]0,1]} \|\langle hD \rangle^\rho v_h\|_{L^\infty(\mathbb{R})} < +\infty \right\}. \tag{3.6}$$

(ii) Let $s \in \mathbb{R}$. We define the semiclassical Sobolev space $H_h^s(\mathbb{R})$ as the space of families $(v_h)_{h\in]0,1]}$ of tempered distributions such that $\langle hD \rangle^s v_h := \mathrm{Op}_h(\langle \xi \rangle^s) v_h$ is a bounded family of L^2, i.e.,

$$H_h^s(\mathbb{R}) := \left\{ v_h \in \mathcal{S}'(\mathbb{R}) \,\Big|\, \sup_{h\in]0,1]} \int_{\mathbb{R}} (1+|h\xi|^2)^s |\hat{v}_h(\xi)|^2\, d\xi < +\infty \right\}. \tag{3.7}$$

For future references, we write down the semiclassical Sobolev injection,

$$\|v_h\|_{W^{\rho,\infty}_h} \leq C_\theta h^{-\frac{1}{2}} \|v_h\|_{H_h^{\rho+\frac{1}{2}+\theta}}, \qquad \forall \theta > 0. \tag{3.8}$$

The following two propositions are stated without proof. They concern the adjoint and the composition of pseudo-differential operators we are considering, and a full detailed treatment is provided in Chapter 7 of [9], or in Chapter 4 of [36].

Proposition 3.6 (Self-Adjointness). — *If a is a real symbol, its Weyl quantization is self-adjoint,*

$$\big(\mathrm{Op}_h^w(a)\big)^* = \mathrm{Op}_h^w(a). \tag{3.9}$$

Proposition 3.7 (Composition for Weyl quantization). — *Let $a, b \in \mathcal{S}(\mathbb{R})$. Then*

$$\mathrm{Op}_h^w(a) \circ \mathrm{Op}_h^w(b) = \mathrm{Op}_h^w(a \sharp b), \tag{3.10}$$

where

$$a \sharp b\,(x,\xi) := \frac{1}{(\pi h)^2} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{2i}{h}\sigma(y,\eta;\,z,\zeta)} a(x+z,\xi+\zeta) b(x+y,\xi+\eta)\; dy d\eta dz d\zeta \tag{3.11}$$

and

$$\sigma(y,\eta;\, z,\zeta) = \eta z - y\zeta.$$

It is often useful to derive an asymptotic expansion for $a \sharp b$, which allows easier computations than the integral Formula (3.11). This expansion is usually obtained by applying the stationary phase argument when $a, b \in S_{\delta,\beta}(M)$, $\delta \in [0, \frac{1}{2}[$ (as shown in [36]). Here we provide an expansion at any order even when one of two symbols belongs to $S_{\frac{1}{2},\beta_1}(M)$ (still having the other in $S_{\delta,\beta_2}(M)$ for $\delta < \frac{1}{2}$, and β_1, β_2 either equal or, if not, one of them equal to zero), whose proof is based on the Taylor development of symbols a, b, and can be found in detail in the appendix.

Proposition 3.8. — *Let $a \in S_{\delta_1,\beta_1}(M_1)$, $b \in S_{\delta_2,\beta_2}(M_2)$, $\delta_1, \delta_2 \in [0, \frac{1}{2}]$, $\delta_1 + \delta_2 < 1$, $\beta_1, \beta_2 \geq 0$ such that*

$$\beta_1 = \beta_2 \geq 0 \qquad \text{or} \qquad \big[\beta_1 \neq \beta_2 \;\text{and}\; \beta_i = 0, \beta_j > 0, i \neq j \in \{1,2\}\big]. \tag{3.12}$$

Then $a \sharp b \in S_{\delta,\beta}(M_1 M_2)$, where $\delta = \max\{\delta_1,\delta_2\}$, $\beta = \max\{\beta_1,\beta_2\}$. Moreover,

$$a \sharp b = ab + \frac{h}{2i}\{a,b\} + \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 2\leq|\alpha|\leq k}} \left(\frac{h}{2i}\right)^{|\alpha|} \frac{(-1)^{\alpha_1}}{\alpha!} \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a\, \partial_x^{\alpha_2}\partial_\xi^{\alpha_1} b + r_k, \tag{3.13}$$

where $\{a,b\} = \partial_\xi a \partial_x b - \partial_\xi b \partial_x a$, $r_k \in h^{(k+1)(1-(\delta_1+\delta_2))} S_{\delta,\beta}(M_1 M_2)$ *and*

(3.14)
$$r_k(x,\xi) = \left(\tfrac{h}{2i}\right)^{k+1} \tfrac{k+1}{(\pi h)^2} \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha|=k+1}} \tfrac{(-1)^{\alpha_1}}{\alpha!} \int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}$$
$$\left\{ \int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x+tz,\xi+t\zeta)(1-t)^k dt \times \partial_y^{\alpha_2}\partial_\eta^{\alpha_1} b(x+y,\xi+\eta)\right\} dy d\eta dz d\zeta.$$

More generally, if $h^{(k+1)\delta_1}\partial^\alpha a \in S_{\delta_1,\beta_1}(M_1^{k+1})$, $h^{(k+1)\delta_2}\partial^\alpha b \in S_{\delta_2,\beta_2}(M_2^{k+1})$, *for* $|\alpha| = k+1$, *for order functions* M_1^{k+1}, M_2^{k+1}, *then*
$$r_k \in h^{(k+1)(1-(\delta_1+\delta_2))} S_{\delta,\beta}(M_1^{k+1} M_2^{k+1}).$$

Remark. — Observe that

(3.15)
$$a \sharp b = ab + \frac{h}{2i}\{a,b\} + \left(\frac{h}{2i}\right)^2 \left[\frac{1}{2}\partial_x^2 a \partial_\xi^2 b + \frac{1}{2}\partial_\xi^2 a \partial_x^2 b - \partial_x\partial_\xi a \partial_x \partial_\xi b\right] + r_2^{a\sharp b},$$

so the contribution of order two (and all other contributions of even order) disappears when we calculate the symbol associated to a commutator

$$a \sharp b - b \sharp a = \frac{h}{i}\{a,b\} + r_2, \tag{3.16}$$

where $r_2 = r_2^{a\sharp b} - r_2^{b\sharp a} \in h^{3(1-(\delta_1+\delta_2))} S_{\delta,\beta}(M_1 M_2)$.

The result of Proposition 3.8 is still true also when one of order functions, or both, has the form $\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-1}$, for a smooth function $f(\xi)$, $f'(\xi)$ bounded, as stated below and proved as well in the appendix.

Lemma 3.9. — *Let* $f(\xi)$ *be a smooth function,* $f'(\xi)$ *bounded. Consider* $a \in S_{\delta_1,\beta_1}(\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d})$, $d \in \mathbb{N}$, *and* $b \in S_{\delta_2,\beta_2}(M)$, *for* M *order function or* $M(x,\xi) = \langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-l}$, $l \in \mathbb{N}$, *some* $\delta_1 \in [0,\frac{1}{2}]$, $\delta_2 \in [0,\frac{1}{2}[$, $\beta_1,\beta_2 \geq 0$ *as in* (3.12). *Then* $a \sharp b \in S_{\delta,\beta}(\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d} M)$, *where* $\delta = \max\{\delta_1,\delta_2\}$, $\beta = \max\{\beta_1,\beta_2\}$, *and the asymptotic expansion* (3.13) *holds, with* r_k *given by* (3.14), $r_k \in h^{(k+1)(1-(\delta_1+\delta_2))} S_{\delta,\beta}(\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d} M)$.
More generally, if $h^{(k+1)\delta_1}\partial^\alpha a \in S_{\delta_1,\beta_1}(\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d'})$ *and* $h^{(k+1)\delta_2}\partial^\alpha b \in S_{\delta_2,\beta_2}(M^{k+1})$, $|\alpha| = k+1$, M^{k+1} *order function or* $M^{k+1}(x,\xi) = \langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-l'}$, *for others* $d', l' \in \mathbb{N}$, *then* $r_k \in h^{(k+1)(1-(\delta_1+\delta_2))} S_{\delta,\beta}(\langle \frac{x+f(\xi)}{\sqrt{h}}\rangle^{-d'} M^{k+1})$.

3.2. Some Technical Estimates. — This subsection is mostly devoted to the introduction of some technical results about symbols and operators we will often use in the entire paper, first of all continuity on Sobolev spaces. We also introduce multi-linear quantizations which will be used in the next section (and which are fully described in [7]), especially because they make our notations easier and clearer at first. Moreover, from now on we follow the notation $p(\xi) := \sqrt{1+\xi^2}$.

The first statement is about continuity on spaces $H^s_h(\mathbb{R})$, and generalizes Theorem 7.11 in [9]. The second statement concerns instead a result of continuity from L^2 to $W^{\rho,\infty}_h$. In the spirit of [17] for the Schrödinger equation, it allows to pass from uniform norms to the L^2 norm losing only a power $h^{-\frac{1}{4}-\sigma}$ for a small $\sigma > 0$, and not a $h^{-\frac{1}{2}}$ as for the Sobolev injection.

Proposition 3.10 (Continuity on H^s_h). — *Let $s \in \mathbb{R}$. Let $a \in S_{\delta,\beta}(\langle\xi\rangle^r)$, $r \in \mathbb{R}$, $\delta \in [0, \frac{1}{2}]$, $\beta \geq 0$. Then $\mathrm{Op}^w_h(a)$ is uniformly bounded: $H^s_h(\mathbb{R}) \to H^{s-r}_h(\mathbb{R})$, and there exists a positive constant C independent of h such that*

$$\|\mathrm{Op}^w_h(a)\|_{\mathcal{L}(H^s_h;H^{s-r}_h)} \leq C, \qquad \forall h \in]0,1]. \tag{3.17}$$

Proposition 3.11 (Continuity from L^2 to $W^{\rho,\infty}_h$). — *Let $\rho \in \mathbb{N}$. Let $a \in S_{\delta,\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $\delta \in [0, \frac{1}{2}]$, $\beta > 0$. Then $\mathrm{Op}^w_h(a)$ is bounded: $L^2(\mathbb{R}) \to W^{\rho,\infty}_h(\mathbb{R})$, and there exists a positive constant C independent of h such that*

$$\|\mathrm{Op}^w_h(a)\|_{\mathcal{L}(L^2;W^{\rho,\infty}_h)} \leq Ch^{-\frac{1}{4}-\sigma}, \qquad \forall h \in]0,1], \tag{3.18}$$

where $\sigma > 0$ depends linearly on β.

Proof. — Firstly, remark that thanks to symbolic calculus of Lemma 3.9, to estimate the $W^{k,\infty}_h$ norm of an operator whose symbol is rapidly decaying in $|h^\beta\xi|$ corresponds actually to estimate the L^∞ norm of an operator associated to another symbol (namely, $\tilde{a}(x,\xi) = \langle\xi\rangle^k a(x,\xi)$) which is still in the same class as a, up to a small loss on h, of order $h^{-k\beta}$.

From the definition of $\mathrm{Op}^w_h(a)v$, and using thereafter integration by part, Cauchy-Schwarz inequality, and Young's inequality for convolutions, we derive what follows:

(3.19)

$$\begin{aligned}
|\mathrm{Op}^w_h(a)v| &= \left|\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\frac{x}{\sqrt{h}}-y)\xi} a(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi) v(\sqrt{h}y)\, dyd\xi\right| \\
&= \left|\frac{1}{(2\pi)^2\sqrt{h}}\int_{\mathbb{R}} \hat{v}(\frac{\eta}{\sqrt{h}})d\eta \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(\frac{x}{\sqrt{h}}-y)\xi+i\eta y} a(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi)\, dyd\xi\right| \\
&= \left|\frac{1}{(2\pi)^2\sqrt{h}}\int_{\mathbb{R}} \hat{v}(\frac{\eta}{\sqrt{h}}) \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{1-i(\frac{x}{\sqrt{h}}-y)\partial_\xi}{1+(\frac{x}{\sqrt{h}}-y)^2}\right)^2 \left(\frac{1+i(\xi-\eta)\partial_y}{1+(\xi-\eta)^2}\right)^2\right. \\
&\qquad \left. \times \left[e^{i(\frac{x}{\sqrt{h}}-y)\xi+i\eta y}\right] a(\frac{x+\sqrt{h}y}{2}, \sqrt{h}\xi)\, dyd\xi d\eta\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{\sqrt{h}}\int_{\mathbb{R}}\left|\hat{v}(\frac{\eta}{\sqrt{h}})\right|\int_{\mathbb{R}}\int_{\mathbb{R}}\langle\frac{x}{\sqrt{h}}-y\rangle^{-2}\langle\xi-\eta\rangle^{-2}\langle h^{\beta}\sqrt{h}\xi\rangle^{-N}\\
&\qquad\times\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}dyd\xi d\eta\\
&\leq \frac{C}{\sqrt{h}}\left\|\hat{v}(\frac{\eta}{\sqrt{h}})\right\|_{L^2_\eta}\|\langle\eta\rangle^{-2}\|_{L^1_\eta}\\
&\qquad\times\left\|\int_{\mathbb{R}}\langle\frac{x}{\sqrt{h}}-y\rangle^{-2}\langle h^{\beta}\sqrt{h}\xi\rangle^{-N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}dy\right\|_{L^2_\xi}\\
&\leq Ch^{-\frac{1}{4}}\|v\|_{L^2}\int_{\mathbb{R}}\langle\frac{x}{\sqrt{h}}-y\rangle^{-2}\left\|\langle h^{\beta}\sqrt{h}\xi\rangle^{-N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}\right\|_{L^2_\xi}dy,
\end{aligned}
$$

where $N>0$ is properly chosen later. We draw attention to two facts, when we integrated by parts: in the third equality in (3.19), we use that

$$
\left(\frac{1-i(\frac{x}{\sqrt{h}}-y)\partial_\xi}{1+(\frac{x}{\sqrt{h}}-y)^2}\right)^2\left(\frac{1+i(\xi-\eta)\partial_y}{1+(\xi-\eta)^2}\right)^2\left[e^{i(\frac{x}{\sqrt{h}}-y)\xi+i\eta y}\right]=e^{i(\frac{x}{\sqrt{h}}-y)\xi+i\eta y}
$$

so, integrating by part, derivatives fall on $\langle\frac{x}{\sqrt{h}}-y\rangle^{-1}$, $\langle\xi-\eta\rangle^{-1}$, giving rise to more decreasing factors, and/or on $a\left(\frac{x+\sqrt{h}y}{2},\sqrt{h}\xi\right)$; the symbol a belongs to $S_{\delta,\beta}(1)$ with a $\delta\leq\frac{1}{2}$, but the loss of $h^{-\delta}$ is offset by the factor $\sqrt{h}$ coming from the derivation of $a(\frac{x+\sqrt{h}y}{2},\sqrt{h}\xi)$ with respect to y and ξ.
To estimate $\|\langle h^\beta\sqrt{h}\xi\rangle^{-N}\big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\big\rangle^{-1}\|_{L^2_\xi}$ we consider a Littlewood-Paley decomposition, i.e.,

$$
1=\sum_{k=0}^{+\infty}\varphi_k(\xi), \tag{3.20}
$$

where $\varphi_k(\xi)\in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\varphi_0\subset B(0,1)$, $\varphi_k(\xi)=\varphi(2^{-k}\xi)$ and $\operatorname{supp}\varphi\subset\{A^{-1}\leq|\xi|\leq A\}$, for a constant $A>0$. Then,

(3.21)

$$
\begin{aligned}
&\left\|\langle h^{\beta}\sqrt{h}\xi\rangle^{-N}\big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\big\rangle^{-1}\right\|^2_{L^2_\xi}\\
&\qquad\qquad=\frac{1}{\sqrt{h}}\sum_{k\geq0}\int_{\mathbb{R}}\langle h^{\beta}\xi\rangle^{-2N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi_k(\xi)d\xi\\
&\qquad\qquad=\frac{1}{\sqrt{h}}\sum_{k\geq0}I_k,
\end{aligned}
$$

where

$$I_0=\int_{\mathbb{R}}\langle h^\beta\xi\rangle^{-2N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi_0(\xi)d\xi, \tag{3.22}$$

and

$$\begin{aligned} I_k&=\int_{\mathbb{R}}\langle h^\beta\xi\rangle^{-2N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi(2^{-k}\xi)d\xi\\ &=2^k\int_{\mathbb{R}}\langle h^\beta 2^k\xi\rangle^{-2N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(2^k\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi(\xi)d\xi, \qquad k\geq 1\\ &\leq A^{2N}2^{(-2N+1)k}h^{-2\beta N}\int_{\mathbb{R}}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(2^k\xi)}{\sqrt{h}}\Big\rangle^{-2}\varphi(\xi)d\xi. \end{aligned} \tag{3.23}$$

For $k\leq k_0$, for a fixed k_0, $p''(2^k\xi)\neq 0$ on the support of φ. As $\xi\to\pm\infty$ we have the expansion

$$p'(\xi)=\frac{\xi}{\sqrt{1+\xi^2}}=\pm1\mp\frac{1}{2\xi^2}+O(|\xi|^{-4}), \tag{3.24}$$

and then

$$p'(2^k\xi)=\pm1\mp\frac{2^{-2k}}{2\xi^2}+O(|2^k\xi|^{-4}). \tag{3.25}$$

For $k\geq k_0$, the function $\xi\to g_k(\xi)=2^{2k}(\frac{x+\sqrt{h}y}{2})+2^{2k}p'(2^k\xi)$ is such that $|g_k'(\xi)|=|\xi|^{-3}|\widetilde{g}_k(\xi)|$, $\widetilde{g}_k(\xi)=1+O(2^{-2k}|\xi|^{-2})$, and $|g_k'(\xi)|\sim 1$ on the support of φ, so for every k we can perform a change of variables $z=g_k(\xi)$ in the last line of (3.23). Hence,

$$\begin{aligned} I_k&\leq A^{2N}2^{(-2N+1)k}h^{-2\beta N}\int\langle\frac{z}{2^{2k}\sqrt{h}}\rangle^{-2}\varphi(g_k^{-1}(z))dz\\ &\leq A^{2N}2^{(-2N+3)k}h^{-2\beta N}\sqrt{h}\int\langle z\rangle^{-2}dz\\ &\leq C2^{(-2N+3)k}h^{-2\beta N}\sqrt{h}, \end{aligned} \tag{3.26}$$

so taking the summation of all I_k for $k\geq 0$ we deduce

$$\Big\|\langle h^\beta\sqrt{h}\xi\rangle^{-N}\Big\langle\frac{\frac{x+\sqrt{h}y}{2}+p'(\sqrt{h}\xi)}{\sqrt{h}}\Big\rangle^{-1}\Big\|_{L^2_\xi}\leq Ch^{-\beta N}\sum_{k\geq 0}2^{(\frac{-2N+3}{2})k}\leq C'h^{-\beta N}, \tag{3.27}$$

if we choose $N>0$ such that $\frac{-2N+3}{2}<0$ (e.g., $N=2$). Finally

$$\|\mathrm{Op}_h^w(a)\|_{\mathcal{L}(L^2;W_h^{\rho,\infty})}=O(h^{-\frac14-\sigma}), \tag{3.28}$$

where $\sigma(\beta)=(N+\rho)\beta$ depends linearly on β. □

The following lemma shows that we have nice upper bounds for operators acting on v whose symbols are supported for $|\xi| \geq h^{-\beta}$, $\beta > 0$, provided that we have an a priori H^s_h estimate on v, with large enough s.

LEMMA 3.12. — *Let $s' \geq 0$. Let $\chi \in C^\infty_0(\mathbb{R})$, $\chi \equiv 1$ in a neighborhood of zero, e.g.,*

$$\begin{aligned} &\chi(\xi) = 1, \qquad \text{for } |\xi| < C_1 \\ &\chi(\xi) = 0, \qquad \text{for } |\xi| > C_2. \end{aligned} \tag{3.29}$$

Then

$$\|\mathrm{Op}_h((1-\chi)(h^\beta\xi))v\|_{H^{s'}_h} \leq C h^{\beta(s-s')}\|v\|_{H^s_h}, \qquad \forall s > s'. \tag{3.30}$$

Proof. — The result is a simple consequence of the fact that $(1-\chi)(h^\beta\xi)$ is supported for $|\xi| \geq C_1 h^{-\beta}$, because

(3.31)
$$\begin{aligned} \|\mathrm{Op}_h((1-\chi)(h^\beta\xi))v\|^2_{H^{s'}_h} &= \int (1+|h\xi|^2)^{s'}|(1-\chi)(h^\beta h\xi)|^2|\hat{v}(\xi)|^2 d\xi \\ &= \int (1+|h\xi|^2)^s(1+|h\xi|^2)^{s'-s}|(1-\chi)(h^\beta h\xi)|^2|\hat{v}(\xi)|^2 d\xi \\ &\leq C h^{2\beta(s-s')}\|v\|^2_{H^s_h}, \end{aligned}$$

where the last inequality follows from an integration on $|h\xi| > C_1 h^{-\beta}$, and from the two following conditions $s'-s<0$, $(1+|h\xi|^2)^{s'-s} \leq C h^{-2\beta(s'-s)}$. □

This result is useful when we want to reduce essentially to symbols rapidly decaying in $|h^\beta\xi|$, for example in the intention of using Proposition 3.11 or when we want to pass from a symbol of a certain positive order to another one of order zero, up to small losses of order $O(h^{-\sigma})$, $\sigma > 0$ depending linearly on β. We can always split a symbol using that $1 = \chi(h^\beta\xi) + (1-\chi)(h^\beta\xi)$, and consider as remainders all contributions coming from the latter.

Define the set $\Lambda := \{(x,\xi) \in \mathbb{R}\times\mathbb{R} \,|\, x + p'(\xi) = 0\}$, i.e., the graph of the function $x \in \,]-1,1[\, \to d\varphi(x)$, $\varphi(x) = \sqrt{1-x^2}$, as drawn in picture 0.1. We will use the following technical lemma, whose proof can be found in Lemma 1.2.6 in [7]:

LEMMA 3.13. — *Let $\gamma \in C^\infty_0(\mathbb{R})$. If the support of γ is sufficiently small, the two functions defined below*

$$\begin{aligned} e_\pm(x,\xi) &= \frac{x+p'(\pm\xi)}{\xi \mp d\varphi(x)}\gamma\big(\langle\xi\rangle^2(x+p'(\pm\xi))\big) \\ \text{and} \quad \tilde{e}_\pm(x,\xi) &= \frac{\xi \mp d\varphi(x)}{x+p'(\pm\xi)}\gamma\big(\langle\xi\rangle^2(x+p'(\pm\xi))\big) \end{aligned} \tag{3.32}$$

verify estimates

$$
\begin{aligned}
&|\partial_x^\alpha \partial_\xi^\beta e_\pm(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{-3+2\alpha-\beta},\\
&|\partial_x^\alpha \partial_\xi^\beta \tilde{e}_\pm(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{3+2\alpha-\beta}.
\end{aligned}
\tag{3.33}
$$

Moreover, if $\operatorname{supp}\gamma$ *is small enough, then on the support of* $\gamma(\langle\xi\rangle^2(x+p'(\pm\xi)))$ *one has* $\langle d\varphi\rangle \sim \langle\xi\rangle$ *and there is a constant* $A>0$ *such that, on that support*

$$
\begin{aligned}
&A^{-1}\langle\xi\rangle^{-2} \leq \pm x+1 \leq A\langle\xi\rangle^{-2}, \qquad \xi\to+\infty\\
&A^{-1}\langle\xi\rangle^{-2} \leq \mp x+1 \leq A\langle\xi\rangle^{-2}, \qquad \xi\to-\infty
\end{aligned}
\tag{3.34}
$$

Finally, as $x\to\pm1$, *for every* $k\in\mathbb{N}$

$$
\partial^k(d\varphi(x)) = O(\langle d\varphi\rangle^{1+2k}).
\tag{3.35}
$$

Lemma 3.14. — *Let* $\gamma\in C_0^\infty(\mathbb{R})$ *such that* $\gamma\equiv1$ *in a neighborhood of zero, and define* $\Gamma(x,\xi)=\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$. *Then* $\Gamma\in S_{\frac12,0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N})$, *for all* $N\geq0$.

Proof. — Let $N\in\mathbb{N}$. Since $\gamma\in C_0^\infty(\mathbb{R})$, $p''\in S_{0,0}(1)$, we have

(3.36)

$$
\begin{aligned}
&|\Gamma(x,\xi)| \leq \|\langle x\rangle^N\gamma(x)\|_{L^\infty}\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N},\\
&|\partial_x\Gamma(x,\xi)| = \big|\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})\frac{1}{\sqrt{h}}\big| \leq h^{-\frac12}\|\langle x\rangle^N\gamma'(x)\|_{L^\infty}\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N},\\
&|\partial_\xi\Gamma(x,\xi)| = \big|\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})\frac{p''(\xi)}{\sqrt{h}}\big| \leq h^{-\frac12}\|p''(\xi)\|_{L^\infty}\|\langle x\rangle^N\gamma'(x)\|_{L^\infty}\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N},
\end{aligned}
$$

and going on one can prove that $|\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}\Gamma| \leq C_{\alpha_1,\alpha_2,N}h^{-\frac12(\alpha_1+\alpha_2)}\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N}$. ☐

Multi-linear Operators. — We briefly generalize some definitions given at the beginning of this section in order to introduce multi-linear operators. As we will consider multi-linear operators with symbols depending only on ξ and, for such symbols, in the linear case, Weyl quantization coincide with classical quantization, for simplicity we will directly talk about the Kohn-Nirenberg quantization.

Let $n\in\mathbb{N}^*$ and set $\xi=(\xi_1,\dots,\xi_n)$. An order function on $\mathbb{R}\times\mathbb{R}^n$ will be a smooth function $(x,\xi)\to M(x,\xi)$ satisfying (3.1), where $\langle\xi-\eta\rangle^{N_0}$ is replaced by

$$
\prod_{i=1}^n\langle\xi_i-\eta_i\rangle^{N_0}.
$$

Equivalently, we define the class $S_{\delta,\beta}(M,n)$, for some $\delta \geq 0$, $\beta \geq 0$ and $M(x,\xi)$ order function on $\mathbb{R}\times\mathbb{R}^n$, to be the set of smooth functions

$$\begin{gathered}(x,\xi_1,\dots,\xi_n,h)\to a(x,\xi,h)\\ \mathbb{R}\times\mathbb{R}^n\times]0,1]\to\mathbb{C}\end{gathered}$$

satisfying the inequality (3.2), $\forall\alpha_1\in\mathbb{N},\alpha_2\in\mathbb{N}^n$, $\forall k,N\in\mathbb{N}$.

Definition 3.15. — Let a be a symbol in $S_{\delta,\beta}(M,n)$ for some order function M, some $\delta\geq 0$, $\beta\geq 0$.

(i) We define the n-linear operator $\mathrm{Op}(a)$ acting on test functions $v_1,\dots,v_n$ by

(3.37)
$$\mathrm{Op}(a)(v_1,\dots,v_n)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix(\xi_1+\dots+\xi_n)}a(x,\xi_1,\dots,\xi_n)\prod_{l=1}^n\hat{v}_l(\xi_l)\,d\xi_1\dots d\xi_n.$$

(ii) We also define the n-linear semiclassical operator $\mathrm{Op}_h(a)$ acting on test functions $v_1,\dots,v_n$ by

(3.38)
$$\mathrm{Op}_h(a)(v_1,\dots,v_n)=\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n}e^{\frac{i}{h}x(\xi_1+\dots+\xi_n)}a(x,\xi_1,\dots,\xi_n)\prod_{l=1}^n\hat{v}_l(\xi_l)\,d\xi_1\dots d\xi_n.$$

For a further need of compactness in our notations, we introduce $I=(i_1,\dots,i_n)$ a n-dimensional vector, $i_k\in\{1,-1\}$ for every $k=1,\dots,n$. We set $|I|=n$ and define

$$w_I=(w_{i_1},\dots,w_{i_n}),\qquad w_1=w,\ w_{-1}=\bar{w}, \tag{3.39}$$

while $m_I(\xi)\in S_{\delta,\beta}(M,n)$ will be always in what follows a symbol of the form

$$m_I(\xi)=m_1^I(\xi_1)\cdots m_n^I(\xi_n). \tag{3.40}$$

Note that, when all variables ξ_j in $m_I(\xi)$ are decoupled, as in (3.40), $\mathrm{Op}(m_I)(w_I)$ is only a compact way of writing $\prod_j\mathrm{Op}(m_j^I)w_{i_j}$. We also warn the reader that in following sections, when we focus on a fixed general symbol $m_I(\xi)$, we will often refer to components $m_k^I(\xi_k)$ as $m_k(\xi_k)$, forgetting the superscript I in order to make notations lighter. Sometimes we will also write $m_k(\xi)$ if this makes no confusion.

