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TORSION AND SYMPLECTIC VOLUME IN SEIFERT MANIFOLDS

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ABSTRACT. — For any oriented Seifert manifold X and compact connected Lie group G with finite center, we relate the Reidemeister density of the moduli space of representations of the fundamental group of X into G to the Liouville measure of some moduli spaces of representations of surface groups into G.

RÉSUMÉ (*Torsion et volume symplectique des variétés de Seifert*). — Pour toute variété de Seifert orientée X et tout groupe de Lie compact connexe G de centre fini, nous calculons la densité de Reidemeister de l'espace des modules des représentations du groupe fondamental de X dans G en fonction de la mesure de Liouville de certains espaces de modules de représentations de groupes de surfaces.

1. Introduction

For any Lie group G and manifold Y, the moduli space $\mathcal{M}(Y)$ of conjugacy classes of representations of $\pi_1(Y)$ in G, has natural differential geometric structures. If Σ is a closed oriented surface, $\mathcal{M}(\Sigma)$ has a symplectic stucture

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defined via intersection pairing [1], [4]. More generally, if Σ is a compact oriented surface and $u \in \mathcal{M}(\partial\Sigma)$, the subspace $\mathcal{M}(\Sigma, u)$ of $\mathcal{M}(\Sigma)$ consisting of the representations restricting to u on the boundary has a natural symplectic structure. If X is a closed 3-dimensional oriented manifold, $\mathcal{M}(X)$ has a natural density μ_X defined from Reidemeister torsion [16].

In this article, we relate these structures for X any oriented Seifert manifold and Σ a convenient oriented surface embedded in X. We will prove that when G is compact with finite center, the subspace $\mathcal{M}^0(X) \subset \mathcal{M}(X)$ of irreducible representations, is a smooth manifold covered by disjoint open subsets O_α, such that each O_α identifies with $\mathcal{M}^0(\Sigma, u_\alpha)$ for some $u_\alpha \in \mathcal{M}(\partial\Sigma)$. Furthermore, on each U_α the canonical density μ_X identifies, up to some multiplicative constant depending on α, with the Liouville measure of the symplectic structure of $\mathcal{M}^0(\Sigma, u_\alpha)$.

Our main motivation is the Witten's asymptotic conjecture, which predicts that the Witten-Reshetikhin-Turaev invariant of a 3-manifold X has a precise asymptotic expansion in the large level limit. This expansion is a sum of oscilatory terms, whose amplitudes are function of the Reidemeister volume of the components of $\mathcal{M}(X)$. In the case where X is a Seifert manifold, some of these amplitudes are actually function of the symplectic volumes of the moduli spaces $\mathcal{M}^0(\Sigma, u)$, [13], [3]. So a relation between Reidemeister and symplectic volumes was expected. At a more general level, it is known that the Chern-Simons theory on a Seifert manifold can be interpreted as two-dimensional Yang-Mills theory [2].

Let us state our results with more detail and then discuss the related literature.

Statement of the main result. — The Seifert manifolds we will consider are the oriented closed connected three manifold equipped with a locally free circle action. Any such manifold may be obtained as follows. Let Σ be an oriented compact surface with $n \geqslant 1$ boundary components $C_1, \ldots, C_n$. Let D be the standard closed disk of $\mathbb{C}$. Let φ_i be an orientation reversing diffeomorphism from $\partial D \times S^1$ to $C_i \times S^1$. Let X be the manifold obtained by gluing n copies of $D \times S^1$ to $\Sigma \times S^1$ through the maps φ_i. We have $[\varphi_i(\partial D)] = -p_i[C_i] + q_i[S^1]$ in $H_1(C_i \times S^1)$ where p_i, q_i are two relatively prime integers. We assume that $p_i \geqslant 1$ for all i.

Let G be a compact connected Lie group with finite center. For $Y = X$, Σ, C_i or S^1, we denote by $\mathcal{M}(Y)$ (resp. $\mathcal{M}^0(Y)$) the set of representations (resp. irreducible representations) of $\pi_1(Y)$ in G up to conjugation. Since C_i and S^1 are oriented circles, we can identify $\mathcal{M}(C_i)$ and $\mathcal{M}(S^1)$ with the set $\mathcal{C}(G)$ of conjugacy classes of G. For any $u \in \mathcal{C}(G)^n$, we denote by $\mathcal{M}^0(\Sigma, u)$ the subset of $\mathcal{M}^0(\Sigma)$ consisting of the representations whose restriction to each C_i is u_i. Recall that $\mathcal{M}^0(\Sigma, u)$ is a smooth symplectic manifold.

For any $(u,v) \in \mathcal{C}(G)^{n+1}$, we denote $\mathcal{M}^0(X,u,v)$ the subset of $\mathcal{M}^0(X)$ consisting of representations whose restriction to each C_i is u_i and to S^1 is v. Let $\mathcal{P}$ be the subset of $\mathcal{C}(G)^{n+1}$ consisting of the (u,v) such that $\mathcal{M}^0(X,u,v)$ is non empty.

Theorem 1.1. — *$\mathcal{M}^0(X)$ is a smooth manifold, whose components may have different dimensions. For any $[\rho] \in \mathcal{M}^0(X)$, the tangent space $T_{[\rho]}\mathcal{M}^0(X)$ is canonically identified with $H^1(X, \mathrm{Ad}\,\rho)$ where $\mathrm{Ad}\,\rho$ is the flat vector bundle associated to ρ via the adjoint representation. Furthermore, $\mathcal{P}$ is finite and for any $(u,v) \in \mathcal{P}$, $\mathcal{M}^0(X,u,v)$ is an open subset of $\mathcal{M}^0(X)$ and the restriction map $R_{u.v}$ from $\mathcal{M}^0(X,u,v)$ to $\mathcal{M}^0(\Sigma,u)$ is a diffeomorphism.*

For any irreducible representation ρ of $\pi_1(X)$ in G, the homology groups $H_0(X,\mathrm{Ad}\,\rho)$ and $H_3(X,\mathrm{Ad}\,\rho)$ are trivial. By Poincaré duality, $H_2(X,\mathrm{Ad}\,\rho)$ is the dual of $H_1(X,\mathrm{Ad}\,\rho)$. So the Reidemeister torsion of $\mathrm{Ad}\,\rho$ is a non vanishing element of $\big(\det H_1(X,\mathrm{Ad}\,\rho)\big)^{-2}$ well-defined up to sign. Consequently, the inverse of the square root of the torsion is a density of $H^1(X,\mathrm{Ad}\,\rho)$. Since $H^1(X,\mathrm{Ad}\,\rho)$ identifies with the tangent space of $\mathcal{M}^0(X)$ at ρ, we define in this way a density μ_X on $\mathcal{M}^0(X)$.

For any $u \in \mathcal{C}(G)$ and $[\rho] \in \mathcal{M}^0(\Sigma,u)$, the tangent space $T_{[\rho]}\mathcal{M}^0(\Sigma,u)$ is identified with the kernel of the morphism $H^1(\Sigma,\mathrm{Ad}\,\rho) \to H^1(\partial\Sigma,\mathrm{Ad}\,\rho)$. The symplectic product of $T_{[\rho]}\mathcal{M}^0(\Sigma,u)$ is induced by the intersection product of $H^1(\Sigma,\mathrm{Ad}\,\rho)$ with $H^1(\Sigma,\partial\Sigma,\mathrm{Ad}\,\rho)$. We denote by μ_u the corresponding Liouville measure of $\mathcal{M}^0(\Sigma,u)$.

As a last definition, let $\Delta : \mathcal{C}(G) \to \mathbb{R}$ be the function given by

$$\Delta(u) = \left|\det_{H_g}(\mathrm{Ad}_g - \mathrm{id})\right|^{1/2},$$

where g is any element in the conjugacy class u and H_g is the orthocomplement of $\ker(\mathrm{Ad}_g - \mathrm{id})$. Equivalently, let $\mathfrak{t}$ be the Lie algebra of a maximal torus of G, $R \subset \mathfrak{t}^*$ be the corresponding set of real roots and $R_+ \subset R$ be a set of positive roots. Then for any $X \in \mathfrak{t}$,

$$\Delta([e^X]) = \prod_{\alpha \in R_+;\ \alpha(X) \neq 0} 2|\sin(\pi\alpha(X))|.$$

Theorem 1.2. — *For any $(u,v) \in \mathcal{P}$, we have on $\mathcal{M}^0(X,u,v)$*

$$\mu_X = \left(\prod_{i=1}^n \frac{\Delta(u_i^{r_i})}{p_i^{(\dim G - \dim u_i)/2}} \right) R^*_{u,v}\mu_u,$$

where $R_{u,v}$ is the restriction map from $\mathcal{M}^0(X,u,v)$ to $\mathcal{M}^0(\Sigma,u)$ and for each i, r_i is any inverse of q_i modulo p_i, and $u_i^{r_i} \in \mathcal{C}(G)$ is the conjugacy class containing the g^{r_i} for $g \in u_i$.

Several definitions require an invariant scalar product on the Lie algebra of G: the symplectic structure of $\mathcal{M}^0(\Sigma, u)$, the Poincaré duality between $H_1(X, \operatorname{Ad}\rho)$ and $H_2(X, \operatorname{Ad}\rho)$ and the Reidemeister torsion of $\operatorname{Ad}\rho$. Our implicit convention is to choose the same invariant scalar product each time.

