COMPACT DOMAINS WITH PRESCRIBED CONVEX BOUNDARY METRICS IN QUASI-FUCHSIAN MANIFOLDS

by Dmitriy Slutskiy

Abstract. — We show the existence of a convex compact domain in a quasi-Fuchsian manifold such that the induced metric on its boundary coincides with a prescribed surface metric of curvature $K \geq -1$ in the sense of A. D. Alexandrov.

This result extends the existence part of the classical result by Alexandrov and Pogorelov on the realization of a convex domain with a prescribed boundary metric in $\mathbb{H}^3$ in the case where $\mathbb{H}^3$ is replaced by a quasi-Fuchsian manifold and therefore the topology of a convex domain is not trivial.

Résumé (*Domaines convexes compacts avec des métriques de bord prescrites dans les variétés quasi-fuchsiennes*). — Nous montrons l'existence d'un tel domaine compact convexe dans une variété quasi-fuchsienne que la métrique induite sur son bord coïncide avec une métrique prescrite de courbure $K \geq -1$ au sens de A. D. Alexandrov.

Ce résultat étend la partie d'existence d'un résultat classique par Alexandrov et Pogorelov sur la réalisation d'un domaine convexe avec une metrique de bord prescrite dans $\mathbb{H}^3$ dans le cas où $\mathbb{H}^3$ est remplacé par une variété quasi-fuchsienne et donc la topologie d'un domaine convexe n'est pas triviale.

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1. Introduction

The problem of existence and uniqueness of an isometric realization of a surface with a prescribed metric in a given ambient space is classical in the metric geometry. Initially stated in the Euclidean case, it can be posed for surfaces in other spaces, in particular, in hyperbolic 3-space $\mathbb{H}^3$.

One of the first fundamental results in this theory is due to A. D. Alexandrov. It concerns the realization of polyhedral surfaces in the spaces of constant curvature.

As in [22], we denote by $M^m(K)$ the m-dimensional complete simply connected space of constant sectional curvature K. So, $M^3(K)$ stands for spherical 3-space of curvature K in the case $K > 0$; $M^3(K)$ stands for hyperbolic 3-space of curvature K when $K < 0$; and in the case $K = 0$, $M^3(K)$ denotes Euclidean 3-space.

Then the result of A. D. Alexandrov reads as follows:

THEOREM 1.1 ([3]). — *Let h be a metric of constant sectional curvature K with cone singularities on a sphere S^2 such that the total angle around every singular point of h does not exceed 2π. Then there exists a closed convex polyhedron in $M^3(K)$ equipped with the metric h which is unique up to the isometries of $M^3(K)$. Here we include the doubly covered convex polygons, which are planar in $M^3(K)$, in the set of convex polyhedra.*

Later, A. D. Alexandrov and A. V. Pogorelov proved the following statement in $\mathbb{H}^3$ [19]:

THEOREM 1.2. — *Let h be a C^∞-regular metric of sectional curvature which is strictly greater than -1 on a sphere S^2. Then there exists an isometric immersion of the sphere (S^2, h) into hyperbolic 3-space $\mathbb{H}^3$ which is unique up to the isometries of $\mathbb{H}^3$. Moreover, this immersion bounds a convex domain in $\mathbb{H}^3$.*

DEFINITION 1.1 ([15, p. 30], [17, p. 11]). — A discrete finitely generated subgroup $\Gamma_F \subset PSL_2(\mathbb{R})$ without torsion and such that the quotient $\mathbb{H}^2/\Gamma_F$ has a finite volume, is called a *Fuchsian group.*

Given a hyperbolic plane $\mathcal{P}$ in $\mathbb{H}^3$ and a Fuchsian group $\Gamma_\mathcal{P} \subset PSL_2(\mathbb{R})$ acting on $\mathcal{P}$, we can canonically extend the action of the group $\Gamma_\mathcal{P}$ on the whole space $\mathbb{H}^3$.

Here we recall another result on the above-mentioned problem considered for a special type of hyperbolic manifolds, namely, for Fuchsian manifolds, which is due to M. Gromov [12]:

THEOREM 1.3. — *Let S be a compact surface of genus greater than or equal to 2, equipped with a C^∞-regular metric h of sectional curvature which is greater*

than -1 everywhere. Then there exists a Fuchsian group Γ_F acting on $\mathbb{H}^3$, such that the surface (S, h) is isometrically embedded in $\mathbb{H}^3/\Gamma_F$.

REMARK 1.4. — The hyperbolic manifold $\mathbb{H}^3/\Gamma_F$ from the statement of Theorem 1.3 is called Fuchsian. Note also that the limit set $\Lambda(\Gamma_F) \subset \partial_\infty\mathbb{H}^3$ of a Fuchsian group Γ_F is a geodesic circle in projective space $\mathbb{CP}^1$ regarded as the boundary at infinity $\partial_\infty\mathbb{H}^3$ of the Poincaré ball model of hyperbolic 3-space $\mathbb{H}^3$.

In 2007 F. Fillastre [9] proved a polyhedral analog of Theorem 1.3, i.e., when h is a hyperbolic metric with cone singularities of angle less than 2π (the term "hyperbolic" means for us "of constant curvature equal to -1 everywhere").

DEFINITION 1.2 ([13]). — A compact hyperbolic manifold M is said to be *strictly convex* if any two points in M can be joined by a minimizing geodesic which lies inside the interior of M. This condition implies that the intrinsic curvature of ∂M is greater than -1 everywhere.

In 1992 F. Labourie [13] obtained the following result which can be considered as a generalization of Theorems 1.2 and 1.3:

THEOREM 1.5. — *Let M be a compact manifold with boundary (different from the solid torus) which admits a structure of a strictly convex hyperbolic manifold. Let h be a C^∞-regular metric on ∂M of sectional curvature which is strictly greater than -1 everywhere. Then there exists a convex hyperbolic metric g on M which induces h on ∂M:*

$$g|_{\partial M} = h.$$

Recall that the *limit set* $\Lambda(\Gamma_F) \subset \partial_\infty\mathbb{H}^3$ of a Fuchsian group Γ_F acting on $\mathbb{H}^3$ is the intersection of some hyperbolic plane with the boundary at infinity of the hyperbolic 3-space $\mathbb{H}^3$, i.e., a circle (in the Poincaré and Klein models of the hyperbolic 3-space).

Particular examples of the varieties considered in Theorem 1.5 are the quasi-Fuchsian manifolds.

DEFINITION 1.3 ([15, p. 120]). — A *quasi-Fuchsian manifold* is a quasiconformal deformation space $QH(\Gamma_F)$ of a Fuchsian group $\Gamma_F \subset PSL_2(\mathbb{R})$.

In other words, a quasi-Fuchsian manifold is a quotient $\mathbb{H}^3/\Gamma_{qF}$ of $\mathbb{H}^3$ by a discrete finitely generated group $\Gamma_{qF} \subset PSL_2(\mathbb{R})$ of hyperbolic isometries of $\mathbb{H}^3$ such that there is a Fuchsian group Γ_F of isometries of $\mathbb{H}^3$ such that the limit set $\Lambda(\Gamma_{qF}) \subset \partial_\infty\mathbb{H}^3$ of Γ_{qF} is a Jordan curve which can be obtained from the circle $\Lambda(\Gamma_{qF}) \subset \partial_\infty\mathbb{H}^3$ by a quasiconformal deformation of $\partial_\infty\mathbb{H}^3$. The group Γ_{qF} is called *quasi-Fuchsian*.

In geometric terms, a quasi-Fuchsian manifold is a complete hyperbolic manifold homeomorphic to $\mathcal{S} \times \mathbb{R}$, where $\mathcal{S}$ is a closed connected surface of genus at least 2, which contains a convex compact subset.

Let us also recall the A. D. Alexandrov notion of curvature which does not require a metric of a surface to be regular.

Let X be a complete locally compact length space and let $d_X(\cdot,\cdot)$ stand for the distance between points in X. For a triple of points $p, q, r \in X$ a geodesic triangle $\triangle(pqr)$ is a triple of geodesics joining these three points. For a geodesic triangle $\triangle(pqr) \subset X$ we denote by $\triangle(\tilde{p}\tilde{q}\tilde{r})$ a geodesic triangle sketched in $M^2(K)$ whose corresponding edges have the same lengths as $\triangle(pqr)$.

DEFINITION 1.4 ([22, p. 7]). — X is said to have *curvature bounded below by* K iff every point $x \in X$ has an open neighborhood $U_x \subset X$ such that for every geodesic triangle $\triangle(pqr)$ whose edges are contained entirely in U_x the corresponding geodesic triangle $\triangle(\tilde{p}\tilde{q}\tilde{r})$ sketched in $M^2(K)$ has the following property: for every point $z \in qr$ and for $\tilde{z} \in \tilde{q}\tilde{r}$ with $d_X(q,z) = d_{M^2(K)}(\tilde{q},\tilde{z})$ we have

$$d_X(p,z) \geq d_{M^2(K)}(\tilde{p},\tilde{z}).$$

In 2016 F. Fillastre, I. Izmestiev, and G. Veronelli [10] proved that for every metric on the torus with curvature bounded from below by -1 in the Alexandrov sense there exists a hyperbolic cusp with convex boundary such that the induced metric on the boundary is the given metric.

Our main goal is to prove the following extension of Theorem 1.5:

THEOREM 1.6. — *Let $\mathcal{M}$ be a compact connected 3-manifold with boundary of the type $\mathcal{S} \times [-1,1]$ where $\mathcal{S}$ is a closed connected surface of genus at least 2. Let h be a metric on $\partial\mathcal{M}$ of curvature $K \geq -1$ in the Alexandrov sense. Then there exists a hyperbolic metric g in $\mathcal{M}$ with a convex boundary $\partial\mathcal{M}$ such that the metric induced on $\partial\mathcal{M}$ is h.*

In particular, the following result proved in [23] immediately follows from Theorem 1.6.

THEOREM 1.7. — *Let $\mathcal{M}$ be a compact connected 3-manifold with boundary of the type $\mathcal{S} \times [-1,1]$ where $\mathcal{S}$ is a closed connected surface of genus at least 2. Let h be a hyperbolic metric with cone singularities of angle less than 2π on $\partial\mathcal{M}$ such that every singular point of h possesses a neighborhood in $\partial\mathcal{M}$ which does not contain other singular points of h. Then there exists a hyperbolic metric g in $\mathcal{M}$ with a convex boundary $\partial\mathcal{M}$ such that the metric induced on $\partial\mathcal{M}$ is h.*

The idea of the proof of Theorem 1.7 is given in [25].

Theorem 1.7 can also be considered as an analog of Theorem 1.1 for the convex hyperbolic manifolds with polyhedral boundary.

DEFINITION 1.5 ([7]). — A *pleated surface* in a hyperbolic 3-manifold $\mathcal{M}$ is a complete hyperbolic surface $\mathcal{S}$ together with an isometric map $f : \mathcal{S} \to \mathcal{M}$ such that every $s \in \mathcal{S}$ is in the interior of some geodesic arc which is mapped by f to a geodesic arc in $\mathcal{M}$.

A pleated surface resembles a polyhedron in the sense that it has flat faces that meet along edges. Unlike a polyhedron, a pleated surface has no corners, but it may have infinitely many edges that form a lamination.

REMARK 1.8. — The surfaces serving as the connected components of the boundary $\partial \mathcal{M}$ of the manifold $\mathcal{M}$ from the statement of Theorem 1.7, which are equipped by assumption with hyperbolic polyhedral metrics, do not necessarily have to be polyhedra embedded in $\mathcal{M}$: these surfaces can be partially pleated, i.e., the universal covers in $\mathbb{H}^3$ of these surfaces can contain pleated 2-dimensional domains situated between several pairwise nonintersecting geodesics which are also geodesics in $\mathbb{H}^3$.

DEFINITION 1.6 ([16]). — Let $\mathcal{M}$ be the interior of a compact manifold with boundary. A complete hyperbolic metric g on $\mathcal{M}$ is convex co-compact if $\mathcal{M}$ contains a compact subset $\mathcal{K}$ which is convex: any geodesic segment c in $(\mathcal{M}, g)$ with endpoints in $\mathcal{K}$ is contained in $\mathcal{K}$.

In 2002 J.-M. Schlenker [21] proved uniqueness of the metric g in Theorem 1.5. Thus, he obtained

THEOREM 1.9. — *Let M be a compact connected 3-manifold with boundary (different from the solid torus) which admits a complete hyperbolic convex co-compact metric. Let g be a hyperbolic metric on M such that ∂M is C^∞-regular and strictly convex. Then the induced metric I on ∂M has curvature $K > -1$. Each C^∞-regular metric on ∂M with $K > -1$ is induced on ∂M for a unique choice of g.*

It would be natural to conjecture that the metric g in the statements of Theorems 1.6 and 1.7 is unique. The methods used in their demonstration do not presently allow to attack this problem.

At last, recalling that the convex quasi-Fuchsian manifolds are special cases of the convex co-compact manifolds, we can guess that Theorems 1.6 and 1.7 remain valid in the case when $\mathcal{M}$ is a convex co-compact manifold. It would be interesting to verify this hypothesis in the future.

2. Construction of a quasi-Fuchsian manifold containing a compact convex domain with a prescribed Alexandrov metric of curvature $K \geq -1$ on the boundary

A compact connected 3-manifold $\mathcal{M}$ of the type $\mathcal{S} \times [-1, 1]$ from the statement of Theorem 1.6, where $\mathcal{S}$ is a closed connected surface of genus at least 2, can be regarded as a convex compact 3-dimensional domain of an unbounded quasi-Fuchsian manifold $\mathcal{M}^\circ = \mathbb{H}^3 / \Gamma_{QF}$ where Γ_{QF} stands for a quasi-Fuchsian group of isometries of hyperbolic space $\mathbb{H}^3$. Note that the boundary $\partial \mathcal{M}$ of such domain $\mathcal{M}$ consists of two distinct locally convex compact 2-surfaces in $\mathcal{M}^\circ$.

Thus, the metric h from the statement of Theorem 1.6 is a pair of Alexandrov metrics of curvature $K \geq -1$ at every point defined on a couple of compact connected surfaces of the same genus as $\mathcal{M}$, and our aim is to find such quasi-Fuchsian subgroup Γ_{QF} of isometries of hyperbolic space $\mathbb{H}^3$ and such convex compact domain $\mathcal{M} \subset \mathcal{M}^\circ$ that the induced metric of its boundary $\partial \mathcal{M}$ coincides with h.

The main idea of the proof of Theorem 1.6 is

(1) to approximate the Alexandrov metric h by a sequence $\{h_n\}_{n \in \mathbb{N}}$ of C^∞-regular metrics for which the Labourie-Schlenker Theorem 1.9 is applicable, and therefore, there are such quasi-Fuchsian groups Γ_n of isometries of $\mathbb{H}^3$ and such convex compact domains $\mathcal{M}_n$ in the quasi-Fuchsian manifolds $\mathcal{M}_n^\circ = \mathbb{H}^3/\Gamma_n$ that the induced metrics of the boundaries $\partial \mathcal{M}_n$ of the sets $\mathcal{M}_n$ are exactly h_n, $n \in \mathbb{N}$;

(2) to find a sequence of positive integers $n_k \xrightarrow[k \to \infty]{} \infty$ such that the subsequences of groups $\{\Gamma_{n_k}\}_{k \in \mathbb{N}}$ and of domains $\{\mathcal{M}_{n_k}\}_{k \in \mathbb{N}}$ converge (the types of convergence will be specified later);

(3) and to show that the induced metric on the boundary of the limit domain $\mathcal{M}$ coincides with h.