4. Semiclassical reduction to an ODE.

In this section we want to reformulate the Cauchy problem (1.1) and to deduce a new equation which can be transformed into an ODE. It is organized in three subsections. In the first one, we introduce semiclassical coordinates, rewrite the problem in this new framework and state the main theorem. The second and third sections are devoted to the proof of the main theorem. In particular, in the second one we introduce some technical lemmas we often refer to and we estimate v when it is away from Λ. In the third one, we first cut symbols in the cubic nonlinearity near Λ and away from points $x = \pm 1$, and develop them at $\xi = d\varphi(x)$, transforming multi-linear pseudo-differential operators in smooth functions of x; then, we repeat the development argument for $\mathrm{Op}_h^w(x\xi + p(\xi))$.

4.1. Semiclassical Coordinates and Statement of the Main Result. — Let u be a solution of (1.1) and set

$$\begin{cases} w = (D_t + \sqrt{1+D_x^2})u \\ \bar{w} = -(D_t - \sqrt{1+D_x^2})u \end{cases}, \qquad \begin{cases} u = \langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2}) \\ D_t u = \frac{w-\bar{w}}{2} \end{cases}. \tag{4.1}$$

With notations introduced in (1.3), the function w satisfies the following equation

(4.2)
$$\begin{aligned}(D_t - \sqrt{1+D_x^2})w = \sum_{k=0}^{3} i^k P'_k &\left(\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2}); \frac{w-\bar{w}}{2}, D_x\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2})\right) \\ + \sum_{k=0}^{2} i^k P''_k &\Big(\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2}), D_x(\frac{w-\bar{w}}{2}), D_x^2\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2}); \\ &\frac{w-\bar{w}}{2}, D_x\langle D_x\rangle^{-1}(\frac{w+\bar{w}}{2})\Big).\end{aligned}$$

Observe that operators which take the place of second derivatives have symbols of order one, while all other symbols are of order zero or negative (-1). Let us simplify the notation for the rest of the section by rewriting the nonlinearity in term of multi-linear pseudo-differential operators introduced in the previous section, namely as

$$\sum_{|I|=3} \mathrm{Op}(m_I)(w_I) + \sum_{|I|=3} \mathrm{Op}(\widetilde{m}_I)(w_I), \tag{4.3}$$

where symbols m_I, $\widetilde{m}_I$ are of the form (3.40). Moreover, m_I will denote symbols of order equal or less than zero, in the sense that all occurring symbols m^I_k are of order equal or less than zero, while in $\widetilde{m}_I$ there will be exactly one symbol of order one, thanks to the quasi-linear nature of the starting equation. Therefore (4.2) is rewritten as

$$(D_t - \sqrt{1+D_x^2})w = \sum_{|I|=3} \mathrm{Op}(m_I)(w_I) + \sum_{|I|=3} \mathrm{Op}(\widetilde{m}_I)(w_I), \tag{4.4}$$

and passing to the semiclassical framework by

$$w(t,x) = \frac{1}{\sqrt{t}} v(t, \frac{x}{t}), \qquad h := \frac{1}{t}, \tag{4.5}$$

we obtain

$$(D_t - \mathrm{Op}_h^w(x\xi + p(\xi))v = h\sum_{|I|=3}\mathrm{Op}_h(m_I)(v_I) + h\sum_{|I|=3}\mathrm{Op}_h(\widetilde{m}_I)(v_I), \tag{4.6}$$

where $p(\xi)=\sqrt{1+\xi^2}$ and where we used the equality $\mathrm{Op}_h(x\xi+p(\xi)+\frac{h}{2i}) = \mathrm{Op}_h^w(x\xi+p(\xi))$ following from

$$\begin{aligned}\mathrm{Op}_h^w(x\xi) &= \frac{h}{2}D_x x + \frac{h}{2}xD_x\\ &= \frac{h}{2i} + x\,hD_x = \frac{h}{2i} + \mathrm{Op}_h(x\xi).\end{aligned}$$

Furthermore, we write explicitly the nonlinearity of the equation, which will be useful hereinafter

$$\begin{aligned}&\big(D_t - \mathrm{Op}_h^w(x\xi+p(\xi)\big)v\\ &\quad= h\sum_{k=0}^{3} i^k P'_k\left(\langle hD\rangle^{-1}(\frac{v+\bar v}{2}); \frac{v-\bar v}{2}, (hD)\langle hD\rangle^{-1}(\frac{v+\bar v}{2})\right)\\ &\qquad+ h\sum_{k=0}^{2} i^k P''_k\left(\langle hD\rangle^{-1}(\frac{v+\bar v}{2}), (hD)(\frac{v-\bar v}{2}), (hD)^2\langle hD\rangle^{-1}(\frac{v+\bar v}{2});\right.\\ &\qquad\qquad\left.\frac{v-\bar v}{2}, (hD)\langle hD\rangle^{-1}(\frac{v+\bar v}{2})\right).\end{aligned}$$

Let us also define the operator $\mathcal{L}$ to be

$$\mathcal{L} := \frac{1}{h}\mathrm{Op}_h^w(x+p'(\xi)). \tag{4.7}$$

The Equation (4.6) represents for us the starting point to deduce an ODE satisfied by v, from which it will be easier to derive an estimate on the L^∞ norm of v. In reality, we will need more than an uniform estimate for v, namely we have to involve also a certain number of its derivatives, and then to control its $W_h^{\rho,\infty}$ norm for a fixed $\rho>0$. With this in mind, we set $\Gamma(x,\xi)=\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, for a function $\gamma\in C_0^\infty(\mathbb{R})$, $\gamma\equiv 1$ in a neighborhood of zero, with a small support. From Lemma 3.14, $\Gamma\in S_{\frac12,0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-N})$ for every $N\in\mathbb{N}^*$, and case by case we will choose the right power we need. We consider also $\Sigma(\xi)=\langle\xi\rangle^\rho$ (in practice, at times we consider $\rho-1\in\mathbb{N}$, with ρ introduced for u in Theorem 1.3, when we prove the bootstrap, or $\rho=-1$ when we develop asymptotics), and define

$$v^\Sigma := \mathrm{Op}_h(\Sigma)v, \tag{4.8}$$

together with

$$\begin{aligned}v^\Sigma_\Lambda &:= \mathrm{Op}_h^w(\Gamma)v^\Sigma,\\ v^\Sigma_{\Lambda^c} &:= \mathrm{Op}_h^w(1-\Gamma)v^\Sigma,\end{aligned} \tag{4.9}$$

and symbols

$$
\begin{aligned}
m_I^\Sigma(\xi) &= \prod_{k=1}^3 m_k^{I,\Sigma}(\xi_k) := \prod_{k=1}^3 m_k^I(\xi_k)\Sigma(\xi_k)^{-1},\\
\widetilde{m}_I^\Sigma(\xi) &= \prod_{k=1}^3 \widetilde{m}_k^{I,\Sigma}(\xi_k) := \prod_{k=1}^3 \widetilde{m}_k^I(\xi_k)\Sigma(\xi_k)^{-1}.
\end{aligned}
\tag{4.10}
$$

The main result we want to prove in this section is the following:

THEOREM 4.1 (Reformulation of the PDE). — *Suppose that we are given constants $A', B' > 0$, some $T > 1$ and a solution*

$$v \in L^\infty([1,T]; H^s_h) \cap L^\infty([1,T]; W^{\rho,\infty}_h)$$

of the Equation (4.6) *(or, equivalently, of* (4.7)*), satisfying the following a priori bounds, for any $\varepsilon \in]0,1]$, $t \in [1,T]$,*

$$\|v(t,\cdot)\|_{W^{\rho,\infty}_h} \leq A'\varepsilon, \tag{4.11}$$

$$\|\mathcal{L}v(t,\cdot)\|_{L^2} + \|v(t,\cdot)\|_{H^s_h} \leq B'h^{-\sigma}\varepsilon, \tag{4.12}$$

for some $\sigma > 0$ small enough. Then, with preceding notations, v^Σ_Λ is solution of

$$
\begin{aligned}
D_t v^\Sigma_\Lambda = {}& \varphi(x)\theta_h(x)v^\Sigma_\Lambda + h\Phi^\Sigma_1(x)\theta_h(x)|v^\Sigma_\Lambda|^2 v^\Sigma_\Lambda\\
&+ h\mathrm{Op}^w_h(\Gamma)\big[\Phi^\Sigma_3(x)\theta_h(x)(v^\Sigma_\Lambda)^3 + \Phi^\Sigma_{-1}(x)\theta_h(x)|v^\Sigma_\Lambda|^2\overline{v^\Sigma_\Lambda}\\
&+ \Phi^\Sigma_{-3}(x)\theta_h(x)(\overline{v^\Sigma_\Lambda})^3\big] + hR(v),
\end{aligned}
\tag{4.13}
$$

with $(\theta_h(x))_h$ a family of smooth functions compactly supported in $]-1,1[$, some smooth coefficients $\Phi^\Sigma_j(x)$, $|\Phi^\Sigma_j(x)| = O(h^{-\sigma'})$ on the support of θ_h, for $j \in \{3,1,-1,-3\}$ and a small $\sigma' > 0$. Moreover, $R(v)$ is a remainder verifying the following estimates

$$\|R(v)\|_{L^2} \leq Ch^{\frac12-\sigma}\,(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h}), \tag{4.14}$$

$$\|R(v)\|_{L^\infty} \leq Ch^{\frac14-\sigma}\,(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h}), \tag{4.15}$$

for a new small $\sigma \geq 0$.

Smooth coefficients $\Phi^\Sigma_j(x)$ in (4.13) may be explicitly calculated starting from the nonlinearity in (4.7), and in particular this will be done for $\Phi^\Sigma_1(x)$ at the beginning of Section 5. Afterwards, we will use the notation $R_1(v)$ to refer to a remainder satisfying the following estimates:

$$\|R_1(v)\|_{H^\rho_h} \leq Ch^{\frac12-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h}), \tag{4.16}$$

$$\|R_1(v)\|_{L^\infty} \leq Ch^{\frac14-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H^s_h}), \tag{4.17}$$

for a small $\sigma \geq 0$.

4.2. Technical Results. — We estimate $v^{\Sigma}_{\Lambda^c}$ as follows:

Lemma 4.2. — *Let* $\widetilde{\Gamma}(\xi)$ *a smooth function such that* $|\partial^\alpha\widetilde{\Gamma}| \lesssim \langle\xi\rangle^{-\alpha}$, χ *as in Lemma 3.12,* $\beta > 0$. *Then*

$$\operatorname{Op}^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))v^{\Sigma} = \operatorname{Op}^w_h\left(\Sigma(\xi)\chi(h^\beta\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\right)v + R_1(v), \tag{4.18}$$

where $R_1(v)$ *is a remainder satisfying* (4.16), (4.17).

Proof. — We consider a function χ as in Lemma 3.12, and we write

$$\begin{aligned}\operatorname{Op}^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))v^{\Sigma} &= \operatorname{Op}^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))\operatorname{Op}^w_h(\Sigma(\xi)\chi(h^\beta\xi))v \\ &\quad + \operatorname{Op}^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))\operatorname{Op}^w_h(\Sigma(\xi)(1-\chi)(h^\beta\xi))v,\end{aligned} \tag{4.19}$$

for $\beta > 0$. The second term in the right hand side represents a remainder $R_1(v)$ satisfying the two inequalities of the statement just because $\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}) \in S_{\frac{1}{2},0}(1)$ (so, for instance, $\|\operatorname{Op}^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))\|_{\mathcal{L}(H^{\rho+1}_h;W^{\rho,\infty}_h)} = O(h^{-\frac{1}{2}})$ by Sobolev inequality (3.8) and Proposition 3.10) and $(1-\chi)(h^\beta\xi)$ is supported for $|\xi| \geq h^{-\beta}$, so that we can use essentially Lemma 3.12.

On the other hand, since $|\partial^\alpha\widetilde{\Gamma}| \leq \langle\xi\rangle^{-\alpha}$ and $\Sigma(\xi)\chi(h^\beta\xi) \in h^{-\sigma}S_{0,\beta}(1)$, with

$$\sigma = \begin{cases}\rho\beta & \textit{if } \rho\in\mathbb{N}\\ 0 & \textit{if } \rho<0\end{cases} \tag{4.20}$$

we use the composition formula of Lemma 3.9 for the first term in the right hand side, i.e.,

$$\begin{aligned}\operatorname{Op}^w_h(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}))&\operatorname{Op}^w_h(\Sigma(\xi)\chi(h^\beta\xi))v \\ &= \operatorname{Op}^w_h\left(\Sigma(\xi)\chi(h^\beta\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\right)v + \operatorname{Op}_h(r_0)v,\end{aligned} \tag{4.21}$$

where $r_0 \in h^{\frac{1}{2}-\sigma}S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$. So $\operatorname{Op}_h(r_0)v$ satisfies inequalities (4.16), (4.17) respectively by Propositions 3.10 and 3.11. ☐

Lemma 4.3. — *Let* $\widetilde{\Gamma}(\xi)$ *be a smooth function such that* $|\partial^\alpha\widetilde{\Gamma}| \lesssim \langle\xi\rangle^{-\alpha}$, $c(x,\xi) \in S_{\delta,\beta}(1)$, $c'(x,\xi) \in S_{\delta',0}(1)$, *with* $\delta,\delta' \in [0,\frac{1}{2}[$, $\beta > 0$. *Then*

$$c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}) \sharp (x+p'(\xi)) = c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi)) + h^{1-2\delta}r \tag{4.22}$$

with $r \in S_{\frac{1}{2},\beta}(1)$, and

(4.23)
$$\left\|\mathrm{Op}_h^w\big(c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)\mathrm{Op}_h^w(c')v\right\|_{L^2} \leq h^{1-\sigma}(\|\mathcal{L}v\|_{L^2}+\|v\|_{H^s_h}),$$

(4.24)
$$\left\|\mathrm{Op}_h^w\big(c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)\mathrm{Op}_h^w(c')v\right\|_{L^\infty} \leq h^{\frac{1}{2}-\sigma}(\|\mathcal{L}v\|_{L^2}+\|v\|_{H^s_h}),$$

with $\sigma=\sigma(\delta,\delta',\beta)\to 0$ as $\delta,\delta',\beta\to 0$.

Moreover, if $\widetilde{\Gamma} = \Gamma_{-1}$, with $|\partial^\alpha\Gamma_{-1}| \lesssim \langle\xi\rangle^{-1-\alpha}$, then in (4.22) $r\in S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, and the L^∞ estimate can be improved

(4.25)
$$\left\|\mathrm{Op}_h^w\big(c(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)\mathrm{Op}_h^w(c')v\right\|_{L^\infty} \leq h^{\frac{3}{4}-\sigma}(\|\mathcal{L}v\|_{L^2}+\|v\|_{H^s_h}).$$

Proof. — The result is immediate if we use the development of Proposition 3.8 at order one,

(4.26)
$$\begin{aligned} c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}) \sharp (x+p'(\xi)) &= c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi)) \\ &\quad + \frac{h}{2i}\left\{c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}),(x+p'(\xi))\right\} + hr_1, \end{aligned}$$

where $r_1\in h^{-2\delta}S_{\frac{1}{2},\beta}(1)$, while by direct calculation the Poisson bracket is equal to:
$$\left\{c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}}),(x+p'(\xi))\right\} = \widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(\partial_\xi c - p''\partial_x c),$$
$\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(\partial_\xi c-p''\partial_x c)\in h^{-\delta}S_{\frac{1}{2},\beta}(1)$. Therefore

$$\begin{aligned} &\mathrm{Op}_h^w\big(c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)\mathrm{Op}_h^w(c')v = \\ &= h\mathrm{Op}_h^w\big(c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\big)\mathcal{L}\mathrm{Op}_h^w(c')v \qquad (4.27)\\ &\quad - \frac{h}{2i}\mathrm{Op}_h^w\big(\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})(\partial_\xi c-p''\partial_x c)+2i\,r_1\big)\mathrm{Op}_h^w(c')v, \end{aligned}$$

and the application of Proposition 3.10, along with Sobolev injection (3.8), immediately implies that the second term in the right hand side satisfies estimates (4.23), (4.24). Moreover, $[\mathcal{L},\mathrm{Op}_h^w(c')] = i(\partial_\xi c'-p''\partial_x c')+h^{1-2\delta'}r_1$, r_1 being a symbol in $S_{\delta',0}(1)$, hence it belongs to $h^{-\delta'}S_{\delta',0}(1)$, and its quantization is a bounded operator from L^2 to L^2 by Proposition 3.10 up to a small loss in $h^{-\delta'}$. This remark, together with $c(x,\xi)\widetilde{\Gamma}(\frac{x+p'(\xi)}{\sqrt{h}})\in S_{\frac{1}{2},\beta}(1)$,

$c' \in S_{\delta',0}(1)$, Proposition 3.10, and Sobolev injection imply that also the first term in the right hand side verifies estimates in (4.23), (4.24). The same reasoning as above can be applied when $\widetilde{\Gamma} = \Gamma_{-1}$ with $|\partial^\alpha \Gamma_{-1}| \lesssim \langle \xi \rangle^{-1-\alpha}$. In this case, the development in (4.26) is justified by Lemma 3.9. Moreover, symbols involving $c(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})$ stay in $S_{\frac{1}{2},\beta}(\langle \frac{x+p'(\xi)}{\sqrt{h}} \rangle^{-1})$, so we can apply Proposition 3.11, instead of Sobolev injection, to control the L^∞ norm, losing only a power $h^{-\frac{1}{4}-\sigma}$, for a small $\sigma > 0$ (and not $h^{-\frac{1}{2}}$ due to Sobolev estimate) and so deriving the improved estimate (4.25). □

Proposition 4.4 (Estimates on $v^\Sigma_{\Lambda^c}$). — *There exist $s \in \mathbb{N}$ and a constant $C > 0$ independent of h such that $v^\Sigma_{\Lambda^c}$ can be considered as a remainder $R(v)$ satisfying* (4.14), (4.15).

Proof. — Firstly, we want to reduce to the study of the action of $\mathrm{Op}_h^w(1-\Gamma)$ on v and not on v^Σ, so we can use Lemma 4.2 with $\widetilde{\Gamma} = 1-\gamma$, obtaining

$$\mathrm{Op}_h^w\Big((1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma = \mathrm{Op}_h^w\Big(\Sigma(\xi)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v + R(v), \tag{4.28}$$

where $R(v)$ satisfies (4.14), (4.15). Then it remains to analyze

$$\mathrm{Op}_h^w\left(\Sigma(\xi)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})\right)v.$$

We write the symbol of the operator as $\Sigma(\xi)\chi(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})$, with $\Gamma_{-1}(\xi) = \frac{(1-\gamma)(\xi)}{\xi}$, and we can apply the previous lemma with $c(x,\xi) = \Sigma(\xi)\chi(h^\beta\xi) \in h^{-\sigma}S_{0,\beta}(1)$, σ as in (4.20), $c'(x,\xi) \equiv 1$, to derive that it is a remainder $R(v)$ satisfying (4.14), (4.15). □

We want to apply first $\mathrm{Op}_h^w(\Sigma(\xi))$ to (4.6). As $\mathrm{Op}_h^w(\Sigma(\xi))$ commutes with $D_t - \mathrm{Op}_h^w(x\xi + p(\xi))$ (because $\Sigma(D)$ commutes with $D_t - p(D)$), we obtain the equation:

$$(D_t - \mathrm{Op}_h^w(x\xi+p(\xi)))v^\Sigma = h\mathrm{Op}_h^w(\Sigma)\Big[\sum_{|I|=3}\mathrm{Op}_h(m_I)(v_I) + \sum_{|I|=3}\mathrm{Op}_h(\widetilde{m}_I)(v_I)\Big]. \tag{4.29}$$

Then, we apply also $\mathrm{Op}_h^w(\Gamma)$ to (4.29), so we have to calculate its commutator with the linear part of the equation, as done in the following:

Lemma 4.5. — *We have*

$$\big[D_t - \mathrm{Op}_h^w(x\xi+p(\xi)), \mathrm{Op}_h^w(\Gamma(x,\xi))\big] = \mathrm{Op}_h^w(b), \tag{4.30}$$

where

$$b(x,\xi) = h\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}}) + h^{\frac{3}{2}}r, \tag{4.31}$$

$r \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, *and* Γ_{-1} *satisfies* $|\partial^\alpha\Gamma_{-1}(\xi)| \lesssim \langle\xi\rangle^{-1-\alpha}$.

Proof. — First we start by calculating $[D_t, \mathrm{Op}_h^w(\Gamma)] = D_t\mathrm{Op}_h^w(\Gamma) - \mathrm{Op}_h^w(\Gamma)D_t$:

(4.32)

$$\begin{aligned}
D_t\mathrm{Op}_h^w(\Gamma)v &= \frac{1}{i}\partial_t\left[\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\gamma(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}})v(t,y)\,dyd\xi\right] \\
&= \frac{-h^2}{2\pi i}\partial_h\left[\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\gamma(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}})v(t,y)\,dyd\xi\right] \\
&= -\frac{h}{2\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\,\gamma'(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}})\frac{p''(h\xi)h\xi}{\sqrt{h}}v(t,y)\,dyd\xi \\
&\quad + \frac{h}{4\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\,\gamma'(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}})(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}})v(t,y)\,dyd\xi \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x-y)\xi}\gamma(\frac{\frac{x+y}{2}+p'(h\xi)}{\sqrt{h}})D_tv(t,y)\,dyd\xi \\
&= ih\,\mathrm{Op}_h^w\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{p''(\xi)\xi}{\sqrt{h}})\right)v \\
&\quad - \frac{ih}{2}\,\mathrm{Op}_h^w\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right)v + \mathrm{Op}_h^w(\Gamma)D_tv.
\end{aligned}$$

Then, using (3.14) and (3.16) we write
(4.33)

$$[\mathrm{Op}_h^w(\Gamma(x,\xi)), \mathrm{Op}_h^w(x\xi+p(\xi))] = \frac{h}{i}\mathrm{Op}_h^w\Big(\big\{\gamma(\frac{x+p'(\xi)}{\sqrt{h}}), x\xi+p(\xi)\big\}\Big) + r_2,$$

with $r_2 \in h^{\frac{3}{2}}S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$ from Lemma 3.9, since $\partial^\alpha\Gamma \in h^{-\frac{|\alpha|}{2}}S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $\partial^\alpha(x\xi+p'(\xi)) \in S_{0,0}(1)$ for $|\alpha| = 3$. On the other hand, developing the braces in (4.33) we find

$$\begin{aligned}
\frac{h}{i}\mathrm{Op}_h^w\Big(\big\{\gamma(\frac{x+p'(\xi)}{\sqrt{h}}), x\xi+p(\xi)\big\}\Big) &= -ih\,\mathrm{Op}_h^w\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})\frac{p''(\xi)\xi}{\sqrt{h}}\right) \\
&\quad + ih\,\mathrm{Op}_h^w\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right),
\end{aligned}$$

so when we add it to $[D_t, \mathrm{Op}_h^w(\Gamma)]$ calculated before, we obtain the result just choosing $\Gamma_{-1}(\xi) = \frac{1}{2}\gamma'(\xi)$. $\square$

We apply $\mathrm{Op}_h^w(\Gamma)$ to Equation (4.29). Using Lemma 4.5, we obtain

(4.34)
$$\begin{aligned}&(D_t-\mathrm{Op}_h^w(x\xi+p(\xi))v^\Sigma_\Lambda\\ &\qquad= h\,\mathrm{Op}_h^w(\Gamma)\mathrm{Op}_h^w(\Sigma)\Big[\sum_{|I|=3}\mathrm{Op}_h(m_I)(v_I)+\sum_{|I|=3}\mathrm{Op}_h(\widetilde{m}_I)(v_I)\Big]\\ &\qquad\quad+h\mathrm{Op}_h^w\left(\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right)v^\Sigma+h^{\frac{3}{2}}\mathrm{Op}_h^w(r)v^\Sigma,\end{aligned}$$

$r\in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, where the second and third term in the right hand side are of the form $hR(v)$, $R(v)$ satisfying the estimates (4.14),(4.15). In fact, using Lemma 4.2 with $\widetilde{\Gamma}(\xi)=\Gamma_{-1}(\xi)\xi$, and Lemma 4.3, we have

$$\begin{aligned}&\mathrm{Op}_h^w\left(\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right)v^\Sigma\\ &\quad=\mathrm{Op}_h^w\left(\Sigma(\xi)\chi(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right)v+R(v)\\ &\quad=R(v),\end{aligned}\tag{4.35}$$

while r can be split via a function χ as in Lemma 3.12, with $\beta>0$, obtaining $r(x,\xi)\chi(h^\beta\xi)\in S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$ to which we can apply Proposition 3.11, and $r(x,\xi)(1-\chi)(h^\beta\xi)$ to which can be instead applied Lemma 3.12. Then also $h^{\frac{3}{2}}\mathrm{Op}_h^w(r)v^\Sigma=hR(v)$. Therefore, the equation we want to deal with becomes

$$\begin{aligned}&(D_t-\mathrm{Op}_h^w(x\xi+p(\xi))v^\Sigma_\Lambda\\ &\quad=h\,\mathrm{Op}_h^w(\Gamma)\mathrm{Op}_h^w(\Sigma)\Big[\sum_{|I|=3}\mathrm{Op}_h(m_I)(v_I)+\sum_{|I|=3}\mathrm{Op}_h(\widetilde{m}_I)(v_I)\Big]+hR(v),\end{aligned}\tag{4.36}$$

with a remainder $R(v)$ which satisfies (4.14), (4.15).

4.3. Development at $\xi=d\varphi(x)$.

— The next step now is to develop the symbol of the *characteristic* term in the nonlinearity, i.e., the one corresponding to $I=(1,1,-1)$, at $\xi=d\varphi(x)$. This will allow us to write it from $|v^\Sigma_\Lambda|^2v^\Sigma_\Lambda$ up to some remainders, transforming the action of pseudo-differential operators acting on it into a product of smooth functions of x. Such development is not essential on *non characteristic* terms, which will be eliminated through a normal form argument in the next section. However, some results, such as Proposition 4.7 and Lemma 4.8, apply suitably also to *non characteristic* terms, so we will freely make use of them to get some simplifications.

We want to prove the following result:

PROPOSITION 4.6. — *Suppose that v satisfies the a priori estimates in* (4.11), (4.12). *Then there exists a family of functions $\theta_h(x) \in C_0^\infty(]-1,1[)$, real valued, equal to one on an interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, $\|\partial^\alpha\theta_h\|_{L^\infty} = O(h^{-2\beta\alpha})$, for a small $\beta > 0$, such that the nonlinearity in* (4.36) *can be written as*

$$\begin{aligned} h\Phi_1^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma &+ h\mathrm{Op}_h^w(\Gamma)\big[\Phi_3^\Sigma(x)\theta_h(x)(v_\Lambda^\Sigma)^3 \\ &+ \Phi_{-1}^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma} + \Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v_\Lambda^\Sigma})^3\big] \\ &+ hR(v), \end{aligned} \tag{4.37}$$

where $\Phi_j^\Sigma(x)$ are smooth functions of x, $|\Phi_j^\Sigma(x)| = O(h^{-\sigma})$ on the support of θ_h, for $j \in \{3,1,-1,-3\}$, and where the remainder $R(v)$ satisfies estimates (4.14), (4.15), *with $\sigma = \sigma(\beta) > 0$ small.*

Before proving this proposition, we need the following general result.