During the proof, we will prove interesting intermediate results:

- for any irreducible representation ρ of $\pi_1(X)$ in G, the cohomology groups $H^1(X, \operatorname{Ad}\rho)$ and $H^2(X, \operatorname{Ad}\rho)$ both identify naturally with the kernel of the restriction morphism $H^1(\Sigma, \operatorname{Ad}\rho) \to H^1(\partial\Sigma, \operatorname{Ad}\rho)$.
- by these identifications, the intersection product of $H^1(X, \operatorname{Ad}\rho)$ with $H^2(X, \operatorname{Ad}\rho)$ is sent to the intersection product of $H^1(\Sigma, \operatorname{Ad}\rho)$ with $H^1(\Sigma, \partial\Sigma, \operatorname{Ad}\rho)$.
- the Reidemeister torsion of $\operatorname{Ad}\rho \to X$ is equal to $C^{-2}\det\psi$ where $\psi : H_1(X, \operatorname{Ad}\rho) \to H_2(X, \operatorname{Ad}\rho)$ is the map induced by the previous identifications and C is the factor appearing in Theorem 1.2.

This results are respectively proved in Sections 4, 5 and 6. Theorem 1.2 is proved in Section 7 and Theorem 1.1 in Section 3.2.

Related results in the litterature. — Witten [17] proved that for S a closed oriented surface, the canonical density μ_S of $\mathcal{M}^0(S)$ defined from Reidemeister torsion, is the Liouville measure of the natural symplectic structure of $\mathcal{M}^0(S)$. He also extended this result to surfaces with boundary. We tried to deduce Theorem 1.2 from this by relating the torsions of $\operatorname{Ad}\rho \to X$ and $\operatorname{Ad}\rho \to \Sigma$, without any success. Our actual proof does not use Witten's result.

Witten also computed explicitely the volumes $\int_{\mathcal{M}^0(\Sigma,u)} \mu_u$, cf. [17], Formula 4.114. For $G = \mathrm{SU}(2)$ and non central conjugacy classes u_i, Park [12] adapted the Witten's method to compute $\int_{\mathcal{M}^0(X,u,v)} \mu_X$, X being our Seifert manifold. Computing the volume of $\mathcal{M}^0(X, u, v)$ with Theorem 1.2 and Witten's formula, we can extend Park's result to any compact connected Lie group G with finite center and any conjugacy classes u_i.

McLellan [8] proved a result similar to Theorem 1.2 for $G = \mathrm{U}(1)$. To do this, he introduced a Sasakian structure on X and used a computation of the corresponding analytic torsion [14]. We will explain in Section 8 how we can recover McLellan's result by adapting our method, providing an elementary proof.

2. The Seifert manifold X

Let $g, n, p_1, q_1, \ldots, p_n, q_n$ be integers such that

$$(1) \qquad g \geqslant 0, \quad n \geqslant 1 \quad \text{and} \quad \forall i, \quad p_i, q_i \text{ are coprime and } p_i \geqslant 1.$$

To such a familly we associate the following manifold X. Let Σ be a compact oriented surface with genus g and n boundary components denoted by $C_1, \ldots, C_n$.

Let D be a closed disk and for any i, let $\varphi_i : \partial D \times S^1 \to C_i \times S^1$ be an orientation reversing diffeomorphism such that we have in $H_1(S^1 \times C_i)$,

$$[\varphi_i(\partial D)] = -p_i[C_i] + q_i[S^1], \tag{2}$$

where ∂D and C_i are oriented as boundaries of D and Σ respectively. Then X is obtained by gluing n copies of $D \times S^1$ to $\Sigma \times S^1$ along its boundary through the maps φ_i,

$$X = (\Sigma \times S^1) \cup_{\varphi_1 \cup \dots \cup \varphi_n} (D \times S^1)^{\cup n}. \tag{3}$$

By construction $\Sigma \times S^1$ is a submanifold of X. In the sequel we often consider Σ and S^1 as submanifolds of X by identifying Σ with $\Sigma \times \{y\}$ and S^1 with $\{x\} \times S^1$, where x and y are some fixed points of Σ and S^1 respectively.

The above definitions are all what we need for this article. Nevertheless, it is interesting to understand this in the context of Seifert manifolds. First, if X is obtained as previously, we can extend the S^1-action on $\Sigma \times S^1$ to X, so that for any i, the action on the i-th copy of $D \times S^1$ is free if $p_i = 1$ and otherwise it has one exceptional orbit with isotropy $\mathbb{Z}_{p_i}$. Conversely, consider any three dimensional closed connected oriented manifold Y equipped with an effective locally free action of S^1. Then choose $n \geqslant 1$ orbits $O_1, \dots, O_n$ of Y including all the exceptional ones. Let $T_1, \cdots, T_n$ be disjoint saturated open tubular neighborhoods of the $O_1, \dots, O_n$ respectively. Let Σ be any cross-section of the action on $Y \setminus (T_1 \cup \dots \cup T_n)$. For any i, set $C_i = (\partial \Sigma) \cap \overline{T}_i$ and define p_i as the order of the isotropy group of O_i and q_i so that $[C_i] = q_i[O_i]$ in $H_1(\overline{T}_i)$, where C_i is oriented as the boundary of Σ and O_i by the S^1-action. Let X be any manifold associated to the data Σ, $(p_1, q_1), \dots, (p_n, q_n)$ as in (3). Then Y is diffeomorphic to X, cf. [5], Theorem 1.5 or the Section 1 of [10] for more details. We can even choose the diffeomorphism between Y and X so that it commutes with the S^1-action and fixes Σ. The collection

$$(g; (p_1, q_1), \dots, (p_n, q_n))$$

is called the unnormalized Seifert invariant of Y.

3. Character space of a Seifert manifold

Notations. — Let G be a Lie group. For any connected topological space Y, we denote by $\mathcal{M}(Y)$ the set of conjugacy classes of representations of $\pi_1(Y)$ into G[1]. A representation $\rho : \pi_1(Y) \to G$ is said to be irreducible if the centraliser of $\rho(\pi_1(Y))$ is reduced to the center of G. We denote by $\mathcal{M}^0(Y)$ the subset of $\mathcal{M}(Y)$ consisting of conjugacy classes of irreducible representations.

1. A representation of $\pi_1(Y)$ into G is a group morphism from $\pi_1(Y)$ to G. Two representations ρ, ρ' are conjugate if there exists $g \in G$, such that $\rho'(h) = g\rho(h)g^{-1}$, $\forall h \in G$.

If Z is a subspace of Y, there is a natural morphism j_* from $\pi_1(Z)$ to $\pi_1(Y)$ and consequently a natural map from $\mathcal{M}(Y)$ to $\mathcal{M}(Z)$, sending $[\varphi]$ into $[\varphi \circ j_*]$. For any representation $\rho : \pi_1(Y) \to G$, we call $\rho \circ j_*$ the restriction of ρ to Y.

3.1. A decomposition of $\mathcal{M}^0(X)$. — From now on, X is the Seifert manifold introduced in Section 2. Recall that we view S^1 and Σ as submanifolds of X.

Proposition 3.1. — *Let ρ be a representation of $\pi_1(X)$ into G. Then ρ is irreducible if and only if its restriction to Σ is irreducible. Furthermore, if ρ is irreducible, then $\rho(S^1)$ is central. Finally, for any i, $\rho(C_i)^{p_i}$ is conjugate to $\rho(S^1)^{q_i}$.*

In the statement we slightly abused notation by applying ρ to oriented circles of X. Since any loop γ of X is homotopic to an element of $\pi_1(X)$ unique up to conjugation, the conjugacy class of $\rho(\gamma)$ is uniquely defined.

Proof. — By Van Kampen theorem, the natural morphism $\pi_1(\Sigma \times S^1) \to \pi_1(X)$ is onto. So $\rho : \pi_1(X) \to G$ is irreducible if and only if its restriction to $\pi_1(\Sigma \times S^1)$ is irreducible. Since $\rho(\pi_1(\Sigma)) \subset \rho(\pi_1(\Sigma \times S^1))$, if $\rho|_{\Sigma}$ is irreducible, then $\rho|_{\Sigma\times S^1}$ is irreducible. Conversely, assume that $\rho|_{\Sigma\times S^1}$ is irreducible. Since $\pi_1(\Sigma \times S^1) \simeq \pi_1(\Sigma) \times \pi_1(S^1)$, $t = \pi_1(S^1)$ is in the centraliser of $\pi_1(\Sigma \times S^1)$, and consequently $\rho(t)$ is central. This implies that $\rho(\pi_1(\Sigma \times S^1))$ and $\rho(\pi_1(\Sigma))$ have the same centraliser. So $\rho|_{\Sigma}$ is irreducible.

By Equation (2), $\rho(C_i)^{p_i}$ and $\rho(S^1)^{q_i}$ are conjugate. □

Let $Z(G)$ be the center of G and $\mathcal{C}(G)$ be the set of conjugacy classes. If $u \in \mathcal{C}(G)$ and p is an integer, $u^p \in \mathcal{C}(G)$ is defined as the conjugacy class of g^p where $g \in u$. Let $\mathcal{P}$ be the subset of $\mathcal{C}(G)^n \times Z(G)$ consisting of the pairs (u, v) such that for any i, $u^{p_i} = v^{q_i}$. Then by the last part of Proposition 3.1,

$$\mathcal{M}^0(X) = \bigcup_{(u,v)\in\mathcal{P}} \mathcal{M}^0(X, u, v), \tag{4}$$

where $\mathcal{M}^0(X, u, v)$ consists of the $[\rho] \in \mathcal{M}^0(X)$ such that $\rho(S^1) \in v$ and $\rho(C_i) \in u_i$ for any i. Denote by $R_{u,v}$ the restriction map

$$R_{u,v} : \mathcal{M}^0(X, u, v) \to \mathcal{M}^0(\Sigma, u), \qquad [\rho] \to [\rho|_{\Sigma}], \tag{5}$$

where $\mathcal{M}^0(\Sigma, u)$ is the subset of $\mathcal{M}^0(\Sigma)$ consisting of the classes $[\rho]$ such that for any i, $\rho(C_i) \in u_i$.