For convenience, let us introduce new notation of some entities that we considered before: we redefine the domain $\mathcal{M}$ and the quasi-Fuchsian manifold $\mathcal{M}^\circ$ by the symbols $\mathcal{M}_\infty$ and $\mathcal{M}_\infty^\circ$, correspondingly. Also, let us denote the connected components of the boundary $\partial \mathcal{M}_\infty$ of the limit domain $\mathcal{M}_\infty$ by $\mathcal{S}_\infty^+$ and $\mathcal{S}_\infty^-$, and the induced metrics on the surfaces $\mathcal{S}_\infty^+$ and $\mathcal{S}_\infty^-$ by h_∞^+ and h_∞^-, respectively. Therefore, to define the metric h from the statement of Theorem 1.6 means to give a pair of Alexandrov metrics h_∞^+ and h_∞^- of curvature $K \geq -1$ at every point.

2.1. Construction of sequences of metrics converging to the prescribed metrics

DEFINITION 2.1. — We say that a sequence of metrics $\{h_n\}_{n \in \mathbb{N}}$ on a compact surface $\mathcal{S}$ converges to a metric h if for any $\varepsilon > 0$ there exists such $N(\varepsilon) \in \mathbb{N}$ that all integers $n \geq N(\varepsilon)$ and for any pair of points x and y on $\mathcal{S}$ the following inequality holds:

$$(2.1) \qquad\qquad |\mathrm{d}_{h_n}(x,y) - \mathrm{d}_h(x,y)| < \varepsilon.$$

First, we shall learn to approximate an Alexandrov metric of curvature $K \geq -1$ on a compact connected surface by a sequence of hyperbolic polyhedral metrics (i.e., of sectional curvature -1 everywhere except at a discrete set of points with conic singularities of angles less than 2π). Next, we shall learn to approximate any hyperbolic polyhedral metric by a sequence of C^∞-regular metrics of curvature $K > -1$. Thus, we will be able to find a sequence of C^∞-regular metrics of curvature $K > -1$ on a compact connected surface

converging to the given metric of curvature $K \geq -1$ at every point in the Alexandrov sense.

2.1.1. *Construction of a sequence of hyperbolic polyhedral metrics converging to a metric in the Alexandrov sense.* — A. D. Alexandrov in [3] developed a way to approximate an Alexandrov metric of curvature $K \geq 0$ on a compact connected surface by a sequence of Euclidean polyhedral metrics. Recently T. Richard [20, Annex A] adapted the Alexandrov method to the case of Alexandrov metrics of curvature $K \geq -1$.

   Here we give a more detailed description of what T. Richard proved in the annex of his PhD thesis.

   In terms of [20, Annex A] let us recall the following definition due to A. D. Alexandrov.

DEFINITION 2.2. — Let (X, d) be an Alexandrov compact surface of curvature $K \geq -1$ everywhere. A *triangulation* $\mathcal{T}$ of (X, d) is a family of geodesic triangles $\{T_i\}_{i \in I}$ with disjoint interiors each homeomorphic to an open disk and such that the family $\{T_i\}_{i \in I}$ covers X. Note that in this definition two triangles can have edges intersecting in more than one point that do not coincide though.

   T. Richard verifies that the following proposition proved in [3, Section 6, p. 88] is valid for an Alexandrov surface of curvature $K \geq -1$.

LEMMA 2.1 (Lemma A.1.2 in [20]). — *For every $\varepsilon > 0$, (X, d) admits a triangulation (in the Alexandrov sense) by convex triangles whose diameters are inferior to ε.*

   After T. Richard let us fix $\varepsilon > 0$, denote by $\mathcal{T}_\varepsilon$ a triangulation of (X, d) provided by Lemma 2.1, and construct a polyhedral surface with hyperbolic faces $(\overline{X}_\varepsilon, \bar{d}_\varepsilon)$ as it follows: for every triangle $T \in \mathcal{T}_\varepsilon$ we associate a comparison triangle $\overline{T}$ sketched on a hyperbolic plane $\mathbb{H}^2$ ($= M^2(-1)$) such that all corresponding edges of T and $\overline{T}$ have equal lengths, then we glue together the collection of hyperbolic comparison triangles following the same combinatorics as that of $\mathcal{T}_\varepsilon$, and thus we obtain a polyhedral surface $\overline{X}_\varepsilon$.

   We must note the following property of $\overline{X}_\varepsilon$:

LEMMA 2.2 (Lemma A.2.1 in [20]). — *$(\overline{X}_\varepsilon, \bar{d}_\varepsilon)$ has curvature $K \geq -1$ everywhere in the Alexandrov sense.*

REMARK 2.3. — By construction, the curvature of $\overline{X}_\varepsilon$ is equal to -1 everywhere with the exception of vertices of the triangles forming $\overline{X}_\varepsilon$. Therefore, Lemma 2.2 means that the above mentioned vertices are conic singularities of angles $\leq 2\pi$ of the hyperbolic polyhedral metric on $\overline{X}_\varepsilon$.

At last, T. Richard [20, pp. 87–91] proves that for any $\varepsilon > 0$ there exists a real number $\varepsilon' > 0$ (depending only on (X, d) and verifying the property $\varepsilon' \to 0$ as $\varepsilon \to 0$) such that for any pair of points v and w in $\overline{X}$ and for a pair of corresponding points $\bar{v}$ and $\bar{w}$ in $\overline{X}_\varepsilon$ the following inequality holds:

$$(2.2) \qquad\qquad |\bar{d}_\varepsilon(\bar{v}, \bar{w}) - d(v, w)| < \varepsilon'.$$

T. Richard calls this way of convergence of hyperbolic polyhedral surfaces $(\overline{X}_\varepsilon, \bar{d}_\varepsilon)$ to the Alexandrov surface (X, d) as $\varepsilon \to 0$ a Gromov-Hausdorff convergence.

Let us rewrite the results of T. Richard described above in the language developed in Section 2. We consider an Alexandrov compact surface (X, d) as a topological surface $\mathcal{S}$ endowed with a metric h of curvature $K \geq -1$ in the Alexandrov sense and we note that the construction of a hyperbolic polyhedral surface $\overline{X}_\varepsilon$ based on a triangulation $\mathcal{T}_\varepsilon$ of (X, d) $(= (\mathcal{S}, h))$ is equivalent to a construction of a hyperbolic polyhedral metric h_ε on $\mathcal{S}$ as follows: leaving the lengths of all edges of the triangulation $\mathcal{T}_\varepsilon$ unchanged, we replace the metric h restricted on the interior of each triangle $T \in \mathcal{T}_\varepsilon$ by a hyperbolic metric (i.e., of curvature -1 everywhere) inside T. Thus, the inequality (2.2) becomes equivalent to the following one:

$$|\mathrm{d}_{h_\varepsilon}(v, w) - \mathrm{d}_h(v, w)| < \varepsilon'$$

for all pairs of points v and w in $\mathcal{S}$ (compare it with (2.1)).

Therefore, choosing a sequence of positive real numbers $\varepsilon_n \to 0$ as $n \to \infty$ and then applying the argument of T. Richard for each ε_n, we state

LEMMA 2.4. — *Let $\mathcal{S}$ be a closed compact surface endowed with a metric h of curvature $K \geq -1$ in the Alexandrov sense, there exists a sequence of hyperbolic polyhedral metrics $\{h_n\}_{n \in \mathbb{N}}$ converging to h (hereinafter we mean by default the convergence of metrics in the sense of inequality (2.1)).*

2.1.2. *Construction of a sequence of C^∞-regular metrics converging to a hyperbolic polyhedral metric.* — In this section, we prove the following

LEMMA 2.5. — *Let $\mathcal{S}$ be a surface with a hyperbolic polyhedral metric h. Then there is a sequence of C^∞-regular metrics $\{h_n\}_{n \in \mathbb{N}}$ with sectional curvatures strictly greater than -1 everywhere, converging to the metric h.*

First, let us state two preliminary results.

LEMMA 2.6. — *Let $\mathcal{S}$ be a surface with a hyperbolic polyhedral metric h. Then there is a sequence of C^∞-regular metrics $\{h_n\}_{n \in \mathbb{N}}$ with sectional curvatures greater than or equal to -1 everywhere, converging to the metric h.*

To prove Lemma 2.6, we construct small conic surfaces in $\mathbb{H}^3$ whose induced metrics coincide with the restrictions of the metric h on neighborhoods of the conic singularities of h, and then we convolute these conic surfaces with

C^∞-smooth functions as in [11]. A full explanation of this idea is given in [23, Lemma 3.10] (see also [24]).

Also, a direct calculation shows the validity of the following statement (see [23, Lemma 3.11] and also [24] for the detailed proof).

LEMMA 2.7. — *Consider a regular metric surface $(\mathcal{S}, h)$, where $\mathcal{S}$ stands for a 2-dimensional surface, h is a metric provided on $\mathcal{S}$, and $K_h(x)$ denotes the sectional curvature of $(\mathcal{S}, h)$ at a point $x \in \mathcal{S}$. If we consider another metric surface $(\mathcal{S}, g)$, where the metric $g = \lambda h$ is a multiple of h and $\lambda > 0$ is a positive constant, then the sectional curvature $K_g(x)$ of $(\mathcal{S}, g)$ at a point $x \in \mathcal{S}$ is related to $K_h(x)$ as follows:*

$$(2.3) \qquad K_g(x) = \frac{1}{\lambda} K_h(x).$$

We are now ready to give a demonstration of Lemma 2.5.

Proof. — Let h be a hyperbolic polyhedral metric on a closed compact surface $\mathcal{S}$ of genus g. By Lemma 2.6, there is a sequence of C^∞-smooth metrics $\{\hbar_n\}_{n \in \mathbb{N}}$ on $\mathcal{S}$, with sectional curvature ≥ -1 everywhere, converging to h as $n \to \infty$.

Next, let us choose a monotonically decreasing sequence of real numbers $\lambda_n \xrightarrow[n \to \infty]{} 1$ and let us define the metrics $h_n \overset{\text{def}}{=} \lambda_n \hbar_n$ on $\mathcal{S}$, $n \in \mathbb{N}$. Thus, by Lemma 2.7, the sectional curvatures of the metrics h_n, $n \in \mathbb{N}$ are strictly greater than -1 everywhere on $\mathcal{S}$, and, by construction, the sequence of C^∞-smooth metrics $\{h_n\}_{n \in \mathbb{N}}$ converges to h as $n \to \infty$. □

2.2. Convergence of convex surfaces in a compact domain in $\mathbb{H}^3$.

— Let h_∞^+ and h_∞^- be two metrics of curvature $K \geq -1$ in the Alexandrov sense everywhere on a closed compact surface $\mathcal{S}$ of genus g. To be able to apply the Labourie-Schlenker Theorem 1.9, we shall construct two sequences of C^∞-regular metrics of curvature strictly greater than -1, converging to h_∞^+ and h_∞^-. By Lemma 2.4, there are two sequences of hyperbolic polyhedral metrics $\{\hbar_n^+\}_{n \in \mathbb{N}}$ and $\{\hbar_n^-\}_{n \in \mathbb{N}}$ on $\mathcal{S}$, converging to h_∞^+ and h_∞^- as $n \to \infty$. Also, by Lemma 2.5, for each $n \in \mathbb{N}$ there are sequences $\{\hbar_{n,k}^+\}_{k \in \mathbb{N}}$ and $\{\hbar_{n,k}^-\}_{k \in \mathbb{N}}$ of C^∞-smooth metrics of curvature $K > -1$ everywhere on $\mathcal{S}$, converging to the hyperbolic polyhedral metrics $\hbar_n^+$ and $\hbar_n^-$, respectively, as $k \to \infty$. Thus, we are now able to extract sequences of C^∞-smooth metrics $\{h_n^+\}_{n \in \mathbb{N}}$ and $\{h_n^-\}_{n \in \mathbb{N}}$ of curvature $K > -1$, converging to the Alexandrov metrics h_∞^+ and h_∞^-, respectively (where $h_n^+ \in \{\hbar_{n,k}^+\}_{k \in \mathbb{N}}$ and $h_n^- \in \{\hbar_{n,k}^-\}_{k \in \mathbb{N}}$, $n \in \mathbb{N}$).

By the Labourie-Schlenker Theorem 1.9, for each $n \in \mathbb{N}$ there is a unique compact convex domain $\mathcal{M}_n$ of a quasi-Fuchsian manifold $\mathcal{M}_n^\circ$ with hyperbolic metric g_n such that the induced metrics of the components $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ of the boundary $\partial \mathcal{M}_n \overset{\text{def}}{=} \mathcal{S}_n^+ \cup \mathcal{S}_n^-$ are equal to h_n^+ and h_n^- (see also Fig. 2.1). It means
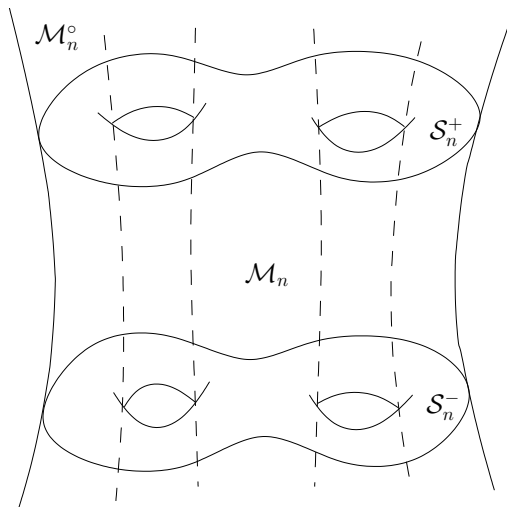
FIGURE 2.1. The surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ in the quasi-Fuchsian manifold $\mathcal{M}_n^\circ$.

that, for each $n \in \mathbb{N}$ there exist isometric embeddings $f_{\mathcal{S}_n^+} : (\mathcal{S}, h_n^+) \to \mathcal{M}_n^\circ$ and $f_{\mathcal{S}_n^-} : (\mathcal{S}, h_n^-) \to \mathcal{M}_n^\circ$ such that $f_{\mathcal{S}_n^+}(\mathcal{S}) = \mathcal{S}_n^+ \subset \mathcal{M}_n^\circ$ and $f_{\mathcal{S}_n^-}(\mathcal{S}) = \mathcal{S}_n^- \subset \mathcal{M}_n^\circ$.