PROPOSITION 4.7. — *Let $a(x,\xi)$ be a real symbol in $S_{\delta,0}(\langle\xi\rangle^q)$, $q \in \mathbb{R}$, for some $\delta > 0$ small. There exists a family $(\theta_h(x))_h$ of C^∞ functions, real valued, supported in some interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, for a small $\beta > 0$, with $(h\partial_h)^k\theta_h$ bounded for every k, such that*

$$\mathrm{Op}_h^w(a)v = \theta_h(x)a(x, d\varphi(x))v + R_1(v), \tag{4.38}$$

where $R_1(v)$ is a remainder satisfying estimates (4.16), (4.17), *with $\sigma = \sigma(\beta,\delta) > 0$, $\sigma \to 0$ as $\beta, \delta \to 0$. The same equality holds replacing v by v^Σ.*

Proof. — In order to develop the symbol $a(x,\xi)$ at $\xi = d\varphi(x)$ we need to stay away from points $x = \pm 1$, so the idea is to truncate near Λ and to estimate different terms coming out.

First, let us consider a function $\chi \in C_0^\infty(\mathbb{R})$ as in Lemma 3.12, $\beta > 0$ small. We decompose $a(x,\xi)$ as follows

$$a(x,\xi) = a(x,\xi)\chi(h^\beta\xi) + a(x,\xi)(1-\chi)(h^\beta\xi). \tag{4.39}$$

It turns out from symbolic calculus, Proposition 3.10, Lemma 3.12 and Sobolev injection (3.8), that $\mathrm{Op}_h^w(a(x,\xi)(1-\chi)(h^\beta\xi))v$ is of the form $R_1(v)$, if we choose $s \gg 1$ sufficiently large, so we have just to deal with $a(x,\xi)\chi(h^\beta\xi)$. Since this symbol is rapidly decaying in $|h^\beta\xi|$, we notice that, to prove that the estimate (4.16) holds for terms of interest, we can turn the H_h^ρ norm into the L^2 norm up to a small loss in h, and then simply estimate the L^2 norm of these terms. This is obvious when $\rho < 0$, for H_h^ρ injects in L^2, while for $\rho \in \mathbb{N}$ this follows using the Definition 3.5 (ii), symbolic calculus, and the fact that $\langle\xi\rangle^\rho\chi(h^\beta\xi) \le h^{-\rho\beta}$. Therefore, it is sufficient for our aim to prove that terms coming out are remainders $R(v)$, in the sense of inequalities (4.14), (4.15).
Secondly, we consider a smooth compactly supported function $\gamma \in C_0^\infty(\mathbb{R})$,

$\gamma \equiv 1$ in a neighborhood of zero, with a sufficiently small support, and we split $a(x,\xi)\chi(h^\beta\xi)$ as
(4.40)
$$a(x,\xi)\chi(h^\beta\xi) = a(x,\xi)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) + a(x,\xi)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}}).$$
Again, the second symbol in the right hand side gives us a remainder. In fact, since $(1-\gamma)(\xi)$ is supported for $|\xi| > \alpha_1$, we can write
(4.41)
$$a(x,\xi)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}}) = a(x,\xi)\chi(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}}),$$
where $\Gamma_{-1}(\xi) = \frac{(1-\gamma)(\xi)}{\xi}$, $|\partial^\alpha\Gamma_{-1}(\xi)| \lesssim \langle\xi\rangle^{-1-\alpha}$. Lemma 4.3 with $c(x,\xi) = a(x,\xi)\chi(h^\beta\xi) \in h^{-\sigma}S_{\delta,\beta}(1)$, $\sigma \geq 0$ small (either equal to $q\beta$ for $q \geq 0$, or to 0 for $q<0$), $c'(x,\xi) \equiv 1$, implies that $\mathrm{Op}_h^w\Big(a(x,\xi)\chi(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v$ satisfies (4.14), (4.15).

The last remaining symbol is $a(x,\xi)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, which is supported in $\{(x,\xi)\in\mathbb{R}\times\mathbb{R} \,\big|\, |\xi| < C_2h^{-\beta}, |\frac{x+p'(\xi)}{\sqrt{h}}| < \alpha_2\}$, so x is bounded in a compact subset of $]-1,1[$ of the form $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, for a suitable positive constant c. We may find a smooth function $\theta_h(x) \in C_0^\infty(]-1,1[)$, $\theta_h \equiv 1$ on $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, satisfying $\|\partial^\alpha\theta_h\|_{L^\infty} = O(h^{-2\beta\alpha})$, and write

$$a(x,\xi)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) = a(x,\xi)\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}). \tag{4.42}$$

Since on the support of θ_h we are away from $x = \pm 1$, we may now develop $a(x,\xi)$ at $\xi = d\varphi(x)$,

$$\begin{aligned} a(x,\xi) &= a(x,d\varphi(x)) + \int_0^1 \partial_\xi a(x,t\xi+(1-t)d\varphi(x))dt\,(\xi - d\varphi(x)) \\ &= a(x,d\varphi(x)) + b(x,\xi)(x+p'(\xi)), \end{aligned} \tag{4.43}$$

where

$$b(x,\xi) = \int_0^1 \partial_\xi a(x,t\xi+(1-t)d\varphi(x))\,dt\,\frac{\xi - d\varphi(x)}{x+p'(\xi)}. \tag{4.44}$$

Then,
(4.45)
$$\begin{aligned} a(x,\xi)\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) &= a(x,d\varphi(x))\theta_h(x) \\ &+ a(x,d\varphi(x))\theta_h(x)\big[\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\big] \\ &+ b(x,\xi)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi)). \end{aligned}$$

Let us call I_1 and I_2 the Weyl quantizations respectively of the second and third term in the right hand side of (4.45). We want to show that they satisfy (4.14), (4.15).
First we analyze the third term in the right hand side of (4.45). We may find another smooth function $\widetilde{\gamma}$, with a small support, such that

$$\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) = \chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\widetilde{\gamma}(\langle\xi\rangle^2(x+p'(\xi))). \tag{4.46}$$

From $a \in S_{\delta,0}(\langle\xi\rangle^q)$ and Lemma 3.13, $B(x,\xi) := b(x,\xi)\chi(h^\beta\xi)\widetilde{\gamma}(\langle\xi\rangle^2(x+p'(\xi)))$ is an element of

$$h^{-\delta}S_{2\beta,\beta}(\langle\xi\rangle^{3+q}) \subset h^{-\sigma}S_{\delta',\beta}(1),$$

for $\delta' = \max\{\delta, 2\beta\}$, $\sigma > 0$ small depending on β and δ. Moreover, $|\partial^\alpha\gamma(\xi)| \leq \langle\xi\rangle^{-1-\alpha}$, so Lemma 4.3 implies that $\mathrm{Op}_h^w\big(B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\big)$ is a remainder $h^{\frac{1}{2}}R(v)$.
On the other hand, I_1 has a symbol whose support is included in the union $\{|\xi| > C_1h^{-\beta}\} \cup \{|\frac{x+p'(\xi)}{\sqrt{h}}| > \alpha_1\}$. Take $\widetilde{\chi} \in C_0^\infty(\mathbb{R})$, $\widetilde{\chi} \equiv 1$ in a neighborhood of zero, $\mathrm{supp}\widetilde{\chi} \subset \{|\xi| < C_1h^{-\beta}\}$, so that $\chi\widetilde{\chi} \equiv \widetilde{\chi}$. We make a further decomposition,

$$\begin{aligned}
&\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1 \\
&= \left(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\right)\widetilde{\chi}(h^\beta\xi) + \left(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\right)(1-\widetilde{\chi})(h^\beta\xi) \\
&= \left(\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\right)\widetilde{\chi}(h^\beta\xi) + \left(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}) - 1\right)(1-\widetilde{\chi})(h^\beta\xi).
\end{aligned} \tag{4.47}$$

The first term in the right hand side is supported for $|\frac{x+p'(\xi)}{\sqrt{h}}| > \alpha_1$, so it can be written as

$$\widetilde{\chi}(h^\beta\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}}),$$

and it is a remainder by Lemma 4.3. Besides, the second term in the right hand side is supported for $|\xi| > C_1h^{-\beta}$, so it is essentially an application of Lemma 3.12 to show that it is a remainder $R(v)$. This shows finally that the development in (4.38) holds. For the last statement of the proposition, one can show that the same proof we did for v can be applied for v^Σ, just changing $a(x,\xi)$ into $a(x,\xi)\Sigma(\xi)$ trough Lemma 4.2 when needed, and for a new $\sigma > 0$ depending also on ρ. $\square$

Proof of Proposition 4.6. — The idea of the proof is to develop all symbols $m_I(\xi), \widetilde{m}_I(\xi)$ occurring in the cubic nonlinearity at $\xi = d\varphi(x)$ using the previous proposition. Remark that, when $i_k = -1$, $v_{i_k} = \bar{v}$ and we have

$$\mathrm{Op}_h(m_k(\xi))\bar{v} = \overline{\mathrm{Op}_h(m_k(-\xi))v} = \overline{\mathrm{Op}_h(m_k(i_k\xi))v}, \tag{4.48}$$

so the development at $\xi = d\varphi(x)$ will give us a term $m_k(i_k d\varphi(x))v_{i_k}$, since $m_k(\xi), d\varphi(x)$ are real valued.

We first write $\mathrm{Op}_h(m_k(\xi))v_{i_k} = \mathrm{Op}_h^w(m_k(\xi))v_{i_k} = \mathrm{Op}_h^w(m_k(\xi)\Sigma(\xi)^{-1})v_{i_k}^\Sigma = \mathrm{Op}_h^w(m_k^\Sigma(\xi))v^\Sigma$ (following the notation introduced in (4.10) - remind that classical quantization coincide with the Weyl one on symbols depending only on ξ) and then we apply Proposition 4.7. From a priori estimates (4.11), (4.12), we have $\|m_k^\Sigma(i_k d\varphi(x))\theta_h(x)v_{i_k}^\Sigma\|_{L^\infty} = O(h^{-\sigma})$, $\|m_k^\Sigma(i_k d\varphi(x))\theta_h(x)v_{i_k}^\Sigma\|_{H_h^\rho} = O(h^{-\sigma})$, for a $\sigma > 0$ depending on β, so we immediately obtain that

$$\mathrm{Op}_h(m_I)(v_I) = \prod_{k=1}^{3} m_k^\Sigma(i_k d\varphi(x))\theta_h(x)v_{i_k}^\Sigma + R_1(v),$$

$R_1(v)$ satisfying estimates (4.16), (4.17), and we can perform the passage from the term

$$\sum_{|I|=3} \mathrm{Op}_h(m_I)(v_I) + \sum_{|I|=3} \mathrm{Op}_h(\widetilde{m}_I)(v_I) \tag{4.49}$$

to its development

$$\sum_{|I|=3} m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma + \sum_{|I|=3} \widetilde{m}_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma + R_1(v). \tag{4.50}$$

The nonlinearity in (4.36) becomes

(4.51)

$$h\,\mathrm{Op}_h^w(\Gamma)\mathrm{Op}_h^w(\Sigma(\xi))\left[\sum_{|I|=3} m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma + \sum_{|I|=3} \widetilde{m}_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma\right]$$
$$+ h\mathrm{Op}_h^w(\Gamma)\mathrm{Op}_h^w(\Sigma(\xi))R_1(v),$$

where $R_1(v)$ satisfies (4.16), so that $\mathrm{Op}_h^w(\Gamma)\mathrm{Op}_h^w(\Sigma(\xi))R_1(v)$ is a remainder of the form $R(v)$, satisfying the estimates (4.14), (4.15), by Propositions 3.10 and 3.11.

The following three lemmas allow us to conclude the proof. In particular, we underline that in Lemma 4.8 we only need an L^2 estimate on what we denote $R(v)$, because we will apply to it the operator $\mathrm{Op}_h^w(\Gamma)$, which is continuous from L^2 to L^∞ with norm $\|\mathrm{Op}_h^w(\Gamma)\|_{\mathcal{L}(L^2;L^\infty)} = O(h^{-\frac{1}{4}-\sigma})$ by Proposition 3.11. □

LEMMA 4.8. — *Let $I=(i_1,i_2,i_3)$, $i_k\in\{1,-1\}$ for $k=1,2,3$, be a fixed vector. Denote by $A(\xi)$ the function $\Sigma(\xi)\chi(h^\beta\xi)$, with χ as in Lemma 3.12, $\beta>0$. Then*

(4.52)

$$\mathrm{Op}_h^w(\Sigma(\xi))\big(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma\big)=A\Big(\sum_{l=1}^3 i_l d\varphi(x)\Big)m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma+h^{\frac12}R(v),$$

$$\mathrm{Op}_h^w(\Sigma(\xi))\big(\widetilde{m}_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma\big)=A\Big(\sum_{l=1}^3 i_l d\varphi(x)\Big)\widetilde{m}_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma+h^{\frac12}R(v),$$

where $R(v)$ satisfies the estimate (4.14).

Proof. — Before proving the result, let us make some observations: first, in all the proof we will use generically the letter σ to denote a small non-negative constant depending on β, that goes to zero when β goes to zero; the symbol $\Sigma(\xi)$ can be reduced to $\Sigma(\xi)\chi(h^\beta\xi)\in h^{-\sigma}S_{0,\beta}(1)$, σ as in (4.20), up to remainders (essentially using Lemma 3.12); from the *a priori* estimates (4.11), (4.12) made on v, we have $\|m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma\|_{L^2}=O(h^{-\sigma})$.

Let us consider a smooth function $\tilde\theta_h(x)\in C_0^\infty(]-1,1[)$, such that $\tilde\theta_h\theta_h\equiv\theta_h$, and let us write

$$m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma=\tilde\theta_h(x)m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma.$$

We enter $\tilde\theta_h(x)$ in $\mathrm{Op}_h^w(\Sigma(\xi)\chi(h^\beta\xi))$ applying symbolic calculus of Proposition 3.8, to be able to develop the symbol at $\xi=\sum_{l=1}^3 i_l d\varphi(x)$. We have

$$\Sigma(\xi)\chi(h^\beta\xi)\,\sharp\,\tilde\theta_h(x)=\Sigma(\xi)\chi(h^\beta\xi)\tilde\theta_h(x)+r_0,\tag{4.53}$$

with $r_0\in h^{1-\sigma}S_{\delta,\beta}(1)$, $\delta>0$ small, so Proposition 3.10 implies that its quantization gives a remainder as in the statement, when applied to $m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma$. Now, since we are away from $x=\pm1$, we can develop $A(\xi)=\Sigma(\xi)\chi(h^\beta\xi)$ at $\xi=\sum_{l=1}^3 i_l d\varphi(x)$ by Taylor's formula, i.e.,

$$A(\xi)=A\Big(\sum_{l=1}^3 i_l d\varphi(x)\Big)+A'(x,\xi)(\xi-\sum_{l=1}^3 i_l d\varphi(x)),\tag{4.54}$$

with

$$A'(x,\xi)=\int_0^1 A'\Big(t\xi+(1-t)\sum_{l=1}^3 i_l d\varphi(x))\Big)\,dt,\tag{4.55}$$

$A'(x,\xi)\tilde{\theta}_h(x)$ belonging to $h^{-\sigma}S_{\delta,0}(1)$. To conclude the proof, we need to show that also $\mathrm{Op}_h^w\Big(A'(x,\xi)\tilde{\theta}_h(x)(\xi-\sum_{l=1}^3 i_l d\varphi(x))\Big)\left(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma\right) = h^{\frac{1}{2}}R(v)$. So let us consider a new function $\tilde{\tilde{\theta}}_h(x)\in C_0^\infty(]-1,1[)$, such that $\tilde{\tilde{\theta}}_h\tilde{\theta}_h\equiv\tilde{\theta}_h$. Since $\tilde{\theta}_h(\xi-\sum_{l=1}^3 i_l d\varphi(x))\in h^{-\sigma}S_{\delta,0}(\langle\xi\rangle)$, and using symbolic calculus of Proposition 3.8, we write

(4.56)
$$A'(x,\xi)\tilde{\tilde{\theta}}_h(x)\sharp\Big(\tilde{\theta}_h(\xi-\sum_{l=1}^3 i_l d\varphi(x))\Big) = A'(x,\xi)\tilde{\theta}_h(x)(\xi-\sum_{l=1}^3 i_l d\varphi(x)) + r_0',$$

where $r_0'\in h^{1-\sigma}S_{\delta,0}(1)$. Again Proposition 3.10 and the uniform bound on v imply that $\mathrm{Op}_h^w(r_0')(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma)$ is a remainder $h^{\frac{1}{2}}R(v)$. We can focus on the term

(4.57)
$$\mathrm{Op}_h^w\big(A'(x,\xi)\tilde{\tilde{\theta}}_h(x)\big)\mathrm{Op}_h^w\Big(\tilde{\theta}_h(x)(\xi-\sum_{l=1}^3 i_l d\varphi(x))\Big)(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^\Sigma),$$

and we can go further, limiting ourselves to consider the action of these operators when v_I^Σ is replaced by

$$v_I^0 := \prod_{k=1}^3 \mathrm{Op}_h^w(\Sigma(\xi)\chi(h^\beta\xi))v_{i_k}, \tag{4.58}$$

again up to terms with symbols supported for $|\xi|\ge h^{-\beta}$, which are remainders from Lemma 3.12. The operator $\mathrm{Op}_h^w\Big(\tilde{\theta}_h(x)(\xi-\sum_{l=1}^3 i_l d\varphi(x))\Big)$ has a symbol linear in ξ, so

(4.59)
$$\begin{aligned}\mathrm{Op}_h^w\Big(\tilde{\theta}_h(x)(\xi-\sum_{l=1}^3 i_l d\varphi(x))\Big) &= \frac{1}{2}hD_x\tilde{\theta}_h(x)+\frac{1}{2}\tilde{\theta}_h(x)hD_x-\tilde{\theta}_h(x)\sum_{l=1}^3 i_l d\varphi(x)\\ &= h\frac{\tilde{\theta}_h'(x)}{2i}+\tilde{\theta}_h(x)(hD_x-\sum_{l=1}^3 i_l d\varphi(x)),\end{aligned}$$

and if we choose $\tilde{\theta}_h$ such that $\tilde{\theta}_h\theta_h\equiv\theta_h$, we have that $\tilde{\theta}_h'\equiv 0$ on the support of θ_h, giving no contributions when $h\frac{\tilde{\theta}_h'(x)}{2}$ is multiplied by $m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^0$. Here $(hD_x-\sum_{l=1}^3 i_l d\varphi(x))$ acts like a derivation on v_I^0, so the Leibniz's rule holds and

$$
\begin{aligned}
&\mathrm{Op}_h^w\Big(\tilde{\theta}_h(x)(\xi-\sum_{l=1}^3 i_l d\varphi(x))\Big)\left(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^0\right)=\\
&=\tilde{\theta}_h(x)(hD_x-\sum_{l=1}^3 i_l d\varphi(x))\left(m_I^\Sigma(d\varphi_I(x))\theta_h(x)v_I^0\right)\\
&=\left[hD_x(m_I^\Sigma(d\varphi_I(x))\theta_h(x))\right]v_I^0\\
&\quad+m_I^\Sigma(d\varphi_I(x))\theta_h(x)\tilde{\theta}_h(x)(hD_x-\sum_{l=1}^3 i_l d\varphi(x))(v_I^0).
\end{aligned}
\tag{4.60}
$$

Then, if for instance $v_I^0=(v^0)^3$ (i.e., $I=(1,1,1)$, and the same idea can be applied with $|v^0|^2v^0$, $|v^0|^2\overline{v^0}$ and $(\overline{v^0})^3$), we derive

$$
\begin{aligned}
\tilde{\theta}_h(x)(hD_x-3d\varphi(x))(v^0)^3 &= 3(v^0)^2\tilde{\theta}_h(x)(hD_x-d\varphi(x))v^0\\
&=3(v^0)^2\mathrm{Op}_h^w(\tilde{\theta}_h(x)(\xi-d\varphi(x)))v^0-\frac{3}{2i}h\tilde{\theta}_h'(x)(v^0)^3,
\end{aligned}
\tag{4.61}
$$

using the relation (4.59) in the last equality (however, the second term in the right hand side disappears when we inject (4.61) in (4.60)). Now we can re-express the first term in the right hand side from $h\mathcal{L}v^0$. In fact, up to further remainders, $\mathrm{Op}_h^w(\tilde{\theta}_h(x)(\xi-d\varphi(x)))v^0$ can be reduced to $\mathrm{Op}_h^w(\tilde{\theta}_h(x)\chi(h^\beta\xi)(\xi-d\varphi(x)))v^0$, and this term can be split with the help of a $\gamma\in C_0^\infty(\mathbb{R})$, $\gamma\equiv 1$ in zero, namely

$$
\begin{aligned}
&\mathrm{Op}_h^w\Big(\tilde{\theta}_h(x)\chi(h^\beta\xi)(\xi-d\varphi(x))\Big)v^0\\
&=\mathrm{Op}_h^w\left(\tilde{\theta}_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(\xi-d\varphi(x))\right)v^0\\
&\quad+\mathrm{Op}_h^w\left(\tilde{\theta}_h(x)\chi(h^\beta\xi)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})(\xi-d\varphi(x))\right)v^0.
\end{aligned}
\tag{4.62}
$$

Both terms have an L^2 norm controlled from above by

$$
Ch^{1-\sigma}(\|\mathcal{L}v\|_{L^2}+\|v\|_{H_h^s}).
$$

In fact, on one hand, we can take up the observation made in (4.46), and rewrite the first term in the right hand side as

$$
\mathrm{Op}_h^w\left(\tilde{\theta}_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\tilde{e}_+(x+p'(\xi))\right)v^0
\tag{4.63}
$$

where $\tilde{e}_+$ is defined in (3.32). On the other hand, also the symbol associated to the second operator in the right hand side can be rewritten highlighting the factor $(x+p'(\xi))$, as follows

$$\tilde{\theta}_h(x)\chi(h^\beta\xi)\left(\frac{\xi-d\varphi(x)}{x+p'(\xi)}\right)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi)),$$

with $\tilde{\theta}_h(x)\chi(h^\beta\xi)\left(\frac{\xi-d\varphi(x)}{x+p'(\xi)}\right)(1-\gamma)(\frac{x+p'(\xi)}{\sqrt{h}}) \in h^{-\sigma}S_{\frac{1}{2},\beta}(1)$. Then, to both operators we can apply Lemma 4.3, for $c(x,\xi)$ respectively equal to $\tilde{\theta}_h(x)\chi(h^\beta\xi)\tilde{e}_+$ and $\tilde{\theta}_h(x)\chi(h^\beta\xi)\left(\frac{\xi-d\varphi(x)}{x+p'(\xi)}\right)$, $c'(x,\xi) = \Sigma(\xi)\chi(h^\beta\xi)$, obtaining that they satisfy (4.23). Summing all up, together with (4.57), (4.60), (4.61), the fact that $A'(x,\xi)\tilde{\tilde{\theta}}_h(x) \in h^{-\sigma}S_{\delta,0}(1)$, and Propositions 3.10, we obtain the result of the lemma. □

From now on, we will denote by $\Phi_3^\Sigma(x), \Phi_1^\Sigma(x), \Phi_{-1}^\Sigma(x), \Phi_{-3}^\Sigma(x)$ respectively the coefficients of $(v^\Sigma)^3, |v^\Sigma|^2v^\Sigma, |v^\Sigma|^2\overline{v^\Sigma}, (\overline{v^\Sigma})^3$.

Since they come from $m_I^\Sigma(d\varphi_I(x))\theta_h(x), \widetilde{m}_I^\Sigma(d\varphi_I(x))\theta_h(x)$ which are $O(h^{-\sigma})$, for a small $\sigma>0$, they are also $O(h^{-\sigma})$.

Lemma 4.9. — *With same notations as before,*

$$\mathrm{Op}_h^w(\Gamma)(\Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2v^\Sigma) = \Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2v^\Sigma + R(v), \tag{4.64}$$

where $R(v)$ satisfies estimates (4.14), (4.15).

Proof. — Let us write $\mathrm{Op}_h^w(\Gamma) = 1-\mathrm{Op}_h^w(1-\Gamma)$. We want to show that the action of $\mathrm{Op}_h^w(1-\Gamma)$ on $\Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2v^\Sigma$ gives us a remainder $R(v)$. First, we can reduce the symbol $1-\Gamma$ to $(1-\Gamma)\chi(h^\beta\xi)$, with χ cut-off function, $\beta>0$, up to remainders from Lemma 3.12. Moreover, we can consider a smooth function $\tilde{\tilde{\theta}}_h(x) \in C_0^\infty(]-1,1[)$ such that $\tilde{\tilde{\theta}}_h\theta_h \equiv \theta_h$, and use symbolic calculus to enter $\tilde{\tilde{\theta}}_h(x)$ in $\mathrm{Op}_h^w((1-\Gamma)\chi(h^\beta\xi))$ (again up to a remainder $R(v)$). Then, we can write

$$(1-\Gamma)\chi(h^\beta\xi)\tilde{\tilde{\theta}}_h(x) = \frac{1}{\sqrt{h}}b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\tilde{\theta}_h(x)(\xi-d\varphi(x)), \tag{4.65}$$

where $b(x,\xi) = \chi(h^\beta\xi)\tilde{\tilde{\theta}}_h(x)(\frac{x+p'(\xi)}{\xi-d\varphi(x)}) \in h^{-\sigma}S_{\delta,\beta}(1)$, $\Gamma_{-1}(\xi) = \frac{(1-\gamma)(\xi)}{\xi}$, $\sigma,\delta>0$ small depending on β, and $\tilde{\theta}_h(x)$ a new smooth function in $C_0^\infty(]-1,1[)$, identically equal to 1 on the support of $\tilde{\tilde{\theta}}_h(x)$. Applying symbolic calculus of

Lemma 3.9, we derive

$$\begin{aligned}
&\frac{1}{\sqrt{h}}b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}}) \sharp \tilde{\theta}_h(x)(\xi - d\varphi(x)) \\
&= \frac{1}{\sqrt{h}}b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\tilde{\theta}_h(x)(\xi - d\varphi(x)) \\
&+ \frac{\sqrt{h}}{2i}\left\{b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}}), \tilde{\theta}_h(x)(\xi - d\varphi(x))\right\} \\
&+ r_1,
\end{aligned} \tag{4.66}$$

with $r_1 \in h^{\frac{1}{2}-\sigma}S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, for a new small $\sigma > 0$. An explicit calculation, and the observation that $\tilde{\theta}'_h \equiv 0$ on the support of $\tilde{\tilde{\theta}}_h$, show that the Poisson bracket is equal to

$$\begin{aligned}
&\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\left[\tilde{\theta}_h(x)(-\partial_\xi b(x,\xi)d^2\varphi(x) - \partial_x b(x,\xi))\right] + \\
&+ \Gamma'_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\chi(h^\beta\xi)\tilde{\theta}_h(x)\left[\frac{-d^2\varphi(x)p''(\xi)-1}{\xi - d\varphi(x)}\right],
\end{aligned} \tag{4.67}$$

and since $x + p'(d\varphi) = 0$, we have $-d^2\varphi(x) = \frac{1}{p''(d\varphi)}$ and

$$\begin{aligned}
&\chi(h^\beta\xi)\tilde{\theta}_h(x)\left[\frac{-d^2\varphi(x)p''(\xi)-1}{\xi - d\varphi(x)}\right] \\
&= \frac{\chi(h^\beta\xi)\tilde{\theta}_h(x)}{p''(d\varphi(x))}\int_0^1 p'''(t\xi + (1-t)d\varphi(x))dt \in h^{-\sigma}S_{\delta,\beta}(1).
\end{aligned} \tag{4.68}$$

Therefore, from $\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})$, $\Gamma'_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}}) \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, other appearing symbols in (4.67) belonging to $h^{-\sigma}S_{\delta,\beta}(1)$, we can rewrite the second and third term in the right hand side of (4.66) as $h^{\frac{1}{2}-\sigma}r$, with $r \in S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, so their action on $\Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2v^\Sigma$ gives us a remainder $R(v)$ by Propositions 3.10, 3.11. In this way, we are reduce to estimate

$$\frac{1}{\sqrt{h}}\mathrm{Op}_h^w\left(b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})\right)\mathrm{Op}_h^w\big(\tilde{\theta}_h(x)(\xi - d\varphi(x))\big)(\Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2v^\Sigma). \tag{4.69}$$

Taking up (4.58), (4.59), (4.60) for $I = (1,1,-1)$, we obtain that $\mathrm{Op}_h^w\big(\tilde{\theta}_h(x)(\xi - d\varphi(x))\big)$ acts like a derivation on its argument and

$$\|\mathrm{Op}_h^w\big(\tilde{\theta}_h(x)(\xi - d\varphi(x))\big)\Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2v^\Sigma\|_{L^2} \le Ch^{1-\sigma}(\|\mathcal{L}v\|_{L^2} + \|v\|_{H_h^s}), \tag{4.70}$$

for a new small $\sigma > 0$ still depending on β, so the fact that $b(x,\xi)\Gamma_{-1}(\frac{x+p'(\xi)}{\sqrt{h}})$ belongs to $S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, along with Propositions 3.10, 3.11, imply that the term in (4.69) is a remainder $R(v)$ satisfying (4.14), (4.15). This concludes the proof. □

Proposition 4.6 allows us to arrive at the following equation

$$\begin{aligned}(4.71)\qquad &(D_t - \mathrm{Op}_h^w(x\xi + p(\xi))v_\Lambda^\Sigma \\ &= h\Phi_1^\Sigma(x)\theta_h(x)|v^\Sigma|^2 v^\Sigma + h\mathrm{Op}_h^w(\Gamma)\left[\Phi_3^\Sigma(x)\theta_h(x)(v^\Sigma)^3\right. \\ &\quad \left.+ \Phi_{-1}^\Sigma(x)\theta_h(x)|v^\Sigma|^2\overline{v^\Sigma} + \Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v^\Sigma})^3\right] + hR(v),\end{aligned}$$

which is not entirely in v_Λ^Σ, so to transform to the right equation we use the following lemma, whose proof comes directly from Proposition 4.4, and this is the reason why we omit the details.