Proposition 3.2. — *For any $(u, v) \in \mathcal{P}$, the map $R_{u,v}$ is a bijection.*

Proof. — It is a consequence of the fact that $\pi_1(\Sigma \times S^1) = \pi_1(\Sigma) \times \pi_1(S^1)$ and that the kernel of the surjective map $\pi_1(\Sigma \times S^1) \to \pi_1(X)$ is the normal subgroup normally generated by the $\varphi_i(\partial D)$'s. □

3.2. Topology and manifold structure. — From now on, assume that G is compact and has a finite center. As explained in Appendix A, for any compact connected manifold Y, $\mathcal{M}(Y)$ has a natural Hausdorff topology and $\mathcal{M}^0(Y)$ is an open subset.

LEMMA 3.3. — *The set $\mathcal{P}$ is finite. For any $(u,v) \in \mathcal{P}$, $\mathcal{M}^0(X,u,v)$ is an open subset of $\mathcal{M}^0(X)$.*

Proof. — By identifying $\mathcal{C}(G)$ with the quotient of a maximal torus by the Weyl group, we easily see that for any $v \in \mathcal{C}(G)$ and $p \in \mathbb{Z}$, the equation $u^p = v$ has only a finite number of solutions. This implies that $\mathcal{P}$ is finite. We deduce that the $\mathcal{M}^0(X,u,v)$'s are open by applying the following fact: for any compact connected manifold Y, for any $x \in \pi_1(Y)$, the map from $\mathcal{M}(Y)$ to $\mathcal{C}(G)$ sending $[\rho]$ into $[\rho(x)]$ is continuous. □

By Appendix A, $\mathcal{M}^0(X)$ has a natural open subset $\mathcal{M}^{\mathrm{s},0}(X)$ which is a manifold. Furthermore, it is known that the spaces $\mathcal{M}^0(\Sigma,u)$ are smooth manifolds.

PROPOSITION 3.4. — *We have $\mathcal{M}^0(X) = \mathcal{M}^{\mathrm{s},0}(X)$. Furthermore, for any $(u,v) \in \mathcal{P}$, $R_{u,v}$ is a diffeomorphism from $\mathcal{M}^0(X,u,v)$ to $\mathcal{M}^0(\Sigma,u)$.*

It is possible that the various $\mathcal{M}^0(X,u,v)$ have different dimensions. Actually,

$$\dim \mathcal{M}^0(\Sigma,u) = 2(g-1)\dim G + \sum_{i=1}^{n} \dim u_i.$$

Proof. — Let $u \in \mathcal{C}(G)^n$ and consider the set M_u of $(a,b,c) \in (G^{2g+n})^0$ satisfying the relations

$$[a_1,b_1]\cdots[a_g,b_g]c_1\cdots c_n = \mathrm{id}, \qquad c_i \in u_i, \quad \forall i.$$

Here we used the same notation $(G^{2g+n})^0$ as in Appendix A. It is known that M_u is a smooth submanifold of G^{2g+n}.

Choose a standard set of generators (x,y,z) of $\pi_1(\Sigma)$ and let $t \in \pi_1(X)$ be isotopic to S^1. The map $\pi_1(\Sigma \times S^1) \to \pi_1(X)$ being onto, (x,y,z,t) is a set of generators of $\pi_1(X)$. Through these generators, $\mathcal{R}^0(\pi_1(X))$ gets identified with a subset A of G^{2g+n+1} as explained in Appendix A. By the decomposition (4), A is the union of the $M_u \times \{v\}$ where (u,v) runs over $\mathcal{P}$. $\mathcal{P}$ being finite, A is a submanifold of G^{2g+n+1}, which shows that $\mathcal{R}^0(\pi_1(X)) = \mathcal{R}^{\mathrm{s},0}(\pi_1(X))$ in the notation of Appendix A and consequently that $\mathcal{M}^{\mathrm{s},0}(X) = \mathcal{M}^0(X)$. For any $(u,v) \in \mathcal{P}$, the projection $M_u \times \{v\} \to M_u$ being a diffeomorphism, we conclude that $R_{u,v}$ is a diffeomorphism. □

Consider again a compact connected manifold and a representation ρ of $\pi_1(Y)$ in G. Composing ρ with the adjoint representation, the Lie algebra $\mathfrak{g}$ becomes a

$\pi_1(Y)$-module. Denote by $H^\bullet(\pi_1(Y), \operatorname{Ad}\rho)$ the group cohomology with coefficient in $\mathfrak{g}$. Alternatively, we may consider the flat vector bundle $\operatorname{Ad}\rho \to Y$ associated to ρ via the adjoint representation. Let $H^\bullet(Y, \operatorname{Ad}\rho)$ be the cohomology of Y with local coefficient. Then for $j = 0$ or 1, $H^j(\pi_1(Y), \operatorname{Ad}\rho) \simeq H^j(Y, \operatorname{Ad}\rho)$.

LEMMA 3.5. — *For any irreducible representation ρ of $\pi_1(X)$, we have a natural identification between $H^1(X, \operatorname{Ad}\rho)$ and $T_{[\rho]}\mathcal{M}^0(X)$.*

Proof. — By Appendix A, $T_{[\rho]}\mathcal{M}^0(X)$ is naturally identified with a subspace of $H^1(X, \operatorname{Ad}\rho)$. Similarly, it is known that $T_{[\rho]}\mathcal{M}^0(\Sigma, u)$ gets identified to the kernel of the morphism $H^1(\Sigma, \operatorname{Ad}\rho) \to H^1(\partial\Sigma, \operatorname{Ad}\rho)$. Furthermore, we easily see that the tangent linear map to $R_{u,v}$ is the restriction of the morphism $H^1(X, \operatorname{Ad}\rho) \to H^1(\Sigma, \operatorname{Ad}\rho)$. As we will see in Theorem 4.2, the following sequence is exact

$$0 \to H^1(X, \operatorname{Ad}\rho) \to H^1(\Sigma, \operatorname{Ad}\rho) \to H^1(\partial\Sigma, \operatorname{Ad}\rho) \to 0.$$

This implies that $H^1(X, \operatorname{Ad}\rho) = T_{[\rho]}\mathcal{M}^0(X)$. □

4. The homology groups $H_1(X, \operatorname{Ad}\rho)$ and $H_1(\Sigma, \operatorname{Ad}\rho)$

As in the previous section, for any compact connected topological space Y and representation $\rho : \pi_1(Y) \to G$, we consider the flat vector bundle $\operatorname{Ad}\rho \to Y$. We are interested in corresponding homology groups $H_\bullet(Y, \operatorname{Ad}\rho)$ for $Y = X$ or Σ. As a first remark, if ρ is irreducible, then by Appendix A, $H^0(Y, \operatorname{Ad}\rho) = H^0(\pi_1(Y), \operatorname{Ad}\rho) = 0$ because the center of G is finite. By duality, $H_0(Y, \operatorname{Ad}\rho) = 0$.

Consider the surface Σ and an irreducible representation $\rho : \pi_1(\Sigma) \to G$. For any boundary component C_i, choose a base point $x_i \in C_i$ and let $V_i = \ker(\operatorname{hol}_i - \operatorname{id})$ where $\operatorname{hol}_i : \operatorname{Ad}\rho|_{x_i} \to \operatorname{Ad}\rho|_{x_i}$ is the holonomy of C_i in $\operatorname{Ad}\rho$. We have two isomorphisms

$$H_0(C_i, \operatorname{Ad}\rho) \simeq V_i, \quad H_1(C_i, \operatorname{Ad}\rho) \simeq V_i.$$

sending $u \in V_i$ into $[x_i] \otimes u$ and $[C_i] \otimes u$ respectively.

LEMMA 4.1. — *We have $H_0(\Sigma, \operatorname{Ad}\rho) = H_2(\Sigma, \operatorname{Ad}\rho) = 0$. Furthermore the natural map $f : H_1(\partial\Sigma, \operatorname{Ad}\rho) \to H_1(\Sigma, \operatorname{Ad}\rho)$ is injective.*

Proof. — Σ being connected with a non empty boundary, $H_0(\Sigma, \partial\Sigma, \operatorname{Ad}\rho) = 0$, so by Poincaré duality, $H_2(\Sigma, \operatorname{Ad}\rho) = 0$. Since ρ is irreducible, $H_0(\Sigma, \operatorname{Ad}\rho) = 0$ and by Poincaré duality, $H_2(\Sigma, \partial\Sigma, \operatorname{Ad}\rho) = 0$. Writing the long exact sequence associated to the pair $(\Sigma, \partial\Sigma)$, we deduce that f is one-to-one. □

Consider now the Seifert manifold X and an irreducible representation $\rho : \pi_1(X) \to G$. Since Σ is a submanifold of X, we have a natural morphism

$$g : H_1(\Sigma, \operatorname{Ad}\rho) \to H_1(X, \operatorname{Ad}\rho)$$

By Lemma 3.1, the restriction of ρ to S^1 is central. So the restriction of the bundle $\operatorname{Ad}\rho$ to $\Sigma \times S^1$ is isomorphic to $\operatorname{Ad}\rho|_X \boxtimes \mathbb{R}_{S^1}$ [2]. Here we denote by $\mathbb{R}_{S^1}$ the trivial vector bundle over S^1 with fiber $\mathbb{R}$. This allows to define a second application

$$h : H_1(\Sigma, \operatorname{Ad}\rho) \to H_2(X, \operatorname{Ad}\rho)$$

which sends $\alpha \in H_1(\Sigma, \operatorname{Ad}\rho)$ into the image of $\alpha \boxtimes [S^1] \in H_2(\Sigma \times S^1, \operatorname{Ad}\rho)$ by the natural morphism $H_2(\Sigma \times S^1, \operatorname{Ad}\rho) \to H_2(X, \operatorname{Ad}\rho)$.