As $\mathcal{M}_n^\circ$ can be retracted by deformation on $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$, we conclude that their fundamental groups are homomorphic:

$$\pi_1(\mathcal{S}_n^+) \simeq \pi_1(\mathcal{M}_n^\circ) \simeq \pi_1(\mathcal{S}_n^-).$$

Also, by construction,

$$\pi_1(\mathcal{S}_n^+) \simeq \pi_1(\mathcal{S}) \simeq \pi_1(\mathcal{S}_n^-).$$

Hence, for all $n \in \mathbb{N}$

$$(2.4) \qquad\qquad \pi_1(\mathcal{M}_n^\circ) \simeq \pi_1(\mathcal{S}).$$

Since the manifolds $\mathcal{M}_n^\circ$, $n \in \mathbb{N}$, are hyperbolic, their universal coverings $\widetilde{\mathcal{M}_n^\circ}$ are actually copies of hyperbolic 3-space $\mathbb{H}^3$. Moreover, as each $\mathcal{M}_n^\circ$ is quasi-Fuchsian, there exists a holonomy representation $\rho_n : \pi_1(\mathcal{M}_n^\circ) \to \mathcal{I}(\widetilde{\mathcal{M}_n^\circ})(= \mathcal{I}(\mathbb{H}^3))$ of the fundamental group of $\mathcal{M}_n^\circ$ in the group of isometries of the universal covering $\widetilde{\mathcal{M}_n^\circ}(= \mathbb{H}^3)$ such that $\mathcal{M}_n^\circ = \widetilde{\mathcal{M}_n^\circ}/[\rho_n(\pi_1(\mathcal{M}_n^\circ))] = \mathbb{H}^3/[\rho_n(\pi_1(\mathcal{M}_n^\circ))]$ and the limit set $\Lambda_{\rho_n} \subset \partial_\infty \mathbb{H}^3$ of $\rho_n(\pi_1(\mathcal{M}_n^\circ))$ is homotopic to a circle. By (2.4), we can also speak about the holonomy representation $\rho_n^\mathcal{S} : \pi_1(\mathcal{S}) \to \mathcal{I}(\widetilde{\mathcal{M}_n^\circ})(= \mathcal{I}(\mathbb{H}^3))$ of the fundamental group of $\mathcal{S}$ in the group of isometries of the universal covering $\widetilde{\mathcal{M}_n^\circ}(= \mathbb{H}^3)$ such that $\rho_n^\mathcal{S}(\pi_1(\mathcal{S})) = \rho_n(\pi_1(\mathcal{M}_n^\circ))$. Thus we have that $\mathcal{M}_n^\circ = \widetilde{\mathcal{M}_n^\circ}/[\rho_n^\mathcal{S}(\pi_1(\mathcal{S}))] = \mathbb{H}^3/[\rho_n^\mathcal{S}(\pi_1(\mathcal{S}))]$ and the limit set

$\Lambda_{\rho_n^\mathcal{S}}$ of $\rho_n^\mathcal{S}(\pi_1(\mathcal{S}))$ is just Λ_{ρ_n}, $n \in \mathbb{N}$. We also suppose that $\pi_1(\mathcal{S})$ is generated by the elements $\{\gamma_1, \dots, \gamma_l\}$.
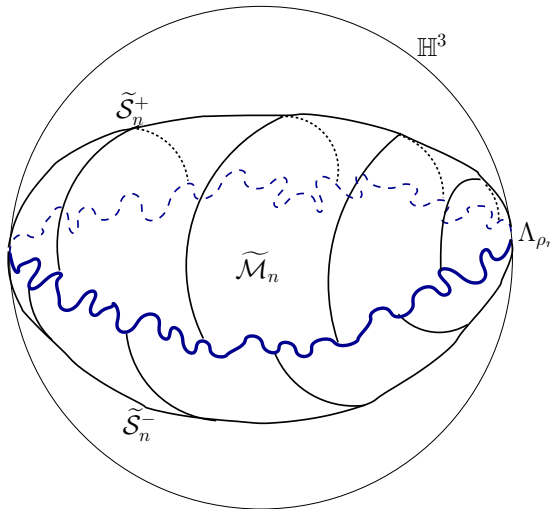

FIGURE 2.2. The universal coverings $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$ in the Kleinian model $\mathbb{K}^3$ of hyperbolic space $\mathbb{H}^3$.

Inside $\widetilde{\mathcal{M}}_n^\circ (= \mathbb{H}^3)$, $n \in \mathbb{N}$, we can find a convex set $\widetilde{\mathcal{M}}_n$ serving as a universal covering of the domain $\mathcal{M}_n \subset \mathcal{M}_n^\circ$, i.e., such that $\mathcal{M}_n = \widetilde{\mathcal{M}}_n / [\rho_n^\mathcal{S}(\pi_1(\mathcal{S}))]$, and a pair of convex surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$ serving as universal coverings of the surfaces $\mathcal{S}_n^+ \subset \mathcal{M}_n^\circ$ and $\mathcal{S}_n^- \subset \mathcal{M}_n^\circ$ (see Fig. 2.2), i.e., such that $\mathcal{S}_n^+ = \widetilde{\mathcal{S}}_n^+ / [\rho_n^\mathcal{S}(\pi_1(\mathcal{S}))]$ and $\mathcal{S}_n^- = \widetilde{\mathcal{S}}_n^- / [\rho_n^\mathcal{S}(\pi_1(\mathcal{S}))]$. By construction, $\partial \widetilde{\mathcal{M}}_n = \widetilde{\mathcal{S}}_n^+ \cup \widetilde{\mathcal{S}}_n^-$ and the boundaries at infinity $\partial_\infty \widetilde{\mathcal{M}}_n = \partial_\infty \widetilde{\mathcal{S}}_n^+ = \partial_\infty \widetilde{\mathcal{S}}_n^- = \Lambda_{\rho_n^\mathcal{S}}$. Indeed, since for every $n \in \mathbb{N}$ the surfaces $\widetilde{\mathcal{S}}_n^+ \subset \mathbb{H}^3$ and $\widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ are invariant under the action of the group $\rho_n^\mathcal{S}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$, we have that the limit sets $\partial_\infty \widetilde{\mathcal{S}}_n^+$ and $\partial_\infty \widetilde{\mathcal{S}}_n^-$ of $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$, respectively, are subsets of the limit set $\Lambda_{\rho_n^\mathcal{S}}$ of $\mathbb{H}^3$ under the action of the group $\rho_n^\mathcal{S}(\pi_1(\mathcal{S}))$. On the other hand, given a point at infinity $z_\infty \in \Lambda_{\rho_n^\mathcal{S}}$, there is a point $z \in \mathbb{H}^3$ and a sequence $\{\gamma_j\}_{j \in \mathbb{N}} \subset \pi_1(\mathcal{S})$ such that the sequence of points $\{\rho_n^\mathcal{S}(\gamma_j).z\}_{j \in \mathbb{N}} \subset \mathbb{H}^3$ tends to z_∞ as j goes to ∞. Consider some points $z^+ \in \widetilde{\mathcal{S}}_n^+$ and $z^- \in \widetilde{\mathcal{S}}_n^-$. Let d_z^+ and d_z^- stand for the hyperbolic distances $\mathrm{d}_{\mathbb{H}^3}(z, z^+)$ and $\mathrm{d}_{\mathbb{H}^3}(z, z^-)$. The elements $\rho_n^\mathcal{S}(\gamma_j)$, $j \in \mathbb{N}$, are isometries of $\mathbb{H}^3$. Hence, $\mathrm{d}_{\mathbb{H}^3}(\rho_n^\mathcal{S}(\gamma_j).z, \rho_n^\mathcal{S}(\gamma_j).z^+) = d_z^+$ and $\mathrm{d}_{\mathbb{H}^3}(\rho_n^\mathcal{S}(\gamma_j).z, \rho_n^\mathcal{S}(\gamma_j).z^-) = d_z^-$ for all $j \in \mathbb{N}$. Therefore, the sequences of points $\{\rho_n^\mathcal{S}(\gamma_j).z^+\}_{j \in \mathbb{N}} \subset \widetilde{\mathcal{S}}_n^+$ and $\{\rho_n^\mathcal{S}(\gamma_j).z^-\}_{j \in \mathbb{N}} \subset \widetilde{\mathcal{S}}_n^-$ also converge to z_∞

as $j \to \infty$, and so, $z_\infty \in \partial_\infty \widetilde{\mathcal{S}}_n^+$ and $z_\infty \in \partial_\infty \widetilde{\mathcal{S}}_n^-$. Thus, $\Lambda_{\rho_n^{\mathcal{S}}} \subset \widetilde{\mathcal{S}}_n^+$ and $\Lambda_{\rho_n^{\mathcal{S}}} \subset \widetilde{\mathcal{S}}_n^-$.

Denote by $p_n : \widetilde{\mathcal{M}}_n \to \mathcal{M}_n$ the projection of $\widetilde{\mathcal{M}}_n$ on $\mathcal{M}_n$, $n \in \mathbb{N}$. By construction, $\mathcal{S}_n^+ = p_n(\widetilde{\mathcal{S}}_n^+)$ and $\mathcal{S}_n^- = p_n(\widetilde{\mathcal{S}}_n^-)$, $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$ we lift the metric g_n of the manifold $\mathcal{M}_n$ to the metric $\tilde{g}_n$ of the universal covering $\widetilde{\mathcal{M}}_n$ in such a way that for any $\gamma \in \pi_1(\mathcal{S})$ and for $x \in \mathcal{M}_n$ and $\tilde{x} \in \widetilde{\mathcal{M}}_n$ satisfying the relation $x = p_n(\tilde{x})$, we have $\tilde{g}_n(\tilde{x}) = p_n{}^* g_n(x)$, i.e., the metric $\tilde{g}_n(\tilde{x}) \in T_{\tilde{x}}^* \widetilde{\mathcal{M}}_n$ is a pull-back of the metric $g_n(x) \in T_x^* \mathcal{M}_n$. We have already remarked that, since g_n is hyperbolic, $\tilde{g}_n$ is hyperbolic too. Denote by $\tilde{h}_n^+$ the restriction of the metric $\tilde{g}_n$ on the surface $\widetilde{\mathcal{S}}_n^+$ and by $\tilde{h}_n^-$ the restriction of the metric $\tilde{g}_n$ on the surface $\widetilde{\mathcal{S}}_n^-$, $n \in \mathbb{N}$. By construction, the metric $\tilde{h}_n^+$ is the lift of h_n^+ from the surface $\mathcal{S}_n^+$ to its universal covering $\widetilde{\mathcal{S}}_n^+$ and the metric $\tilde{h}_n^-$ is the lift of h_n^- from $\mathcal{S}_n^-$ to $\widetilde{\mathcal{S}}_n^-$, $n \in \mathbb{N}$.

DEFINITION 2.3. — The *diameter* δ of a set S with a metric h is the following quantity: $\delta \overset{\text{def}}{=} \sup\{\mathrm{d}_h(u, v) | u, v \in S\}$ where $\mathrm{d}_h(u, v)$ stands for the distance between points u and v in the metric h.

LEMMA 2.8. — *There exists a positive constant $\delta_{\mathcal{S}} < \infty$ which bounds from above the diameters δ_n^+ and δ_n^- of the surfaces $(\mathcal{S}, h_n^+)$ and $(\mathcal{S}, h_n^-)$ for all $n \in \mathbb{N}$.*

Proof. —   Recall the way of construction of the metric h_n^+ on $\mathcal{S}$, $n \in \mathbb{N}$.

First we applied Lemma 2.4, and thus obtained the sequence of hyperbolic polyhedral metrics $\{\hbar_n^+\}_{n \in \mathbb{N}}$ converging to the Alexandrov metric h_∞^+. Every metric $\hbar_n^+$ is obtained from h_∞^+ by choosing a geodesic triangulation on $(\mathcal{S}, h_\infty^+)$ and by replacing the metric h_∞^+ of curvature $K \geq -1$ in the interior of each triangle by a hyperbolic plane metric (i.e., of curvature $K = -1$) while keeping the lengths of the edges of a considered triangulation unchanged. Therefore, by construction, the length of any curve on $\mathcal{S}$ measured in the metric $\hbar_n^+$ does not exceed the corresponding length measured in h_∞^+.

Next, for each $n \in \mathbb{N}$ we constructed the sequence of C^∞-regular metrics $\{\hbar_{n,k}^+\}_{k \in \mathbb{N}}$ of curvature $K > -1$ converging to the hyperbolic polyhedral metric $\hbar_n^+$ by applying Lemma 2.5, and the metric h_n^+ belongs to the set $\{\hbar_{n,k}^+\}_{k \in \mathbb{N}}$. The application of Lemma 2.5 consists of two stages. The first step is the construction of a sequence of C^∞-regular metrics $\{\overline{\hbar}_{n,k}^+\}_{k \in \mathbb{N}}$ of curvature $K \geq -1$ converging to $\hbar_n^+$ due to Lemma 2.6, smoothing the conic singularities of $\hbar_n^+$ by convolution. This procedure does not increase the distance between any two points on the surface $\mathcal{S}$. At the second stage, we considered a sequence of positive real numbers $\{\lambda_k\}_{k \in \mathbb{N}}$ decreasing to 1 and then, by multiplying the metric $\overline{\hbar}_{n,k}^+$ by the constant $\lambda_k(> 1)$, we obtained the metric $\hbar_{n,k}^+$ for each $k \in \mathbb{N}$ and for every $n \in \mathbb{N}$, and thus, we increased all distances on $\mathcal{S}$ by $\sqrt{\lambda_k}$.

Since $\lambda_1 \geq \lambda_k$ for every $k \in \mathbb{N}$, the distances on $\mathcal{S}$ measured in the metric $h_\lambda^+ \stackrel{\text{def}}{=} \lambda_1 h_\infty^+$ are not smaller than the corresponding distances measured in the metrics h_n^+ for all $n \in \mathbb{N}$. Similarly, the distances on $\mathcal{S}$ measured in the metric $h_\lambda^- \stackrel{\text{def}}{=} \lambda_1 h_\infty^-$ are not smaller than the corresponding distances measured in the metrics h_n^- for all $n \in \mathbb{N}$.

The diameters δ_λ^+ and δ_λ^- of the surfaces $(\mathcal{S}, h_\lambda^+)$ and $(\mathcal{S}, h_\lambda^-)$ are finite numbers because $\mathcal{S}$ is compact. We can pose $\delta_\mathcal{S} = \max(\delta_\lambda^+, \delta_\lambda^-)$. $\qquad \square$

LEMMA 2.9. — *There exists a positive constant $\delta_\mathcal{M} < \infty$ such that for each $n \in \mathbb{N}$ and for every pair of points $u \in \mathcal{S}_n^+ \subset \mathcal{M}_n^\circ$ and $v \in \mathcal{S}_n^- \subset \mathcal{M}_n^\circ$ the distance $\mathrm{d}_{g_n}(u,v)$ between u and v in the manifold $\mathcal{M}_n^\circ$ is less than $\delta_\mathcal{M}$.*

Proof. — By Theorem 3.1 in Section 3, the distances $\sigma_n^\mathcal{S}$ between the surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$, $n \in \mathbb{N}$, are uniformly bounded by a constant $\sigma_\mathcal{S}$. Also, by Lemma 2.8, the diameters of $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ are both bounded by a constant $\delta_\mathcal{S}$ which does not depend on n. Hence, our assertion is valid if we take $\delta_\mathcal{M}$ to be equal to $\sigma_\mathcal{S} + 2\delta_\mathcal{S}$. $\qquad \square$

G. McShane remarked that the existence of a constant $\delta_\mathcal{M} > 0$ which serves as a common upper bound for the distances between the boundary components $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ of the domains $\mathcal{M}_n$, $n \in \mathbb{N}$ does not guarantee that the diameters of $\mathcal{M}_n$ are uniformly bounded from above.

Indeed, J. Brock in his PhD thesis (see also [6]) studied the following example.

Given a pair of homeomorphic Riemann surfaces X and Y of finite type and a "partial pseudo Anosov" mapping class ϕ, by the Ahlfors-Bers simultaneous uniformization theorem there is a sequence of quasi-Fuchsian manifolds $\{Q(\phi^n X, Y)\}_{n=1}^\infty$. The diameters of each the boundary components of the convex hull of $Q(\phi^n X, Y)$ is uniformly bounded in n and so is the distance between the two boundary components but the diameter of the convex hull of $Q(\phi^n X, Y)$ goes to infinity because of a "cusp growing there" as $n \to \infty$.