Lemma 4.10. — *Under the same hypothesis as in Theorem 4.1, there exists $s > 0$ sufficiently large, and a constant $C > 0$ independent of h, such that*

$$(4.72)\qquad \|v_I^\Sigma - (v_\Lambda^\Sigma)_I\|_{L^2} \le Ch^{\frac{1}{2}-\sigma}\left(\|\mathcal{L}v\|_{L^2} + \|v\|_{H_h^s}\right),$$

$$(4.73)\qquad \|v_I^\Sigma - (v_\Lambda^\Sigma)_I\|_{L^\infty} \le Ch^{\frac{1}{4}-\sigma}\left(\|\mathcal{L}v\|_{L^2} + \|v\|_{H_h^s}\right),$$

for a small $\sigma > 0$ depending on β.

Therefore v_Λ^Σ is solution of the following equation:

$$\begin{aligned}(4.74)\qquad &(D_t - \mathrm{Op}_h^w(x\xi + p(\xi))v_\Lambda^\Sigma \\ &= h\Phi_1^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma + h\mathrm{Op}_h^w(\Gamma)\left[\Phi_3^\Sigma(x)\theta_h(x)(v_\Lambda^\Sigma)^3\right. \\ &\quad \left.+ \Phi_{-1}^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma} + \Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v_\Lambda^\Sigma})^3\right] + hR(v),\end{aligned}$$

$R(v)$ being a remainder satisfying estimates (4.14), (4.15).

To conclude this section, we develop $\mathrm{Op}_h^w(x\xi + p(\xi))v_\Lambda^\Sigma$ at $\xi = d\varphi(x)$.

Proposition 4.11. — *Under the same hypothesis as in Theorem 4.1,*

$$(4.75)\qquad \mathrm{Op}_h^w(x\xi + p(\xi))v_\Lambda^\Sigma = \varphi(x)\theta_h(x)v_\Lambda^\Sigma + hR(v),$$

where $R(v)$ satisfies the estimates in (4.14), (4.15). Then, v_Λ^Σ is solution of the following equation:

(4.76)

$$\begin{aligned}D_t v_\Lambda^\Sigma &= \varphi(x)\theta_h(x)v_\Lambda^\Sigma + h\Phi_1^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma \\ &\quad + h\mathrm{Op}_h^w(\Gamma)\left[\Phi_3^\Sigma(x)\theta_h(x)(v_\Lambda^\Sigma)^3 + \Phi_{-1}^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma} + \Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v_\Lambda^\Sigma})^3\right] \\ &\quad + hR(v),\end{aligned}$$

Proof. — Consider a cut-off function χ as in Lemma 3.12, and $\beta > 0$. One can split as follows

$$(4.77)\qquad v_\Lambda^\Sigma = \mathrm{Op}_h^w(\chi(h^\beta\xi)\Gamma(x,\xi))v^\Sigma + \mathrm{Op}_h^w((1-\chi)(h^\beta\xi)\Gamma(x,\xi))v^\Sigma,$$

and easily show that $\mathrm{Op}_h^w(x\xi+p(\xi))\mathrm{Op}_h^w((1-\chi)(h^\beta\xi)\Gamma(x,\xi))v^\Sigma$ is a remainder of the form $hR(v)$, $R(v)$ satisfying estimates (4.14), (4.15), just using symbolic calculus and Lemma 3.12.

Therefore, we have to deal with $\mathrm{Op}_h^w(x\xi+p(\xi))\mathrm{Op}_h^w(\chi(h^\beta\xi)\Gamma(x,\xi))v^\Sigma$. We have already observed that for (x,ξ) in the support of $\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, x is bounded on a compact set $[-1+ch^{2\beta},1-ch^{2\beta}]$, which allows us to consider a smooth function $\theta_h(x)\in C_0^\infty(]-1,1[)$, identically equal to one on this interval, and then on the support of $\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, so that:

$$\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})=\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}). \tag{4.78}$$

Moreover, the derivatives of θ_h are zero on the support of $\partial^\alpha(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}}))$, for every multi-index α. This implies that products of $\theta_h'(x)$ with $\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$ and all its derivatives are always zero so, by Lemma 3.9,

$$\theta_h(x)\sharp\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})=\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})+r_\infty, \tag{4.79}$$

where $r_\infty\in h^NS_{\frac{1}{2},\beta}(\langle x\rangle^{-\infty})$, for N as large as we want. In this way we can factor out $\theta_h(x)$, write the equality

$$\begin{aligned}&\mathrm{Op}_h^w(x\xi+p(\xi))\mathrm{Op}_h^w\left(\theta_h(x)\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\right)v^\Sigma=\\&=\mathrm{Op}_h^w(x\xi+p(\xi))\theta_h(x)\mathrm{Op}_h^w\left(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\right)v^\Sigma+hR(v),\end{aligned} \tag{4.80}$$

and return to v^Σ_Λ by

$$\mathrm{Op}_h^w\Big(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma=v^\Sigma_\Lambda-\mathrm{Op}_h^w\Big((1-\chi(h^\beta\xi))\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma. \tag{4.81}$$

Then,

$$\begin{aligned}&\mathrm{Op}_h^w(x\xi+p(\xi))\theta_h(x)\mathrm{Op}_h^w\Big(\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma=\\&\qquad=\mathrm{Op}_h^w(x\xi+p(\xi))\theta_h(x)v^\Sigma_\Lambda\\&\qquad\quad-\mathrm{Op}_h^w(x\xi+p(\xi))\theta_h(x)\mathrm{Op}_h^w\Big((1-\chi(h^\beta\xi))\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma,\end{aligned} \tag{4.82}$$

and one can show that the second term in the right hand side is a remainder $hR(v)$ essentially using symbolic calculus, Lemma 3.12, and Sobolev injection.

Symbolic calculus enables us also to put $\theta_h(x)$ in $\mathrm{Op}_h^w(x\xi+p(\xi))$, as the following deduction shows,

$$\begin{aligned}\mathrm{Op}_h^w(x\xi+p(\xi))\theta_h(x)v_\Lambda^\Sigma &= \mathrm{Op}_h^w\big((x\xi+p(\xi))\theta_h(x)\big)v_\Lambda^\Sigma\\ &\quad+\frac{h}{2i}\mathrm{Op}_h^w\big(\theta_h'(x)(x+p'(\xi))\big)v_\Lambda^\Sigma+hR(v)\\ &=\mathrm{Op}_h^w\big((x\xi+p(\xi))\theta_h(x)\big)v_\Lambda^\Sigma+hR(v),\end{aligned}\tag{4.83}$$

with $R(v)$ satisfying (4.14), (4.15), using Proposition 3.10 and Sobolev injection. In the last equality, $\frac{h}{2i}\mathrm{Op}_h^w\big(\theta_h'(x)(x+p'(\xi))\big)v_\Lambda^\Sigma$ enters in the remainder, for $\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\in S_{\frac12,0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-2})$ by Lemma 3.14, $\theta_h'(x)(x+p'(\xi))\in h^{-\delta}S_{\delta,0}(1)$ for a small $\delta>0$, and using symbolic calculus. Actually, we first write

$$\begin{aligned}&\frac{h}{2i}\mathrm{Op}_h^w\big(\theta_h'(x)(x+p'(\xi))\big)v_\Lambda^\Sigma\\ &\quad=\frac{h^{\frac32}}{2i}\mathrm{Op}_h^w\Big(\theta_h'(x)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\Big)v^\Sigma+h^{\frac32}\mathrm{Op}_h^w(r_0)v^\Sigma,\end{aligned}\tag{4.84}$$

where $r_0\in h^{-2\delta}S_{\frac12,0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, and then we use Lemma 4.2 with $\tilde\Gamma(\xi)=\gamma(\xi)\xi$, and Lemma 4.3 to deduce that it is a remainder $hR(v)$.

We can now analyze $\mathrm{Op}_h^w((x\xi+p(\xi))\theta_h(x))v_\Lambda^\Sigma$. As we are away from points $x=\pm1$, we can develop the symbol $x\xi+p(\xi)$ at $\xi=d\varphi(x)$, and since $\partial_\xi(x\xi+p(\xi))|_{\xi=d\varphi(x)}=0$ we have

$$\begin{aligned}x\xi+p(\xi)&=xd\varphi(x)+p(d\varphi(x))\\ &\quad+\int_0^1 p''(t\xi+(1-t)d\varphi(x))(1-t)dt\,(\xi-d\varphi(x))^2\\ &=xd\varphi(x)+p(d\varphi(x))+b(x,\xi)(x+p'(\xi))^2,\end{aligned}\tag{4.85}$$

where

$$b(x,\xi)=\int_0^1 p''(t\xi+(1-t)d\varphi(x))(1-t)dt\left(\frac{\xi-d\varphi(x)}{x+p'(\xi)}\right)^2.$$

Observe that $xd\varphi(x)+p(d\varphi(x))=\varphi(x)$. To conclude the proof, we need to show that

$$\mathrm{Op}_h^w\left(b(x,\xi)\theta_h(x)(x+p'(\xi))^2\right)v_\Lambda^\Sigma$$

gives rise to a remainder, too. First, we may consider a function χ as in Lemma 3.12, $\beta>0$, and split $b(x,\xi)\theta_h(x)(x+p'(\xi))^2$ as the sum of $b(x,\xi)\theta_h(x)(x+p'(\xi))^2(1-\chi(h^\beta\xi))\in h^{-\sigma}S_{\delta,0}(\langle\xi\rangle^2)$, for small $\delta,\sigma>0$, whose quantization acts on v_Λ^Σ as a remainder because of Lemma 3.12, and $b(x,\xi)\theta_h(x)(x+p'(\xi))^2\chi(h^\beta\xi)$. For $b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2$, we can perform a further splitting through a function $\tilde\gamma\in C_0^\infty(\mathbb{R})$, such that

$\tilde{\gamma}\left(\langle\xi\rangle^2(x+p'(\xi))\right)\equiv 1$ on the support of $\chi(h^\beta\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})$, i.e.,

$$\begin{aligned}(4.86)\qquad & b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2\tilde{\gamma}\left(\langle\xi\rangle^2(x+p'(\xi))\right)\\ &\quad + b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2(1-\tilde{\gamma})\left(\langle\xi\rangle^2(x+p'(\xi))\right).\end{aligned}$$

As discussed before, this implies that $(1-\tilde{\gamma})(\langle\xi\rangle^2(x+p'(\xi)))$ and all its derivatives are equal to zero on that support. Since $b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2$ $(1-\tilde{\gamma})\left(\langle\xi\rangle^2(x+p'(\xi))\right)\in h^{-\sigma}S_{\delta,\beta}(1)$ for $\sigma,\delta>0$ small depending on β, one can apply symbolic calculus (up to a large enough order) to obtain

(4.87)

$$b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2(1-\tilde{\gamma})\left(\langle\xi\rangle^2(x+p'(\xi))\right)\sharp\,\gamma(\frac{x+p'(\xi)}{\sqrt{h}})=r'_\infty,$$

with $r'_\infty=h^N S_{\frac{1}{2},\beta}(1)$, N sufficiently large, to have

$$\mathrm{Op}_h^w\left(b(x,\xi)\theta_h(x)\chi(h^\beta\xi)(x+p'(\xi))^2(1-\tilde{\gamma})\left(\langle\xi\rangle^2(x+p'(\xi))\right)\right)v^\Sigma_\Lambda=hR(v).$$

On the other hand, $B(x,\xi):=b(x,\xi)\theta_h(x)\chi(h^\beta\xi)\tilde{\gamma}(\langle\xi\rangle^2(x+p'(\xi)))$ belongs to $h^{-\sigma}S_{\delta,\beta}(1)$, for $\delta\geq 2\beta$, by Lemma 3.13. Using twice Lemma 3.9, together with the fact that $\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-3})$ and $B(x,\xi)(x+p'(\xi))^2\in h^{1-\sigma}S_{\delta,\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^2)$, we derive

(4.88)

$$\left(B(x,\xi)(x+p'(\xi))^2\right)\sharp\,\gamma(\frac{x+p'(\xi)}{\sqrt{h}})=B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))^2+hr_0,$$

and

$$\begin{aligned}(4.89)\qquad & \left(B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\right)\sharp\,(x+p'(\xi))\\ &\qquad = B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))^2+hr'_0,\end{aligned}$$

where $r_0,r'_0\in h^{\frac{1}{2}-\sigma}S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$. Therefore

$$\begin{aligned}(4.90)\qquad & \left(B(x,\xi)(x+p'(\xi))^2\right)\sharp\,\gamma(\frac{x+p'(\xi)}{\sqrt{h}})\\ &\quad=\left(B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\right)\sharp\,(x+p'(\xi))+h(r_0-r'_0),\end{aligned}$$

and

$$\begin{aligned}(4.91)\qquad & \mathrm{Op}_h^w(B(x,\xi)(x+p'(\xi))^2)v^\Sigma_\Lambda\\ &\quad= h\mathrm{Op}_h^w\left(B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi))\right)\mathcal{L}v^\Sigma+h\mathrm{Op}_h^w(r_0-r'_0)v^\Sigma,\end{aligned}$$

so one can show that the right hand side is a remainder $hR(v)$, commutating $\mathcal{L}$ with $\mathrm{Op}_h^w(\Sigma(\xi))$, using that $B(x,\xi)\gamma(\frac{x+p'(\xi)}{\sqrt{h}})(x+p'(\xi)), r_0 - r_0' \in h^{\frac{1}{2}-\sigma}S_{\frac{1}{2},\beta}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, and Propositions 3.10, 3.11. We finally obtain

$$\mathrm{Op}_h^w(x\xi + p(\xi))v_\Lambda^\Sigma = \varphi(x)\theta_h(x)v_\Lambda^\Sigma + hR(v), \tag{4.92}$$

and according to (4.74), v_Λ^Σ is solution of

(4.93)

$$\begin{aligned} D_t v_\Lambda^\Sigma = \varphi(x)\theta_h(x)v_\Lambda^\Sigma &+ h\Phi_1^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma \\ &+ h\mathrm{Op}_h^w(\Gamma)\Big[\Phi_3^\Sigma(x)\theta_h(x)(v_\Lambda^\Sigma)^3 + \Phi_{-1}^\Sigma(x)\theta_h(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma} + \Phi_{-3}^\Sigma(x)\theta_h(x)(\overline{v_\Lambda^\Sigma})^3\Big] \\ &+ hR(v), \end{aligned}$$

where $R(v)$ is a remainder satisfying estimates (4.14), (4.15). □

5. Study of the ODE and end of the proof

5.1. The L^∞ estimate. — The goal of this subsection is to the derive from the Equation (4.76) an ODE for a new function f_Λ^Σ obtained from v_Λ^Σ, from which we can deduce uniform bounds for v_Λ^Σ, and for the starting function v, with a certain number $\rho \in \mathbb{N}$ of its derivatives. The idea is to get rid of contributions of *non characteristic* terms (i.e., of cubic terms different from $|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma$) by a reasoning of normal forms. This will allow us to eliminate all terms still containing pseudo-differential operators, to finally write an ODE, and to prove the required L^∞ estimate, if the *null condition* is satisfied.

In the previous section, we denoted by $\Phi_3^\Sigma(x)$, $\Phi_1^\Sigma(x)$, $\Phi_{-1}^\Sigma(x)$ and $\Phi_{-3}^\Sigma(x)$ (modulo some new smooth terms) respectively the coefficients of $(v_\Lambda^\Sigma)^3$, $|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma$, $|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma}$, $(\overline{v_\Lambda^\Sigma})^3$ in the right hand side of (4.76). One can calculate them explicitly, using both the expression of the nonlinearity obtained in Proposition 4.6 and its polynomial representation as in Equation (4.7). In the latter, after the development at $\xi = d\varphi(x)$, we essentially replaced hD by $d\varphi(x)$ when it is applied to v_Λ^Σ, and by $-d\varphi(x)$ when it is applied to $\overline{v_\Lambda^\Sigma}$, modulus some new smooth coefficients $a_I(x) := A(\sum_{l=1}^{3} i_l d\varphi(x))\Sigma(d\varphi(x))^{-3}$, for every $I = (i_1, i_2, i_3)$ (the factor $\Sigma(d\varphi(x))^{-3}$ coming out from $m_I^\Sigma(d\varphi_I(x)) = m_I(d\varphi(x))\Sigma(d\varphi(x))^{-3}$, according to the notation introduced in (4.10), $A(\xi) = \Sigma(\xi)\chi(h^\beta\xi)$).

We are interested in particular in $\Phi_1^\Sigma(x)$ or, to be more precise, to its real part. In fact, the *null condition* introduced in Definition 1.1 at the very beginning is the same as requiring for the coefficient of $|v_\Lambda^\Sigma|^2 v_\Lambda^\Sigma$ to be real, i.e., its imaginary part must be equal to zero. Since polynomials P_k', P_k'' are real as well as $d\varphi(x)$, $\langle d\varphi(x)\rangle$, the only contribution to the imaginary part comes from P_k', P_k'' for $k = 1, 3$ (which have a factor i^k) and produces a multiple of the function

$\Phi(x)$ defined in (1.5). Therefore, if we suppose that the nonlinearity satisfies this *null condition* (as demanded in Theorem 1.2) then we find for $\Phi_1^\Sigma(x)$ that

$$\begin{aligned}\Phi_1^\Sigma(x) = \frac{1}{8}a_{(1,1,-1)}(x)\langle d\varphi\rangle^{-3}\big[&3P_0(1, d\varphi\langle d\varphi\rangle, (d\varphi)^2;\, \langle d\varphi\rangle, d\varphi) \\ &+P_2(1, d\varphi\langle d\varphi\rangle, (d\varphi)^2;\, \langle d\varphi\rangle, d\varphi)\big]\,.\end{aligned} \tag{5.1}$$

PROPOSITION 5.1. — *Suppose we are given two constants $A'', B'' > 0$, some $T > 1$ and a $\sigma > 0$ small. Let v_Λ^Σ be a solution of the Equation* (4.76) *on the interval* $[1, T]$, *v_Λ^Σ satisfying the a priori estimates*

$$\|v_\Lambda^\Sigma(t,\cdot)\|_{L^\infty(\mathbb{R})} \le A''\varepsilon, \tag{5.2}$$

$$\|v_\Lambda^\Sigma(t,\cdot)\|_{L^2(\mathbb{R})} \le B''\varepsilon h^{-\sigma}, \tag{5.3}$$

for all $t \in [1, T]$. Let $\tilde\theta_h(x) \in C_0^\infty(]-1,1[)$, such that $\tilde\theta_h\theta_h \equiv \theta_h$, and define

$$\begin{aligned} f_\Lambda^\Sigma := v_\Lambda^\Sigma + \mathrm{Op}_h^w(\Gamma)\Big[&-\frac{h}{2}\frac{\tilde\theta_h(x)}{\varphi(x)}\Phi_3^\Sigma(x)(v_\Lambda^\Sigma)^3 \\ &+\frac{h}{2}\frac{\tilde\theta_h(x)}{\varphi(x)}\Phi_{-1}^\Sigma(x)|v_\Lambda^\Sigma|^2\overline{v_\Lambda^\Sigma} + \frac{h}{4}\frac{\tilde\theta_h(x)}{\varphi(x)}\Phi_{-3}^\Sigma(x)(\overline{v_\Lambda^\Sigma})^3\Big].\end{aligned} \tag{5.4}$$

Then f_Λ^Σ is well defined and it is solution of the ODE:

$$D_t f_\Lambda^\Sigma = \varphi(x)\theta_h(x) f_\Lambda^\Sigma + h\theta_h(x)\Phi_1^\Sigma(x)|f_\Lambda^\Sigma|^2 f_\Lambda^\Sigma + hR(v), \tag{5.5}$$

where $R(v)$ is a remainder satisfying estimates (4.14), (4.15).

Proof. — Firstly, we would like to underline that, if we suppose bounds in (4.11) and (4.12) on v, then Hypothesis (5.2) and (5.3) follow immediately, because of the definition of v_Λ^Σ as $\mathrm{Op}_h^w(\Gamma)v^\Sigma$. In fact, estimate (5.3) follows from Proposition 3.10 and the *a priori* estimate (4.12), with $B'' = B'$. Regarding the estimate (5.2), we can write

$$v_\Lambda^\Sigma = v^\Sigma - v_{\Lambda^c}^\Sigma, \tag{5.6}$$

and since $\|v^\Sigma(t,\cdot)\|_{L^\infty} = \|v(t,\cdot)\|_{W_h^{\rho,\infty}}$,

$$\begin{aligned}\|v_\Lambda^\Sigma(t,\cdot)\|_{L^\infty} &\le \|v^\Sigma(t,\cdot)\|_{L^\infty} + \|v_{\Lambda^c}^\Sigma(t,\cdot)\|_{L^\infty} \\ &= \|v(t,\cdot)\|_{W_h^{\rho,\infty}} + \|v_{\Lambda^c}^\Sigma(t,\cdot)\|_{L^\infty},\end{aligned} \tag{5.7}$$

where we estimated $\|v_{\Lambda^c}^\Sigma(t,\cdot)\|_{L^\infty}$ in Proposition 4.4. Therefore, using that for $\sigma > 0$ sufficiently small $h^{\frac14-\sigma} \le h^{\frac18}$, we have

$$\begin{aligned}\|v_\Lambda^\Sigma(t,\cdot)\|_{L^\infty} &\le \|v(t,\cdot)\|_{W_h^{\rho,\infty}} + Ch^{\frac18}(\|\mathcal{L}v(t,\cdot)\|_{L^2} + \|v(t,\cdot)\|_{H_h^s}) \\ &\le A'\varepsilon + CB'\varepsilon\, h^{\frac18-\sigma} \\ &\le A''\varepsilon,\end{aligned} \tag{5.8}$$

if we choose $A'' > 0$ sufficiently large to have $A', CB' \le \frac{A''}{2}$.

Secondly, $\varphi(x) \neq 0$ for all x in the support of $\tilde{\theta}_h$. In fact, we consider $\tilde{\theta}_h$ such that $\tilde{\theta}_h\theta_h \equiv \theta_h$, so we can suppose that its support is of the form $[-1 + C'h^{2\beta}, 1 - C'h^{2\beta}]$, for a suitable small positive constant C'. On this interval $x^2 \leq (1 - C'h^{2\beta})^2 = 1 + C'^2h^{4\beta} - 2C'h^{2\beta}$, so

$$\varphi(x) = \sqrt{1-x^2} \geq \sqrt{C'h^{2\beta}(2 - C'h^{2\beta})} \gtrsim h^{\beta}, \tag{5.9}$$

which implies that the quotient $\frac{\tilde{\theta}_h(x)}{\varphi(x)}$ is well defined and $|\frac{\tilde{\theta}_h(x)}{\varphi(x)}| \leq h^{-\beta}$. Then, set

$$\begin{aligned} f^{\Sigma}_{\Lambda} := v^{\Sigma}_{\Lambda} + \mathrm{Op}^w_h(\Gamma)\Big[h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Big(&k_1\Phi^{\Sigma}_3(x)(v^{\Sigma}_{\Lambda})^3 \\ &+ k_2\Phi^{\Sigma}_{-1}(x)|v^{\Sigma}_{\Lambda}|^2\overline{v^{\Sigma}_{\Lambda}} + k_3\Phi^{\Sigma}_{-3}(x)(\overline{v^{\Sigma}_{\Lambda}})^3\Big)\Big], \end{aligned} \tag{5.10}$$

with $k_1, k_2, k_3 \in \mathbb{R}$ to be properly chosen, and apply D_t to this expression. We have already calculated $D_t\mathrm{Op}^w_h(\Gamma)$ in (4.32), obtaining that the commutator is

$$\begin{aligned} [D_t, \mathrm{Op}^w_h(\Gamma)] = ih^{\frac{1}{2}}\mathrm{Op}^w_h &\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})p''(\xi)\xi\right) \\ &- \frac{ih}{2}\mathrm{Op}^w_h\left(\gamma'(\frac{x+p'(\xi)}{\sqrt{h}})(\frac{x+p'(\xi)}{\sqrt{h}})\right), \end{aligned} \tag{5.11}$$

where both appearing symbols belong to $S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$. The truncation of these symbols through a function $\chi(h^{\beta}\xi)$ as in Lemma 3.12, and Propositions 3.10, 3.11, together with estimates (5.2), (5.3) on v^{Σ}_{Λ}, show that the action of the commutator on brackets in (5.10) gives rise to a remainder $hR(v)$.
Denoting by $O(5)$ all terms of order 5 in $(v^{\Sigma}_{\Lambda}, \overline{v^{\Sigma}_{\Lambda}})$, and using (4.76), we can compute

$$\begin{aligned} D_tf^{\Sigma}_{\Lambda} = D_tv^{\Sigma}_{\Lambda} + \mathrm{Op}^w_h(\Gamma)\Big[&k_1h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^{\Sigma}_3(x)[3\varphi(x)\theta_h(x)(v^{\Sigma}_{\Lambda})^3 + h^2O(5)] \\ &+k_2h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^{\Sigma}_{-1}(x)[-\varphi(x)\theta_h(x)|v^{\Sigma}_{\Lambda}|^2\overline{v^{\Sigma}_{\Lambda}} + h^2O(5)] \\ &+k_3h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^{\Sigma}_{-3}(x)[-3\varphi(x)\theta_h(x)(\overline{v^{\Sigma}_{\Lambda}})^3 + h^2O(5)]\Big] \\ &+ hR(v), \end{aligned} \tag{5.12}$$

where $hR(v)$ includes also terms coming out from $D_t(h\tilde{\theta}_h(x))$, and