THEOREM 4.2. — *We have $H_0(X, \operatorname{Ad}\rho) = H_3(X, \operatorname{Ad}\rho) = 0$. Furthermore the following sequences are exact:*

$$0 \to H_1(\partial\Sigma, \operatorname{Ad}\rho) \xrightarrow{f} H_1(\Sigma, \operatorname{Ad}\rho) \xrightarrow{g} H_1(X, \operatorname{Ad}\rho) \to 0,$$

$$0 \to H_1(\partial\Sigma, \operatorname{Ad}\rho) \xrightarrow{f} H_1(\Sigma, \operatorname{Ad}\rho) \xrightarrow{h} H_2(X, \operatorname{Ad}\rho) \to 0.$$

Proof. — Since ρ is irreducible, $H_0(X, F) = 0$. By Poincaré duality, $H_3(X, F) = 0$. To prove that the sequences are exact, we will consider the Mayer-Vietoris long exact sequence associated to the decomposition (3) of X.

Since the restriction of $\operatorname{Ad}\rho$ to $\Sigma \times S^1$ is isomorphic to $\operatorname{Ad}\rho|_\Sigma \boxtimes \mathbb{R}_{S^1}$, we can compute by applying the Künneth theorem to the maps

$$H_j(\partial\Sigma \times S^1, \operatorname{Ad}\rho) \to H_j(\Sigma \times S^1, \operatorname{Ad}\rho), \qquad j = 3, 2, 1, 0. \tag{6}$$

We have that $H_j(S^1, \mathbb{R}) = \mathbb{R}$ for $j = 0, 1$ and by Lemma 4.1, $H_j(\Sigma, \operatorname{Ad}\rho) = 0$ for $j = 0, 2$. We deduce that

$$H_3(\Sigma \times S^1, \operatorname{Ad}\rho) = H_0(\Sigma \times S^1, \operatorname{Ad}\rho) = 0$$

and $H_0(\partial\Sigma \times S^1, \operatorname{Ad}\rho) \simeq H_0(\partial\Sigma, \operatorname{Ad}\rho)$, which determines (6) for $j = 0$ and 3. For $j = 2$, the map (6) identifies with the map $f : H_1(\partial\Sigma, \operatorname{Ad}\rho) \to H_1(\Sigma, \operatorname{Ad}\rho)$ and for $j = 1$ with

$$f \oplus 0 : H_1(\partial\Sigma, \operatorname{Ad}\rho) \oplus H_0(\partial\Sigma, \operatorname{Ad}\rho) \to H_1(\Sigma, \operatorname{Ad}\rho),$$

because $H_0(\Sigma, \operatorname{Ad}\rho) = 0$. Applying again the Künneth theorem, the maps $H_j(\Sigma \times S^1, \operatorname{Ad}\rho) \to H_j(X, \operatorname{Ad}\rho)$ identify with g and h for $j = 1$ and 2 respectively.

It remains to compute the maps $H_j(C_i \times S^1, \operatorname{Ad}\rho) \to H_j(D \times S^1, \tilde{\varphi}_i^* \operatorname{Ad}\rho)$. Here we denote by $\tilde{\varphi}_i$ the embedding of $D \times S^1$ into X extending φ_i. Since D is contractible, $H_j(D \times S^1, \tilde{\varphi}_i^* \operatorname{Ad}\rho) = 0$ for $j = 2, 3$. Let us determine the holonomy of S^1 in the bundle $\tilde{\varphi}_i^* \operatorname{Ad}\rho \to D \times S^1$. It is equal to the holonomy of $\varphi_i(S^1)$ in $\operatorname{Ad}\rho \to C_i \times S^1$. For any loop γ of $C_i \times S^1$ based at $(x_i, 0)$, we

2. If $E \to B$ and $E' \to B'$ are two vector bundles, we denote by $E \boxtimes E'$ the vector bundle $(\pi^* E) \otimes ((\pi')^* E')$ where π and π' are the projection from $B \times B'$ onto B and B'.

denote by $\mathrm{hol}_\gamma : \mathrm{Ad}\,\rho|_{x_i} \to \mathrm{Ad}\,\rho|_{x_i}$ the holonomy of γ in $\mathrm{Ad}\,\rho \to C_i \times S^1$. Since D is contractible, $\mathrm{hol}_{\varphi_i(\partial D)}$ is trivial, so that

$$\begin{aligned}\ker(\mathrm{hol}_{\varphi_i(S^1)} - \mathrm{id}) &= \ker(\mathrm{hol}_{\varphi_i(\partial D)} - \mathrm{id}) \cap \ker(\mathrm{hol}_{\varphi_i(S^1)} - \mathrm{id})\\ &= \ker(\mathrm{hol}_{C_i} - \mathrm{id}) \cap \ker(\mathrm{hol}_{S^1} - \mathrm{id})\\ &= \ker(\mathrm{hol}_{C_i} - \mathrm{id})\\ &= V_i,\end{aligned}$$

where we have used first that φ_i is a diffeomorphism and second that hol_{S^1} is trivial. We deduce that

$$H_j(D \times S^1, \tilde{\varphi}_i^* \,\mathrm{Ad}\,\rho) \simeq V_i$$

for $j = 0$ or 1. As above, let us identify $H_1(C_i \times S^1, \mathrm{Ad}\,\rho)$ with $H_1(C_i, \mathrm{Ad}\,\rho) \oplus H_0(C_i, \mathrm{Ad}\,\rho) = V_i \oplus V_i$. Then by Equation (2), the map $H_1(C_i \times S^1, \mathrm{Ad}\,\rho) \to H_1(D \times S^1, \tilde{\varphi}_i^* \,\mathrm{Ad}\,\rho)$ corresponds to

$$V_i \oplus V_i \to V_i, \qquad (u, v) \to q_i u + p_i v.$$

Putting everything together and setting $V = \bigoplus V_i$, we obtain the following long exact sequence

$$\begin{aligned}0 \to V \xrightarrow{f} H_1(\Sigma, \mathrm{Ad}\,\rho) \xrightarrow{h} H_2(X, \mathrm{Ad}\,\rho) \to V \oplus V\\ \xrightarrow{\tilde{f}} H_1(\Sigma, \mathrm{Ad}\,\rho) \oplus V \xrightarrow{[g,\tilde{g}]} H_1(X, \mathrm{Ad}\,\rho) \to V \xrightarrow{\mathrm{id}} V \to 0,\end{aligned}$$

where $\tilde{g} : V \to H_1(X, \mathrm{Ad}\,\rho)$ is unknown and $\tilde{f}$ is the map $\left(\begin{smallmatrix} f & 0 \\ q & p \end{smallmatrix}\right)$ with $q, p : V \to V$ the maps whose restriction to V_i are the multiplications by q_i, p_i respectively.

We recover the fact that f is injective. Since f is injective and the p_i don't vanish, $\tilde{f}$ is injective too. Furthermore the identity of V is certainly injective. So the Mayer-Vietoris long exact sequences breaks into three exact sequences:

$$0 \to V \xrightarrow{f} H_1(\Sigma, \mathrm{Ad}\,\rho) \xrightarrow{h} H_2(X, \mathrm{Ad}\,\rho) \to 0 \tag{7}$$

$$0 \to V \oplus V \xrightarrow{\tilde{f}} H_1(\Sigma, \mathrm{Ad}\,\rho) \oplus V \xrightarrow{[g,\tilde{g}]} H_1(X, \mathrm{Ad}\,\rho) \to 0 \tag{8}$$

$$0 \to V \xrightarrow{\mathrm{id}} V \to 0. \tag{9}$$

Here, (7) is the second exact sequence in the statement of the theorem. Finally, it is an easy exercise to deduce from the exact sequence (8) that the first sequence in the statement of the theorem is exact. □

5. Poincaré duality on X and Σ

Choose an invariant scalar product on the Lie algebra of G. For any topological space Y and representation ρ of $\pi_1(Y)$ in G, the flat vector bundle $\mathrm{Ad}\,\rho$ inherits a flat metric. This allows to define a cup product $H^k(Y, Z, \mathrm{Ad}\,\rho) \times$

$H^\ell(Y,Z,\mathrm{Ad}\,\rho) \to H^{k+\ell}(Y,Z,\mathbb{R})$ for any closed subspace Z of Y. We will us these products for X and Σ.

Consider an irreducible representation ρ of $\pi_1(\Sigma)$ in G. We have a bilinear map

$$H^1(\Sigma,\partial\Sigma,\mathrm{Ad}\,\rho)\times H^1(\Sigma,\mathrm{Ad}\,\rho)\to\mathbb{R},\quad (\alpha,\beta)\to\alpha\cdot\beta \tag{10}$$

sending (α,β) to the evaluation of the cup product $\alpha\cup\beta$ on the fundamental class of $(\Sigma,\ \partial\Sigma)$. Consider the following portion of the long exact sequence associated to the pair $(\Sigma,\partial\Sigma)$

$$\cdots\to H^1(\Sigma,\partial\Sigma,\mathrm{Ad}\,\rho)\xrightarrow{\pi} H^1(\Sigma,\mathrm{Ad}\,\rho)\xrightarrow{f^*} H^1(\partial\Sigma,\mathrm{Ad}\,\rho)\to\cdots$$

and introduce the space $K:=\ker f^*=\mathrm{Im}\,\pi\subset H^1(\Sigma,\mathrm{Ad}\,\rho)$. For any $\alpha,\ \beta\in K$, we set

$$\Omega(\alpha,\beta):=\tilde\alpha\cdot\beta, \tag{11}$$

where $\tilde\alpha$ is any element of $H^1(\Sigma,\partial\Sigma,\mathrm{Ad}\,\rho)$ such that $\pi(\tilde\alpha)=\alpha$.

Lemma 5.1. — *The bilinear map Ω is well-defined, antisymmetric and non degenerate.*

So (K,Ω) is a symplectic vector space.