However, the diameters of the domains $\mathcal{M}_n$, $n \in \mathbb{N}$ do not play a role in the demonstration of Theorem 1.6; only the distances between the surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$, $n \in \mathbb{N}$, are of importance here.

Let us now return to the proof of Theorem 1.6.

Let us fix an arbitrary point $x \in \mathcal{S}$, which is not, however, a point of singularity for the metrics h_∞^+ and h_∞^- on $\mathcal{S}$, and let us denote $x_n^+ \stackrel{\text{def}}{=} f_{\mathcal{S}_n^+}(x) \in \mathcal{S}_n^+ \subset \mathcal{M}_n^\circ$ and $x_n^- \stackrel{\text{def}}{=} f_{\mathcal{S}_n^-}(x) \in \mathcal{S}_n^- \subset \mathcal{M}_n^\circ$, $n \in \mathbb{N}$. Denote also the distance between the points x_n^+ and x_n^- in $\mathcal{M}_n^\circ$ by σ_n^x, $n \in \mathbb{N}$. By Lemma 2.9, $\sigma_n^x < \delta_\mathcal{M}$ for all $n \in \mathbb{N}$.

Let us consider two copies $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$ of the universal covering of the surface $\mathcal{S}$ with the projections $p^+ : \widetilde{\mathcal{S}}^+ \to \mathcal{S}$ and $p^- : \widetilde{\mathcal{S}}^- \to \mathcal{S}$ and let us fix some points

$\tilde{x}^+ \in \widetilde{\mathcal{S}}^+$ and $\tilde{x}^- \in \widetilde{\mathcal{S}}^-$ such that $p^+(\tilde{x}^+) = x$ and $p^-(\tilde{x}^-) = x$. Without loss of generality we may think that the fundamental group $\pi_1(\mathcal{S})$ acts on $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$ in the sense that $\mathcal{S} \simeq \widetilde{\mathcal{S}}^+/\pi_1(\mathcal{S})$ and $\mathcal{S} \simeq \widetilde{\mathcal{S}}^-/\pi_1(\mathcal{S})$. For every $n \in \mathbb{N}$ we fix an arbitrary pair of points $\tilde{x}_n^+ \in \widetilde{\mathcal{S}}_n^+ \subset \widetilde{\mathcal{M}}_n^\circ(= \mathbb{H}^3)$ and $\tilde{x}_n^- \in \widetilde{\mathcal{S}}_n^- \subset \widetilde{\mathcal{M}}_n^\circ$ verifying the conditions $p_n(\tilde{x}_n^+) = x_n^+$ and $p_n(\tilde{x}_n^-) = x_n^-$, and such that the distance in $\mathcal{M}_n^\circ$ between $\tilde{x}_n^+$ and $\tilde{x}_n^-$ is equal to σ_n^x. The functions $f_{\mathcal{S}_n^+} : \mathcal{S} \to \mathcal{S}_n^+$ and $f_{\mathcal{S}_n^-} : \mathcal{S} \to \mathcal{S}_n^-$ defined above induce the canonical bijective developing maps $\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \widetilde{\mathcal{S}}_n^+$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widetilde{\mathcal{S}}^- \to \widetilde{\mathcal{S}}_n^-$ with the properties $\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+) = \tilde{x}_n^+$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\tilde{x}^-) = \tilde{x}_n^-$ and such that for any $\gamma \in \pi_1(\mathcal{S})$ it is true that $\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\gamma.\tilde{x}^+) = \rho_n^{\mathcal{S}}(\gamma).\tilde{x}_n^+$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\gamma.\tilde{x}^-) = \rho_n^{\mathcal{S}}(\gamma).\tilde{x}_n^-$, $n \in \mathbb{N}$.

REMARK 2.10. — The above-mentioned property of developing maps holds for any points $\tilde{y}^+ \in \widetilde{\mathcal{S}}^+$, $\tilde{y}^- \in \widetilde{\mathcal{S}}^-$ and for every $\gamma \in \pi_1(\mathcal{S})$:

$$\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\gamma.\tilde{y}^+) = \rho_n^{\mathcal{S}}(\gamma).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}^+) \quad \text{and} \quad \tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\gamma.\tilde{y}^-) = \rho_n^{\mathcal{S}}(\gamma).\tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\tilde{y}^-), \quad n \in \mathbb{N}.$$

Let the metrics $\tilde{h}_\lambda^+$ and $\tilde{h}_\lambda^-$ on the universal coverings $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$ of the surface $\mathcal{S}$ be the pull-backs of the metrics h_λ^+ and h_λ^- on $\mathcal{S}$ defined in the proof of Lemma 2.8. We are now able to construct the Dirichlet domains $\Delta^+ \subset \widetilde{\mathcal{S}}^+$ and $\Delta^- \subset \widetilde{\mathcal{S}}^-$ of $\mathcal{S}$ with respect to the metrics h_λ^+ and h_λ^- based in the points $\tilde{x}^+ \in \widetilde{\mathcal{S}}^+$ and $\tilde{x}^- \in \widetilde{\mathcal{S}}^-$, respectively. In what follows we will work with the fundamental domains $\Delta^+ \subset \widetilde{\mathcal{S}}^+$ and $\Delta^- \subset \widetilde{\mathcal{S}}^-$ of $\mathcal{S}$.

LEMMA 2.11. — For each $n \in \mathbb{N}$ the domains $\Delta_n^+ \overset{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\Delta^+) \subset \widetilde{\mathcal{S}}_n^+ \subset \mathbb{H}^3$ and $\Delta_n^- \overset{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\Delta^-) \subset \widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ are included in the hyperbolic balls $B(\tilde{x}_n^+, \delta_{\mathcal{S}})$ and $B(\tilde{x}_n^-, \delta_{\mathcal{S}})$ of radius $\delta_{\mathcal{S}}$ centered at the points $\tilde{x}_n^+$ and $\tilde{x}_n^-$ respectively.

Proof. — It suffices to prove this statement for the domain Δ_n^+.

Assume that the surface $\widetilde{\mathcal{S}}^+$ is equipped with the metric $\tilde{h}_\lambda^+$. It follows from the definition of the Dirichlet domain that the distance from any point $x \in \Delta^+ \subset \widetilde{\mathcal{S}}^+$ to the center $\tilde{x}^+$ of Δ^+ is not greater than the diameter of the surface $(\mathcal{S}, h_\lambda^+)$ which is less than or equal to $\delta_{\mathcal{S}}$ (see the proof of Lemma 2.8). Recall that the developing map $\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \widetilde{\mathcal{S}}_n^+$ can be viewed as the identical application from one copy of the surface $\widetilde{\mathcal{S}}^+$ equipped with the metric $\tilde{h}_\lambda^+$ to another copy of $\widetilde{\mathcal{S}}^+$ equipped with the metric $\tilde{h}_n^+$. Also, by the construction made in the proof of Lemma 2.8, all distances on the surface $\mathcal{S}$ measured in the metric h_n^+ do not exceed the corresponding distances on $\mathcal{S}$ in the metric h_λ^+. Hence, this property is valid for the pull-backs $\tilde{h}_n^+$ and $\tilde{h}_\lambda^+$ on $\widetilde{\mathcal{S}}^+$ of the metrics $\tilde{h}_n^+$ and h_λ^+ on $\mathcal{S}$. Therefore, the distance from any point $v \in \Delta_n^+ = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\Delta^+) \subset \widetilde{\mathcal{S}}_n^+$ to the center $\tilde{x}_n^+ = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+)$ of Δ_n^+ is not greater than $\delta_{\mathcal{S}}$.

To complete the proof we remark that for any couple of points $v_1, v_2 \in \widetilde{\mathcal{S}}_n^+$ the distance between them in the hyperbolic metric of 3-space $\mathbb{H}^3$ does not exceed the distance between v_1 and v_2 in the induced metric $\tilde{h}_n^+$ on the 2-surface $\widetilde{\mathcal{S}}_n^+$:
$$\mathrm{d}_{\mathbb{H}^3}(v_1, v_2) \leq \mathrm{d}_{\tilde{h}_n^+}(v_1, v_2). \qquad \square$$

Denote by $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ the union of Δ^+ with all "neighbor" fundamental domains of $\mathcal{S}$ of the form $\gamma.\Delta^+$ for all $\gamma \in \pi_1(\mathcal{S})$ such that $\mathrm{cl}\Delta^+ \cap \mathrm{cl}\gamma.\Delta^+ \neq \emptyset$. Similarly we define the set $\widehat{\Delta}^- \subset \widetilde{\mathcal{S}}^-$.

LEMMA 2.12. — *For each $n \in \mathbb{N}$ the domains $\widehat{\Delta}_n^+ \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\widehat{\Delta}^+) \subset \widetilde{\mathcal{S}}_n^+ \subset \mathbb{H}^3$ and $\widehat{\Delta}_n^- \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_n^-}(\widehat{\Delta}^-) \subset \widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ are included in the hyperbolic balls $B(\tilde{x}_n^+, 3\delta_\mathcal{S})$ and $B(\tilde{x}_n^-, 3\delta_\mathcal{S})$ of radius $3\delta_\mathcal{S}$ centered at the points $\tilde{x}_n^+$ and $\tilde{x}_n^-$ correspondingly.*

Proof. — It suffices to prove this statement for the domain $\widehat{\Delta}_n^+$.

First, by Lemma 2.11, the domain Δ_n^+ is inscribed in the ball $B(\tilde{x}_n^+, \delta_\mathcal{S})$. Similarly, for each $\gamma \in \pi_1(\mathcal{S})$ the domain $\rho_n^\mathcal{S}(\gamma).\Delta_n^+$ (isometric to Δ_n^+) is inscribed in the ball $B(\rho_n^\mathcal{S}(\gamma).\tilde{x}_n^+, \delta_\mathcal{S})$. Note that $\widehat{\Delta}_n^+$ is the union of Δ_n^+ with the domains of the form $\rho_n^\mathcal{S}(\gamma).\Delta_n^+$ such that $\mathrm{cl}\Delta_n^+ \cap \mathrm{cl}\rho_n^\mathcal{S}(\gamma).\Delta_n^+ \neq \emptyset$, where $\gamma \in \pi_1(\mathcal{S})$. Thus, the set $\widehat{\Delta}_n^+$ is contained in the union $\mathcal{U}_B$ of the ball $B(\tilde{x}_n^+, \delta_\mathcal{S})$ and all balls of the type $B(\rho_n^\mathcal{S}(\gamma).\tilde{x}_n^+, \delta_\mathcal{S})$ such that $B(\rho_n^\mathcal{S}(\gamma).\tilde{x}_n^+, \delta_\mathcal{S}) \cap B(\tilde{x}_n^+, \delta_\mathcal{S}) \neq \emptyset$. Clearly, $\mathcal{U}_B$ lies entirely inside the ball $B(\tilde{x}_n^-, 3\delta_\mathcal{S})$. $\square$

The following statement is an immediate corollary of Lemmas 2.9 and 2.12.

LEMMA 2.13. — *For each $n \in \mathbb{N}$ the domains $\widehat{\Delta}_n^+ \stackrel{\text{def}}{=} \tilde{f}_{\widehat{\mathcal{S}}_n^+}(\widehat{\Delta}^+) \subset \widetilde{\mathcal{S}}_n^+ \subset \mathbb{H}^3$ and $\widehat{\Delta}_n^- \stackrel{\text{def}}{=} \tilde{f}_{\widehat{\mathcal{S}}_n^-}(\widehat{\Delta}^-) \subset \widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ are both included in the hyperbolic balls $B(\tilde{x}_n^+, 3\delta_\mathcal{S} + \delta_\mathcal{M})$ and $B(\tilde{x}_n^-, 3\delta_\mathcal{S} + \delta_\mathcal{M})$ of radius $3\delta_\mathcal{S} + \delta_\mathcal{M}$ centered at the points $\tilde{x}_n^+$ and $\tilde{x}_n^-$.*

It is high time to identify the universal coverings $\widetilde{\mathcal{M}}_n^\circ$ (which are copies of $\mathbb{H}^3$) by supposing that the points $\tilde{x}_n^+$ coincide for all $n \in \mathbb{N}$. Let us temporarily forget the 3-dimensional domains $\widetilde{\mathcal{M}}_n$ of hyperbolic space $\mathbb{H}^3$ in order to concentrate our attention on the study of properties of the sequences of surfaces $\{\widetilde{\mathcal{S}}_n^+\}_{n\in\mathbb{N}}$ and $\{\widetilde{\mathcal{S}}_n^-\}_{n\in\mathbb{N}}$.

Recall the statement of the classical Arzelà-Ascoli Theorem.

THEOREM 2.14 (Theorem 7.5.7 in [8], p. 137). — *Suppose F is a Banach space and E a compact metric space. In order that a subset H of the Banach space $\mathcal{C}_F(E)$ of continuous functions from E to F be relatively compact, necessary and sufficient conditions are that H be equicontinuous and that, for each $x \in E$ the set H_x of all $f(x)$ such that $f \in H$ be relatively compact in F.*

We will apply it in the following

LEMMA 2.15. — *There exist subsequences of functions* $\{\tilde{f}_{\widetilde{\mathcal{S}}_{n_k}^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{k\in\mathbb{N}}$
and $\{\tilde{f}_{\widetilde{\mathcal{S}}_{n_k}^-} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ *that converge to continuous functions* $\tilde{f}_{\widetilde{\mathcal{S}}_{\infty}^+} : \widehat{\Delta}^+ \to$
$\mathbb{H}^3$ *and* $\tilde{f}_{\widetilde{\mathcal{S}}_{\infty}^-} : \widehat{\Delta}^- \to \mathbb{H}^3$ *correspondingly.*

Proof. — It suffices to find a converging subsequence of the sequence of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$. To this purpose we will apply the Arzelà-Ascoli Theorem 2.14.

Let us equip the domain $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ with the restriction $\tilde{h}_\lambda^+|_{\widehat{\Delta}^+}$ of the metric $\tilde{h}_\lambda^+$. Consider the domain $(\widehat{\Delta}^+, \tilde{h}_\lambda^+|_{\widehat{\Delta}^+})$ as a compact metric space E from the statement of Theorem 2.14; hyperbolic space $\mathbb{H}^3$ as a Banach space F; the sequence of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ in the space of continuous functions from $(\widehat{\Delta}^+, \tilde{h}_\lambda^+|_{\widehat{\Delta}^+})$ to $\mathbb{H}^3$ as the set $H \subset \mathcal{C}_F(E)$.

By Lemma 2.13, the images $\widehat{\Delta}_n^+ = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\widehat{\Delta}^+) \subset \widetilde{\mathcal{S}}_n^+ \subset \mathbb{H}^3$ of the maps $\tilde{f}_{\widetilde{\mathcal{S}}_n^+}$, $n \in \mathbb{N}$, are all included in the ball $B(\tilde{x}_n^+, 3\delta_{\mathcal{S}} + \delta_{\mathcal{M}})$ (recall that we identified all points $\tilde{x}_n^+ \in \mathbb{H}^3$, $n \in \mathbb{N}$). Thus, for each $x \in E$ the set H_x is relatively compact in F.