(5.13)
$$\begin{aligned}D_t f^{\Sigma}_{\Lambda} &= \varphi(x)\theta_h(x)v^{\Sigma}_{\Lambda} + h\theta_h(x)\Phi^{\Sigma}_1(x)|v^{\Sigma}_{\Lambda}|^2 v^{\Sigma}_{\Lambda}\\ &+ \mathrm{Op}^w_h(\Gamma)\Big[h\theta_h(x)\Big((3k_1+1)\Phi^{\Sigma}_3(x)(v^{\Sigma}_{\Lambda})^3 + (-k_2+1)\Phi^{\Sigma}_{-1}(x)|v^{\Sigma}_{\Lambda}|^2\overline{v^{\Sigma}_{\Lambda}}\\ &\qquad + (-3k_3+1)\Phi^{\Sigma}_{-3}(x)(\overline{v^{\Sigma}_{\Lambda}})^3\Big)\Big] + hR(v),\end{aligned}$$

where $h^2O(5)$ entered in $hR(v)$ from Propositions 3.10, 3.11, estimates (5.2), (5.3), and the fact that involved coefficients are $O(h^{-\sigma})$, for a small $\sigma>0$. We use again the definition of f^{Σ}_{Λ} to replace v^{Σ}_{Λ} in the linear and in the *characteristic* part. We have $h\theta_h(x)\Phi^{\Sigma}_1(x)|v^{\Sigma}_{\Lambda}|^2v^{\Sigma}_{\Lambda} = h\theta_h(x)\Phi^{\Sigma}_1(x)|f^{\Sigma}_{\Lambda}|^2 f^{\Sigma}_{\Lambda} + h^2O(5)$ and

(5.14)
$$\begin{aligned}\varphi(x)\theta_h(x)v^{\Sigma}_{\Lambda} &= \varphi(x)\theta_h(x)f^{\Sigma}_{\Lambda}\\ &- \varphi(x)\theta_h(x)\mathrm{Op}^w_h(\Gamma)\\ &\times\left[h\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Big(k_1\Phi^{\Sigma}_3(x)(v^{\Sigma}_{\Lambda})^3 + k_2\Phi^{\Sigma}_{-1}(x)|v^{\Sigma}_{\Lambda}|^2\overline{v^{\Sigma}_{\Lambda}} + k_3\Phi^{\Sigma}_{-3}(x)(\overline{v^{\Sigma}_{\Lambda}})^3\Big)\right]\\ &= \varphi(x)\theta_h(x)f^{\Sigma}_{\Lambda} - \mathrm{Op}^w_h(\Gamma)\\ &\times\Big[h\theta_h(x)\Big(k_1\Phi^{\Sigma}_3(x)(v^{\Sigma}_{\Lambda})^3 + k_2\Phi^{\Sigma}_{-1}(x)|v^{\Sigma}_{\Lambda}|^2\overline{v^{\Sigma}_{\Lambda}} + k_3\Phi^{\Sigma}_{-3}(x)(\overline{v^{\Sigma}_{\Lambda}})^3\Big)\Big]\\ &+ hR(v),\end{aligned}$$

where the last equality is consequence of the fact that, by Lemma 3.9, $[\varphi(x)\theta_h(x), \mathrm{Op}^w_h(\Gamma)] = h^{\frac{1}{2}-\sigma}\mathrm{Op}^w_h(r_0)$, $r_0 \in S_{\frac{1}{2},0}(\langle\frac{x+p'(\xi)}{\sqrt{h}}\rangle^{-1})$, $\sigma>0$ small. Again a truncation through $\chi(h^{\beta}\xi)$, and the application of Propositions 3.10, 3.11, together with estimates on v^{Σ}_{Λ}, ensure that the contribution coming from the action of the commutator on its argument enters in the remainder. We finally obtain

(5.15)
$$\begin{aligned}D_t f^{\Sigma}_{\Lambda} &= \varphi(x)\theta_h(x)f^{\Sigma}_{\Lambda} + h\theta_h(x)\Phi^{\Sigma}_1(x)|f^{\Sigma}_{\Lambda}|^2 f^{\Sigma}_{\Lambda}\\ &+ \mathrm{Op}^w_h(\Gamma)\Big[h\theta_h(x)\Big((2k_1+1)\Phi^{\Sigma}_3(x)(v^{\Sigma}_{\Lambda})^3 + (-2k_2+1)\Phi^{\Sigma}_{-1}(x)|v^{\Sigma}_{\Lambda}|^2\overline{v^{\Sigma}_{\Lambda}}\\ &\qquad + (-4k_3+1)\Phi^{\Sigma}_{-3}(x)(\overline{v^{\Sigma}_{\Lambda}})^3\Big)\Big] + hR(v),\end{aligned}$$

and we get rid of *non-characteristic* terms by requiring

$$\begin{cases}2k_1+1 &= 0\\ -2k_2+1 &= 0\\ -4k_3+1 &= 0\end{cases} \quad\Rightarrow\quad \begin{cases}k_1 = -\frac{1}{2}\\ k_2 = \frac{1}{2}\\ k_3 = \frac{1}{4},\end{cases}$$

from which we obtain the statement. □

Proposition 5.2. — *Let f^Σ_Λ be the function defined in* (5.4), *solution of the ODE* (5.5) *under the a priori estimates* (5.2), (5.3). *Then the following inequality holds:*

(5.16)
$$\|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \leq \|f^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} + C\int_1^t \tau^{-\frac{5}{4}+\sigma}(\|\mathcal{L}v(\tau,\cdot)\|_{L^2} + \|v(\tau,\cdot)\|_{H^s_h})\,d\tau,$$

for $\sigma>0$ small, and a positive constant $C>0$.

Proof. — Using the Equation (5.5), we can compute

(5.17)
$$\begin{aligned}\frac{\partial}{\partial t}|f^\Sigma_\Lambda(t,x)|^2 &= 2\Im(f^\Sigma_\Lambda\overline{D_t f^\Sigma_\Lambda})(t,x)\\ &= 2\Im(\varphi(x)\theta_h(x)|f^\Sigma_\Lambda|^2 + h\theta_h(x)\Phi^\Sigma_1(x)|f^\Sigma_\Lambda|^4 + hR(v)f^\Sigma_\Lambda)(t,x)\\ &= 2\Im(hR(v)f^\Sigma_\Lambda)(t,x) \leq 2h|f^\Sigma_\Lambda(t,x)||R(v)|,\end{aligned}$$

from which follows an integral inequality

$$\|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \leq \|f^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} + \int_1^t \frac{\|R(v)(\tau,\cdot)\|_{L^\infty}}{\tau}\,d\tau. \tag{5.18}$$

Using the estimate (4.15) for $R(v)$, we obtain the result

(5.19)
$$\|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty} \leq \|f^\Sigma_\Lambda(1,\cdot)\|_{L^\infty} + C\int_1^t \tau^{-\frac{5}{4}+\sigma}(\|\mathcal{L}v(\tau,\cdot)\|_{L^2}) + \|v(\tau,\cdot)\|_{H^s_h})\,d\tau.$$

□

Finally, the L^∞ estimate we found for f^Σ_Λ in the previous proposition enables us to propagate the uniform estimate on v, as showed in the following:

Proposition 5.3 (Propagation of the uniform estimate). — *Let v be a solution of the Equation* (4.7) *on some interval $[1,T]$, $T>1$ and $\sigma>0$ small. Then, for a fixed constant $K>1$, there exist two constants $A',B'>0$ sufficiently large, $\varepsilon_0>0$ sufficiently small, $s,\rho\in\mathbb{N}$ with $s\gg\rho$, such that, if $0<\varepsilon<\varepsilon_0$, and v satisfies*

$$\begin{aligned}&(A.1)\ \|v(t,\cdot)\|_{W^{\rho,\infty}_h} \leq A'\varepsilon,\\ &(B.1)\ \|v(t,\cdot)\|_{H^s_h} \leq B'\varepsilon\, h^{-\sigma},\\ &(B.2)\ \|\mathcal{L}v(t,\cdot)\|_{L^2} \leq B'\varepsilon\, h^{-\sigma},\end{aligned} \tag{5.20}$$

for every $t\in[1,T]$, then it satisfies also

$$(A.1')\ \|v(t,\cdot)\|_{W^{\rho,\infty}_h} \leq \frac{A'}{K}\varepsilon, \qquad \forall t\in[1,T]. \tag{5.21}$$

Proof. — The proof of the proposition comes directly from Proposition 5.2 and from the equivalence between $\|v^\Sigma_\Lambda\|_{L^\infty}$ and $\|f^\Sigma_\Lambda\|_{L^\infty}$. In fact, functions $\Phi^\Sigma_j(x)$ are cubic expressions in $d\varphi(x)$ and $\langle d\varphi(x)\rangle$, so they are bounded up to a loss $h^{-\delta}$, $\delta>0$ depending on β, on the support of $\tilde{\theta}_h(x)$, where also $\varphi(x)\gtrsim h^\beta>0$. This implies that $|\frac{\tilde{\theta}_h(x)}{\varphi(x)}\Phi^\Sigma_j(x)|\le Ch^{-\delta}$, $j\in\{3,-1,-3\}$, with a new $\delta>0$ depending linearly on β, so that by the definition of f^Σ_Λ, Proposition 3.11 and estimates (5.2), (5.3) (which follow from (5.20), as already observed in Proposition 5.1), we find

$$\frac{1}{2}\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty}\le\|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty}\le 2\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty}. \tag{5.22}$$

Furthermore, the *a priori* estimate on the $W^{\rho,\infty}_h$ norm of v extends to the L^∞ norm of v^Σ_Λ just by the decomposition

$$v^\Sigma_\Lambda=v^\Sigma-v^\Sigma_{\Lambda^c}, \tag{5.23}$$

and by Proposition 4.4, so for example at time $t=1$ we have

$$\begin{aligned}\|v^\Sigma_\Lambda(1,\cdot)\|_{L^\infty}&\le\|v^\Sigma(1,\cdot)\|_{L^\infty}+\|v^\Sigma_{\Lambda^c}(1,\cdot)\|_{L^\infty}\\&\le\|v(1,\cdot)\|_{W^{\rho,\infty}_h}+C(\|\mathcal{L}v(1,\cdot)\|_{L^2}+\|v(1,\cdot)\|_{H^s_h})\\&\le\frac{A'}{32K}\varepsilon+CB'\varepsilon\\&\le\frac{A'}{16K}\varepsilon,\end{aligned} \tag{5.24}$$

where we choose $A'>0$ sufficiently large such that $\|v(1,\cdot)\|_{W^{\rho,\infty}_h}\le\frac{A'}{32K}\varepsilon$ and $CB'<\frac{A'}{32K}$. Therefore

$$\|f^\Sigma_\Lambda(1,\cdot)\|_{L^\infty}\le 2\|v^\Sigma_\Lambda(1,\cdot)\|_{L^\infty}\le\frac{A'}{8K}\varepsilon. \tag{5.25}$$

Using Proposition 5.2, (5.25) and the *a priori* estimates (B.1), (B.2), we find that

$$\begin{aligned}\|f^\Sigma_\Lambda(t,\cdot)\|_{L^\infty}&\le\frac{A'}{8K}\varepsilon+CB'\varepsilon\int_1^t\tau^{-\frac{5}{4}+\sigma}\,d\tau\\&\le\frac{A'}{8K}\varepsilon+C'B'\varepsilon\\&\le\frac{A'}{4K}\varepsilon,\end{aligned} \tag{5.26}$$

where again the last inequality follows from the choice of $A'>0$ large enough to have $C'B'<\frac{A'}{8K}$. Then we have

$$\|v^\Sigma_\Lambda(t,\cdot)\|_{L^\infty}\le\frac{A'}{2K}\varepsilon, \tag{5.27}$$

and

$$\begin{aligned}
\|v^{\Sigma}(t,\cdot)\|_{L^\infty} &\le \|v^{\Sigma}_{\Lambda}(t,\cdot)\|_{L^\infty} + \|v^{\Sigma}_{\Lambda^c}(t,\cdot)\|_{L^\infty} \\
&\le \frac{A'}{2K}\varepsilon + CB'\varepsilon h^{\frac{1}{4}-\sigma'} \\
&\le \frac{A'}{K}\varepsilon.
\end{aligned} \tag{5.28}$$

□

5.2. Asymptotics. — We want now to derive the asymptotic expansion for the function $\langle hD\rangle^{-1}v$, v being the solution of (4.7), when it exists on $[1,+\infty[$. The reader can refer to the next subsection to find the proof of the global existence of v, which implies also the global existence of the solution u of the starting problem (1.1).

PROPOSITION 5.4. — *Under the same hypothesis as Theorem 4.1, with $T = +\infty$, there exists a family $(\theta_h(x))_h$ of C^∞ functions, real valued, supported in some interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$, $\theta_h \equiv 1$ on an interval of the same form, such that $(h\partial_h)^k\theta_h(x)$ is bounded for any k, and a family $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ of $\mathbb{C}$-valued functions on $\mathbb{R}$, supported in $[-1,1]$, uniformly bounded, such that*

$$\begin{aligned}
\langle hD\rangle^{-1}v = \varepsilon a_\varepsilon(x)\exp\Big[i\varphi(x)\int_1^t \theta_{1/\tau}(x)d\tau \\
+ i\varepsilon^2|a_\varepsilon(x)|^2\Phi^{\Sigma}_1(x)\int_1^t \theta_{1/\tau}(x)\frac{d\tau}{\tau}\Big] + t^{-\frac{1}{4}+\sigma}r(t,x),
\end{aligned} \tag{5.29}$$

where $h=\frac{1}{t}$, $\sigma>0$ is small and $\sup_{t\ge 1}\|r(t,\cdot)\|_{L^2\cap L^\infty} \le C\varepsilon$.

Proof. — Let us take $\Sigma(\xi) = \langle\xi\rangle^{-1}$, so that $v^{\Sigma} = \langle hD\rangle^{-1}v$. Summing all prevoius results, we have obtained that under the *a priori* estimates (4.11), (4.12), the function f^{Σ}_{Λ} defined in (5.4) satisfies (5.5), with a remainder $R(v) = O_{L^\infty\cap L^2}(\varepsilon t^{-\frac{1}{4}+\sigma})$, for a sufficiently small $\sigma>0$. Inequality (5.17) and the bound (4.15) show that

$$\|f^{\Sigma}_{\Lambda}(t,\cdot) - f^{\Sigma}_{\Lambda}(t',\cdot)\|_{L^\infty} \le C\int_{t'}^t \tau^{-\frac{5}{4}+\sigma}(\|\mathcal{L}v(\tau,\cdot)\|_{L^2} + \|v(\tau,\cdot)\|_{H^s_h})\,d\tau.$$

Combining with the *a priori* estimate (4.12), there is a continuous function $x\to|\tilde{a}(x)|$ such that $\big||f^{\Sigma}_{\Lambda}(t,x)|^2 - |\tilde{a}(x)|^2\big| = O(\varepsilon t^{-\frac{1}{2}+\sigma})$, for a new small $\sigma>0$, and replacing this new function in (5.5) we obtain the equation

$$D_t f^{\Sigma}_{\Lambda} = \theta_h(x)\left[\varphi(x) + h\Phi^{\Sigma}_1(x)|\tilde{a}(x)|^2\right] f^{\Sigma}_{\Lambda} + h\,r(t,x), \tag{5.30}$$

for $r = O_{L^\infty \cap L^2}(\varepsilon t^{-\frac{1}{4}+\sigma})$, which is a linear non homogeneous ODE for f^Σ_Λ. This implies that there is a $O(\varepsilon)$ continuous function $\tilde{a}$ such that

(5.31)
$$f^\Sigma_\Lambda(t,x) = \tilde{a}(x) \exp\left[i\varphi(x)\int_1^t \theta_{1/\tau}(x)d\tau + i|\tilde{a}(x)|^2\Phi^\Sigma_1(x)\int_1^t \theta_{1/\tau}(x)\frac{d\tau}{\tau}\right]$$
(5.32)
$$+ t^{-\frac{1}{4}+\sigma} r(t,x),$$

for a new r. Finally, using the definition of f^Σ_Λ and Proposition 4.4, we have $\|f^\Sigma_\Lambda - v^\Sigma_\Lambda\|_{L^2\cap L^\infty} = O(\varepsilon t^{-\frac{3}{4}+\sigma})$ and $\|v^\Sigma_\Lambda - v^\Sigma\|_{L^2\cap L^\infty} = O(\varepsilon t^{-\frac{1}{4}+\sigma})$, so we can deduce from (5.31) the asymptotic expansion for $v^\Sigma = \langle hD\rangle^{-1}v$. Since (4.38) for $a \equiv 1$ shows that v^Σ vanishes to main order when $x \notin [-1,1]$ and $t \to +\infty$, we get that $\tilde{a}(x)$ is supported for $x \in [-1,1]$, and we conclude the proof choosing $\tilde{a}(x) = \varepsilon a_\varepsilon(x)$ for a bounded $a_\varepsilon(x)$ as in the statement. □

5.3. End of the Proof

Proof of Theorem 1.2. — Let us prove that, for small enough data, the solution of the initial Cauchy problem (1.1) is global. We show that we can propagate some convenient *a priori* estimates on u, as stated in Theorem 1.3, namely we want to show that there are some integers $s \gg \rho \gg 1$, some constants $A, B > 0$ large enough, $\varepsilon_0 \in]0,1]$ and $\sigma > 0$ small enough such that, if $u \in C^0([1,T[; H^{s+1}) \cap C^1([1,T[; H^s)$ is solution of (1.1) for some $T > 1$, and satisfies

$$\begin{aligned} &\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \le A\varepsilon t^{-\frac{1}{2}}, \\ &\|Zu(t,\cdot)\|_{H^1} \le B\varepsilon t^\sigma, \qquad \|\partial_t Zu(t,\cdot)\|_{L^2} \le B\varepsilon t^\sigma \\ &\|u(t,\cdot)\|_{H^s} \le B\varepsilon t^\sigma, \qquad \|\partial_t u(t,\cdot)\|_{H^{s-1}} \le B\varepsilon t^\sigma, \end{aligned}$$

for every $t \in [1,T]$, then in the same interval it verifies improved estimates,

$$\begin{aligned} &\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \le \frac{A}{2}\varepsilon t^{-\frac{1}{2}}, \\ &\|Zu(t,\cdot)\|_{H^1} \le \frac{B}{2}\varepsilon t^\sigma, \qquad \|\partial_t Zu(t,\cdot)\|_{L^2} \le \frac{B}{2}\varepsilon t^\sigma \\ &\|u(t,\cdot)\|_{H^s} \le \frac{B}{2}\varepsilon t^\sigma, \qquad \|\partial_t u(t,\cdot)\|_{H^{s-1}} \le \frac{B}{2}\varepsilon t^\sigma. \end{aligned}$$

We can immediately observe that from (1.6), these bounds are verified at time $t = 1$. In Theorem 2.2 in Section 2, we proved that we can improve the energy bounds $\|Zu(t,\cdot)\|_{H^1}$, $\|\partial_t Zu(t,\cdot)\|_{L^2}$, $\|u(t,\cdot)\|_{H^s}$ and $\|\partial_t u(t,\cdot)\|_{H^{s-1}}$. To show that the propagation of the uniform bound $\|u(t,\cdot)\|_{W^{t,\rho,\infty}}$ holds, we passed from Equation (1.1) to (4.2) at the beginning of Section 4, and then we showed that the function v is solution of (4.7). The *a priori* assumptions made on u

imply the following estimates on v,

$$\begin{aligned}
&\|v(t,\cdot)\|_{W^{\rho-1,\infty}_h} \le C_1 A\varepsilon,\\
&\|\mathcal{L}v(t,\cdot)\|_{H^1_h} \le 5B\varepsilon h^{-\sigma}, \qquad \|v(t,\cdot)\|_{H^{s-1}_h} \le B\varepsilon h^{-\sigma},
\end{aligned} \tag{5.33}$$

for $h^{-1} := t$ in $[1,T]$. In fact, from (4.1), the Definition (4.5) of v in semiclassical coordinates and the Equation (1.1),

$$C_2\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \le t^{-\frac{1}{2}}\|v(t,\cdot)\|_{W^{\rho-1,\infty}_h} \le C_1\|u(t,\cdot)\|_{W^{t,\rho,\infty}},$$

$$\|v(t,\cdot)\|_{H^{s'}_h} = \|w(t,\cdot)\|_{H^{s'}} \le \|\partial_t u(t,\cdot)\|_{H^{s'}} + \|u(t,\cdot)\|_{H^{s'+1}},$$

for some positive constants C_1, C_2, so the first and third inequality in (5.33) are satisfied. Moreover, $\mathcal{L}v$ can be expressed in term of w, Zw, as showed below using Equation (4.7),

$$\begin{aligned}
\frac{1}{i}Zw(t,y) &= h^{\frac{1}{2}}\left[(1-x^2)D_x + txD_t + i\frac{x}{2}\right]v(t,x)|_{x=\frac{y}{t}}\\
&= \left(h^{\frac{1}{2}}\left[(1-x^2)D_x + tx\operatorname{Op}^w_h(x\xi+p(\xi)) + i\frac{x}{2}\right]v + h^{\frac{1}{2}}x\widetilde{P}\right)|_{x=\frac{y}{t}}\\
&= \left(h^{\frac{1}{2}}\left[D_x + tx\operatorname{Op}^w_h(p(\xi))\right]v + h^{\frac{1}{2}}x\widetilde{P}\right)|_{x=\frac{y}{t}},
\end{aligned} \tag{5.34}$$

where $\widetilde{P}$ denotes the right hand side of (4.7) multiplied by h^{-1}. Using symbolic calculus of Proposition 3.8,

$$\begin{aligned}
\frac{1}{i}Zw(t,y) &= \left(h^{\frac{1}{2}}\left[h^{-1}\operatorname{Op}^w_h(xp(\xi)+\xi) - \frac{1}{2i}\operatorname{Op}^w_h(p'(\xi))\right]v + h^{\frac{1}{2}}x\widetilde{P}\right)|_{x=\frac{y}{t}}\\
&= \left(h^{\frac{1}{2}}\left[\operatorname{Op}^w_h(p(\xi))\mathcal{L}v - \frac{1}{i}\operatorname{Op}^w_h(p'(\xi))v + x\widetilde{P}\right]\right)|_{x=\frac{y}{t}},
\end{aligned} \tag{5.35}$$

where we used that $p(\xi)=\sqrt{1+\xi^2}$, $p'(\xi)=\xi/p(\xi)$. Therefore, since $\operatorname{Op}^w_h(p'(\xi)) : H^s_h \to H^s_h$ are uniformly bounded for all $s\in\mathbb{R}$ by Proposition 3.10, and from $\|v(t,\cdot)\|_{L^2} = \|w(t,\cdot)\|_{L^2}$, we derive $\|\mathcal{L}v(t,\cdot)\|_{H^1_h} \le \|Zw(t,\cdot)\|_{L^2} + \|w(t,\cdot)\|_{L^2} + \|x\widetilde{P}\|_{L^2}$, where

$$\|x\widetilde{P}(t,\cdot)\|_{L^2} \le C\|v(t,\cdot)\|^2_{W^{\rho-1,\infty}_h}(\|\mathcal{L}v(t,\cdot)\|_{L^2} + \|w(t,\cdot)\|_{H^s}).$$

Then, from Definition (4.1),

$$\|w(t,\cdot)\|_{L^2} \le \|\partial_t u(t,\cdot)\|_{L^2} + \|u(t,\cdot)\|_{H^1},$$

$$\|Zw(t,\cdot)\|_{L^2} \le \|\partial_t Zu(t,\cdot)\|_{L^2} + \|Zu(t,\cdot)\|_{H^1} + \|\partial_t u(t,\cdot)\|_{L^2} + \|u(t,\cdot)\|_{H^1},$$

so we can use the uniform estimate $\|v(t,\cdot)\|_{W^{\rho-1,\infty}_h} \le C_1 A\varepsilon$, choose $\varepsilon_0 \ll 1$ small enough such that $CC_1A^2\varepsilon_0^2 < \frac{1}{2}$, and use the *a priori* energy bounds on u in (1.11), to have

$$\|\mathcal{L}v(t,\cdot)\|_{H^1_h} \le 2\|Zu(t,\cdot)\|_{L^2} + 2\|u(t,\cdot)\|_{L^2} + \|u(t,\cdot)\|_{H^s} \le 5B\varepsilon h^{-\sigma}.$$

Under these bounds on v, in Proposition 5.3 we proved that, for $A' = C_1A$ and $B' = 5B$, the uniform estimate on v can be propagated, choosing for instance $K = \frac{2C_1}{C_2}$ to obtain $\|v(t,\cdot)\|_{W_h^{\rho-1,\infty}} \le \frac{AC_2}{2}\varepsilon$, and then $\|u(t,\cdot)\|_{W^{t,\rho,\infty}} \le \frac{A}{2}\varepsilon t^{-\frac{1}{2}}$, which concludes the proof of the boostrap and of global existence.

We prove now the asymptotics. We consider $\Sigma(\xi) = \langle\xi\rangle^{\rho+1}$ and we write

$$\langle hD\rangle^{-1}v = \mathrm{Op}_h^w(\langle\xi\rangle^{-1}\langle\xi\rangle^{-\rho-1})v^\Sigma.$$

Using Proposition 4.7, we develop the symbol $\langle\xi\rangle^{-\rho-2}$ at $\xi = d\varphi(x)$,

$$\mathrm{Op}_h^w(\langle\xi\rangle^{-\rho-2})v^\Sigma = \theta_h(x)\langle d\varphi(x)\rangle^{-\rho-2}v^\Sigma + O_{L^\infty\cap L^2}(\varepsilon h^{\frac{1}{4}-\sigma}),$$

and using the expression obtained in (5.29), along with the uniform bound on v^Σ, we derive that in the limit $t\to+\infty$ the function $\tilde{a}(x) = \varepsilon a_\varepsilon(x)$ verifies

$$|\tilde{a}(x)| \le |\theta_h(x)\langle d\varphi(x)\rangle^{-\rho-2}v^\Sigma| + O(\varepsilon t^{-\frac{1}{4}+\sigma}) \overset{t\to+\infty}{\le} C\varepsilon\langle d\varphi(x)\rangle^{-\rho-2}. \tag{5.36}$$

For points x in $]-1,1[$ such that $\langle d\varphi(x)\rangle \ge \alpha h^{-\beta}$, for a small $\alpha > 0$, we have $|\tilde{a}(x)| = O(\varepsilon h^{\beta(\rho+2)})$ and then the corresponding contribution to the right hand side of (5.29) is $O(\varepsilon t^{-\min(\beta(\rho+2),\frac{1}{4}-\sigma)})$ in $L^\infty\cap L^2$.

Let us now consider points x in $]-1,1[$ such that $\langle d\varphi(x)\rangle \le \alpha h^{-\beta}$, and remind that the function $\theta_h(x)$ in (5.29) is identically equal to one on some interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$. We can write

$$\int_1^t \theta_{1/\tau}(x)d\tau = t-1+\int_1^\infty(\theta_{1/\tau}(x)-1)d\tau - \int_t^\infty(\theta_{1/\tau}(x)-1)d\tau, \tag{5.37}$$

observing that on the support of $\theta_{1/\tau}(x)-1$, $\tau < \max c^{\frac{1}{2\beta}}(1-x,x+1)^{-\frac{1}{2\beta}}$. Therefore the last integral is taken on a finite interval and since $|x\pm 1| \sim \langle d\varphi(x)\rangle^{-2}$ as $x\to\mp 1$ by (3.34), this implies that at the same time we have $\tau \le c\langle d\varphi(x)\rangle^{\frac{1}{\beta}}$ and $\langle d\varphi(x)\rangle^{\frac{1}{\beta}} \le \alpha t$. For $t\le\tau$ and $\alpha > 0$ small, this leads to a contradiction and to the fact that the last integral in (5.37) is equal to zero. Then in (5.29) we can write

$$a_\varepsilon(x)\exp\left[i\varphi(x)\int_1^t\theta_{1/\tau}(x)d\tau\right] = a_\varepsilon(x)\exp[i\varphi(x)t + ig(x)],$$

with $g(x) = \varphi(x)\left[\int_1^\infty(\theta_{1/\tau}(x)-1)d\tau - 1\right]$, and similarly, for x satisfying $\langle d\varphi(x)\rangle \le \alpha h^{-\beta}$,

$$|a_\varepsilon(x)|^2\Phi_1^\Sigma(x)\int_1^t\theta_{1/\tau}(x)\frac{d\tau}{\tau} = |a_\varepsilon(x)|^2\Phi_1^\Sigma(x)\log t + \tilde{g}(x),$$

for $\tilde{g}(x) = |a_\varepsilon(x)|^2\Phi_1^\Sigma(x)\left[\int_1^\infty(\theta_{1/\tau}(x)-1)\frac{d\tau}{\tau} - 1\right]$. Moreover, for $\langle d\varphi(x)\rangle \le \alpha h^{-\beta}$ the coefficient $a_{(1,1,-1)}(x)$ appearing in $\Phi_1^\Sigma(x)$ is equal to $\langle d\varphi(x)\rangle^{-1}$, since

$\chi(h^\beta d\varphi(x))\gamma(\frac{x+p'(d\varphi(x))}{\sqrt{h}}) \equiv 1$ if α is chosen sufficiently small, which implies that $\Phi_1^\Sigma(x)$ is exactly $\Phi_1(x)$ introduced in (1.8). Modifying the function $a_\varepsilon(x)$ by a factor of modulus one, we derive from (5.29) the asymptotic behavior for $\langle hD\rangle^{-1}v$:

$$\langle hD\rangle^{-1}v = \varepsilon a_\varepsilon(x)\exp\left[i\varphi(x)t + i(\log t)\varepsilon^2|a_\varepsilon(x)|^2\Phi_1(x)\right] + t^{-\theta}r(t,x), \tag{5.38}$$

for some $\theta > 0$ and $\|r(t,\cdot)\|_{L^\infty} = O(\varepsilon)$, and reminding the relationship between v and w in (4.5), and between w and u in (4.1), we finally obtain the asymptotics for u in (1.7). $\square$

Appendix

This appendix is devoted to the detailed proof of Proposition 3.8 and Lemma 3.9, which are technical.