Proof. — For any $\tilde\alpha,\ \tilde\beta\in H^1(\Sigma,\partial\Sigma,\mathrm{Ad}\,\rho)$, $\tilde\alpha\cdot\pi(\tilde\beta)+\tilde\beta\cdot\pi(\tilde\alpha)=0$. Assuming that $\pi(\tilde\alpha)=\alpha$ and $\pi(\tilde\beta)=\beta$, we get that

$$\Omega(\alpha,\beta)=\tilde\alpha\cdot\beta=-\tilde\beta\cdot\alpha,$$

which proves that $\Omega(\alpha,\beta)$ does not depend on the choice of $\tilde\alpha$ and that $\Omega(\alpha,\beta)=-\Omega(\beta,\alpha)$. By Poincaré duality, the pairing (10) is non degenerate, so the same holds for Ω. $\square$

Consider now an irreducible representation ρ of $\pi_1(X)$ in G. By Poincaré duality, we have a nondegenerate pairing

$$H^1(X,\mathrm{Ad}\,\rho)\times H^2(X,\mathrm{Ad}\,\rho)\to\mathbb{R} \tag{12}$$

sending (α,β) to the evaluation of $\alpha\cup\beta\in H^3(X)$ on the fundamental class. By Theorem 4.2, the maps g^* and h^* induce isomorphisms from $H^1(X,\mathrm{Ad}\,\rho)$ and $H^2(X,\mathrm{Ad}\,\rho)$ to K.

Theorem 5.2. — *For any $\alpha\in H^1(X,\mathrm{Ad}\,\rho)$ and $\beta\in H^2(X,\mathrm{Ad}\,\rho)$, we have*

$$\alpha\cdot\beta=\Omega(g^*\alpha,h^*\beta).$$

Proof. — We will use de Rham cohomology. First let us prove that any element in $H^2(X,\mathrm{Ad}\,\rho)$ has a representative $\beta\in\Omega^2(X,\mathrm{Ad}\,\rho)$ which vanishes identically on a neighborhood of $Z = \tilde{\varphi}_1(D \times S^1) \cup \cdots \cup \tilde{\varphi}_n(D \times S^1)$ and such that

$$\beta = p_\Sigma^* \tilde{\beta} \wedge p_{S^1}^* \tau + d\gamma \quad \text{ on } \Sigma \times S^1, \tag{13}$$

where p_Σ and p_{S^1} are the projection from $\Sigma \times S^1$ onto Σ and S^1 respectively, $\tilde{\beta} \in \Omega^1(\Sigma, \operatorname{Ad}\rho)$ is closed, $\tau \in \Omega^1(S^1)$ satisfies $\int_{S^1} \tau = 1$ and γ belongs to $\Omega^1(\Sigma \times S^1, \operatorname{Ad}\rho)$.

To check that, let us start with any representative $\beta \in \Omega^2(X, \operatorname{Ad}\rho)$. Since $H^2(Z, \operatorname{Ad}\rho) = 0$, we have $\beta = d\mu$ on Z. We can even assume that this holds on a neighborhood U of Z. Let $\varphi \in \mathcal{C}^\infty(X)$ with support contained in U and identically equal to 1 on Z. Replacing β with $\beta - d(\varphi\mu)$, we have that $\beta \equiv 0$ on a neighborhood of Z. By Künneth theorem, $H^2(\Sigma \times S^1, \operatorname{Ad}\rho) = H^1(\Sigma, \operatorname{Ad}\rho) \otimes H^1(S^1, \mathbb{R})$, which implies that β has the form (13) on $\Sigma \times S^1$.

Let us prove that any element in $H^1(X, \operatorname{Ad}\rho)$ has a representative $\alpha \in \Omega^1(X, \operatorname{Ad}\rho)$ such that

$$\alpha = p_\Sigma^* \tilde{\alpha} \qquad \text{on } \Sigma \times S^1, \tag{14}$$

where $\tilde{\alpha} \in \Omega^1(\Sigma, \operatorname{Ad}\rho)$ is closed and vanishes identically on $\partial\Sigma$.

To check that, we start with any representative $\alpha \in \Omega^1(X, \operatorname{Ad}\rho)$. By Künneth theorem, $H^1(\Sigma \times S^1, \operatorname{Ad}\rho) = H^1(\Sigma, \operatorname{Ad}\rho)$ so that we have on $\Sigma \times S^1$ the equality $\alpha = p_\Sigma^* \tilde{\alpha} + d\gamma$ with $\tilde{\alpha} \in \Omega^1(\Sigma, \operatorname{Ad}\rho)$ closed and $\gamma \in \Omega^0(\Sigma \times S^1, \operatorname{Ad}\rho)$. Observe that $[\tilde{\alpha}] = g^*[\alpha]$, so by Theorem 4.2, $f^*[\tilde{\alpha}] = 0$. Thus adding to $\tilde{\alpha}$ an exact form (which modifies γ), the restriction of $\tilde{\alpha}$ to $\partial\Sigma$ vanishes. Finally, extending γ to X, and replacing α by $\alpha - d\gamma$, we obtain Equation (14).

Now consider α and β as above. Then

$$[\alpha] \cdot [\beta] = \int_X \alpha \wedge \beta = \int_{\Sigma \times S^1} \alpha \wedge \beta,$$

because β vanishes identically on a neighborhood of Z. To evaluate this last integral, we replace α and β by their expressions (14), (13). By Stokes' theorem,

$$\int_{\Sigma \times S^1} p_\Sigma^* \tilde{\alpha} \wedge d\gamma = \int_{\partial\Sigma \times S^1} p_\Sigma^* \tilde{\alpha} \wedge \gamma = 0$$

because $\tilde{\alpha}$ vanishes identically on $\partial\Sigma$. By Fubini theorem and because $\int_{S^1} \tau = 1$,

$$\int_{\Sigma \times S^1} p_\Sigma^* \tilde{\alpha} \wedge p_\Sigma^* \tilde{\beta} \wedge p_{S^1}^* \tau = \int_\Sigma \tilde{\alpha} \wedge \tilde{\beta}.$$

Since $\tilde{\alpha}$ vanishes on $\partial\Sigma$, it is the representative of a class in $H^1(\Sigma, \partial\Sigma, \operatorname{Ad}\rho)$. So this last integral is equal to $\Omega([\tilde{\alpha}], [\tilde{\beta}])$. Furthermore $[\tilde{\alpha}] = g^*[\alpha]$ and $[\tilde{\beta}] = h^*[\beta]$, which concludes the proof. $\square$

6. Torsion of X

Let ρ be an irreducible representation ρ of $\pi_1(X)$ in G. Since $H_0(X, \operatorname{Ad}\rho) = H_3(X, \operatorname{Ad}\rho) = 0$, the torsion of the flat euclidean vector bundle $\operatorname{Ad}\rho$ is a non vanishing vector of the line

$$\det H_\bullet(X, \operatorname{Ad}\rho) \simeq \big(\det(H_1(X, \operatorname{Ad}\rho))\big)^{-1} \otimes \det(H_2(X, \operatorname{Ad}\rho))$$

well-defined up to sign. In the Appendix B, we recall its definition and the properties we will need to compute it. By Theorem 4.2, we have an isomorphism

$$\psi : H_1(X, \operatorname{Ad}\rho) \to H_2(X, \operatorname{Ad}\rho)$$

sending $g(\beta)$ into $h(\beta)$ for any $\beta \in H_1(\Sigma, \operatorname{Ad}\rho)$. The determinant of ψ belongs to $\det H_\bullet(X, \operatorname{Ad}\rho)$.

Let $\Delta : \mathcal{C}(G) \to \mathbb{R}$ be the function given by

$$\Delta(u) = \big|\det_{H_g}(\operatorname{Ad}_g - \operatorname{id})\big|^{1/2},$$

where g is any element in the conjugacy class u and H_g is the orthocomplement of $\ker(\operatorname{Ad}_g - \operatorname{id})$.

Theorem 6.1. — *For any irreducible representation ρ of $\pi_1(X)$ in G, the torsion of $\operatorname{Ad}\rho \to X$ is given by*

$$\tau(\operatorname{Ad}\rho) = \prod_{i=1}^{n} \frac{p_i^{\dim V_i}}{\Delta^2(\rho(C_i)^{r_i})} \det\psi, \tag{15}$$

where r_i is any inverse of q_i modulo p_i and $V_i = \ker(\operatorname{Ad}_{\rho(C_i)} - \operatorname{id})$.

Let us make a few remark on the left hand side of (15).

1. It follows from the relation (2) and the fact that $\rho(S^1)$ is central by Lemma 3.1, that $(\operatorname{Ad}_{\rho(C_i)})^{p_i}$ is the identity. So the right hand side of (15) does not depend on the choice of r_i.
2. V_i is the Lie algebra of the centralizer of $\rho(C_i)$ in G. So the dimension of V_i is equal to $\dim G - \dim u_i$ where u_i is the conjugacy class of $\rho(C_i)$.