As it was already done in the proof of Lemma 2.11, we consider every developing map $\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \widetilde{\mathcal{S}}_n^+$ as the inclusion of the domain $\widehat{\Delta}^+$ equipped with the metric $\tilde{h}_\lambda^+|_{\widehat{\Delta}^+}$ to the surface $\widetilde{\mathcal{S}}^+$ with the metric $\tilde{h}_n^+$, $n \in \mathbb{N}$. So, for any $\varepsilon > 0$ if we pose $\delta := \varepsilon$ then for every pair of points $x, y \in \widehat{\Delta}^+$ such that $\mathrm{d}_{\tilde{h}_\lambda^+}(x, y) < \delta$ it is true that $\mathrm{d}_{\mathbb{H}^3}(\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(x), \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(y)) \leq \mathrm{d}_{\tilde{h}_n^+}(\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(x), \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(y)) < \varepsilon$ (recall that, by construction, distances measured in the metric $\tilde{h}_\lambda^+$ are not smaller than the corresponding distances measured in the metric $\tilde{h}_n^+$), $n \in \mathbb{N}$. Thus, the functions $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ are equicontinuous.

Therefore, by the Arzelà-Ascoli Theorem 2.14, there exists a subsequence of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}_{n_k}^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ that converges to some continuous function $\tilde{f}_{\widetilde{\mathcal{S}}_{\infty}^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$. Similarly we obtain that there exists a subsequence of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}_{n_k}^-} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ that converges to some continuous function $\tilde{f}_{\widetilde{\mathcal{S}}_{\infty}^-} : \widehat{\Delta}^- \to \mathbb{H}^3$. $\qquad\square$

ASSUMPTION 1. — Further we assume that the sequences of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}_{\infty}^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_{\infty}^-} : \widehat{\Delta}^- \to \mathbb{H}^3$.

2.3. Convergence of the holonomy representations $\{\rho_n^{\mathcal{S}}\}_{n\in\mathbb{N}}$ and of the developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$. —

Now we need to derive several properties of the holonomy representations $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$, $n \in \mathbb{N}$.

LEMMA 2.16. — *Given two points $y^1, y^2 \in \mathbb{H}^3$ together with orthogonal bases $\{e^1, e^2, e^3\}$ and $\{\hat{e}^1, \hat{e}^2, \hat{e}^3\}$ of the tangent spaces $T_{y^1}\mathbb{H}^3$ and $T_{y^2}\mathbb{H}^3$, there is a unique isometry $\vartheta \in \mathcal{I}(\mathbb{H}^3)$ such that $y^2 = \vartheta.y^1$ and $\hat{e}^i = d_{y^1}\vartheta(e^i)$, $i = 1, \ldots, 3$.*

Proof. — Following Chapter 1, § 1.5 in [1, p. 13] let us recall the construction of the hyperboloid model $\mathbb{I}^3$ of hyperbolic space $\mathbb{H}^3$. Denoting the coordinates in space $\mathbb{R}^4$ by x_0, x_1, x_2, x_3, we introduce the Minkowski scalar product in $\mathbb{R}^4$ by the formula

$$(2.5) \qquad (x, y)_M = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3,$$

which turns $\mathbb{R}^4$ into a pseudo-Euclidean vector space, denoted by $\mathbb{R}^{3,1}$.

A basis $\{u^0, u^1, u^2, u^3\} \subset \mathbb{R}^{3,1}$ is said to be *orthonormal* if $(u^0, u^0)_M = -1$, $(u^i, u^i)_M = 1$ for $i \neq 0$, and $(u^i, u^j)_M = 0$ for $i \neq j$. For example, the standard basis

$$(2.6) \qquad \{\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^{3,1}$$

is orthonormal.

Each pseudo-orthogonal (i.e., preserving the above scalar product) transformation of $\mathbb{R}^{3,1}$ takes an open cone of time-like vectors

$$\mathfrak{C} = \{x \in \mathbb{R}^{3,1} : (x, x)_M < 0\}$$

consisting of two connected components

$$\mathfrak{C}^+ = \{x \in \mathfrak{C} : x_0 > 0\}, \quad \mathfrak{C}^- = \{x \in \mathfrak{C} : x_0 < 0\}$$

onto itself. Denote by $O(3, 1)$ the group of all pseudo-orthogonal transformations of space $\mathbb{R}^{3,1}$, and by $O'(3, 1)$ its subgroup of index 2 consisting of those pseudo orthogonal transformations which map each connected component of the cone $\mathfrak{C}$ onto itself.

Using notation developed in § A.1 [5, p. 1] we remind that the manifold

$$\mathbb{I}^3 = \{x \in \mathbb{R}^{3,1} : (x, x)_M = -1, x_0 > 0\}$$

with the metric induced by the pseudo-Euclidean metric (2.5) is called the hyperboloid model $\mathbb{I}^3$ of hyperbolic space $\mathbb{H}^3$, and the restrictions of the elements of $O'(3, 1)$ on $\mathbb{I}^3$ form the group $\mathcal{I}(\mathbb{H}^3)$ of all isometries of $\mathbb{H}^3$.

Again, by Chapter 1, § 1.5 in [1, p. 13], for any $x \in \mathbb{I}^3$ we can naturally identify the tangent space $T_x \mathbb{I}^3$ with the orthogonal complement of the vector x in space $\mathbb{R}^{3,1}$, which is a 3-dimensional Euclidean space (with respect to the same scalar product). If $\{u^1, u^2, u^3\}$ is an orthonormal basis in it, then $\{x, u^1, u^2, u^3\}$ is an orthonormal basis in the space $\mathbb{R}^{3,1}$.

Obviously, the vector ϵ^0 of the standard basis (2.6) $\mathbb{R}^{3,1}$ lies in $\mathbb{I}^3$ and the vectors $\{\epsilon^1, \epsilon^2, \epsilon^3\}$ defined in (2.6) form an orthonormal basis of the tangent space $T_{\epsilon^0}\mathbb{I}^3$. Also, according to a fact mentioned in the previous paragraph, the sets of four vectors $\{y^1, e^1, e^2, e^3\} \subset \mathbb{R}^{3,1}$ and $\{y^2, \hat{e}^1, \hat{e}^2, \hat{e}^3\} \subset \mathbb{R}^{3,1}$ from

the statement of Lemma 2.16 are orthonormal bases of $\mathbb{R}^{3,1}$. Define the linear transformations ϑ_1 and ϑ_2 of $\mathbb{R}^{3,1}$ determined by their 4×4-real matrices $M_1^\vartheta \overset{\text{def}}{=} (y^1, e^1, e^2, e^3)$ and $M_2^\vartheta \overset{\text{def}}{=} (y^2, \hat{e}^1, \hat{e}^2, \hat{e}^3)$ with the columns consisting of the coordinates of the corresponding vectors in the standard basis of $\mathbb{R}^{3,1}$. A direct calculation shows the transformations ϑ_1 and ϑ_2 send the standard base to the orthonormal bases $\{y^1, e^1, e^2, e^3\}$ and $\{y^2, \hat{e}^1, \hat{e}^2, \hat{e}^3\}$ of $\mathbb{R}^{3,1}$, respectively. Moreover, we know that the vectors ϵ^0, y^1, and y^2 belong to the upper cone $\mathfrak{C}^+$. Hence, ϑ_1 and ϑ_2 are elements of the group $O'(3,1)$, and we can take the transformation ϑ from the statement of Lemma 2.16 to be equal to $\vartheta_2[\vartheta_1]^{-1}$. $\qquad \square$

DEFINITION 2.4. — Given a sequence of hyperbolic isometries $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n \in \mathbb{N}}$ determined by points $y_n^1, y_n^2 \in \mathbb{H}^3$ and orthogonal bases $\{e_n^1, e_n^2, e_n^3\}$, $\{\hat{e}_n^1, \hat{e}_n^2, \hat{e}_n^3\}$ of the tangent spaces $T_{y_n^1}\mathbb{H}^3$ and $T_{y_n^2}\mathbb{H}^3$, we say that the isometries $\{\vartheta_n\}_{n \in \mathbb{N}}$ *converge* to an isometry $\vartheta_\infty \in \mathcal{I}(\mathbb{H}^3)$ *in the sense of Lemma* 2.16 if the sequences of base points $\{y_n^1\}_{n \in \mathbb{N}}$, $\{y_n^2\}_{n \in \mathbb{N}}$ converge to points $y_\infty^1, y_\infty^2 \in \mathbb{H}^3$ and the sequences of orthogonal bases $\{e_n^1, e_n^2, e_n^3\}_{n \in \mathbb{N}}$, $\{\hat{e}_n^1, \hat{e}_n^2, \hat{e}_n^3\}_{n \in \mathbb{N}}$ converge to orthogonal bases $\{e_\infty^1, e_\infty^2, e_\infty^3\}$, $\{\hat{e}_\infty^1, \hat{e}_\infty^2, \hat{e}_\infty^3\}$ of the tangent spaces $T_{y_\infty^1}\mathbb{H}^3$ and $T_{y_\infty^2}\mathbb{H}^3$, and the above-mentioned limits define uniquely the isometry ϑ_∞. Denote a convergence of isometries in the sense of Lemma 2.16 by $\vartheta_n \Rightarrow \vartheta_\infty$ as $n \to \infty$.

DEFINITION 2.5. — We say that hyperbolic isometries $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n \in \mathbb{N}}$ *converge* to an isometry $\vartheta_\infty \in \mathcal{I}(\mathbb{H}^3)$ *in a "weak" sense* if for any point $y \in \mathbb{H}^3$ the sequence $\{\vartheta_n.y\}_{n \in \mathbb{N}}$ converges to the point $\vartheta_\infty.y \in \mathbb{H}^3$ as $n \to \infty$. Denote a "weak" convergence of isometries by $\vartheta_n \xrightarrow[n \to \infty]{} \vartheta_\infty$.

LEMMA 2.17. — *Given a collection of hyperbolic isometries* $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n=1}^\infty$, $\vartheta_n \Rightarrow \vartheta_\infty$ *as* $n \to \infty$ *if and only if* $\vartheta_n \xrightarrow[n \to \infty]{} \vartheta_\infty$.

Proof. — A hyperbolic isometry $\vartheta : \mathbb{H}^3 \to \mathbb{H}^3$ which sends any $y \in \mathbb{H}^3$ to the point $\vartheta.y \in \mathbb{H}^3$ can be interpreted as a linear transformation of Minkowski space $\mathbb{R}^{3,1}$ as it was mentioned in the proof of Lemma 2.16. Therefore, $\vartheta(y)$ depends continuously on $y \in \mathbb{H}^3$.

Suppose that $\vartheta_n \Rightarrow \vartheta_\infty$ as $n \to \infty$. By construction, a transformation $\vartheta \in \mathcal{I}(\mathbb{H}^3)$ from Lemma 2.16 depends continuously on the parameters $y^1, y^2 \in \mathbb{H}^3$, $\{e^1, e^2, e^3\} \subset T_{y^1}\mathbb{H}^3$, and $\{\hat{e}^1, \hat{e}^2, \hat{e}^3\} \subset T_{y^2}\mathbb{H}^3$. Hence, for any point $y \in \mathbb{H}^3$ the sequence $\{\vartheta_n.y\}_{n \in \mathbb{N}}$ converges to the point $\vartheta_\infty.y \in \mathbb{H}^3$ as $n \to \infty$, which means that the convergence of the isometries $\{\vartheta_n\}_{n \in \mathbb{N}}$ in the sense of Lemma 2.16 implies also the "weak" convergence of these isometries to ϑ_∞.

Suppose now that $\vartheta_n \xrightarrow[n \to \infty]{} \vartheta_\infty$. Being a linear transformation of Minkowski space $\mathbb{R}^{3,1}$, the hyperbolic isometries $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n=1}^\infty$ are represented in the

standard basis of $\mathbb{R}^{3,1}$ by the 4×4-real matrices $M^{\vartheta_n} \stackrel{\text{def}}{=} (\vartheta_n^0, \vartheta_n^1, \vartheta_n^2, \vartheta_n^3)$, where ϑ_n^k, $k = 0, 1, 2, 3$, are the columns of M_n^{ϑ}.

Let $P_0 \stackrel{\text{def}}{=} (1, 0, 0, 0)^T \in \mathbb{I}^3 \subset \mathbb{R}^{3,1}$. The "weak" convergence of the isometries $\{\vartheta_n\}_{n \in \mathbb{N}}$ at the point P_0 means that $M^{\vartheta_n}.P_0 \xrightarrow[n \to \infty]{} M^{\vartheta_\infty}.P_0$, i.e.,

$$(2.7) \qquad \vartheta_n^0 \xrightarrow[n \to \infty]{} \vartheta_\infty^0.$$

Let $P_1 \stackrel{\text{def}}{=} (\sqrt{2}, 1, 0, 0)^T \in \mathbb{I}^3 \subset \mathbb{R}^{3,1}$. The "weak" convergence of the isometries $\{\vartheta_n\}_{n \in \mathbb{N}}$ at the point P_1 means that $M^{\vartheta_n}.P_1 \xrightarrow[n \to \infty]{} M^{\vartheta_\infty}.P_1$, i.e., $\sqrt{2}\vartheta_n^0 + \vartheta_n^1 \xrightarrow[n \to \infty]{} \sqrt{2}\vartheta_\infty^0 + \vartheta_\infty^1$. Taking into account (2.7), we obtain that $\vartheta_n^1 \xrightarrow[n \to \infty]{} \vartheta_\infty^1$. Similarly we get that $\vartheta_n^2 \xrightarrow[n \to \infty]{} \vartheta_\infty^2$ and $\vartheta_n^3 \xrightarrow[n \to \infty]{} \vartheta_\infty^3$. Thus, the "weak" convergence of the isometries $\{\vartheta_n\}_{n \in \mathbb{N}}$ to ϑ_∞ as $n \to \infty$ implies also their convergence in the sense of Lemma 2.16. $\qquad \square$

LEMMA 2.18. — *For each $n \in \mathbb{N}$ let a pair of surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^- \subset \mathbb{H}^3$ (which are the images of developing maps $\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \mathcal{S}^+ \to \widetilde{\mathcal{S}}_n^+$ and $\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \mathcal{S}^- \to \widetilde{\mathcal{S}}_n^-$) be invariant under the actions of a quasi-Fuchsian group $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$. Suppose in addition that the restrictions of the developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n \in \mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{n \in \mathbb{N}}$ on the domains $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ and $\widehat{\Delta}^- \subset \widetilde{\mathcal{S}}^-$ defined in Section 2.2 converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widehat{\Delta}^- \to \mathbb{H}^3$. Then there is a sequence of positive integers $n_k \xrightarrow[k \to \infty]{} \infty$ such that the morphisms $\{\rho_{n_k}^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)\}_{k \in \mathbb{N}}$ converge to a morphism $\rho_\infty^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)$ in the sense of Lemma 2.16, i.e., for every $\gamma \in \pi_1(\mathcal{S})$ there exists a hyperbolic isometry which we denote by $\rho_\infty^{\mathcal{S}}(\gamma)$ such that $\rho_{n_k}^{\mathcal{S}}(\gamma) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma)$ as $k \to \infty$.*

Proof. — First, we prove that there is a sequence of positive integers $n_k \xrightarrow[k \to \infty]{} \infty$ such that for any generator γ_i of the group $\pi_1(\mathcal{S})$ together with its inverse element $\gamma_i^{-1} \in \pi_1(\mathcal{S})$, $i = 1, \ldots, l$, the subsequences of isometries $\rho_{n_k}^{\mathcal{S}}(\gamma_i) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma_i)$ and $\rho_{n_k}^{\mathcal{S}}(\gamma_i^{-1}) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma_i^{-1})$ converge as $k \to \infty$.