Proof of Proposition 3.8. — Let us expand $a(x+z,\xi+\zeta)$ at (x,ξ) with Taylor's formula:

$$\begin{aligned} a(x+z,\xi+\zeta) = a(x,\xi) &+ \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{1}{\alpha!}\,\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x,\xi)z^{\alpha_1}\zeta^{\alpha_2} \\ &+ \sum_{\substack{\beta=(\beta_1,\beta_2)\\ |\beta|=k+1}} \frac{k+1}{\beta!}\,z^{\beta_1}\zeta^{\beta_2}\int_0^1 \partial_x^{\beta_1}\partial_\xi^{\beta_2}a(x+tz,\xi+t\zeta)(1-t)^k\,dt, \end{aligned}$$

and replace this development in (3.11), obtaining:

$$\begin{aligned} a\,\sharp\,b = &\frac{1}{(\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}a(x,\xi)b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\ &+ \frac{1}{(\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)} \\ &\times \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{1}{\alpha!}\,\partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x,\xi)b(x+y,\xi+\eta)\,z^{\alpha_1}\zeta^{\alpha_2}\,dyd\eta dzd\zeta \\ &+ \frac{1}{(\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)} \\ &\times \Big\{ \sum_{\substack{\beta=(\beta_1,\beta_2)\\ |\beta|=k+1}} \frac{k+1}{\beta!}\,z^{\beta_1}\zeta^{\beta_2}\int_0^1 \partial_x^{\beta_1}\partial_\xi^{\beta_2}a(x+tz,\xi+t\zeta)(1-t)^k\,dt\Big\} \\ &\times b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\ := &\ I_1 + I_2 + I_3. \end{aligned}$$

From a direct calculation and using that the inverse Fourier transform of the complex exponential is the delta function, i.e.,

$$\text{(A)} \qquad \frac{1}{\pi h}\int_{\mathbb{R}} e^{\frac{2i}{h}XY}\,dY = \delta_0(X),$$

we derive

$$\begin{aligned} I_1 &= \frac{1}{(\pi h)^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)} a(x,\xi)b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\ &= a(x,\xi)\int_{\mathbb{R}^2} b(x+y,\xi+\eta)\delta_0(y)\delta_0(\eta)\,dyd\eta = a(x,\xi)b(x,\xi), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \\ &= \frac{1}{(\pi h)^2}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{1}{\alpha!}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}\,\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)b(x+y,\xi+\eta)\,z^{\alpha_1}\zeta^{\alpha_2}\,dyd\eta dzd\zeta \\ &= \frac{1}{(\pi h)^2}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{1}{\alpha!}\left(\frac{h}{2i}\right)^{|\alpha|} \\ &\quad\times\int_{\mathbb{R}^4} \partial_\eta^{\alpha_1}(-\partial_y^{\alpha_2})e^{\frac{2i}{h}(\eta z - y\zeta)}\,\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\ &= \frac{1}{(\pi h)^2}\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{(-1)^{\alpha_1}}{\alpha!}\left(\frac{h}{2i}\right)^{|\alpha|} \\ &\quad\times\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)}\,\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)\,\partial_y^{\alpha_2}\partial_\eta^{\alpha_1} b(x+y,\xi+\eta)\,dyd\eta dzd\zeta \\ &= \sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ 1\le|\alpha|\le k}} \frac{(-1)^{\alpha_1}}{\alpha!}\left(\frac{h}{2i}\right)^{|\alpha|}\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x,\xi)\,\partial_x^{\alpha_2}\partial_\xi^{\alpha_1} b(x,\xi). \end{aligned}$$

The same calculation shows that I_3 is given by

$$\begin{aligned} I_3 = \frac{k+1}{(\pi h)^2}\left(\frac{h}{2i}\right)^{k+1} &\sum_{\substack{\alpha=(\alpha_1,\alpha_2)\\ |\alpha|=k+1}} \frac{(-1)^{\alpha_1}}{\alpha!}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z - y\zeta)} \\ &\times\left\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2} a(x+tz,\xi+t\zeta)(1-t)^k dt\partial_y^{\alpha_2}\partial_\eta^{\alpha_1} b(x+y,\xi+\eta)\right\} dyd\eta dzd\zeta, \end{aligned}$$

and it belongs to $h^{(k+1)(1-(\delta_1+\delta_2))}S_{\delta,\beta}(M_1M_2)$ since

$$\begin{aligned}
&\frac{1}{h^2}\int_{\mathbb{R}^4} e^{\frac{2i}{h}(\eta z-y\zeta)}\\
&\quad\times\Big\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x+tz,\xi+t\zeta)(1-t)^k dt\partial_y^{\alpha_2}\partial_\eta^{\alpha_1}b(x+y,\xi+\eta)\Big\}\,dyd\eta dzd\zeta\\
&=\int_{\mathbb{R}^4} e^{2i(\eta z-y\zeta)}\\
&\quad\times\Big\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x+t\sqrt{h}z,\xi+t\sqrt{h}\zeta)(1-t)^k dt\,\partial_y^{\alpha_2}\partial_\eta^{\alpha_1}b(x+\sqrt{h}y,\xi+\sqrt{h}\eta)\Big\}\\
&\quad\times dyd\eta dzd\zeta\\
&=\int_{\mathbb{R}^4}\left(\frac{1+2iy\partial_\zeta}{1+4y^2}\right)^N\left(\frac{1-2i\eta\partial_z}{1+4\eta^2}\right)^N\left(\frac{1-2iz\partial_\eta}{1+4z^2}\right)^N\left(\frac{1+2i\zeta\partial_y}{1+4\zeta^2}\right)^N e^{2i(\eta z-y\zeta)}\\
&\quad\times\Big\{\int_0^1 \partial_x^{\alpha_1}\partial_\xi^{\alpha_2}a(x+t\sqrt{h}z,\xi+t\sqrt{h}\zeta)(1-t)^k dt\,\partial_y^{\alpha_2}\partial_\eta^{\alpha_1}b(x+\sqrt{h}y,\xi+\sqrt{h}\eta)\Big\}\\
&\quad\times dyd\eta dzd\zeta
\end{aligned}$$

so integrating by parts,

$$\begin{aligned}
&\le Ch^{-(\delta_1+\delta_2)(\alpha_1+\alpha_2)}\int_{\mathbb{R}^4}\langle y\rangle^{-N}\langle\eta\rangle^{-N}\langle z\rangle^{-N}\langle\zeta\rangle^{-N}\\
&\quad\times\Big\{\int_0^1 M_1(x+t\sqrt{h}z,\xi+t\sqrt{h}\zeta)dt\,M_2(x+\sqrt{h}y,\xi+\sqrt{h}\eta)\Big\}\,dyd\eta dzd\zeta\\
&\le Ch^{-(\delta_1+\delta_2)(k+1)}\int_{\mathbb{R}^4}\langle y\rangle^{-N+N_0}\langle\eta\rangle^{-N+N_0}\langle z\rangle^{-N+N_0}\langle\zeta\rangle^{-N+N_0}\,dyd\eta dzd\zeta\\
&\quad\times M_1(x,\xi)M_2(x,\xi)\\
&\le Ch^{-(\delta_1+\delta_2)(k+1)}M_1(x,\xi)M_2(x,\xi).
\end{aligned}$$

Equivalently, one can show that $|\partial^\alpha I_3|\le Ch^{(k+1)(1-(\delta_1+\delta_2))-\delta|\alpha|}M_1(x,\xi)M_2(x,\xi)$. The last statement of the proposition follows immediately if we replace in previous inequalities M_1 and M_2 respectively by M_1^{k+1}, M_2^{k+1}. □

Proof of Lemma 3.9. — The proof of the lemma is the same as the previous one if, when we calculate to which class the remainder r_k belongs, we remark that

$$\begin{aligned}
\left\langle\frac{x+t\sqrt{h}z+f(\xi+t\sqrt{h}\zeta)}{\sqrt{h}}\right\rangle^{-d}&=\left\langle\frac{x+f(\xi)}{\sqrt{h}}+tz+tb(\xi,\zeta)\zeta\right\rangle^{-d}\\
&\lesssim\langle tz\rangle^N\langle t\zeta\rangle^N\left\langle\frac{x+f(\xi)}{\sqrt{h}}\right\rangle^{-d}
\end{aligned}$$

$$\left\langle \frac{x+\sqrt{h}y+f(\xi+\sqrt{h}\eta)}{\sqrt{h}} \right\rangle^{-l} = \left\langle \frac{x+f(\xi)}{\sqrt{h}} + y + b'(\xi,\eta)\eta \right\rangle^{-l}$$
$$\lesssim \langle y\rangle^N \langle\eta\rangle^N \left\langle \frac{x+f(\xi)}{\sqrt{h}} \right\rangle^{-l},$$

with $b(\xi,\zeta)=\int_0^1 f'(\xi+st\sqrt{h}\zeta)ds \lesssim 1$, $b'(\xi,\eta)=\int_0^1 f'(\xi+s\sqrt{h}\eta)ds \lesssim 1$, for a certain $N\in\mathbb{N}$. □

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LARGE DEVIATIONS AND PATH PROPERTIES OF THE TRUE SELF-REPELLING MOTION

by Laure Dumaz

Abstract. — We derive some large deviation bounds for events related to the "true self-repelling motion," a one-dimensional self-interacting process introduced by Tóth and Werner, that has very different path properties than usual diffusion processes. We then use these estimates to study certain of these path properties such as its law of iterated logarithms for both small and large times.

Résumé (*Grandes déviations et propriétés trajectorielles du « vrai » processus auto-répulsif*). — Nous montrons dans cet article certaines bornes de grandes déviations pour des événements liés au « vrai » processus auto-répulsif, un processus unidimensionnel introduit par Toth et Werner, qui a des propriétés trajectorielles très différentes de celles des diffusions usuelles. Nous utilisons ensuite ces estimées pour étudier certaines de ces propriétés trajectorielles concernant la loi du logarithme itéré pour les petits temps ainsi que les grands temps.

1. Introduction

In the present paper, we study some features of a self-interacting one-dimensional process called the true self-repelling motion, defined by Tóth and

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Werner in [9]. Let us first very briefly recall the intuitive definition of this process and describe the motivations that lead to our study.

The true self-repelling motion is a continuous real-valued process $(X_t, t \geq 0)$ that is locally self-interacting with its past occupation-time. More precisely, for each positive time t, define its occupation-time measure μ_t that assigns to each interval $I \subset \mathbb{R}$, the time spent in it by X before time t:

$$\mu_t(I) = \int_0^t 1_{X_s \in I}\, ds.$$

It turns out that for this particular process X, almost surely for each t, the measure μ_t has a continuous density $L_t(x)$. By analogy with semi-martingales, where such occupation-time densities also exist, the curve $x \mapsto L_t(x)$ is called the "local-time" profile of X at time t. Heuristically, the dynamics of X_t is such that the TSRM is locally pushed in the direction of the negative "gradient" of the local time at its current position. Loosely formulated, one can write $dX_t = -\nabla_x L_t(X_t)dt$ (even if $(X_t, t \geq 0)$ is a random process). For more details and comments on this description, we refer to [9]. It turns out that this process is of a very different type than diffusions. For example (see again [9]), its quadratic variation almost surely vanishes whereas its variation of power $3/2$ is positive and finite. Similarly, it does not have the Brownian scaling property, it has instead a $2/3$ scaling behavior i.e., for any positive λ, $(X_{\lambda t}, t \geq 0)$ has the same law as $(\lambda^{2/3} X_t, t \geq 0)$.

This same exponent $2/3$ appears in various other models that can be interpreted as continuous height-fluctuations of $1+1$-dimensional models in the Khardar-Parisi-Zhang universality class (such as the Tracy-Widom distribution for eigenvalues of large random matrices, the movement of the second-class particle in a TASEP etc.). TSRM seems however at present to be one of the few such "non-diffusive" continuous processes that probabilists can define (see also [2] for related questions). All this gives us some motivation to study in more detail its behavior, in order to see what features it shares with the other previously-mentioned models, and also for its own independent interest.

Let us now describe briefly the results of the present paper: Both for the process $(X_t, t \geq 0)$ itself as for the height process $(H_t, t \geq 0)$, we give upper and lower bounds for the probability that their value at a given time is very large. Combined with $0-1$-law arguments, this enables us to derive almost sure fluctuation results (of the type of the law of the iterated logarithm) for these two processes. For instance, we shall see that $\limsup_{t\to\infty} X_t/(t^{2/3}(\log\log t)^{1/3})$ is almost surely equal to a finite positive constant, and a similar result when $t \to 0$.

The construction of the process X_t is based on a family of coalescing one-dimensional Brownian motions starting from all points in the plane. Such families had been constructed by Arratia in [1], and further studied in [9, 8, 3, 6] and are called "Brownian web" in the latter papers. As a consequence, the estimates on the TSRM will follow from results concerning this Brownian web. In Section 2, we will recall some aspects of the construction of TSRM and some features of the Brownian web. In Section 3, we will focus on the large deviation estimates concerning X_1, we then derive the LIL for X in Section 4, and we finally focus on the fluctuations of the height-process in the final Section 5.

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2. Preliminaries and notations

In this section, we put down some notation, and collect some elementary estimates that will be useful later on.

2.1. Versions of the Brownian web. — The true self-repelling motion (TSRM) is a deterministic function of a certain family of coalescing one-dimensional Brownian motions. There are two natural variants of TSRM, that respectively correspond to such Brownian families in the entire plane (this is the "stationary" TSRM, this version has stationary increments) or in the upper half-plane (this is the TSRM with "zero-initial conditions"). Other initial conditions are also possible, see Section 4 of [8] for examples.

Let us briefly first recall the construction in the *stationary* case which will be the main focus of this paper. To start with, choose any deterministic countable dense family Q of points $(\tilde{x}, \tilde{h})$ in the plane, say $Q = \mathbb{Q}^2$. It is then possible to define the joint law of a family $(\Lambda_{\tilde{x},\tilde{h}}(\cdot), (\tilde{x}, \tilde{h}) \in Q)$ in such a way that, for each $(\tilde{x}, \tilde{h}) \in Q$, $\Lambda_{\tilde{x},\tilde{h}}$ is a function from $[\tilde{x}, \infty)$ into $\mathbb{R}$, that is distributed like a Brownian motion started from height $\tilde{h}$ at time $\tilde{x}$. Furthermore (see e.g., [9] for details), different curves are "independent until their first meeting time" and they coalesce after this meeting time (and follow the same Brownian evolution). Recall that Q is dense in the plane, so that the picture of all these lines is dense in the plane. The coalescent structure nevertheless defines a tree-like structure rooted "at $x = +\infty$". This family of curves Λ is often referred to as the "forward lines".

If we are given a countable dense family $\tilde{Q}$ in the plane, then one can almost surely define the family of "backward" lines $(\Lambda_{\tilde{x},\tilde{h}}(\cdot), (\tilde{x}, \tilde{h}) \in \tilde{Q})$ such that each $\Lambda_{\tilde{x},\tilde{h}}$ is now a function defined on $(-\infty, \tilde{x}]$ in such a way that the backward lines can be viewed as the "dual tree" of the previous dense tree (it is therefore a deterministic function of all forward lines). It is proved in [9] that this family of backward lines has the same law as the reversed image (changing x into $-x$) of the law of the forward lines (choosing $\tilde{Q}$ to be the symmetric image of Q).

There is an alternative construction where one does not have to first define the whole dense family of forward lines to construct the backward ones: instead, one can construct the forward and the backward paths one by one for each $(\tilde{x}, \tilde{h}) \in Q$ inductively, applying a reflection/coalescence rule explained in the Section 3.1.4 of [8]. Roughly, the rule is that when two curves meet, there is coalescence if they are of the same type (both backward or both forward), and otherwise, the two curves are "reflected on each other". Note that the proofs in [8] use the discrete model (with reflecting/coalescing random walks) and an invariance principle.

Both constructions define (for each Q) a family of curves $\Lambda_{\tilde{x},\tilde{h}}(\cdot)$ (from $\mathbb{R}$ to $\mathbb{R}$) indexed by $(\tilde{x}, \tilde{h}) \in Q$, such that for each $(\tilde{x}, \tilde{h})$ in Q, $\Lambda_{\tilde{x},\tilde{h}}(\tilde{x}) = \tilde{h}$ almost surely. It is then natural to wonder whether there exists certain "versions" of the process $(\Lambda_{x,h},\ (x,h) \in \mathbb{R}^2)$, defined simultaneously for all points (x, h) in the plane, with some additional regularity properties. It turns out that the situation is reminiscent of that of real-valued Lévy processes, where one can choose a right-continuous or a left-continuous version, except that time is here replaced by the h-variable.

In [9], the authors choose to define the forward line starting at $(x, h) \in \mathbb{R}^2$ denoted $\Lambda_{x,h}(y), y \geq x$ by taking the supremum of all $\Lambda_{\tilde{x},\tilde{h}}(y)$ over the countable family of lines

$$\{(\tilde{x}, \tilde{h}) \in Q\ :\ \tilde{x} < x,\ \Lambda_{\tilde{x},\tilde{h}}(x) < h\}$$

that is to say over the lines in the countable family that are starting before x and passing below h at time x. Their Theorem 2.1 states that this family Λ then verifies:

- for any finite set $(x_1, h_1), \ldots, (x_n, h_n) \in \mathbb{R}^2$, a.s. $(\Lambda_{x_i,h_i},\ i \in \{1, \ldots, n\})$ is distributed as independent coalescing Brownian motions,
- a.s., for all $(x, h) \in \mathbb{R}^2$, $\Lambda_{x,h}(x) = h$,
- a.s., for all $(x_1, h_1), (x_2, h_2) \in \mathbb{R}^2$, Λ_{x_1,h_1} and Λ_{x_2,h_2} do not cross each other,
- a.s., for all $x < y$, the mapping $h \mapsto \Lambda_{x,h}(y)$ is left-continuous,

and that those four properties characterize its distribution. Note that the first one tells us that the choice of Q does not change the distribution of Λ. The last "left-continuity" means that for those (x, h) where there might be some choice, one chooses the lowest one. Throughout our paper, the notation $(\Lambda_{x,h})_{(x,h)\in\mathbb{R}^2}$ corresponds to this version of the coalescing family.

Clearly, there is another natural choice, that one can obtain by considering the symmetric picture (upwards down) i.e., to define

$$\Lambda^+_{x,h} = \inf\{\Lambda_{\tilde{x},\tilde{h}}(y),\ \tilde{x} < x,\ \Lambda_{\tilde{x},\tilde{h}}(x) > h,\ (\tilde{x}, \tilde{h}) \in Q\}.$$

This family Λ^+ verifies the same properties as Λ, except that left-continuity with respect to h is replaced by right-continuity.

Another option proposed by Fontes, Isopi, Newman, and Ravishankar in [3] is to define a metric on a natural space on which the coalescing family lives and to consider the closure of $(\Lambda_{x,h}(y), y \geq x\ ;\ (x,h) \in Q)$ in this topological space. Note that you can now have more than one curve starting from certain (exceptional) points. In fact, the curves of the families Λ and Λ^+ correspond to the two extremal choices for the curves of their family. This construction is useful in order to state the convergence of the discrete model with coalescing random walks towards the coalescing Brownian motions. The family is called in their paper Brownian Web (Double Brownian Web if you add the backward lines). By a slight abuse of terminology, we will just call our family $(\Lambda_{x,h}(y),\ y \in \mathbb{R}\ ;\ (x,h) \in \mathbb{R}^2)$ "Brownian Web" (BW).

2.2. TSRM and the Brownian web. — The intuitive link between the TSRM and the BW goes as follows: Let us consider the process (X_t, H_t) started at $(0,0)$ which traces the contour of the "forward tree" moving upwards, that is to say above $\Lambda_{0,0}$ and towards $+\infty$. It is in fact the same contour as that of the "backward tree". This process visits all the points above the curve $\Lambda_{0,0}$ (it is plane-filling). The time-parametrization will be chosen in such a way that the area swept by (X,H) during the interval $[0,t]$ is exactly t and its first coordinate X will be the "true" self-repelling motion.

In order to be more precise, we need some additional notations. For each $(x,h) \in \mathbb{R}^2$, let $S_{x,h}$ denotes the (algebraic) area between $\Lambda_{x,h}$ and $\Lambda_{0,0}$:

$$S_{x,h} := \int_{-\infty}^{+\infty} (\Lambda_{x,h}(y) - \Lambda_{0,0}(y))\, dy.$$

Almost surely, for every (x,h) above the initial curve $\Lambda_{0,0}$, the process (X,H) is equal to (x,h) at the random time $S_{x,h}$ and has visited all the points between $\Lambda_{x,h}$ and $\Lambda_{0,0}$. Tóth and Werner proved that this indeed defines a continuous process $(X_t,H_t)_{t\geq 0}$ (see Lemma 3.4 of [9]). Thanks to the Brownian structure of the tree and the correspondence between area in the tree and time for the process, one can then easily deduce basic properties for (X,H) such as the recurrence of X in $\mathbb{R}$, or the scaling property: for every $a > 0$, $(X_{at}, H_{at})_{t\geq 0}$ and $(a^{2/3}X_t, a^{1/3}H_t)_{t\geq 0}$ are identical in law (Proposition 3.5 of [9]).

Another important observation is that together with the initial profile $\Lambda_{0,0}$, the first coordinate X contains enough information in order to recover both the process H and the upper part of the BW $(\Lambda_{x,h},\ x \in \mathbb{R},\ h \geq \Lambda_{0,0}(x))$. Indeed, as we already mentioned in the introduction, the occupation-time measure of X turns out (for each time t) to have a continuous density with respect to Lebesgue measure, denoted by $L_t(\cdot)$. Moreover, the definition of (X,H) readily shows that when $t = S_{x,h}$, then

$$\Lambda_{0,0}(\cdot) + L_t(\cdot) = \Lambda_{x,h}(\cdot)$$

i.e., that the random area $S_{x,h}$ corresponds to the first time t at which the local time at x, $L_t(x)$, reaches the level $h - \Lambda_{0,0}(x)$, and the curve of the BW from $\Lambda_{x,h} - \Lambda_{0,0}$ is the local time curve at $S_{x,h}$. It is a stronger analog to Ray-Knight Theorems for Brownian motion.

For each fixed (deterministic) $x \in \mathbb{R}$, we will denote by σ_x the first hitting time of x by the TSRM X. It is easy to see that $a.s$, this time equals the infimum of the set of times at which $L_\cdot(x)$, is positive. That is to say, for every given $x \neq 0$, σ_x is almost surely equal to the infimum of $S_{x,h}$ over all $h > \Lambda_{0,0}(x)$ (note that this is not true for all x simultaneously because of the existence of "fast points" or of local maxima).

In the sequel, we shall simply denote by $\Gamma_x(\cdot)$, the profile at this time σ_x:

$$\Gamma_x(\cdot) := L_{\sigma_x}(\cdot) + \Lambda_{0,0}(\cdot).$$

Remark that almost surely for every $x \in \mathbb{R}$ this curve is equal to $\Lambda^+_{x,\Lambda_{0,0}(x)}(\cdot)$, coming from the right-continuous version of the BW (this is contained in Theorem 4.3 (ii) in [9]). Note also that with this definition Γ_0 is just the same as the initial profile $\Lambda_{0,0}$.

The following lemma describes the joint law of Γ_0 and Γ_x. In fact, we will use a slightly stronger version and describe the law of Γ_Y, when Y is a for some Γ_0-measurable random variable Y:

Lemma 2.1. — *Let Y denote a Γ_0-measurable random variable. Then, conditionally on Γ_0, the distribution of Γ_Y is that of a coalescing-reflecting Brownian motion started from $(Y, \Gamma_0(Y))$, that is reflected on Γ_0 in the interval between 0 and Y and coalescing with it outside of this interval.*

As the "starting point" $(Y, \Gamma_0(Y))$ of Γ_x is random, this fact is not totally straightforward. Our proof uses features of the BW established in [9].

Proof. — We already know that for a *fixed* point (x, h) in the plane and conditionally on Γ_0, $\Lambda_{x,h}$ has the distribution of a Brownian motion reflected on Γ_0 between 0 and x and coalescing with it outside this interval. As the point $(Y, \Gamma_0(Y))$ is Γ_0-measurable, conditionally on Γ_0, the distribution of the increments of $\Lambda_{Y,\Gamma_0(Y)}$ remains those of a Brownian motion starting at this point, reflected on Γ_0 between 0 and Y and coalescing with it outside this interval. It remains to use Proposition 2.2 (v) in [9] which tells us that $\Lambda_{Y,\Gamma_0(Y)}$ is continuous to deduce that the distribution of this process corresponds indeed to the above description. □

A consequence of this lemma is that the distribution of σ_x itself can be simply expressed in terms of areas under Brownian curves:

$$\sigma_x \overset{(d)}{=} \sqrt{2}\left(\int_0^x |B_t| dt + \int_x^{\tau'} |B_t| dt\right) \tag{1}$$

where B is a Brownian motion started at the origin and τ' denotes its first hitting time of 0 after time x. Indeed, the initial curve $\Gamma_0(x-\cdot)$ has the distribution of a Brownian motion starting at $\Gamma_0(x)$ and the distribution of $\Gamma_x(x-\cdot)$ conditionally on Γ_0 is given by Lemma 2.1, thus the difference $\Gamma_x(x-\cdot)-\Gamma_0(x-\cdot)$ has the distribution of a reflected Brownian motion multiplied by $\sqrt{2}$, absorbed at its first hitting of 0 after time x.

2.3. Brownian estimates. — As shown by the example of the law of σ_x, the construction of the TSRM via the Brownian web makes it possible to express the probability of TSRM-events in terms of Brownian motions and areas under Brownian curves. We now collect some results concerning the law of Brownian motion integrals that we will need later in the paper.

Throughout this paper, B will denote a standard Brownian motion, and $\tilde{B}$ a reflected Brownian motion (that has the same law as $|B|$), P_x will denote the law of these processes started at x. When $x=0$, we will sometimes simply write P instead of P_0. For each $y\in\mathbb{R}$, the first hitting time of the level y by B (respectively $\tilde{B}$) after time t will be denoted by $\tau_y^{(t)}$ (resp. $\tilde{\tau}_y^{(t)}$), when $t=0$, we simply write τ_y (resp. $\tilde{\tau}_y$).

In order to derive our estimates about the tail of X_1 and H_1, we will build on the following rather classical asymptotics about the areas under a Brownian motion and a Brownian bridge. The first two results can for instance be found in [4] and the very classical third one in [5]. Here and throughout the paper,

$$\kappa := 2|a_1'|^3/27$$

where a_1' denotes the first (negative) zero of the derivative of the Airy function Ai.