Proof. — By the proof of Theorem 4.2, the Mayer-Vietoris long exact sequence breaks into three short exact sequences: (7), (8) and (9). Choose $\alpha \in \det V$ and $\beta \in \bigwedge^{\dim H_1(\Sigma, \operatorname{Ad}\rho) - \dim V} H_1(\Sigma, \operatorname{Ad}\rho)$ such that $f(\alpha) \wedge \beta \in \det H_1(\Sigma, \operatorname{Ad}\rho)$ does not vanish. By (7), we have an isomorphism

$$\mathbb{R} \simeq \det V \otimes \big(\det H_1(\Sigma, \operatorname{Ad}\rho)\big)^{-1} \otimes \det H_2(X, \operatorname{Ad}\rho) \tag{16}$$

sending 1 into $\alpha \otimes \big(f(\alpha) \wedge \beta\big)^{-1} \otimes h(\beta)$. By (8), we have an isomorphism

$$\mathbb{R} \simeq (\det V)^{-2} \otimes \big(\det H_1(\Sigma, \operatorname{Ad}\rho) \otimes \det V\big) \otimes \big(\det H_1(X, \operatorname{Ad}\rho)\big)^{-1} \tag{17}$$

sending 1 into $\alpha^{-2} \otimes \big((f(\alpha) \wedge \beta) \otimes (\det p)\alpha\big) \otimes g(\beta)^{-1}$ where p is the map introduced in the proof of Theorem 4.2. We easily compute that:

$$\det p = \prod_{i=1}^{n} p_i^{\dim V_i}.$$

By (9), we have an isomorphism

$$\mathbb{R} \simeq \det V \otimes \big(\det V\big)^{-1} \tag{18}$$

sending 1 into $\alpha \otimes \alpha^{-1}$. Taking the tensor product of (16), (17) and (18), we get the isomorphism associated to the Mayer-Vietoris long exact sequence:

$$\mathbb{R} \simeq \big(\det H_1(X, \operatorname{Ad}\rho)\big)^{-1} \otimes \det H_2(X, \operatorname{Ad}\rho). \tag{19}$$

It sends 1 into $\big(\det p\big) h(\beta)/g(\beta) = \big(\det p\big)\big(\det \psi\big)$.

Let us compute the torsion of the restrictions of $\operatorname{Ad}\rho$ to $C_i \times S^1$, $\Sigma \times S^1$ and $\tilde{\varphi}_i(D \times S^1)$ respectively. We will use the identifications made previously for the various cohomology groups. First, the torsion of $\operatorname{Ad}\rho \to C_i \times S^1$ is

$$1 \in \mathbb{R} \simeq \det V_i \otimes \big(\det V_i\big)^{-1} \otimes \det V_i \otimes \big(\det V_i\big)^{-1}.$$

Indeed, the bundle $\operatorname{Ad}\rho|_{C_i \times S^1}$ is isomorphic to $\operatorname{Ad}\rho|_{C_i} \boxtimes \mathbb{R}_{S^1}$. Furthermore, $\chi(C_i) = \chi(S^1) = 0$. By property 2 of the Appendix B, this implies that the torsion of $\operatorname{Ad}\rho \to C_i \times S^1$ is 1.

Second the torsion of $\operatorname{Ad}\rho \to \Sigma \times S^1$ is

$$1 \in \mathbb{R} \simeq \det H_1(\Sigma, \operatorname{Ad}\rho) \otimes \big(\det H_1(\Sigma, \operatorname{Ad}\rho)\big)^{-1}.$$

Indeed, the bundle $\operatorname{Ad}\rho|_{\Sigma \times S^1}$ is isomorphic to $\operatorname{Ad}\rho|_{\Sigma} \boxtimes \mathbb{R}_{S^1}$. Since $\chi(S^1) = 0$, we deduce from properties 2 and 4 of the Appendix B that the torsion of $\operatorname{Ad}\rho \to \Sigma \times S^1$ is equal to $\tau(\mathbb{R}_{S^1})^{\chi(\operatorname{Ad}\rho|_{\Sigma})} = 1$.

Third the torsion of $\tilde{\varphi}_i^* \operatorname{Ad}\rho \to D \times S^1$ belongs to $\mathbb{R} \simeq \det V_i \otimes \big(\det V_i\big)^{-1}$. Since φ_i is a diffeomorphism from $\partial D \times S^1$ to $C_i \times S^1$ reversing the orientation and satisfying (2), we have the following relation in $H_1(C_i \times S^1)$

$$\varphi_i([S^1]) = r_i[C_i] + s_i[S^1],$$

where r_i, s_i are such that such that $p_i s_i + q_i r_i = 1$. Since $\rho(S^1)$ is central, $\operatorname{Ad}_{\rho(S^1)}$ is the identity, so

$$\operatorname{Ad}_{\rho(\varphi_i(S^1))} = \operatorname{Ad}^{r_i}_{\rho(C_i)}.$$

By Property 4 of the Appendix B, we conclude that the torsion of $\tilde{\varphi}_i^* \operatorname{Ad}\rho$ is equal to the square of $\Delta(\rho(C_i)^{r_i})$.

By Property 3 of the Appendix B, we deduce from the previous computations that

$$\big(\det p\big)\big(\det \psi\big) = \tau(\operatorname{Ad}\rho) \prod_{i=1}^{n} \Delta^2(\rho(C_i)^{r_i}),$$

which concludes the proof. □

Using the duality between homology and cohomology and Poincaré duality, we have

$$\begin{aligned}\det H_\bullet(X,\mathrm{Ad}\,\rho) &\simeq \det(H^1(X,\mathrm{Ad}\,\rho))\otimes\big(\det(H^2(X,\mathrm{Ad}\,\rho))\big)^{-1}\\ &\simeq \big(\det(H^1(X,\mathrm{Ad}\,\rho))\big)^2.\end{aligned}\tag{20}$$

So $(\det\psi)^{-1}$ may be viewed as the square of a volume element of $H^1(X,\mathrm{Ad}\,\rho)$. Recall that g^* induces an isomorphism from $H^1(X,\mathrm{Ad}\,\rho)$ to a symplectic vector space (K,Ω). The following lemma is an easy consequence of Theorem 5.2.

Lemma 6.2. — *g^* sends $(\det\psi)^{-1}$ to the square of the Liouville form $\Omega^N/N!$, where $N=\frac{1}{2}\dim K$.*

7. Application to moduli spaces

Let us apply the previous results to the character manifold $\mathcal{M}^0(X)$. Recall that a density of a n dimensional manifold M is a section of the line bundle, whose fiber at x is the space of applications $f:(T_xM)^n\to\mathbb{R}$ satisfying $f(Ax_1,\dots,Ax_n)=|\det A|f(x_1,\dots,x_n)$ for any endomorphism A of T_xM. Here, we have natural densities on $\mathcal{M}^0(X)$ and $\mathcal{M}^0(\Sigma,u)$ defined as follows:

- Since the tangent space $T_{[\rho]}\mathcal{M}^0(X)$ is $H^1(X,\mathrm{Ad}\,\rho)$, by the isomorphism (20), the torsion $\tau(\mathrm{Ad}\,\rho)$ is the inverse of the square of a density of $T_{[\rho]}\mathcal{M}^0(X)$. This defines a density μ_X of $\mathcal{M}^0(X)$ whose value at $[\rho]$ is $\tau(\mathrm{Ad}\,\rho)^{-1/2}$.
- For any $u\in\mathcal{C}(G)^n$, $\mathcal{M}^0(\Sigma,u)$ is a symplectic manifold, so it has a canonical density μ_u. The symplectic structure of $T_\rho\mathcal{M}^0(\Sigma,u)$ is the form Ω considered in Lemma 5.1 and if $N=\frac{1}{2}\dim\mathcal{M}^0(\Sigma,u)$, $\mu_u([\rho])=|\Omega^{\wedge N}|/N!$.

For any $(u,v)\in\mathcal{P}$, we defined a diffeomorphism $R_{u.v}$ from $\mathcal{M}^0(X,u,v)$ to $\mathcal{M}^0(\Sigma,u)$. By the proof of Lemma 3.5, the linear tangent map of $R_{u,v}$ at $[\rho]$ is the map $g^*:H^1(X,\mathrm{Ad}\,\rho)\to K$. We deduce from Theorem 6.1 and Theorem 5.2 via Lemma 6.2 our main result.

Theorem 7.1. — *For any $(u,v)\in\mathcal{P}$, we have on $\mathcal{M}^0(X,u,v)$*

$$\mu_X=\left(\prod_{i=1}^n\frac{\Delta(u_i^{r_i})}{p_i^{\dim V_i/2}}\right)R^*_{u,v}\mu_u$$

with r_i any inverse of q_i modulo p_i.

8. Abelian case

In this section, we adapt the previous result to the constant coefficient case. We consider the same Seifert manifold X as above and we assume that the Euler number

$$\chi = -\sum_{i=1}^{n} \frac{q_i}{p_i}$$

does not vanish.

For $Y = X, \Sigma, \partial\Sigma$, we let $H_j(Y) := H_j(Y, \mathbb{R})$. In contrast to the previous case, the groups $H_0(X)$ and $H_3(X)$ do not vanish. Introduce the three maps

$$f : H_1(\partial\Sigma) \to H_1(\Sigma), \quad g : H_1(\Sigma) \to H_1(X), \quad h : H_1(\Sigma) \to H_2(X)$$

defined as follows. f and g are the morphisms corresponding to the inclusions $\partial\Sigma \subset \Sigma$ and $\Sigma \subset X$ respectively. h sends $\gamma \in H_1(\Sigma)$ to the image of $\gamma \boxtimes [S^1] \in H_2(\Sigma \times S^1)$ in $H_2(X)$.