Indeed, since for any $i = 1, \ldots, l$ points $\tilde{x}^+$, $\gamma_i.\tilde{x}^+$, and $\gamma_i^{-1}.\tilde{x}^+$ lie inside $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ by construction, and because of convergence of the developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n \in \mathbb{N}}$ to a continuous function $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$, we know that the sequences of points $\tilde{x}_n^+ (= \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+)) \xrightarrow[n \to \infty]{} \tilde{x}_\infty^+ (= \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{x}^+))$, $\rho_n^{\mathcal{S}}(\gamma_i).\tilde{x}_n^+ (= \rho_n^{\mathcal{S}}(\gamma_i).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+) = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\gamma_i.\tilde{x}^+)) \xrightarrow[n \to \infty]{} \rho_\infty^{\mathcal{S}}(\gamma_i).\tilde{x}_\infty^+ (= \rho_\infty^{\mathcal{S}}(\gamma_i).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{x}^+) = \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\gamma_i.\tilde{x}^+))$, and $[\rho_n^{\mathcal{S}}(\gamma_i)]^{-1}.\tilde{x}_n^+ (= \rho_n^{\mathcal{S}}(\gamma_i^{-1}).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{x}^+) = \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\gamma_i^{-1}.\tilde{x}^+)) \xrightarrow[n \to \infty]{} [\rho_\infty^{\mathcal{S}}(\gamma_i)]^{-1}.\tilde{x}_\infty^+ (= \rho_\infty^{\mathcal{S}}(\gamma_i^{-1}).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{x}^+) = \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\gamma_i^{-1}.\tilde{x}^+))$ converge in $\mathbb{H}^3$.

Also we know that for each $n \in \mathbb{N}$ and for every $i = 1, \ldots, l$, the differential $d_{\tilde{x}_n^+} \rho_n^{\mathcal{S}}(\gamma_i)$ sends an orthonormal base $\{e_1^{n,i}, e_2^{n,i}, e_3^{n,i}\}$ of the tangent space $T_{\tilde{x}_n^+} \mathbb{H}^3$ to an orthonormal base $\{\hat{e}_1^{n,i}, \hat{e}_2^{n,i}, \hat{e}_3^{n,i}\}$ of $T_{\rho_n^{\mathcal{S}}(\gamma_i).\tilde{x}_n^+} \mathbb{H}^3$ (recall that, by constructions all the points $\tilde{x}_n^+$, $n \in \mathbb{N}$ coincide). Since the subsequences $\{e_j^{n,i}\}_{n\in\mathbb{N}}$, $\{\hat{e}_j^{n,i}\}_{n\in\mathbb{N}}$, $j = 1, 2, 3$, $i = 1, \ldots, l$, of unitary vectors are bounded, there exists a sequence of positive integers $n_k \xrightarrow{k\to\infty} \infty$ such that the pairs of subsequences of orthonormal bases $\{e_1^{n_k,i}, e_2^{n_k,i}, e_3^{n_k,i}\}_{k\in\mathbb{N}}$ and $\{\hat{e}_1^{n_k,i}, \hat{e}_2^{n_k,i}, \hat{e}_3^{n_k,i}\}_{k\in\mathbb{N}}$ converge all together ($i = 1, \ldots, l$) ensemble to orthonormal bases $\{e_1^{\infty,i}, e_2^{\infty,i}, e_3^{\infty,i}\}$ and $\{\hat{e}_1^{\infty,i}, \hat{e}_2^{\infty,i}, \hat{e}_3^{\infty,i}\}$. Hence, by Lemma 2.16, there exists a hyperbolic isometry that we denote by $\rho_\infty^{\mathcal{S}}(\gamma_i)$ which sends the point $\tilde{x}_\infty^+$ to the point $\rho_\infty^{\mathcal{S}}(\gamma_i).\tilde{x}_\infty^+$ defined above, and which differential $d_{\tilde{x}_\infty^+} \rho_\infty^{\mathcal{S}}(\gamma_i)$ sends an orthonormal base $\{e_1^{\infty,i}, e_2^{\infty,i}, e_3^{\infty,i}\}$ of the tangent space $T_{\tilde{x}_\infty^+} \mathbb{H}^3$ to an orthonormal base $\{\hat{e}_1^{\infty,i}, \hat{e}_2^{\infty,i}, \hat{e}_3^{\infty,i}\}$ of $T_{\rho_\infty^{\mathcal{S}}(\gamma_i).\tilde{x}_\infty^+} \mathbb{H}^3$ such that $\rho_{n_k}^{\mathcal{S}}(\gamma_i) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma_i)$ as $k \to \infty$.

Secondly, we derive that for any element $\gamma \in \pi_1(\mathcal{S})$ the subsequences of isometries $\rho_{n_k}^{\mathcal{S}}(\gamma) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma)$ converges as $k \to \infty$. Indeed, every $\gamma \in \pi_1(\mathcal{S})$ can be decomposed in a product of generators of $\pi_1(\mathcal{S})$ together with their inverse elements, for which the demanded convergence has already been shown. $\square$

ASSUMPTION 2. — Further we assume that the sequence of holonomy representations $\{\rho_n^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)\}_{n\in\mathbb{N}}$ (where the groups $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$ are quasi-Fuchsian) converges to a holonomy representation $\rho_\infty^{\mathcal{S}} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)$ (where $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ is a discrete group of isometries of $\mathbb{H}^3$) in the sense of Lemma 2.16 as $n \to \infty$.

Let us now prove the following property of the functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widehat{\Delta}^- \to \mathbb{H}^3$ with respect to the group of isometries $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of space $\mathbb{H}^3$.

REMARK 2.19. — If for a pair of points $\tilde{y}_1^+, \tilde{y}_2^+ \in \widehat{\Delta}^+$ there exists a transformation $\gamma^+ \in \pi_1(\mathcal{S})$ such that $\tilde{y}_2^+ = \gamma^+.\tilde{y}_1^+$, then the following equality holds:

$$(2.8) \qquad \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_2^+) = \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+).$$

Similarly, if for a pair of points $\tilde{y}_1^-, \tilde{y}_2^- \in \widehat{\Delta}^-$ there exists a transformation $\gamma^- \in \pi_1(\mathcal{S})$ such that $\tilde{y}_2^- = \gamma^-.\tilde{y}_1^-$, then

$$\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\tilde{y}_2^-) = \rho_\infty^{\mathcal{S}}(\gamma^-).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\tilde{y}_1^-).$$

Proof. —   It suffices to prove the formula (2.8).

By Remark 2.10, the relation

$$(2.9) \qquad \tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_2^+) = \rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+)$$

is valid for all $n \in \mathbb{N}$.

By Assumption 1, the sequence $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_2^+)\}_{n\in\mathbb{N}} \subset \mathbb{H}^3$ converges to the point $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_2^+) \in \mathbb{H}^3$. Hence, taking into account the formula (2.9) we see that in order to prove the equality (2.8) we need to demonstrate the convergence of the sequence $\{\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+)\}_{n\in\mathbb{N}} \subset \mathbb{H}^3$ to the point $\rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)$, i.e., fixing $\varepsilon > 0$, we ought to find such $n_0 \in \mathbb{N}$ that

(2.10)
$$\forall n > n_0 \quad \text{the inequality} \quad \mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) < \varepsilon \quad \text{holds.}$$

First, by the above-mentioned Assumption 1, the sequence $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+)\}_{n\in\mathbb{N}} \subset \mathbb{H}^3$ converges to the point $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+) \in \mathbb{H}^3$. Therefore,

(2.11)
$$\exists n_1 \in \mathbb{N} : \forall n > n_1 \quad \text{the inequality} \quad \mathrm{d}_{\mathbb{H}^3}(\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) < \frac{\varepsilon}{2} \quad \text{is valid.}$$

Also, by Assumption 2, $\rho_n^{\mathcal{S}}(\gamma^+) \Rightarrow \rho_\infty^{\mathcal{S}}(\gamma^+)$ as $n \to \infty$. Hence, by Lemma 2.17, the sequence of points $\{\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)\}_{n\in\mathbb{N}} \subset \mathbb{H}^3$ converges to the point $\rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+) \in \mathbb{H}^3$, i.e.,

(2.12) $\exists n_2 \in \mathbb{N} : \forall n > n_2$ the inequality

$$\mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) < \frac{\varepsilon}{2} \quad \text{is true.}$$

Applying the triangle inequality, we get:

$$\mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+))$$

(2.13)
$$\leq \mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) + \mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)).$$

The fact that $\rho_n^{\mathcal{S}}(\gamma^+)$ is an isometry of $\mathbb{H}^3$ implies the equality:

(2.14) $\mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) = \mathrm{d}_{\mathbb{H}^3}(\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)).$

Therefore, substituting (2.14) in (2.13), we obtain:

$$\mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) \leq$$

(2.15) $\mathrm{d}_{\mathbb{H}^3}(\tilde{f}_{\widetilde{\mathcal{S}}_n^+}(\tilde{y}_1^+), \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)) + \mathrm{d}_{\mathbb{H}^3}(\rho_n^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+), \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_1^+)).$

Hence, by (2.15), (2.11), and (2.12), we conclude that it is sufficient to pose $n_0 = \max(n_1, n_2)$ to satisfy the condition (2.10). $\square$

Now we are able to extend the functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widehat{\Delta}^- \to \mathbb{H}^3$ to the whole domains $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$. Let us do it as follows: for arbitrary points $\tilde{y}^+ \in \widetilde{\mathcal{S}}^+$ and $\tilde{y}^- \in \widetilde{\mathcal{S}}^-$ we find such points $\tilde{y}_\Delta^+$ and $\tilde{y}_\Delta^-$ in the fundamental domains $\Delta^+ \subset \widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ and $\Delta^- \subset \widehat{\Delta}^- \subset \widetilde{\mathcal{S}}^-$ of the surface $\mathcal{S}$ and such elements $\gamma^+, \gamma^- \in \pi_1(\mathcal{S})$ that $\tilde{y}^+ = \gamma^+.\tilde{y}_\Delta^+$ and $\tilde{y}^- = \gamma^-.\tilde{y}_\Delta^-$, then we define

$\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}^+) \stackrel{\text{def}}{=} \rho_\infty^{\mathcal{S}}(\gamma^+).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\tilde{y}_\Delta^+)$ and $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\tilde{y}^-) \stackrel{\text{def}}{=} \rho_\infty^{\mathcal{S}}(\gamma^-).\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\tilde{y}_\Delta^-)$. By construction, the surfaces $\widetilde{\mathcal{S}}_\infty^+ \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}(\widetilde{\mathcal{S}}^+)$ and $\widetilde{\mathcal{S}}_\infty^- \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}_\infty^-}(\widetilde{\mathcal{S}}^-)$ are invariant under the actions of the group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$.

Repeating almost literally the demonstration of Remark 2.19, we can prove

LEMMA 2.20. — *The sequences of developing maps* $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ *and* $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ *converge to continuous functions* $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3$ *and* $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^-} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3$.

Finally, we show

REMARK 2.21. — The boundaries at infinity $\partial_\infty \widetilde{\mathcal{S}}_\infty^+ \subset \partial_\infty \mathbb{H}^3$ and $\partial_\infty \widetilde{\mathcal{S}}_\infty^- \subset \partial_\infty \mathbb{H}^3$ of the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$ coincide with the limit set $\Lambda_{\rho_\infty^{\mathcal{S}}}$ of the group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$. Moreover, the group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$ from Lemma 2.18 is quasi-Fuchsian.

Proof. — By Lemma 2.20, the sequences of surfaces $\{\widetilde{\mathcal{S}}_n^+\}_{n\in\mathbb{N}}$ and $\{\widetilde{\mathcal{S}}_n^-\}_{n\in\mathbb{N}}$ bounding the convex connected hyperbolic domains $\{\widetilde{\mathcal{M}}_n\}_{n\in\mathbb{N}}$ converge to the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$ in $\mathbb{H}^3$. Hence, the sets $\{\widetilde{\mathcal{M}}_n\}_{n\in\mathbb{N}}$ converge to a convex connected hyperbolic domain $\widetilde{\mathcal{M}}_\infty$. Moreover, the boundaries at infinity $\{\partial_\infty \widetilde{\mathcal{S}}_n^+\}_{n\in\mathbb{N}}$ and $\{\partial_\infty \widetilde{\mathcal{S}}_n^-\}_{n\in\mathbb{N}}$ converge to the curves $\partial_\infty \widetilde{\mathcal{S}}_\infty^+ \subset \partial_\infty \mathbb{H}^3$ and $\partial_\infty \widetilde{\mathcal{S}}_\infty^- \subset \partial_\infty \mathbb{H}^3$. Indeed, our surfaces in the Poincaré disk model of $\mathbb{H}^3$ considered as Euclidean surfaces inside a unitary ball converge together with their boundaries.

Recall that, by the Labourie-Schlenker Theorem 1.9, for each $n \in \mathbb{N}$ the curves $\partial_\infty \widetilde{\mathcal{S}}_n^+$ and $\partial_\infty \widetilde{\mathcal{S}}_n^-$ coincide with the limit set $\Lambda_{\rho_n^{\mathcal{S}}}$ of the quasi-Fuchsian holonomy representations $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S}))$ which is homotopic to a circle in $\partial_\infty \mathbb{H}^3$. On the other hand, by Assumption 2, $\rho_n^{\mathcal{S}}(\pi_1(\mathcal{S})) \Rightarrow \rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ as $n \to \infty$, which implies that the sequence of the limit sets $\{\Lambda_{\rho_n^{\mathcal{S}}}\}_{n\in\mathbb{N}}$ converges to the limit set $\Lambda_{\rho_\infty^{\mathcal{S}}}$ (see, for instance, [14, p. 323]).

Thus, the boundaries at infinity $\partial_\infty \widetilde{\mathcal{S}}_\infty^+$ and $\partial_\infty \widetilde{\mathcal{S}}_\infty^-$ of the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$ coincide with the limit set $\Lambda_{\rho_\infty^{\mathcal{S}}}$ of the group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$. Furthermore, we conclude that the boundary $\partial \widetilde{\mathcal{M}}_\infty$ of the domain $\widetilde{\mathcal{M}}_\infty$ consists of the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$, and the boundary at infinity $\partial_\infty \widetilde{\mathcal{M}}_\infty$ of $\widetilde{\mathcal{M}}_\infty$ also coincides with $\Lambda_{\rho_\infty^{\mathcal{S}}}$.

Since the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$ are topological disks embedded in $\mathbb{H}^3$, their common boundary at infinity is homotopic to a circle. Therefore, by definition, the group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ is quasi-Fuchsian. $\square$

Note that the domain $\widetilde{\mathcal{M}}_\infty$ which appeared during the demonstration of Remark 2.21, is invariant under the actions of the quasi-Fuchsian group $\rho_\infty^{\mathcal{S}}(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$.

2.4. Adaptation of a classical theorem of A. D. Alexandrov to the hyperbolic case

Recall a classical result due to A. D. Alexandrov:

THEOREM 2.22. — *If a sequence of closed convex surfaces $\mathcal{F}_n$ converges to a closed convex surface $\mathcal{F}$ and if two sequences of points X_n and Y_n on $\mathcal{F}_n$ converge to two points X and Y of $\mathcal{F}$, respectively, then the distances between the points X_n and Y_n measured on the surfaces $\mathcal{F}_n$ converge to the distance between the points X and Y measured on $\mathcal{F}$, i.e., $\mathrm{d}_{\mathcal{F}}(X,Y) = \lim_{n\to\infty} \mathrm{d}_{\mathcal{F}_n}(X_n, Y_n)$.*

For the proof of Theorem 2.22 in Euclidean 3-space see Theorem 1 in Sec. 1 of Chapter III [3]. An adaptation of this proof of Theorem 2.22 to the hyperbolic case is given in [23] (see also [24]). Another proof of Theorem 2.22 in hyperbolic space $\mathbb{H}^3$ is given by A. D. Alexandrov in his paper [2, Theorem 3] (in Russian). We will largely use this result in Section 2.5.