PROPOSITION 2.2. — 1. *For some positive constant γ, when $\varepsilon\to 0$,*

$$P_0\left(\int_0^1 |B_t|dt \le \varepsilon\right) \sim \gamma\,\varepsilon \exp\left(-\frac{\kappa}{\varepsilon^2}\right).$$

2. *In the case of the Brownian bridge,*

$$P_0\left(\int_0^1 |B_t - tB_1|dt \le \varepsilon\right) \sim \gamma' \exp\left(-\frac{\kappa}{\varepsilon^2}\right)$$

as $\varepsilon\to 0$ for some positive constant γ'.

3. *The law of the area under a Brownian motion starting at 1 stopped at its first hitting of 0 is given by*

$$P_1\left(\int_0^{\tau_0} B_t dt \le u^{-3}\right) = \frac{\int_u^\infty e^{-2y^3/9}dy}{\int_0^\infty e^{-2y^3/9}dy}.$$

This last statement follows in fact directly from the fact that the function $F(x,A) := P_x(\int_0^{\tau_0} B_t dt \leq A)$ is a function of $x/A^{1/3}$ that satisfies the PDE $(\partial_x^2 - 2x\partial_A)F = 0$ (because $F(B_{t\wedge\tau_0}, A - \int_0^{t\wedge\tau_0} B_s ds)$ is a martingale).

Suppose that U_1 and U_2 are independent copies of the random variable $\int_0^{\tau_0} B_t dt$ in statement 3. A simple consequence of that estimate that will shall use at some point is that when $x \to 0$,

$$P_1(U_1 + U_2 \leq x) = \exp(-8/(9x) + O(\log(1/x))). \tag{2}$$

Indeed a lower bound of $P_1(U_1 + U_2 \leq x)$ is simply given by $(P_1(U_1 \leq x/2))^2$. For the upper bound, a possible proof consists in dividing the interval $[0,x]$ into $[1/x]+1$ intervals of length x^2 and to examine the probability that $U_1+U_2 \leq x$ according to which portion U_1 belongs to:

$$P_1(U_1 + U_2 \leq x) \leq \sum_{j=0}^{[1/x]+1} P_1(U_1 \in [jx^2, (j+1)x^2])\, P_1(U_2 \leq x - jx^2 + x^2).$$

Using Proposition 2.2-3, we deduce:

$$P_1(U_1 + U_2 \leq x) \leq \sum_{j=0}^{[1/x]+1} \exp\left(-\frac{2}{9\,x}\left[\frac{1}{(j+1)x} + \frac{1}{1+x(1-j)}\right] + O(\log(1/x))\right).$$

The minimum over $j \in \{0, \dots, [1/x]+1\}$ of the function between the brackets takes the form $4(1+O(x))$. It gives the desired upper bound. □

3. Tail estimates for the distribution of X_1

The main goal of the present section is to derive the following fact:

Proposition 3.1. — *When $x \to \infty$,*

$$\mathbb{P}(X_1 > x) = \exp\left(-2\kappa\, x^3 + O(\ln(x))\right).$$

Note that X_1 and $-X_1$ have the same distribution, so that this also describes the behavior of $\mathbb{P}(X_1 < -x)$ when $x \to +\infty$ We would like to also point out that our proof can be easily adapted to the case when the initial condition is flat. The only difference is that the coefficient 2κ in front of x^3 is replaced by κ (because the corresponding Brownian motion is not multiplied by $\sqrt{2}$).

Proof. — Recall the representation of the law of σ_x from the end of Section 2.2. It follows that

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in[0,1]} X_s \geq x\right) &= \mathbb{P}(\sigma_x \leq 1) \\ &\leq P\left(\sqrt{2}\int_0^x \tilde{B}_u du \leq 1\right) = P\left(\int_0^1 \tilde{B}_u du \leq \frac{1}{\sqrt{2x^3}}\right).\end{aligned}$$

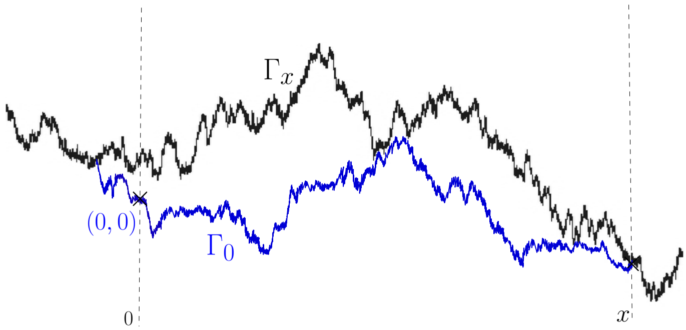

FIGURE 3.1. The two reflected-coalescing curves Γ_x and Γ_0

Combined with Proposition 2.2-1, this proves immediately the upper bound.

For the lower bound, it is sufficient to estimate the probability of a well-chosen subset of the event $\{X_1 > x\}$, that can be easily described using the Brownian web. In order to ensure that $X_1 > x$, it would for instance suffice that $\sigma_{x+1/x^2} < 1$ and that X stays to the right of x during a time-interval of length 1 after σ_{x+1/x^2}. We will use a slight variation of this idea: Let $\tilde{\Gamma}_x$ denote the line corresponding to the first time at which the local time $L_t(x)$ of X at x exceeds $1/x$. Let $\hat{\Gamma}_{x+1/x^2}$ denote the line corresponding to the first time at which the local time at $x + 1/x^2$ exceeds $1/(2x)$, and finally let Γ'_0 be the line corresponding to the first time at which the local time at 0 reaches $1/x$. We will evaluate the probability that the following four events hold simultaneously (see Figure 3.2. for a representation of those events):

- The integral of $\tilde{\Gamma}_x - \Gamma_0$ over $[0, x]$ does not exceed $1 - 2/x^3$ and $\tilde{\Gamma}_x(0) < 1/x^4$.
- The integral of $\Gamma'_0 - \Gamma_0$ on $(-\infty, 0)$ does not exceed $1/x^3$.
- $\hat{\Gamma}_{x+1/x^2}(x) - \Gamma_0(x) \leq 1/x$ and $\hat{\Gamma}_{x+1/x^2} - \Gamma_0$ hits 0 on $[x, x + 1/x^2]$, and the integral of this function on $[x, x + 1/x^2]$ does not exceed $1/x^3$.
- The integral of $\hat{\Gamma}_{x+1/x^2} - \Gamma_0$ on $[x + 1/x^2, \infty)$ is greater than one.

It is easy to check just using monotonicity of the BW that if these four events hold then X_1 will be bigger than x – i.e., to the right of x in the two-dimensional picture (the first, second and third one imply $\sup_{s\leq 1} X_s \geq x+1/x^2$, the third and last one ensure that X_1 stays above x during the time-interval $[\sigma_{x+1/x^2}, \sigma_{x+1/x^2} + 1]$). Notice also that these four events are independent as the processes defining them (restricted to the appropriate time-intervals) correspond to different parts of the BW (and this is why we chose to work with these events). Let us evaluate the probability of each of them. Thanks to Brownian scaling, the second and the third one are equal to positive constants independent of x.

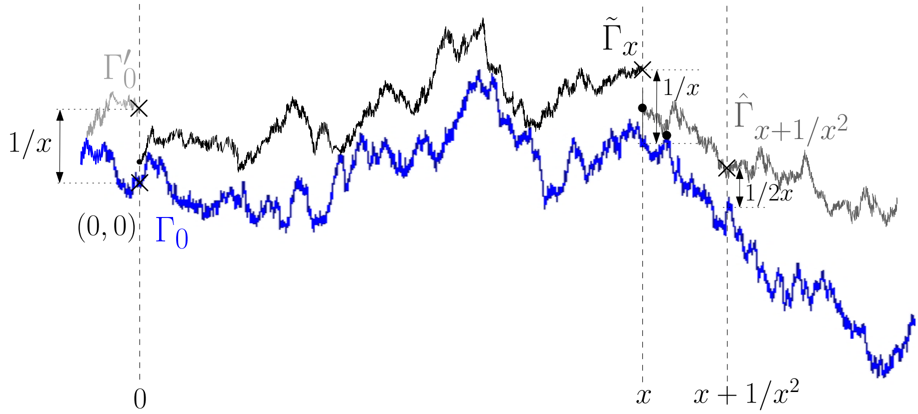

FIGURE 3.2. The BW-curves Γ_0, Γ'_0, $\tilde{\Gamma}_x$ and $\hat{\Gamma}_{x+1/x^2}$

If the process $\hat{\Gamma}_{x+1/x^2} - \Gamma_0$ stays above $1/(4x)$ in the time interval of length $4x$ starting at $x + 1/x^2$, then the fourth event is satisfied. It implies that the probability of the fourth event is bounded from below by

$$(3) \qquad P_0\left(\inf_{s \le 4x} B_s \ge -1/(4\sqrt{2}x)\right) = P_0\left(|B_1| \le 1/(8\sqrt{2}x^{3/2})\right) \ge c/x^{3/2}$$

for some absolute constant c.

The probability of the first one is responsible for the main exponential term: The strong Markov property shows that it is bounded from below by

$$P_{1/(\sqrt{2}x)}\Big(\tau_0 < 1/x^2,\ \int_0^{\tau_0} B_t dt \le 1/(\sqrt{2}x^3)\Big) \\ \cdot P_0\left(\sqrt{2}\int_0^x |B_t| dt \le 1 - 3/x^3,\ \sqrt{2}|B_x| \le 1/x^4\right).$$

The scaling property shows again that the first term in this product does not depend on x. The second term can be evaluated thanks to the Brownian bridge. Scaling shows that it is bounded from below by:

$$P_0\left(\int_0^1 |B_t - tB_1| dt \le \frac{1}{\sqrt{2}x^{3/2}}\left(1 - 3/x^3\right) - \frac{|B_1|}{2},\ |B_1| \le \frac{1}{\sqrt{2}x^{9/2}}\right) \\ \ge P_0\left(\int_0^1 |B_t - tB_1| dt \le \frac{1}{\sqrt{2}x^{3/2}}\left(1 - 4/x^3\right)\right) \times P_0\left(|B_1| \le \frac{1}{\sqrt{2}x^{9/2}}\right)$$

because of the independence between $(B_t - tB_1,\ t \in [0,1])$ and B_1. Putting the pieces together, we get finally that

$$P(X_1 \geq x) \geq \frac{c'}{x^6} \times P_0\left(\int_0^1 |B_t - tB_1| dt \leq \frac{1}{\sqrt{2} x^{3/2}} \left(1 - 4/x^3\right)\right) \tag{4}$$

where c' is some absolute constant. Proposition 2.2-2 then allows to conclude. □

4. Law of the iterated logarithm for X

4.1. Statement and proof of the upper bounds. — The main goal of this section is to use the previous estimates in order to derive the analog for X of the law of the iterated logarithm:

PROPOSITION 4.1. — *Almost surely*

$$\limsup_{t\to+\infty} \frac{X_t}{t^{2/3}\left(\ln\ln(t)\right)^{1/3}} = \limsup_{t\to 0+} \frac{X_t}{t^{2/3}\left(\ln\ln(1/t)\right)^{1/3}} = 1/(2\kappa)^{1/3}.$$

Stationarity shows that this also describes the almost sure fluctuations at any given positive time t_0 i.e., that almost surely,

$$\limsup_{t\to 0+} \frac{X_{t_0+t} - X_{t_0}}{t^{2/3}\left(\ln\ln(1/t)\right)^{1/3}} = 1/(2\kappa)^{1/3}.$$

The same type of local result will hold for the TSRM with flat initial condition at any given positive time. However, if X is the TSRM with flat initial conditions, then the result stated in the proposition does not hold anymore. The proof can however be directly adapted and then shows that one just has to replace the constant $1/(2\kappa)^{1/3}$ by $1/\kappa^{1/3}$.

Let us now first briefly derive the upper bounds in this proposition i.e., the fact that these limsups are not greater than $1/(2\kappa)^{1/3}$. This part of the proof will go along similar lines as the standard proof of the LIL for the Brownian motion (see e.g., Chapter II p. 56 of [7]) based on Borel-Cantelli Lemmas. Let us first focus on the $t \to \infty$ part. Clearly, it suffices to show that for some given $\lambda > 1$ and $\varepsilon > 0$, there almost surely exists some N such that for all $n \geq N$,

$$\sup_{t\in[0,\lambda^n]} X_t \leq \frac{1+\varepsilon}{(2\kappa)^{1/3}} \lambda^{2n/3} \left(\ln\ln(\lambda^n)\right)^{1/3}.$$

If we define

$$x_n := \frac{1+\varepsilon}{(2\kappa)^{1/3}} \left(\ln\ln(\lambda^n)\right)^{1/3},$$

we get (because $\sup_{t\in[0,\lambda^n]} X_t/\lambda^{2n/3}$ and $\sup_{t\in[0,1]} X_t$ have the same law) from Proposition 3.1 that

$$\mathbb{P}\left(\lambda^{-2n/3}\sup_{t\in[0,\lambda^n]} X_t \geq x_n\right) = \mathbb{P}\left(\sup_{t\in[0,1]} X_t \geq x_n\right) = e^{-2\kappa\, x_n^3 + O(\ln(x_n))}.$$

Our choice for x_n ensures that

$$\sum_n \mathbb{P}\left(\lambda^{-2n/3}\sup_{t\in[0,\lambda^n]} X_t \geq x_n\right) < \infty.$$

Note that ε can be chosen arbitrarily small which implies the result when $t \to \infty$.

The proof for $t \to 0$ is almost identical, except that we now have to choose $\lambda \in (0,1)$ and that the events we will consider are:

$$\sup_{t\in[\lambda^n,\lambda^{n-1}]} X_t \leq \frac{1+\varepsilon}{(2\kappa)^{1/3}}\,\lambda^{2(n-1)/3}\left(\ln\ln(1/\lambda^{n-1})\right)^{1/3}.$$

The result follows again using scaling.

4.2. Proof of the lower bounds. — The purpose of this subsection is to derive the lower bounds in Proposition 4.1. Let us stress that some caution is needed because the process X does not have independent increments, so that one has the standard proof of the LIL for Brownian motion can not be adapted directly.

We again first focus on the case where $t \to +\infty$. Let us fix any small δ. Our goal is to show that for $c := 1/(2\kappa)^{1/3}$ one can almost surely find a sequence of times $t_n \to +\infty$, such that

$$X_{t_n} \geq (c-\delta)\, t_n^{2/3}\left(\ln\ln(t_n)\right)^{1/3}. \tag{5}$$

We will choose t_n to be some first hitting times. More precisely, let us choose $\lambda > 1$, $\varepsilon \in (0, 2/3)$ and define for each $n \geq 1$,

$$\lambda_n = \lambda^{n^{1+\varepsilon}},$$

and let

$$\tilde\sigma_n := \sigma_{\lambda_n} = \inf\{t \geq 0 \ :\ X_t = \lambda_n\}.$$

Our sequence (t_n) will be a subsequence of $(\tilde\sigma_n)$.

Note that $\lambda_n/\lambda_{n-1} \sim \lambda^{(1+\varepsilon)n^\varepsilon}$ increases quite rapidly when $n \to \infty$, but not too fast either (both facts will be useful in our proof). Define

$$\gamma_n := c'\,\lambda_n^{3/2} \Big/ \sqrt{\ln\ln(\lambda_n^{3/2})}$$

where the positive constant c' will be chosen later.

Our goal is to prove that $\tilde\sigma_n \leq \gamma_n$ (i.e., that the area between Γ_{λ_n} and Γ_0 does not exceed γ_n) infinitely often as soon as $c' > \sqrt{2\kappa}$, which indeed implies (5).

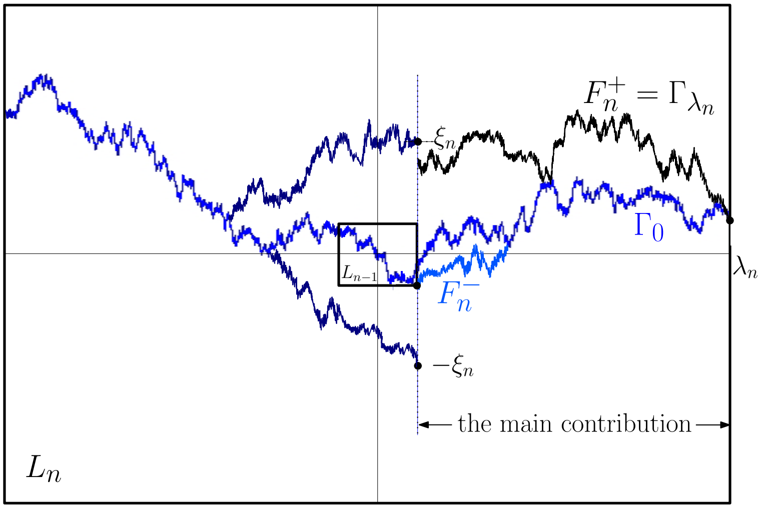

FIGURE 4.1. Boxes L_n and L_{n-1} and the curves involved in $\mathcal{A}_n$ and $\mathcal{B}_n$

Let us define the boxes $L_n := [-\lambda_n, \lambda_n] \times [-\eta_n, \eta_n]$ with $\eta_n := 3\sqrt{\lambda_n \ln(n)}$. As the sequence (λ_n) increases fast, the box L_{n-1} is really small compared to L_n when n is large.

Our choice for η_n ensures that if we define

$$\mathcal{D}_n := \{\Gamma_0([-\lambda_n, \lambda_n]) \in [-\eta_n, \eta_n]\}$$

then

$$\sum_n \mathbb{P}(\mathcal{D}_n^c) < \infty,$$

so that almost surely, $\mathcal{D}_n$ holds for all large enough n. Similarly, one can also for instance see that

$$\Gamma_{\lambda_n}([\lambda_{n-1}, \lambda_n]) \in [-\eta_n, \eta_n]$$

almost surely for all large enough n.

The fact that the events $\{\tilde{\sigma}_n \leq \gamma_n\}$ for $n \geq 1$ are not independent leads us to define closely related events that happen to be independent, so that we will be able to apply Borel-Cantelli arguments. The events that we are going to focus on will be defined in terms of the Brownian Web in the disjoint portions $(L_n \setminus L_{n-1})$. One minor technical difficulty is that in order to recognize where Γ_0 is in $L_n \setminus L_{n-1}$, one needs information about the Brownian web in L_{n-1}. We will circumvent this problem by considering instead the forward line in the web denoted by F_n^- started from the bottom right corner of L_{n-1}. Then, we define F_n^+ to be the backward line in the web that is started from $F_n^-(\lambda_n)$ reflected above this curve F_n^-.

Now, we define the event $\mathcal{A}_n$ that the following three events hold:

- The area between F_n^+ and F_n^- is small i.e.,
$$\int_{\lambda_{n-1}}^{\lambda_n} (F_n^+(u) - F_n^-(u))\, du \leq (1-\varepsilon)\gamma_n.$$
- $F_n^+(\lambda_{n-1}) \in [\eta_{n-1}, \xi_n]$ with $\xi_n := \varepsilon c' \sqrt{\lambda_n / \ln(\ln(\lambda_n^{3/2})}$.
- F_n^- and F_n^+ stay in L_n during the interval $[\lambda_{n-1}, \lambda_n]$.

The last event ensures that $\mathcal{A}_n$ is indeed measurable with respect to the Brownian web in L_n. Note that, as before, the probability of this third event is very close to 1 for n large, and in fact equal to $1 - a_n$ for some summable a_n.

We can use the same trick as in the proof of the lower bound of the tail of X_1 in order to get a lower bound for the probability that the first two events involved in this definition happen: Indeed, using scaling and then the independence between $t \in [0,1] \mapsto B_t - tB_1$ and B_1, we get that

$$\begin{aligned}\mathbb{P}(\mathcal{A}_n) + a_n &\geq P\left(\int_0^1 |B_u|\, du \leq (1-\varepsilon)c'\alpha_n,\ \sqrt{2}\eta_{n-1}/\sqrt{\lambda_n - \lambda_{n-1}} \leq |B_1| \leq \varepsilon c'\alpha_n\right) \\ &\geq P\left(\int_0^1 |B_u - uB_1|\, du \leq (1 - \frac{3}{2}\varepsilon)c'\alpha_n\right) \\ &\quad \times P\left(|B_1| \in \left[\frac{\sqrt{2}\eta_{n-1}}{\sqrt{\lambda_n - \lambda_{n-1}}}, \varepsilon c'\alpha_n\right]\right),\end{aligned}$$

where $\alpha_n := 1/\sqrt{2\,\ln\ln(\lambda_n^{3/2})}$. Part 2 of Proposition 2.2 then shows that

$$\sum \mathbb{P}(\mathcal{A}_n) = \infty$$

as soon as $c' \geq (1+\varepsilon)^{1/2}/(1 - 3\varepsilon/2) \times \sqrt{2\kappa}$ (this is where we use that the sequence (λ_n) is not increasing too fast).

Consider now the two backward lines started at (λ_{n-1}, ξ_n) and $(\lambda_{n-1}, -\xi_n)$. Define the event $\mathcal{B}_n$ that the area between these two curves does not exceed ξ_n^3, that they coalesce in the interval $[\lambda_{n-1} - 2\xi_n^2, \lambda_{n-1} - \xi_n^2]$, that they do not enter the box $[\lambda_{n-1} - \xi_n^2, \lambda_{n-1}] \times [-\xi_n/3, \xi_n/3]$ and do not exit the box $[\lambda_{n-1} - 2\xi_n^2, \lambda_{n-1}] \times [-2\xi_n, 2\xi_n]$. Clearly, scaling shows that the probability of this event does not depend on n. Furthermore, our definition of ξ_n ensures that for large enough n, one can check whether this event holds by just looking at the Brownian web in the part of $L_n \setminus L_{n-1}$ that is to the left of λ_{n-1}, which implies in particular that $\mathcal{B}_n$ is independent of $\mathcal{A}_n$.

Hence, it also follows that the events $(\mathcal{A}_n \cap \mathcal{B}_n)$ are independent, so that almost surely, $\mathcal{A}_n \cap \mathcal{B}_n$ holds for infinitely many values of n. As $\mathcal{D}_n$ holds almost surely for all large n, we conclude that almost surely $\mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{D}_{n-1}$ occurs infinitely often. But we can notice that when this last event holds, then, due to the monotonicity properties of the Brownian web, we get that $F_n^- \leq \Gamma_0$, F_n^- coalesces with Γ_0 in the interval $[\lambda_{n-1}, \lambda_n]$ (because $F_n^+(\lambda_{n-1})$ is bigger than η_{n-1}) and thus $F_n^+ = \Gamma_{\lambda_n}$. Moreover, $\{F_n^+(\lambda_{n-1}) \leq \xi_n\} \cap \mathcal{D}_{n-1}$ implies that the backward lines involved in $\mathcal{B}_n$ enclose Γ_0 and Γ_{λ_n}. As ξ_n^3 is much smaller than $\varepsilon\gamma_n$, it permits to conclude that $\mathcal{A}_n \cap \mathcal{B}_n \cap \mathcal{D}_{n-1}$ is included in $\tilde{\sigma}_n \leq \gamma_n$ as soon as c' is greater than $(1+\varepsilon)^{1/2}/(1-3\varepsilon/2) \times \sqrt{2\kappa}$. Taking the limit $\varepsilon \to 0$ gives the result.

The proof of the lower bound when $t \to 0$ is almost identical. The very same proofs goes through without modification, one just has to take λ smaller than 1 instead of larger than 1

5. Fluctuations of the height

5.1. Statement of tail-estimates. — In this section, we will mostly study the tails of the distribution of the height H_t. Again, we can restrict ourselves to $t = 1$ thanks to the scaling property. The estimates that we will derive are the following:

PROPOSITION 5.1. — *There exists two positive constants η and η' such that for all large h,*

$$\begin{aligned} \exp(-\eta h^{3/2}) &\leq \mathbb{P}(H_1 \leq -h) \leq \mathbb{P}\left(\inf_{t\in[0,1]} H_t \leq -h\right) \leq \exp(-\tfrac{1}{\eta} h^{3/2}) \\ \exp(-\eta' h^{3/2}) &\leq \mathbb{P}(H_1 \geq h) \leq \mathbb{P}\left(\sup_{t\in[0,1]} H_t \geq h\right) \leq \exp(-\tfrac{1}{\eta'} h^{3/2}). \end{aligned}$$

We use here two different constants η and η' to stress that, unlike the case of X, the distribution of H is *not* symmetric (i.e., the distributions of $-H_1$ and H_1 are quite different). See Fig. 5.1.

In fact the derivation of the tail-estimates for H_1 are very different than those for X_1, because the initial profile will now play a key-role. Roughly speaking, the exceptional events that we will focus on will require a combination of a very favorable initial profile Γ_0 and a particular behavior of the TSRM between time 0 and 1.

The next three subsections are devoted to the proof of Proposition 5.1.

5.2. Lower bounds. — We will first derive the lower bound for the probability that $H_1 \leq -h$ and we will in fact focus on the sub-event $\{H_1 \leq -h \text{ and } X_1 > 0\}$. To guess what configuration to consider, we can imagine that for the initial profile, the random variable

$$Y_{-h} := \inf\{y \geq 0 \;:\; \Gamma_0(y) \leq -h\}$$

is exceptionally small. Then, on $[0, Y_{-h}]$, Γ_0 will at first glance look like a non-horizontal line with negative strong slope $-\alpha$ (to be determined), and one can compute the cost for another Brownian motion going backwards and reflected

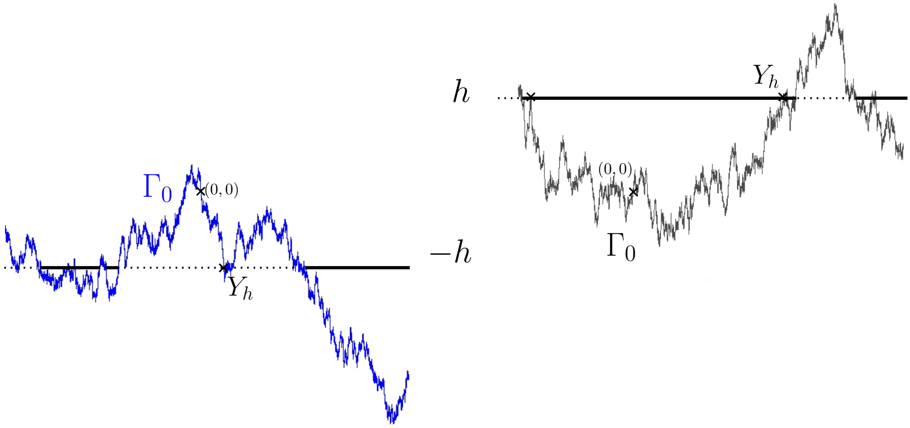

FIGURE 5.1. The initial configuration Γ_0 and the lines $-h$ and h (the thick lines represent the possible places (X_1, H_1) where one can have $H_1 < -h$ (on the left hand side) and $H_1 > h$ (on the right hand side))

on this slope, starting at the point $(h/\alpha, -h)$ in order to create an area less than 1. One has to find a compromise between the cost of creating this initial configuration (which is roughly $P(\tau_{-h} < h/\alpha)$) and the cost of creating this small area with this slope. A back-of-the envelope calculation shows that a slope α of the order of $\sqrt{h}$ is close to optimal. In other words, we will roughly ask the initial profile Γ_0 to go down to level $-h$ during the interval $[0, \sqrt{h}]$ (recall that the "natural" Brownian scaling would give an interval of length h^2), and then the TSRM to run exceptionally fast down this slope.