Proposition 8.1. — *The morphisms g and h are surjective and their kernel is the image of f.*

Proof. — The proof is similar to the one of Theorem 4.2, with the additional difficulty that $H_0(X) \simeq \mathbb{R}$, $H_0(\Sigma) \simeq \mathbb{R}$ and $H_3(X) \simeq \mathbb{R}$. The Mayer-Vietoris long exact sequence splits into three exact sequences:

$$\begin{aligned} 0 \to \mathbb{R} \to \mathbb{R}^n \xrightarrow{f} H_1(\Sigma) \xrightarrow{h} H_2(X) \to 0 \\ 0 \to \mathbb{R}^{2n} \xrightarrow{B} \mathbb{R}^n \oplus H_1(\Sigma) \oplus \mathbb{R} \xrightarrow{A} H_1(X) \to 0 \\ 0 \to \mathbb{R}^n \xrightarrow{C} \mathbb{R}^n \oplus \mathbb{R} \to \mathbb{R} \to 0, \end{aligned} \tag{21}$$

where C and B are given by $C(x) = (x, x_1 + \cdots + x_n)$ and $B(x,y) = (z, f(x), y_1 + \cdots + y_n)$ with $z \in \mathbb{R}^n$ given by $z_i = q_i x_i + p_i y_i$. By the first sequence in (21), h is surjective and its kernel is the image of f. One checks that $\operatorname{Im} B$ and $0 \oplus H_1(\Sigma) \oplus 0$ are transversal subspaces, their intersection being $0 \oplus \operatorname{Im} f \oplus 0$. Using that for any $\gamma \in H_1(\Sigma)$, $A(0, \gamma, 0) = g(\gamma)$, one deduces from the second sequence of (21) that g is surjective with kernel the image of f. □

Let $\overline{\Sigma}$ be the closed surface obtained by gluing a disk to each boundary component of Σ. The inclusion $\Sigma \subset \overline{\Sigma}$ induces an isomorphism $H_1(\overline{\Sigma}) \simeq H_1(\Sigma)/\operatorname{Im} f$. So by Proposition 8.1, we have two isomorphisms

$$\tilde{g} : H_1(\overline{\Sigma}) \to H_1(X), \qquad \tilde{h} : H_1(\overline{\Sigma}) \to H_2(X).$$

Proposition 8.2. — *For any $\alpha \in H^1(X)$ and $\beta \in H^2(X)$, we have*

$$\alpha \cdot_X \beta = (\tilde{g}^* \alpha) \cdot_{\overline{\Sigma}} (\tilde{h}^* \beta),$$

where $\cdot_X$ and $\cdot_{\overline{\Sigma}}$ denote the Poincaré pairings of X and $\overline{\Sigma}$ respectively.

Proof. — The proof is similar to the one of Theorem 5.2. First, since the image of B in (21) contains $0 \oplus 0 \oplus H_0(\Sigma)$, the image of $H^2(X) \to H^1(\Sigma \times S^1) \simeq H^1(\Sigma) \oplus H^0(\Sigma)$ is contained in $H^1(\Sigma)$. Consequently any class α of $H^1(X)$ has a representative $a \in \Omega^1(X)$ such that

$$a = p^* \tilde{a} \qquad \text{on } \Sigma \times S^1$$

with $p : \Sigma \times S^1 \to \overline{\Sigma}$ the projection and $\tilde{a} \in \Omega^1(\overline{\Sigma})$ a representative of $\tilde{g}^* \alpha$. Second, any class $\beta \in H^2(X)$ has a representative $b \in \Omega^2(X)$ whose support is contained in an open subset of $\Sigma \times S^1$ and such that

$$b = p^* \tilde{b} \wedge q^* \tau + d\gamma,$$

where $\tilde{b} \in \Omega^1(\overline{\Sigma})$ is a representative of $\tilde{h}^* \beta$, supported in an open subset of Σ, q is the projection $\Sigma \times S^1 \to S^1$, $\tau \in \Omega^1(S^1)$ is such that $\int_{S^1} \tau = 1$ and $\gamma \in \Omega^1(\Sigma \times S^1)$. Finally, one checks that

$$\int_X a \wedge b = \int_{\overline{\Sigma}} \tilde{a} \wedge \tilde{b}.$$

using Stokes' formula. □

We can also compute the torsion of X as in Theorem 6.1. Since $H_0(X)$ and $H_3(X)$ have rank one, the torsion belongs to $H_0(X) \otimes \big(\det H_1(X)\big)^{-1} \otimes \det H_2(X) \otimes \big(H_3(X)\big)^{-1}$.

PROPOSITION 8.3. — *The Reidemeister torsion of X is given by*

$$\tau(X) = \chi \prod_{i=1}^{n} p_i \; [x] \otimes \det \psi \otimes [X]^{-1},$$

where $\chi = -\sum q_i/p_i$ is the Euler number of X, $x \in X$ and $[x] \in H_0(X)$ is the corresponding class, ψ is the map $\tilde{h} \circ \tilde{g}^{-1} : H_1(X) \to H_2(X)$ and $[X] \in H_3(X)$ is the fundamental class.

Proof. — We adapt the proof of Theorem 6.1. Let $e = 1 \in \mathbb{R}$, (e_i) be the canonical basis of $\mathbb{R}^n$, $\delta = e_1 \wedge \cdots \wedge e_n$, $\rho = f(e_1) \wedge \cdots \wedge f(e_{n-1})$ and $\sigma \in \bigwedge^{2g} H_1(\Sigma)$ such that $\rho \wedge \sigma$ is a generator of $\bigwedge^{2g+n-1} H_1(\Sigma)$. Then one checks that the isomorphisms corresponding to the three exact sequences in (21) send 1 into $e \otimes \delta^{-1} \otimes (\rho \wedge \sigma) \otimes h(\sigma)^{-1}$, $\chi^{-1}(\prod p_i)^{-1} e \otimes (\delta \otimes (\rho \wedge \sigma) \otimes e)^{-1} \otimes h(\sigma)^{-1}$ and $\delta^{-1} \otimes (\delta \otimes e) \otimes e^{-1}$ respectively. The factor $\chi(\prod p_i)$ appears because $B(\delta \otimes \delta) = \chi(\prod p_i) \delta \otimes \rho \otimes e$. The torsions of $\partial\Sigma \times S^1$, $\partial\Sigma \times D$ and $\Sigma \times S^1$ are respectively $\delta \otimes (\delta \otimes \delta)^{-1} \otimes \delta$, $\delta \otimes \delta^{-1}$ and $(\rho \wedge \sigma) \otimes (\delta \otimes (\rho \wedge \sigma) \otimes e)^{-1} \otimes e$. We conclude with Property 3 of Appendix B. □

Trivializing $H_0(X)$ and $H_3(X)$ by sending $[x]$ and $[X]$ to 1, and identifying $H_1(X)$ with the dual of $H_2(X)$ by Poincaré duality, the inverse of square root of the torsion gets identified with an element of $\det H_1(X)$. By Propositions 8.2 and 8.3, the torsion satisfies

$$\tilde{g}^*(\tau(X))^{-1/2} = \Big|\chi \prod_{i=1}^{n} p_i\Big|^{-1/2} \mu, \tag{22}$$

where $\mu \in \det H_1(\overline{\Sigma})$ is the Liouville density of $H^1(\overline{\Sigma})$.

This may be applied to the space $\mathcal{J}(X)$ consisting of representation of $\pi_1(X)$ in $\mathrm{U}(1)$ as follows. First, for any connected compact manifold Y, $\mathcal{J}(Y)$ is an abelian Lie group, the product being the pointwise multiplication. The Lie algebra of $\mathcal{J}(Y)$ is the space of morphisms from $\pi_1(Y)$ to $\mathbb{R}$, which identifies with $H^1(Y)$. In particular for the Seifert manifold X, the Lie algebra of $\mathcal{J}(X)$ being $H^1(X)$, $(\tau(X))^{-1/2}$ determines an invariant density of $\mathcal{J}(X)$. Furthermore, the Lie algebra of $\mathcal{J}(\overline{\Sigma})$ being $H^1(\overline{\Sigma})$, $\mathcal{J}(\overline{\Sigma})$ has an invariant symplectic structure and a corresponding Liouville density.

For any $(u,v) \in \mathrm{U}(1)^{n+1}$, let $\mathcal{J}(X,u,v)$ be the subset of $\mathcal{J}(X)$ consisting of the representations ρ such that $\rho(C_i) = u_i$ for any i and $\rho(S^1) = v$. Then

$$\mathcal{J}(X) = \bigcup_{(u,v)\in \mathcal{Q}} \mathcal{J}(X,u,v), \tag{23}$$

where $\mathcal{Q}$ is the set of $(u,v) \in \mathrm{U}(1)^{n+1}$ such that $u_1 \cdots u_n = 1$ and for any i, $u_i^{p_i} = v^{q_i}$. Since the Euler number χ does not vanish, $\mathcal{Q}$ is finite. Furthermore, for any $(u,v) \in \mathcal{Q}$, $\mathcal{J}(X,u,v)$ is connected. So (23) is the decomposition of $\mathcal{J}(X)$ into connected components.

Let $\mathbf{1} = (1,\dots,1) \in \mathrm{U}(1)$. $\mathcal{J}(X,\mathbf{1},1)$ is the component of the identity of $\mathcal{J}(X)$. We have a natural Lie group isomorphism Φ from $\mathcal{J}(X,\mathbf{1},1)$ to $\mathcal{J}(\overline{\Sigma})$, such that for any $\rho \in \mathcal{J}(X,\mathbf{1},1)$, the restrictions of ρ and $\Phi(\rho)$ to Σ are the same. The linear tangent map at the identity to Φ is the adjoint map to the map $\tilde{g} : H_1(\overline{\Sigma}) \to H_1(X)$. Thus Equation (22) computes the invariant density of $\mathcal{J}(X,\mathbf{1},1)$ in terms of the pull back by Φ of the Liouville density. We recover in this way Theorem 9 of [8].

Appendix A. Representation space

The general theory describing the smooth structure of a representation space is rather involved and belongs more to algebraic geometry, [7]. In this appendix, we summarize the basic general facts we need, remaining in the context of differential geometry.

Let G be a connected Lie group and π be a finitely generated group. Let $\mathcal{R}(\pi)$ be the space of representations of π in G. For any set of generators $a = (a_1,\dots,a_N)$ of π, the map

$$\xi_a : \mathcal{R}(\pi) \to G^N, \qquad \rho \to (\rho(a_1),\dots,\rho(a_N)),$$

is injective, and allows us to identify $\mathcal{R}(\pi)$ with $\xi_a(\mathcal{R}(\pi))$. If $a=(a_1,\dots,a_N)$ and $b=(b_1,\dots,b_M)$ are two sets of generators, the bijection $\xi_b\circ\xi_a^{-1}$ from $\xi_a(\mathcal{R}(\pi))$ onto $\xi_b(\mathcal{R}(\pi))$ is a homeomorphism. Indeed, expressing the a_i's in terms of the b_j's, we obtain a smooth map $\varphi: G^N\to G^M$ extending $\xi_b\circ\xi_a^{-1}$. We endow $\mathcal{R}(\pi)$ with the topology such that for any set of generators a of π, ξ_a is a homeomorphism onto its image.