2.5. Induced metrics of the surfaces $\widetilde{\mathcal{S}}_\infty^+$ and $\widetilde{\mathcal{S}}_\infty^-$.
— Return to consideration of the family of convex domains $\{\widetilde{\mathcal{M}}_n\}_{n=1}^\infty$ with the boundaries $\partial\widetilde{\mathcal{M}}_n = \widetilde{\mathcal{S}}_n^+ \cup \widetilde{\mathcal{S}}_n^-$ (see Sections 2.2 and 2.3) in hyperbolic space $\mathbb{H}^3$. Assume in addition that the marked points $\tilde{x}_n^+ \in \widetilde{\mathcal{S}}_n^+$, $n = 1, \ldots, \infty$, are all identified with an arbitrary point $O_\mathbb{H} \in \mathbb{K}^3$.

Consider a ball $\hat{\mathcal{B}} \subset \mathbb{H}^3$ centered at $O_\mathbb{H}$ of a sufficiently big hyperbolic radius $\hat{\rho}$ (it will be enough to put $\hat{\rho} = 9\delta_{\mathcal{S}} + \delta_{\mathcal{M}}$, where the constants $\delta_{\mathcal{S}}$ and $\delta_{\mathcal{M}}$ are defined in Lemmas 2.8 and 2.9). Define the convex compact hyperbolic sets $\mathcal{M}_n^{\mathcal{B}} \overset{\text{def}}{=} \widetilde{\mathcal{M}}_n \cap \hat{\mathcal{B}}$, and denote by $\hat{\mathcal{S}}_n^+ \overset{\text{def}}{=} \partial\mathcal{M}_n^{\mathcal{B}} \cap \widetilde{\mathcal{S}}_n^+$ and $\hat{\mathcal{S}}_n^- \overset{\text{def}}{=} \partial\mathcal{M}_n^{\mathcal{B}} \cap \widetilde{\mathcal{S}}_n^-$ the intersections of the boundary $\partial\mathcal{M}_n^{\mathcal{B}}$ of the domain $\mathcal{M}_n^{\mathcal{B}}$ with the surfaces $\widetilde{\mathcal{S}}_n^+$ and $\widetilde{\mathcal{S}}_n^-$, $n = 1, \ldots, \infty$. By construction, the sets $\widehat{\Delta}_n^+$ and $\widehat{\Delta}_n^-$ defined in Lemma 2.12 are subsets of $\hat{\mathcal{S}}_n^+$ and $\hat{\mathcal{S}}_n^-$ correspondingly, $n = 1, \ldots, \infty$.

REMARK 2.23. — The ball $\hat{\mathcal{B}}$ is taken big enough in order to provide the following property: for an arbitrary pair of points $A^+, B^+ \in \widehat{\Delta}_n^+$ there exists a path $\zeta^+ \subset \widehat{\Delta}_n^+$ connecting A^+ and B^+ which is shorter than any path $\xi^+ \subset \partial\mathcal{M}_n^{\mathcal{B}}$ connecting A^+ and B^+ and such that $\xi^+ \cap (\partial\mathcal{M}_n^{\mathcal{B}} \setminus \hat{\mathcal{S}}_n^+) \neq \emptyset$. Similarly, for points $A^-, B^- \in \widehat{\Delta}_n^-$ there exists a path $\zeta^- \subset \widehat{\Delta}_n^-$ connecting A^- and B^- which is shorter than any path $\xi^- \subset \partial\mathcal{M}_n^{\mathcal{B}}$ connecting A^- and B^- and such that $\xi^- \cap (\partial\mathcal{M}_n^{\mathcal{B}} \setminus \hat{\mathcal{S}}_n^-) \neq \emptyset$. For this purpose, radius $\hat{\rho} = 9\delta_{\mathcal{S}} + \delta_{\mathcal{M}}$ of the ball $\hat{\mathcal{B}}$ is sufficient although not optimal.

Recall that, by Lemma 2.20, the sequences of developing maps $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^+} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}_n^-} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}_\infty^+}$:

$\widetilde{\mathcal{S}}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}^-_\infty} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3$, and the images of the maps $\tilde{f}_{\widetilde{\mathcal{S}}^+_n}$ and $\tilde{f}_{\widetilde{\mathcal{S}}^-_n}$ are convex surfaces $\widetilde{\mathcal{S}}^+_n$ and $\widetilde{\mathcal{S}}^+_n$ respectively, $n = 1, \ldots, \infty$. Therefore, by construction, the surfaces $\{\widehat{\Delta}^+_n\}_{n\in\mathbb{N}}$ and $\{\widehat{\Delta}^-_n\}_{n\in\mathbb{N}}$ converge to $\widehat{\Delta}^+_\infty$ and $\widehat{\Delta}^-_\infty$, and moreover, the sequence of closed convex nondegenerate surfaces $\{\partial\mathcal{M}^{\mathcal{B}}_n\}_{n\in\mathbb{N}}$ converges to the closed convex nondegenerate surface $\partial\mathcal{M}^{\mathcal{B}}_\infty$ in $\mathbb{H}^3$. Applying the hyperbolic version of Theorem 2.22 to the family of surfaces $\{\partial\mathcal{M}^{\mathcal{B}}_n\}_{n\in\mathbb{N}}$ which converges to $\partial\mathcal{M}^{\mathcal{B}}_\infty$ we conclude that the sequence of induced metrics on $\partial\mathcal{M}^{\mathcal{B}}_n$ tends to the induced metric on $\partial\mathcal{M}^{\mathcal{B}}_\infty$ as $n \to \infty$. In particular, given any two sequences of points A^+_n and B^+_n in $\widehat{\Delta}^+_n \subset \partial\mathcal{M}^{\mathcal{B}}_n$ converging to two points A^+_∞ and B^+_∞ in $\widehat{\Delta}^+_\infty \subset \partial\mathcal{M}^{\mathcal{B}}_\infty$, respectively, the distances between the points A^+_n and B^+_n measured on the surfaces $\partial\mathcal{M}^{\mathcal{B}}_n$ converge to the distance between the points A^+_∞ and B^+_∞ measured on $\partial\mathcal{M}^{\mathcal{B}}_\infty$, i.e.,

$$(2.16) \qquad \mathrm{d}_{\partial\mathcal{M}^{\mathcal{B}}_\infty}(A^+_\infty, B^+_\infty) = \lim_{n\to\infty}\mathrm{d}_{\partial\mathcal{M}^{\mathcal{B}}_n}(A^+_n, B^+_n).$$

By Remark 2.23, the distance between the points A^+_n and B^+_n measured on $\partial\mathcal{M}^{\mathcal{B}}_n$ is equal to the distance between these points measured on $\hat{\mathcal{S}}^+_n$; also, by construction, $\hat{\mathcal{S}}^+_n$ is a convex subset of the surface $\widetilde{\mathcal{S}}^+_n$ with the induced metric $\tilde{h}^+_n$, therefore

$$(2.17) \qquad \mathrm{d}_{\partial\mathcal{M}^{\mathcal{B}}_n}(A^+_n, B^+_n) = \mathrm{d}_{\tilde{h}^+_n}(A^+_n, B^+_n),$$

$n = 1, \ldots, \infty$. Substituting (2.17) in (2.16), we get:

$$\mathrm{d}_{\tilde{h}^+_\infty}(A^+_\infty, B^+_\infty) = \lim_{n\to\infty}\mathrm{d}_{\tilde{h}^+_n}(A^+_n, B^+_n).$$

Hence, the sequence of the induced metrics $\tilde{h}^+_n$ of the surfaces $\widetilde{\mathcal{S}}^+_n$ restricted on the sets $\widehat{\Delta}^+_n$ converges to the induced metric $\tilde{h}^+_\infty$ of the surface $\widetilde{\mathcal{S}}^+_\infty$ restricted on $\widehat{\Delta}^+_\infty$ as $n \to \infty$. By analogy, the sequence of the induced metrics $\{\tilde{h}^-_n|_{\widehat{\Delta}^-_n}\}_{n\in\mathbb{N}}$ converges to the induced metric $\tilde{h}^-_\infty|_{\widehat{\Delta}^-_\infty}$.

In Sections 2.2 and 2.3 we constructed the surfaces $\widetilde{\mathcal{S}}^+_n$ and $\widetilde{\mathcal{S}}^-_n$ to be invariant under the actions of the discrete group $\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$ for each $n = 1, \ldots, \infty$. Hence, the induced metrics $\tilde{h}^+_n$ and $\tilde{h}^-_n$ on the surfaces $\widetilde{\mathcal{S}}^+_n$ and $\widetilde{\mathcal{S}}^-_n$, respectively, are periodic with respect to the group $\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))$, $n = 1, \ldots, \infty$. We have just proved that the metrics $\tilde{h}^+_n$ and $\tilde{h}^-_n$ converge to $\tilde{h}^+_\infty$ and $\tilde{h}^-_\infty$, correspondingly, in the neighborhoods $\widehat{\Delta}^+_n \subset \widetilde{\mathcal{S}}^+_n$ and $\widehat{\Delta}^-_n \subset \widetilde{\mathcal{S}}^-_n$ of the fundamental domains $\Delta^+_n \subset \widetilde{\mathcal{S}}^+_n$ and $\Delta^-_n \subset \widetilde{\mathcal{S}}^-_n$ of the surfaces $\mathcal{S}^+_n$ and $\mathcal{S}^-_n$. Since, by Assumption 2 and Remark 2.21, the sequence of quasi-Fuchsian groups $\{\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))\}_{n\in\mathbb{N}}$ converges to a quasi-Fuchsian group $\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$, we now conclude that the metrics $\tilde{h}^+_n$ and $\tilde{h}^-_n$ converge to $\tilde{h}^+_\infty$ and $\tilde{h}^-_\infty$ everywhere on $\widetilde{\mathcal{S}}^+_n$ and $\widetilde{\mathcal{S}}^-_n$ as $n \to \infty$.

To complete the proof of Theorem 1.6 let us consider the convex compact hyperbolic domain $\mathcal{M}_\infty \overset{\text{def}}{=} \widetilde{\mathcal{M}}_\infty/[\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))]$ with the boundary

$$\partial\mathcal{M}_\infty \overset{\text{def}}{=} \mathcal{S}^+_\infty \cup \mathcal{S}^-_\infty \overset{\text{def}}{=} \left(\widetilde{\mathcal{S}}^+_\infty/[\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))]\right) \bigcup \left(\widetilde{\mathcal{S}}^-_\infty/[\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))]\right)$$

in the unbounded hyperbolic manifold $\mathcal{M}^\circ_\infty \overset{\text{def}}{=} \mathbb{H}^3/[\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))]$. The metric $\tilde{h}^+_\infty$ on the universal covering $\widetilde{\mathcal{S}}^+_\infty$ of the boundary component $\mathcal{S}^+_\infty$ of the domain $\mathcal{M}_\infty$ induces the metric $\breve{h}^+_\infty$ on the compact surface $\mathcal{S}^+_\infty$. We have recently showed that the pull-backs $\tilde{h}^+_n$ of the metrics h^+_n (see Section 2.2) converge to the pull-back $\tilde{h}^+_\infty$ of the metric $\breve{h}^+_\infty$. Hence, the sequence of metrics $\{h^+_n\}_{n\in\mathbb{N}}$ tends to the metric $\breve{h}^+_\infty$ as $n \to \infty$. But in the very beginning of Section 2.2 the C^∞-smooth metrics $\{h^+_n\}_{n\in\mathbb{N}}$ were constructed in order to approximate the Alexandrov metric h^+_∞. Therefore, the induced metric $\breve{h}^+_\infty$ on $\mathcal{S}^+_\infty$ coincides with the prescribed metric h^+_∞. Similarly we obtain that the metric on the surface $\mathcal{S}^-_\infty$ is exactly h^-_∞.

We sum up that the convex hyperbolic bounded domain $\mathcal{M}_\infty$ with the boundary $\partial\mathcal{M}_\infty = \mathcal{S}^+_\infty \cup \mathcal{S}^-_\infty$ in the quasi-Fuchsian manifold $\mathcal{M}^\circ_\infty$ was constructed in such a way that the induced metrics of the boundary components $\mathcal{S}^+_\infty$ and $\mathcal{S}^-_\infty$ coincide with the prescribed Alexandrov metrics h^+_∞ and h^-_∞. Theorem 1.6 is proved. $\qquad\square$

3. Distance between boundary components of a convex compact domain in a quasi-Fuchsian manifold.

Consider a sequence of convex bounded domains $\mathcal{M}_n$ with the upper boundaries $\mathcal{S}^+_n$ and the lower boundaries $\mathcal{S}^-_n$ in quasi-Fuchsian manifolds $\mathcal{M}^\circ_n$, such that for all n the convex regular metric surfaces $\mathcal{S}^+_n$ and $\mathcal{S}^-_n$ with the induced metrics h^+_n and h^-_n, respectively, are topologically the same surface $\mathcal{S}$.

DEFINITION 3.1. — The distance $d(\mathcal{K}, \mathcal{L})$ between subsets $\mathcal{K}$ and $\mathcal{L}$ of a set $\mathcal{N}$ is defined as follows: $d(\mathcal{K}, \mathcal{L}) \overset{\text{def}}{=} \inf\{d_\mathcal{N}(u,v) | u \in \mathcal{K}, v \in \mathcal{L}\}$, where $d_\mathcal{N}(u,v)$ stands for the distance between points u and v in $\mathcal{N}$.

In this section, we prove the following result which is essentially used in the demonstration of Theorem 1.7 from the first part of this paper:

THEOREM 3.1. — *Let the metrics h^+_n tend to some metric h^+_∞ (correspondingly, h^-_n tend to h^-_∞) as n goes to ∞. Then there is a common upper bound for the distances between $\mathcal{S}^+_n$ and $\mathcal{S}^-_n$ in $\mathcal{M}^\circ_n$ which does not depend on n.*

The proof of Theorem 3.1 is essentially based on

THEOREM 3.2. — *Given a convex bounded domain $\mathcal{M}$ with the upper boundary $\mathcal{S}^+$ and the lower boundary $\mathcal{S}^-$ in a quasi-Fuchsian manifold $\mathcal{M}^\circ$. If the metric surface $\mathcal{S}^+$ possesses two homotopically different nontrivial closed simple intersecting curves c_1^+ and c_2^+ of the lengths l_1^+ and l_2^+, and $\mathcal{S}^-$ possesses two homotopically different nontrivial closed simple intersecting curves c_1^- and c_2^- of the lengths l_1^- and l_2^- such that c_1^+ and c_1^-, as well as c_2^+ and c_2^-, are homotopically equivalent pairs of curves in $\mathcal{M}$, then the distance $d(\mathcal{S}^+, \mathcal{S}^-)$ between $\mathcal{S}^+$ and $\mathcal{S}^-$ is bounded from above by the constant*

$$
d(\mathcal{S}^+, \mathcal{S}^-) < \max \left\{ \left(l_1^+ + l_1^- + \ln \frac{2l_1^+}{l_1^-} \right), \left(l_1^+ + l_1^- + \ln \frac{2l_1^-}{l_1^+} \right), \right.
$$

$$
\left(l_2^+ + l_2^- + \ln \frac{2l_2^+}{l_2^-} \right), \left(l_2^+ + l_2^- + \ln \frac{2l_2^-}{l_2^+} \right),
$$

$$
2\operatorname{arcosh}\left[\cosh l_1^+ \cosh\left(l_1^+ + \operatorname{arcosh}\frac{e^{l_1^+} (l_1^+)^2}{\varepsilon_3^2} \right) \right],
$$

$$
2\operatorname{arcosh}\left[\cosh l_1^- \cosh\left(l_1^- + \operatorname{arcosh}\frac{e^{l_1^-} (l_1^-)^2}{\varepsilon_3^2} \right) \right],
$$

$$
2\operatorname{arcosh}\left[\cosh l_2^+ \cosh\left(l_2^+ + \operatorname{arcosh}\frac{e^{l_2^+} (l_2^+)^2}{\varepsilon_3^2} \right) \right],
$$

$$
\left. 2\operatorname{arcosh}\left[\cosh l_2^- \cosh\left(l_2^- + \operatorname{arcosh}\frac{e^{l_2^-} (l_2^-)^2}{\varepsilon_3^2} \right) \right] \right\},
$$

where the symbol ε_3 stands for the Margulis constant of hyperbolic space $\mathbb{H}^3$ (this constant will be defined shortly).