More precisely, let us describe the events that we will require to hold (Figures 5.2. and 5.3. can help to quickly see what is going on). Define

$$\varepsilon_h := 1/(5\sqrt{h+2}),$$

and the function $f_h(\cdot)$ to be the linear function defined on $[0, \sqrt{h+2}]$ such that $f_h(0) = 0$ and $f_h(\sqrt{h+2}) = -h-2$. Define U_h to be the tube of vertical width ε_h around f_h, and V_h to be the same tube, but shifted vertically by $2\varepsilon_h$ so that V_h lies just above U_h. In other words,

$$U_h = \{(x,l) \ : \ x \in [0, \sqrt{h+2}] \text{ and } |l - f_h(x)| < \varepsilon_h\}$$
$$V_h = \{(x,l) \ : \ x \in [0, \sqrt{h+2}] \text{ and } |l - f_h(x) - 2\varepsilon_h| < \varepsilon_h\}.$$

Then, we will require that

- The initial profile Γ_0 stays within U_h for all $x \in [0, \sqrt{h+2}]$.

- The backward line starting at $(\sqrt{h+2}, -h-2+2\varepsilon_h)$ stays in V_h for all $x \in [0, \sqrt{h+2}]$.
- The backward lines starting at $(0,0)$ and at $(0,1)$ coalesce in such a way that the area between them is less than $1/10$, i.e., $\int_{-\infty}^{0}(\Lambda_{(0,1)}(y) - \Lambda_{(0,0)}(y))dy \le 1/10$.
- The forward lines starting at $(\sqrt{h+2}, -h-3)$, $(\sqrt{h+2}, -h-1)$ and at $(\sqrt{h+2}, -h-1/2)$ do coalesce before reaching the height $-h$, and the area between the last two curves is greater than 1.
- The backward line starting at $(\sqrt{h+2}, -h-1/2)$ and at $(\sqrt{h+2}, \Gamma_0(\sqrt{h+2}))$ do coalesce before reaching the height $-h$, and in a horizontal time-span smaller than one.

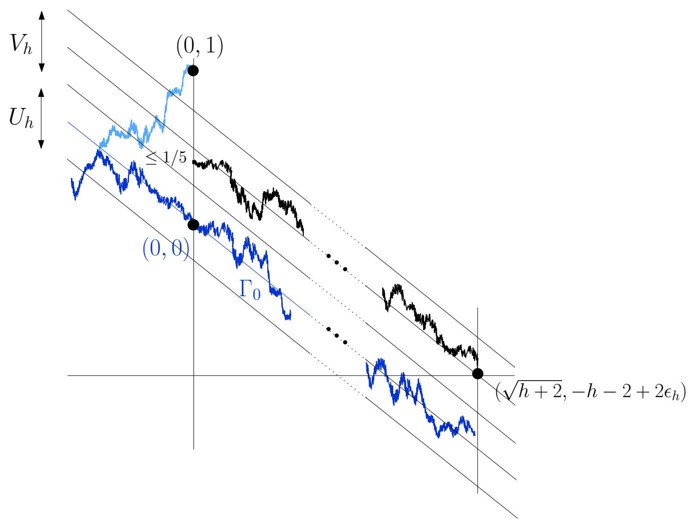

FIGURE 5.2. Realization of the first three events

All these definitions may seem somewhat messy, but it is easy to check, just using the monotonicity properties of the Brownian web, that if all these events occur, then H_t will hit $-h-2$ before time 1, and that the process (X_t, H_t) will stay under the horizontal line $-h$ for a time-interval of length at least one after this time. In particular, if the five events hold simultaneously, then $H_1 \le -h$.

The first four events are independent, because they correspond to events dealing with the Brownian web in disjoint domains. The conditional probability of the last one given the first four turns out to be bounded from below by a positive constant that does not depend on h. Indeed, it is independent on the third and fourth events. Moreover, the first and second events imply that the backward line started at $(\sqrt{h+2}, \Gamma_0(\sqrt{h+2}))$ stays in the tube $U_h \cup V_h$.

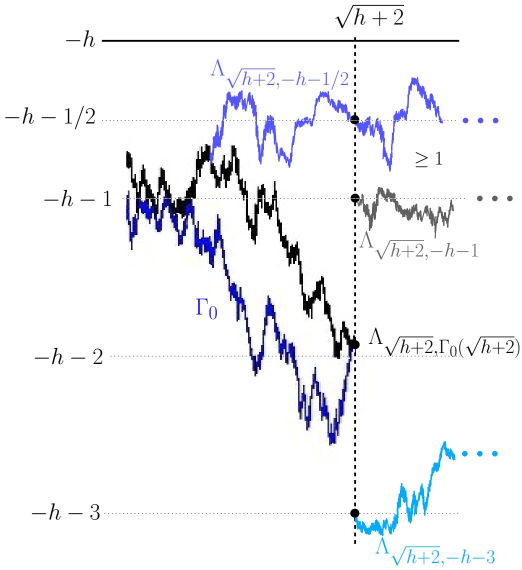

FIGURE 5.3. Zoom around $t = \sqrt{h+2}$ with the lines involved for the realization of the fourth and fifth events

Therefore the conditional probability is bounded from below by the probability that a standard Brownian motion hits the affine function $-f_h - 2$ before $1/2$ which clearly is positive (and bounded from below independently from h).

It remains to evaluate the probabilities of the first four events separately. The third and the fourth are positive and independent of h. The first two probabilities are equal. Note that $(\Gamma_0(x), x \leq \sqrt{h+2})$ is a Brownian motion and therefore $\Gamma^B(x) := \Gamma_0(x) - x\Gamma_0(\sqrt{h+2})/\sqrt{h+2}$ is a Brownian bridge independent from $\Gamma_0(\sqrt{h+2})$. Furthermore, if

$$\Gamma_0(\sqrt{h+2}) \in [-h-2-\varepsilon_h/2, -h-2+\varepsilon_h/2]$$

and

$$\sup_{x \in [0,\sqrt{h+2}]} |\Gamma^B(x)| \leq \varepsilon_h/2,$$

then the first event holds. The probabilities of each of these two independent events turns out of the type $\exp\big(-ch^{3/2}\big)$, which concludes the proof of the lower bound of $\mathbb{P}(H_1 < -h)$.

The proof of the lower bound for $\mathbb{P}(H_1 > h)$ is almost identical. The only difference is that the tubes now go upwards instead of downwards, and the reader can easily check that the same arguments work.

5.3. Upper bound for $\mathbb{P}(\inf_{s\in[0,1]} H_s < -h)$. — We define again for $l > 0$

$$Y_{-l} = \inf\{y > 0 \ : \ \Gamma_0(y) \leq -l\},$$

and we simply use Y for Y_{-h}. Clearly, by symmetry,

$$\mathbb{P}\left(\inf_{s\in[0,1]} H_s < -h\right) = 2\mathbb{P}(\sigma_Y < 1) \leq 2\mathbb{P}\left(\int_0^Y (\Gamma_Y(y) - \Gamma_0(y))dy < 1\right).$$

Recall from Lemma 2.1 that conditionally on Γ_0, the law of Γ_Y is that of a backward Brownian motion started at $(Y, \Gamma_0(Y))$ and reflected on Γ_0.

Note that Williams decomposition theorem (see for instance chapter 4 Corollary (4.6) p. 317 in [7]) states that the law of $(\Gamma_0(Y-y)+h, y\in[0,Y])$ is that of a three-dimensional Bessel process up to its last passage time at level h.

Recall that by the strong Markov property for the Brownian motion, if one defines (for a given $\epsilon > 0$)

$$\Gamma^j := (\Gamma_0(t + Y_{-j\epsilon}) + j\epsilon, t \in [Y_{-j\epsilon}, Y_{-(j+1)\epsilon}])$$

then $\Gamma^0, \Gamma^1, \Gamma^2, \ldots$ are i.i.d. Let us choose $\epsilon = c/h^{1/2}$ for some large c, and denote by N the integer part of h/ϵ.

Monotonicity properties readily imply (by comparing Γ_Y with the process where at each $Y_{-j\epsilon}$, the Brownian motion has to jump down to the actual location of Γ_0) that one can compare $\int_0^{Y_{-N\epsilon}} (\Gamma_Y(y) - \Gamma_0(y))dy$ with the sum of N i.i.d. copies of $\int_0^{Y_{-\epsilon}} (\Gamma_{Y_{-\epsilon}}(y) - \Gamma_0(y))dy$ (the latter being stochastically dominated by the former). Hence, it finally suffices to evaluate the probability that the sum of N copies of

$$\int_0^{Y_{-\epsilon}} (\Gamma_{Y_{-\epsilon}}(y) - \Gamma_0(y))dy$$

is smaller than 1. By scaling, this is exactly the same as the probability that the sum of N copies of

$$\int_0^{Y_{-c}} (\Gamma_{Y_{-c}}(y) - \Gamma_0(y))dy$$

is smaller than $h^{3/2}$ (which is smaller than $2cN$). Note that if we have chosen c sufficiently large, we made sure (because of scaling) that

$$E\left(\int_0^{Y_{-c}} (\Gamma_{Y_{-c}}(y) - \Gamma_0(y))dy\right) = c^3 E\left(\int_0^{Y_{-1}} (\Gamma_{Y_{-1}}(y) - \Gamma_0(y))dy\right) > 4c$$

and it therefore follows from the standard Cràmer Theorem for sums of i.i.d. positive random variables that for some positive constant a, the probability in question is bounded from above by $\exp(-ah^{3/2})$ for all large h, which concludes this part of the proof. □

5.4. Upper bound for $\mathbb{P}(\sup_{s\in[0,1]} H_s > h)$. — Our goal is now to derive the upper bound for the probability that H_t reaches a large positive h before time 1. In other words, we want to evaluate the probability that there exists an x for which $S_{x,h} \leq 1$. Note that for symmetry reasons, this probability is bounded from above by twice the probability that there exists a positive x for which $S_{x,h} \leq 1$.

Note that the situation is different than in the previous section. Indeed, for H_t to be negative before time 1, the strategy had to be to find quickly a position where the initial profile was negative. Here, it could a priori happen that the H_t is very large just because the TSRM spent some time in a tiny interval. So, the position at which this can happen is not a priori prescribed (see Figures 5.4 and 5.5).

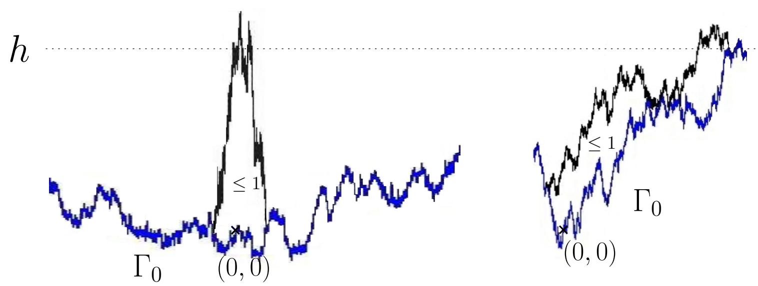

FIGURE 5.4. Possible configurations for $H_1 > h$

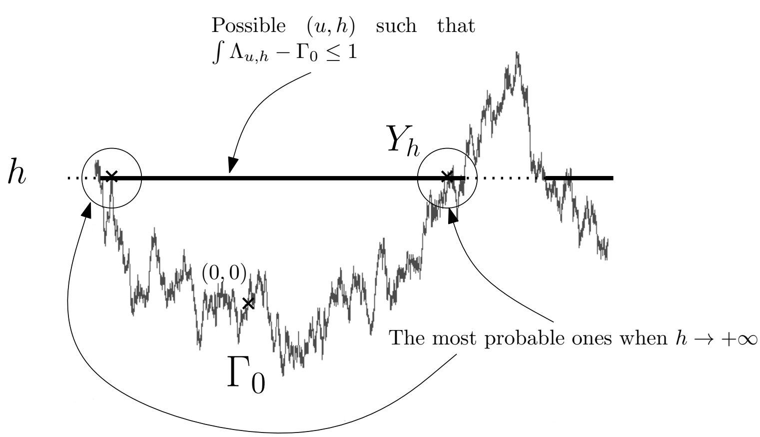

FIGURE 5.5. The initial local time with the possible positions for X when H first hits the level h

Recall from our earlier estimates that the probability that the TSRM X reaches $\sqrt{h}$ before time 1 is bounded by $\exp(-\kappa h^{3/2})$. It will therefore be sufficient to evaluate

$$\mathbb{P}(\exists x \in [0, \sqrt{h}],\ S_{x,h} \leq 1).$$

Also, it is easy to check that the probability that $\Gamma[-\sqrt{h}, \sqrt{h}] \notin [-h/4, h/4]$ is also very small, and bounded from above by $\exp(-ch^{3/2})$ for some constant c and all large h.

It remains to bound the probability that $\Gamma[-\sqrt{h}, \sqrt{h}] \in [-h/4, h/4]$ and $S_{x,h} \leq 1$ for some $x \in [0, \sqrt{h}]$. In fact, we shall see that it is smaller than $\exp(-ch^3)$ for some constant c.

Indeed, if this holds for some $x \in [0, \sqrt{h}]$, it means that the backward line in the BW starting from (x, h) has to hit level $h/2$ in the interval $[x - 4/h, x]$. Indeed, otherwise, the domain in-between the initial profile and this backward line would contain a rectangle with area $(4/h) \times (h/4) = 1$.

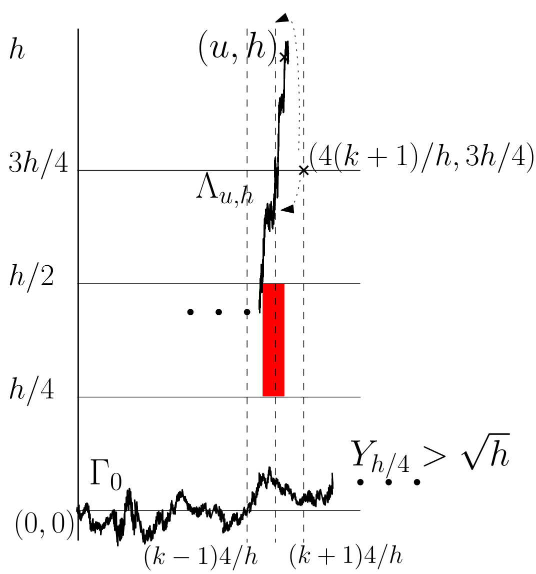

FIGURE 5.6. Representation of (u, h) verifying $S_{u,h} \leq 1$

Let us now suppose that for some $x \in [0, \sqrt{h}]$, the backward line in the BW starting from (x, h) has to hit level $h/2$ in the interval $[x - 4/h, x]$. Let us define j to be the smallest integer such that $\tilde{x} := 4j/h \geq x$. Then, the backwards line starting from $(\tilde{x}, 3h/4)$ has to either hit h or $h/2$ in the interval $[\tilde{x} - 2/h, \tilde{x}]$ (indeed, if it stays in the interval $[h/2, h]$, it would coalesce with the backward line starting from (x, h) and therefore hit $h/2$). See Fig. 5.6.

The probability that a Brownian motion started from the origin hits level $h/4$ before time $2/h$ decays very fast when $h \to \infty$ (a possible upper bound is of the type $\exp(-ch^3)$). Note that there are of the order of $\sqrt{h} \times h/2$ possibilities for $\tilde{x}$.

Putting the pieces together, we obtain an upper bound of the type

$$\mathbb{P}(\Gamma[-\sqrt{h}, \sqrt{h}] \in [-h/4, h/4] \text{ and } \exists x \in [0, \sqrt{h}],\ S_{x,h} \leq 1) \leq C' h^{3/2} e^{-ch^3}.$$

The upper bound for $\mathbb{P}(\sup_{s\in[0,1]} H_s > h)$ follows. □

5.5. Flat initial condition. — It is worthwhile to note that for the flat initial conditions i.e., when $\Gamma_0 = 0$, the situation is completely different. Indeed, clearly (H_t) is a non-negative process (so that there is no tail on the negative side...), and it is not possible to use a "favorable" initial profile to help constructing an event where H_1 becomes very large. In fact, the decay rate of the probability that H_1 is large is very different:

Proposition 5.2. — *When $h \to \infty$, $\mathbb{P}(H_1 \geq h) = \exp\left(-8h^3/9 + O(\ln(h))\right)$*

Proof. — Let us start with the lower bound. Let us study the stopping times S_{0,h_1} and S_{0,h_2} for $h_1 = h + 1/h^2$ and $h_2 = h + 5/h^2$ by (X, H). We would like to find an event that ensures that $S_{0,h_1} < 1$ and that H_t remains above h during a time at least 1 after this moment. We will consider the following four events (here Λ_1 and Λ_2 denote the BW lines that go through $(0, h_1)$ and $(0, h_2)$):

- $\Lambda_1[-2/h^4, 2/h^4] \subset [h + 1/(2h^2), h + 3/(2h^2)]$.
- $\Lambda_2[-1/h^4, 1/h^4] \subset [h + 9/(2h^2), h + 11/(2h^2)]$.
- Λ_1 and Λ_2 coalesce in the vertical strip above $[1/h^4, 2/h^4]$ and in the vertical strip above $[-2/h^4, -1/h^4]$ (combined with the previous conditions, this implies that the area between Λ_1 and Λ_2 is greater than $6/h^6$ and that $H_t \geq h$ during the corresponding time-interval $[S_{0,h_1}, S_{0,h_2}]$).
- The integral of Λ_1 on the interval $[2/h^4, +\infty)$ and on the interval $(-\infty, -2/h^4]$ both belong to $[1/2 - 2/h^3 - 3/h^6, 1/2 - 2/h^3 - 2/h^6]$.

It is easy to see that $H_1 \geq h$ if those events hold simultaneously. Scaling shows that the probability that the first three are satisfied simultaneously is a constant that does not depend on h. Using the simple Markov property, conditionally on the first events, $(\Lambda_1(2/h^2 + u), u \geq 0)$ is a Brownian motion starting at some level in $[1/(2h^2), 3/(2h^2)]$. With the expression of the density of the area under a Brownian motion until its first passage time at 0 given in Proposition 2.2-3, we have:

$$P_{h+u/h^2}\left(\int_0^{\tau_0} B_t dt \in [1/2 - 2/h^3 - 3/h^6, 1/2 - 2/h^3 - 2/h^6]\right) \geq \exp\left(-4h^3/9 + O(\ln(h))\right)$$

which is valid uniformly for every $u \in [1/2, 3/2]$.
Therefore,

$$\mathbb{P}(H_1 \geq h) \geq \left(\exp\left(-4h^3/9 + O(\ln(h))\right)\right)^2 = \exp\left(-8h^3/9 + O(\ln(h))\right).$$

For the upper bound, we can adapt the proof of the corresponding bound in the stationary case. The situation is at first sight simpler here, because we do not have to worry about the initial line.

Let us denote the Brownian web with flat initial data by $(\Lambda'_{x,h}, (x,h) \in \mathbb{R} \times \mathbb{R}_+^*)$ and $S'_{x,h}$ the integral of $\Lambda'_{x,h}(\cdot)$ over $\mathbb{R}$. First, notice that symmetry and the tail estimates for X_1 show that it is sufficient to find an upper bound for

$$\mathbb{P}(\exists y \in [0, Ch] \,:\, S'_{y,h} \leq 1)$$

for some given large enough C.

We now divide the interval $[0, Ch]$ into circa Ch^{10} smaller intervals $I_k := [x_k, x_{k+1}] = [k/h^9, (k+1)/h^9]$ and we wish to bound $\mathbb{P}(y \in I_k,\, S'_{y,h} \leq 1)$ for each k.

We are going to consider two cases depending on whether $\Lambda'_{y,h}(I_k) \subset [h - 1/h^2, \infty)$ or not:

- If $\Lambda'_{y,h}(I_k) \subset [h - 1/h^2, \infty)$, then $\Lambda'_{x_{k+1}, h-1/h^2}$ is below $\Lambda'_{y,h}$, so that $S'_{x_{k+1}, h-1/h^2} \leq 1$.
- If $\Lambda'_{y,h}(I_k) \not\subset [h - 1/h^2, \infty)$, then either the backward line started at $(x_{k+1}, h - 1/(2h^2))$ or the forward line started at $(x_k, h - 1/(2h^2))$ does not stay in $[h - 1/h^2, h)$ during the interval I_k.

The probability of the second case is very small, and can be bounded by a constant times $\exp(-Ch^6)$. The probability of the first case is bounded by the probability that the area under a two-sided Brownian motion starting at the level $h - 1/h^2$ until the first hitting times of 0 (on both sides) is less than 1. One can then conclude using the estimate (2), and summing over the Ch^{10} values of k (that correspond to another $e^{O(\log(h))}$ term). $\square$

5.6. Almost sure fluctuations. — Our tail-estimates for H are less precise than those we obtained for X. However, let us say a few words on how to nevertheless deduce information about the almost sure behavior of the process $(H_t, t \geq 0)$:

Corollary 5.3. — *There exists four constants $l^+ > 0$, $l_0^+ > 0$, $l^- < 0$ and $l_0^- < 0$ such that*

$$\limsup_{t\to\infty} t^{-1/3}(\ln\ln(t))^{-2/3}\, H_t = l^+,$$
$$\liminf_{t\to\infty} t^{-1/3}(\ln\ln(t))^{-2/3}\, H_t = l^-,$$
$$\limsup_{t\to 0} t^{-1/3}(\ln\ln(1/t))^{-2/3}\, H_t = l_0^+,$$
$$\liminf_{t\to 0} t^{-1/3}(\ln\ln(1/t))^{-2/3}\, H_t = l_0^-.$$

Carefully adapting the proofs that we presented for X, using our tail estimates for the process H yields statements of the following type: There exists two positive finite constants $\tilde{l}$ and $\hat{l}$ such that almost surely

$$\hat{l} \le \limsup_{t\to\infty} t^{-1/3}(\ln\ln(t))^{-2/3}\, H_t \le \tilde{l}.$$

The upper bound is a direct consequence of the Borel-Cantelli Lemma, and the lower bound is obtained as in the case of X by considering event that are measurable with respect to the information provided by Brownian web restricted to disjoint domains. Let us briefly give the outline of the proof. As it is very similar to the fluctuations of X, we omit the details and just outline the proof:

Let us choose a sequence (λ_n) increasing fast, but not too fast either ($\lambda_n := \lambda^{n(n-1)}$ with $\lambda > 1$ is suitable). It suffices to prove that there exists some absolute constant $c > 0$ such that almost surely, the process H reaches the height λ_n before time $c\,\lambda_n^3/(\ln(\lambda_n))^2$ for infinitely many values of n. For each given n, we will focus on the first time at which the TSRM X reaches the position $Y_{\lambda_n} := \inf\{y \ge 0 \;:\; \Gamma_0(y) = \lambda_n\}$. Clearly, at that random time $\sigma_{Y_{\lambda_n}}$, the height H is equal to λ_n.

Set $l_n := \lambda_n^2/\ln(n)$ and consider the boxes $L'_n := [-l_n, l_n] \times [-2\lambda_n, 2\lambda_n]$. It is easy to see via the Borel-Cantelli Lemma that almost surely, for all but finitely many n, the event

$$\mathcal{D}'_n := \{\Gamma_0[-l_n, l_n,] \subset [-2\lambda_n, \lambda_n]\}$$

does hold.

We introduce also $\xi_n := \alpha\,\lambda_n/\ln(n)$. As for the proof of the lower bound, we define two parallel upwards-going tubes U_n and V_n such that the bottom line of U_n is the segment joining the points $(l_{n-1}, -\xi_n + 2\lambda_{n-1})$ and (l_n, λ_n) and the vertical width of U_n is ξ_n. The tube V_n is simply the same tube as U_n but translated vertically by ξ_n. We consider the following three BW-curves: F_n^- the BW-curve starting from $(l_{n-1}, -\xi_n/2)$, F_n^+ the backward BW-curve starting from $(l_n, \lambda_n + 2\xi_n)$ and G_n^+ the backward BW-curve starting at $(l_{n-1}, 2\xi_n)$. We will now study $\mathcal{A}'_n$ that the following events occurs:

- F_n^- stays in the tube U_n and F_n^+ stays in the tube V_n.

- The integral of $G_n^+ - F_n^-$ over $(-\infty, l_{n-1}]$ is less than ξ_n^3 and G_n^+ and F_n^- do not enter in L'_{n-1}.

Notice that the events $\mathcal{A}'_n$ depend only on BW-curves in $L'_n \backslash L'_{n-1}$ and are therefore independent. Note also that the first event in $\mathcal{A}'_n$ is independent of the second one. The probability of the second event is bounded below by a constant. A similar computation to the proof of the lower bound permits to deal with the first one and shows that the series $\sum \mathbb{P}(\mathcal{A}'_n)$ diverges. Thus almost surely $\mathcal{A}'_n$ holds infinitely often. Moreover, the values of the sequences ξ_n and l_n and BW monotonicity imply that $\mathcal{A}'_n \cap \mathcal{D}'_{n-1}$ is a sub-event of $\sigma_{Y_{\lambda_n}} \leq c\,\lambda_n^3/(\ln(\lambda_n))^2$ for some $c > 0$ which does not depend on n. It proves the desired bound.

Let us now describe how to use a $0-1$ type argument in order to conclude. We want for instance to show that

$$Z := \limsup_{t\to\infty} \frac{H_t}{t^{1/3}(\ln\ln t)^{2/3}}$$

is almost surely constant (the previous estimates then show that this constant is positive and finite).

Consider for any positive h, the curve $\Lambda_{(0,h)}$ started at height h on the vertical axis. It is the profile at the stopping time corresponding to the first time at which $L_t(0)$ reaches h. Let us denote this random time by ρ_h. We know that $\rho_h \to \infty$ almost surely as $h \to \infty$.

For all $h > 0$, we denote by $\mathcal{G}_h$ the σ-field that contains all the information about the Brownian web above the line $\Lambda_{(0,h)}$. In other words, it is the σ-field generated by this line and by $((X_{t+\rho_h}, H_{t+\rho_h}), t \geq 0)$. Note that Z is therefore $\mathcal{G}_h$ measurable (for all $h > 0$). As $\mathcal{G}_h$ is decreasing with h, it follows that Z is measurable with respect to $\mathcal{G}_\infty := \bigcap_h \mathcal{G}_h$.

For all positive N, let us now denote $\mathcal{V}_N$ the σ-field generated by the process (X,H) up to the first time at which $\max(|X|,|H|)$ reaches N. Clearly, this stopping time is almost surely finite and when $N \to \infty$, it converges almost surely to ∞ because (X,H) is a continuous process. Furthermore, any event in $\mathcal{V}_N$ can be read off by looking at the Brownian web lines inside the square $A_N = [-N,N]^2$.

Suppose that N is fixed, that U is a $\sigma(Z)$-measurable event, and that V is $\mathcal{V}_N$ measurable. Suppose furthermore that $W_{h,N}$ is the event that the line $\Lambda_{(0,h)}$ does not intersect $[-N,N]^2$. Clearly, the events $W_{h,N} \cap U$ and V are independent as the former can be read off by looking only at the Brownian web outside of $[-N,N]^2$. On the other hand, we know that $P(W_{h,N}) \to 1$ as $h \to \infty$. Hence,

$$P(U\cap V) = \lim_{h\to\infty} P(U \cap V \cap W_{h,N}) = P(V) \lim_{h\to\infty} P(U \cap W_{h,N}) = P(U)P(V).$$

It follows that U is independent of the σ-field generated by $\bigcup_N \mathcal{V}_N$, that contains $\sigma(H_t, t \geq 0)$ and therefore also U. Hence, $P(U) = 0$ or $P(U) = 1$.

The proof of the fact that

$$Z' := \limsup_{t\to 0} \frac{H_t}{t^{1/3}(\ln\ln(1/t))^{2/3}}$$

is almost surely constant is similar. We know that almost surely $\rho_h \to 0$ as $h \to 0$, and that H is continuous. It follows that the process $(H_t, t \geq 0)$ is measurable with respect to $\sigma(\bigcup_h \mathcal{G}_h)$. But for any fixed $h_0 > 0$, the probability that $\Lambda_{(0,h_0)}$ intersects the box $[-1/N, 1/N]^2$ goes to 0 as $N \to \infty$, and on the other hand, we know that Z' is measurable with respect to each $\mathcal{V}_{1/N}$ (because $\rho_{1/N} > 0$). Hence, it follows readily that Z' is independent of $\mathcal{G}_{h_0}$, and then, letting $h_0 \to 0$ that the random variable Z' is independent of itself and therefore constant.

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