Let $\mathcal{R}^{\mathrm{s}}(\pi)$ be the set of representations ρ of π in G admitting an open neighborhood U and a set of generators $(a_1,\dots,a_N)$ such that $\xi_a(U)$ is a smooth submanifold of G^N. $\mathcal{R}^{\mathrm{s}}(\pi)$ has a unique manifold structure such that for any such pairs (U,a), the map $\xi_a: U\to\xi_a(U)$ is a diffeomorphism. Indeed, arguing as above, we see that for any two pairs (U,a) and (V,b) the map $\xi_b\circ\xi_a^{-1}:\xi_a(U\cap V)\to\xi_b(U\cap V)$ is a diffeomorphism.

For any representation ρ of π in G, composing ρ with the adjoint representation, the Lie algebra $\mathfrak{g}$ becomes a left G-module. Consider the corresponding cochain complex in degrees 0 and 1: $C^0(\pi,\operatorname{Ad}\rho)=\mathfrak{g}$, $C^1(\pi,\operatorname{Ad}\rho)=\operatorname{Map}(\pi,\mathfrak{g})$, the differential in degree 0 is

$$d_\rho:\mathfrak{g}\to C^1(\pi,\operatorname{Ad}\rho),\quad \xi\to\big(\gamma\to\operatorname{Ad}_{\rho(\gamma)}\xi-\xi\big)$$

and the space of 1-cocycle

$$Z^1(\pi,\operatorname{Ad}\rho)=\big\{\tau:\pi\to\mathfrak{g};\ \forall\gamma_1,\gamma_2\in\pi,\ \tau(\gamma_1\gamma_2)=\tau(\gamma_1)+\operatorname{Ad}_{\rho(\gamma_1)}\tau(\gamma_2)\big\}.$$

For any $\gamma\in\pi$, the map $e_\gamma:\mathcal{R}(\pi)\to G$ sending ρ into $\rho(\gamma)$ is continuous. Its restriction to $\mathcal{R}^{\mathrm{s}}(\pi)$ is smooth. If $\rho\in\mathcal{R}^{\mathrm{s}}(\pi)$, we have a natural map from $T_\rho\mathcal{R}^{\mathrm{s}}(\pi)$ to $Z^1(\pi,\operatorname{Ad}\rho)$ sending $\dot\rho$ to the cocycle τ given by

$$\tau(\gamma)=R_{\rho(\gamma)^{-1}}T_\rho e_\gamma(\dot\rho),\qquad\forall\gamma\in\pi,$$

where for any $g\in G$, $R_{g^{-1}}:T_gG\to\mathfrak{g}$ is the linear map tangent to the right multiplication by g^{-1}. It is easily seen that this map is well-defined and injective, so we consider the tangent space $T_\rho\mathcal{R}^{\mathrm{s}}(\pi)$ as a subspace of $Z^1(\pi,\operatorname{Ad}\rho)$.

G acts on $\mathcal{R}(\pi)$ by conjugation. The action preserves $\mathcal{R}^{\mathrm{s}}(\pi)$. A straightforward computation shows that the infinitesimal action at $\rho\in\mathcal{R}^{\mathrm{s}}(\pi)$ is the differential d_ρ introduced above.

Assume from now on that G is compact. The subset $\mathcal{R}^0(\pi)$ of $\mathcal{R}(\pi)$ consisting of irreducible representations is open. Indeed, if $(a_1,\dots,a_n)$ is any set of generators, then $\xi_a(\mathcal{R}^0(\pi))=\xi_a(\mathcal{R}(\pi))\cap(G^N)^0$ where $(G^N)^0$ consists of the N-uplets whose centralizer in G is the center. By the slice theorem for action of compact Lie group, $(G^N)^0$ is open in G^N, because it is either empty or the principal stratum for the diagonal action of G on G^N by conjugation.

Set $\mathcal{R}^{\mathrm{s},0}(\pi):=\mathcal{R}^0(\pi)\cap\mathcal{R}^{\mathrm{s}}(\pi)$. The quotient space $\mathcal{M}^{\mathrm{s},0}(\pi):=\mathcal{R}^{\mathrm{s},0}(\pi)/G$ is a smooth manifold because it is the quotient of a smooth manifold by a smooth action of the compact Lie group G with constant isotropy $Z(G)$. Furthermore, for any $\rho\in\mathcal{R}^{\mathrm{s},0}(\pi)$, the infinitesimal action at ρ being d_ρ, we have

$$H^0(\pi,\operatorname{Ad}\rho)=\ker d_\rho=\mathfrak{z}(\mathfrak{g}),$$

where $\mathfrak{z}(\mathfrak{g})$ is the Lie algebra of the center of G. Furthermore $T_{[\rho]}\mathcal{M}^{\mathrm{s},0}(\pi)$ identifies with a subspace of $H^1(\pi, \operatorname{Ad}\rho) = Z^1(\pi, \operatorname{Ad}\rho)/\operatorname{Im} d_\rho$.

Appendix B. Reidemeister Torsion

Let M be a compact manifold possibly with a non empty boundary. Let $E \to M$ be a flat real vector bundle equipped with a flat metric. Denote by $\det H_\bullet(E)$ the line

$$\det H_\bullet(E) = \det H_0(E) \otimes (\det H_1(E))^{-1} \otimes \cdots \otimes (\det H_n(E))^{(-1)^n},$$

where n is the dimension of M and for any finite dimensional vector space V, $\det V = \bigwedge^{\mathrm{top}} V$. In the acyclic case, $\det H_\bullet(E) = \mathbb{R}$. The Reidemeister torsion of E is a non-vanishing vector $\tau(E) \in \det H_\bullet(E)$ well-defined up to sign. Let us recall briefly its definition.

Let K be the simplicial complex of a smooth triangulation of X. For any cell σ of K, let E_σ be the space of flat sections of the restriction of E to σ. Introduce the complex $C_\bullet(K, E)$ where $C_k(K, E) = \bigoplus_{\dim \sigma = k} E_\sigma$ with the usual differential. Then the $H_k(E)$ are the homology groups of $C_\bullet(K, E)$. Consequently, we have an isomorphism $\det C_\bullet(K, E) \simeq \det H_\bullet(E)$. Furthermore, for any cell σ, E_σ is an Euclidean space. So $C_k(K, E)$ has a natural scalar product where the E_σ are mutually orthogonal, and $\det H_\bullet(E)$ inherits an Euclidean product by the previous isomorphism. The Reidemeister torsion $\tau(E)$ is by definition a unit vector of $\det H_\bullet(E)$. It does not depend on the choice of the triangulation, cf. [9], Section 9.

The torsion satisfies the following properties, cf. [6] for 1, 2 and [9], Section 3 for 3.

1. Let $E = E_1 \oplus E_2$ where E_1 and E_2 are two flat Euclidean vector bundles with base M. Then we have a natural isomorphism $H_\bullet(E) \simeq H_\bullet(E_1) \oplus H_\bullet(E_2)$. The corresponding isomorphism $\det H_\bullet(E) \simeq \det H_\bullet(E_1) \otimes \det H_\bullet(E_2)$ sends $\tau(E)$ into $\tau(E_1) \otimes \tau(E_2)$.
2. Let $E_1 \to M_1$ and $E_2 \to M_2$ be two flat Euclidean vector bundles. Assume that M_1 is closed. Set $M = M_1 \times M_2$ and $E = E_1 \boxtimes E_2$. By Künneth theorem, we have $H_\bullet(E) \simeq H_\bullet(E_1) \otimes H_\bullet(E_2)$. The corresponding isomorphism
$$\det H_\bullet(E) \simeq \big(\det H_\bullet(E_1)\big)^{\chi(E_2)} \otimes \big(\det H_\bullet(E_2)\big)^{\chi(E_1)}$$
sends $\tau(E)$ into $\tau(E_1)^{\chi(E_2)} \otimes \tau(E_2)^{\chi(E_1)}$.
3. Let E be a flat Euclidean vector bundle whose base M is obtained by gluing two manifolds M_1, M_2 along their boundary N. By the Mayer-Vietoris exact sequence, we have an isomorphism
$$\det H_\bullet(E) \otimes \det H_\bullet(E|_N) \simeq \det H_\bullet(E|_{M_1}) \otimes \det H_\bullet(E|_{M_2}).$$

This isomorphism sends $\tau(E) \otimes \tau(E|_N)$ to $\tau(E|_{M_1}) \otimes \tau(E|_{M_2})$.

Finally, it is a classical exercise to compute the torsion of a bundle over a circle.

4. Let E be a flat Euclidean vector bundle E on an oriented circle C. Let $p \in C$ and let $\varphi : E_p \to E_p$ be the holonomy of C. Let $H = \ker(\varphi - \mathrm{id})$. We have two isomorphisms $H_0(E) \simeq H$ and $H_1(E) \simeq H$ sending $u \in H$ into $[p] \otimes u$ and $[C] \otimes u$ respectively. Thus $\det H_\bullet(E) \simeq \det H \otimes (\det H)^{-1} \simeq \mathbb{R}$, so that the torsion may be considered as a real number. With this convention, we
$$\tau(E) = \det{}^{-1}\big((\varphi - \mathrm{id})|_{H^\perp}\big),$$
where $H^\perp$ is the orthogonal complement of H.

Further references on Reidemeister torsion are the monographs [15] and [11].

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