This result is of independent interest as well. Note that we do not require the regularity of surface metrics in Theorems 3.1 and 3.2.

Let us show how Theorem 3.2 implies Theorem 3.1.

Proof of Theorem 3.1. — Consider two homotopically different nontrivial closed curves c_1 and c_2 on the surface $\mathcal{S}$ such that they intersect each other but do not intersect with the singular points of the metrics h_∞^+ and h_∞^- on $\mathcal{S}$. Since the sequence of metrics $\{h_n^+\}_{n \in \mathbb{N}}$ converges to the metric h_∞^+, the lengths $l_1^{+,n}$ of the curve $c_1 \in \mathcal{S}$ measured in the metrics h_n^+, $n \in \mathbb{N}$, tend to the length $l_1^{+,\infty} > 0$ of c_1 measured in the metric h_∞^+ as $n \to \infty$. The converging sequence of the positive real numbers $\{l_1^{+,n}\}_{n \in \mathbb{N}}$ is bounded from below by a real number $\omega_1^+ > 0$ and from above by a real number $\Omega_1^+ > 0$. Similarly, the lengths $l_1^{-,n}$ of the curve $c_1 \in \mathcal{S}$ measured in the metrics h_n^-, $n \in \mathbb{N}$, are bounded from below by some $\omega_1^- > 0$ and from above by some $\Omega_1^- > 0$; the lengths $l_2^{+,n}$ of the curve $c_2 \in \mathcal{S}$ measured in the metrics h_n^+, $n \in \mathbb{N}$, are bounded from below by some $\omega_2^+ > 0$ and from above by some $\Omega_2^+ > 0$; and the lengths $l_2^{-,n}$ of the

curve $c_2 \in \mathcal{S}$ measured in the metrics h_n^-, $n \in \mathbb{N}$, are bounded from below by some $\omega_2^- > 0$ and from above by some $\Omega_2^- > 0$.

By Theorem 3.2, the distance $d(\mathcal{S}_n^+, \mathcal{S}_n^-)$ between the surfaces $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ in the quasi-Fuchsian manifold $\mathcal{M}_n^\circ$ is uniformly bounded from above for any $n \in \mathbb{N}$:

$$d(\mathcal{S}_n^+, \mathcal{S}_n^-) < \max\left\{ \left(\Omega_1^+ + \Omega_1^- + \ln\frac{2\Omega_1^+}{\omega_1^-}\right), \left(\Omega_1^+ + \Omega_1^- + \ln\frac{2\Omega_1^-}{\omega_1^+}\right),\right.$$

$$\left(\Omega_2^+ + \Omega_2^- + \ln\frac{2\Omega_2^+}{\omega_2^-}\right), \left(\Omega_2^+ + \Omega_2^- + \ln\frac{2\Omega_2^-}{\omega_2^+}\right),$$

$$2\mathrm{arcosh}\left[\cosh\Omega_1^+\cosh\left(\Omega_1^+ + \mathrm{arcosh}\frac{e^{\Omega_1^+}(\Omega_1^+)^2}{\varepsilon_3^2}\right)\right],$$

$$2\mathrm{arcosh}\left[\cosh\Omega_1^-\cosh\left(\Omega_1^- + \mathrm{arcosh}\frac{e^{\Omega_1^-}(\Omega_1^-)^2}{\varepsilon_3^2}\right)\right],$$

$$2\mathrm{arcosh}\left[\cosh\Omega_2^+\cosh\left(\Omega_2^+ + \mathrm{arcosh}\frac{e^{\Omega_2^+}(\Omega_2^+)^2}{\varepsilon_3^2}\right)\right],$$

$$\left. 2\mathrm{arcosh}\left[\cosh\Omega_2^-\cosh\left(\Omega_2^- + \mathrm{arcosh}\frac{e^{\Omega_2^-}(\Omega_2^-)^2}{\varepsilon_3^2}\right)\right]\right\}. \quad \square$$

Our aim now is to demonstrate Theorem 3.2. We will widely use the Margulis lemma to prove this fact. In the most general case the Margulis lemma reads as follows [5, Theorem D.1.1, p. 134]:

GENERAL MARGULIS LEMMA. — *For every $m \in \mathbb{N}$ there exists a constant $\varepsilon_m \geq 0$ such that for any properly discontinuous subgroup Γ of the group $\mathcal{I}(\mathbb{H}^m)$ of isometries of $\mathbb{H}^m$ and for any $x \in \mathbb{H}^m$, the group $\Gamma_{\varepsilon_m}(x)$ generated by the set $F_{\varepsilon_m}(x) = \{\gamma \in \Gamma : \mathrm{d}_{\mathbb{H}^m}(x, \gamma(x)) \leq \varepsilon_m\}$ is almost-nilpotent, where $\mathrm{d}_{\mathbb{H}^m}(\cdot, \cdot)$ stands for the distance in hyperbolic space $\mathbb{H}^m$.*

If we restrict the General Margulis Lemma to the case of the quasifuchsian isometries of hyperbolic 3-space $\mathbb{H}^3$ which is interesting to us, then the lemma can be rewritten in this way [18, Theorem B, p. 100]:

MARGULIS LEMMA. — *There is a universal constant $\varepsilon_3 > 0$ such that for any properly discontinuous subgroup Γ of the group $\mathcal{I}(\mathbb{H}^3)$ of isometries of $\mathbb{H}^3$ if two closed simple intersecting curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ of the manifold $\mathbb{H}^3/\Gamma$ have lengths less than ε_3, then $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are homotopically equivalent in $\mathbb{H}^3/\Gamma$.*

Hence, the main idea of the proof of Theorem 3.2 is to find a pair of closed simple intersecting curves inside $\mathcal{M}$ of lengths less than the Margulis constant ε_3 and such that they are not homotopically equivalent once the distance between $\mathcal{S}^+$ and $\mathcal{S}^-$ is big enough. Then, by the Margulis lemma, the curves

under consideration ought to be homotopically equivalent, which leads us to a contradiction. Let us now give a more detailed plan of the proof of Theorem 3.2:
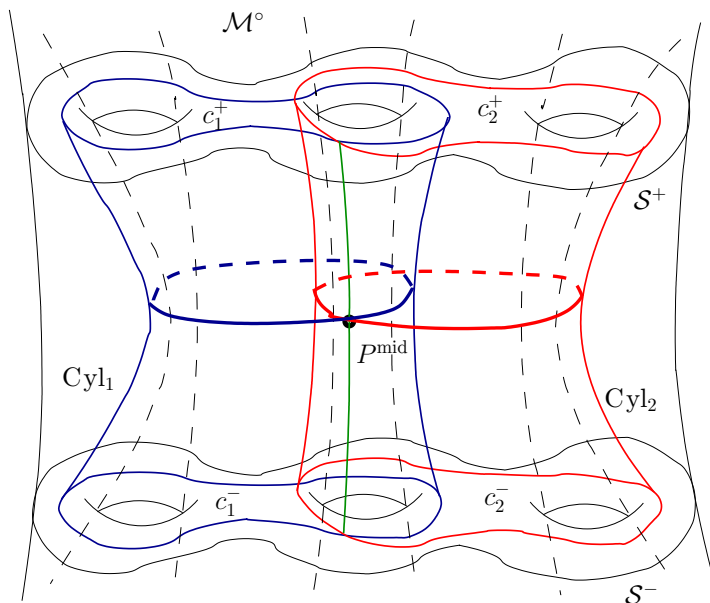

FIGURE 3.1. The cylinders Cyl_1 and Cyl_2 in the manifold $\mathcal{M}^\circ$.

- Suppose that the curves c_1^+ and c_2^+ intersect at a point P^+ (this point is not necessarily unique), and the curves c_1^- and c_2^- intersect at a point P^-. We will construct cylinders Cyl_1 and Cyl_2 in $\mathcal{M}$ that realize homotopies between c_1^+ and c_1^- and between c_2^+ and c_2^- correspondingly. Then the intersection of Cyl_1 and Cyl_2 contains a (curved) line with ends P^+ and P^-. Denote the midpoint of this line by P^{mid}.
- We will find a constant based on l_1^+, l_1^-, l_2^+, l_2^-, and ε_3, and we will construct curves on Cyl_1 and Cyl_2 (see Fig. 3.1) passing through P^{mid} such that if the distance between $\mathcal{S}^+$ and $\mathcal{S}^-$ is greater than the constant mentioned above then both constructed curves are shorter than ε_3.

3.1. Construction of the cylinders Cyl_1 and Cyl_2. — We consider a quasifuchsian manifold $\mathcal{M}^\circ$. By definition, it means that $\mathcal{M}^\circ$ is a quotient $\mathbb{H}^3/\Gamma^\circ$ where Γ° is a quasifuchsian subgroup of the group $\mathcal{I}(\mathbb{H}^3)$ of isometries of hyperbolic 3-space. Note that Γ° is homomorphic to the fundamental group $\pi_1(\mathcal{M}^\circ)$.

Denote by γ_1 the closed geodesic of $\mathcal{M}^\circ$ homotopically equivalent to c_1^+ and c_1^-. Similarly, denote by γ_2 the closed geodesic of $\mathcal{M}^\circ$ homotopically

equivalent to c_2^+ and c_2^-. By abuse of notation, we denote by γ_1 and γ_2 the elements of $\pi_1(\mathcal{M}^\circ)$ corresponding to the closed geodesics under consideration. The universal covering of the domain $\mathcal{M} \subset \mathcal{M}^\circ$ is a convex simply connected subset $\widetilde{\mathcal{M}}$ of $\mathbb{H}^3$. Denote by $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ the isometries of $\mathbb{H}^3$ corresponding to the elements γ_1 and γ_2 of $\pi_1(\mathcal{M}^\circ)$.
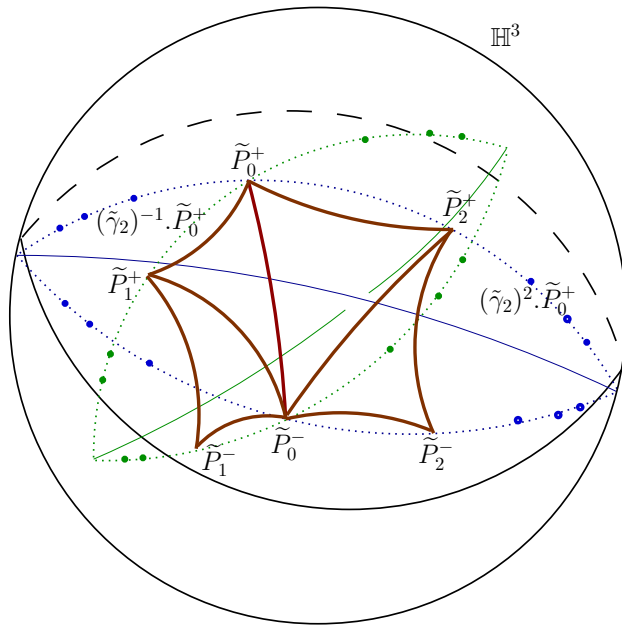

FIGURE 3.2. Construction of fundamental domains of the cylinders Cyl_1 and Cyl_2 in the Poincaré model of $\mathbb{H}^3$.

Let us now consider any single point $\widetilde{P}_0^+ \in \mathbb{H}^3$ serving as a pre-image of $P^+ \in c_1^+ \cap c_2^+$ in the universal covering $\widetilde{\mathcal{M}}$. Among all the points in the pre-image of $P^- \in c_1^- \cap c_2^-$ in $\widetilde{\mathcal{M}}$, we choose $\widetilde{P}_0^- \in \mathbb{H}^3$ to be the closest to $\widetilde{P}_0^+$ (in case there are several points realizing the minimal distance to $\widetilde{P}_0^+$, we choose one of them arbitrarily). Denote $\widetilde{P}_1^+ \stackrel{\text{def}}{=} \tilde{\gamma}_1.\widetilde{P}_0^+$, $\widetilde{P}_1^- \stackrel{\text{def}}{=} \tilde{\gamma}_1.\widetilde{P}_0^-$, $\widetilde{P}_2^+ \stackrel{\text{def}}{=} \tilde{\gamma}_2.\widetilde{P}_0^+$, $\widetilde{P}_2^- \stackrel{\text{def}}{=} \tilde{\gamma}_2.\widetilde{P}_0^-$ (recall that for every point $T \in \mathbb{H}^3$ and for every $\tilde{\gamma} \in \mathcal{I}(\mathbb{H}^3)$ the symbol $\tilde{\gamma}.T$ stands for the image of T under the isometry $\tilde{\gamma}$). Then we set the unions of flat hyperbolic triangles $\triangle \widetilde{P}_0^+ \widetilde{P}_0^- \widetilde{P}_1^+ \cup \triangle \widetilde{P}_1^+ \widetilde{P}_1^- \widetilde{P}_0^-$ and $\triangle \widetilde{P}_0^+ \widetilde{P}_0^- \widetilde{P}_2^+ \cup \triangle \widetilde{P}_2^+ \widetilde{P}_2^- \widetilde{P}_0^-$ in $\mathbb{H}^3$ to be fundamental domains of the cylinders Cyl_1 and Cyl_2 (see Fig. 3.2).

The fundamental domain $\tilde{c}_1^+ \subset \mathbb{H}^3$ of the curve c_1^+ has the same length l_1^+ as c_1^+. We can choose $\tilde{c}_1^+$ to connect $\widetilde{P}_0^+$ and $\widetilde{P}_1^+$. Hence, the length of the straight (hyperbolic) segment $\widetilde{P}_0^+ \widetilde{P}_1^+$ is less than or equal to l_1^+. Similarly,

$\mathrm{d}_{\mathbb{H}^3}(\widetilde{P}_0^-, \widetilde{P}_1^-) \leq l_1^-$, $\mathrm{d}_{\mathbb{H}^3}(\widetilde{P}_0^+, \widetilde{P}_2^+) \leq l_2^+$, and $\mathrm{d}_{\mathbb{H}^3}(\widetilde{P}_0^-, \widetilde{P}_2^-) \leq l_2^-$. Also, by construction, the midpoints $\widetilde{P}_0^{\mathrm{mid}}$, $\widetilde{P}_1^{\mathrm{mid}}$, and $\widetilde{P}_2^{\mathrm{mid}}$ of the segments $\widetilde{P}_0^+ \widetilde{P}_0^-$, $\widetilde{P}_1^+ \widetilde{P}_1^-$, and $\widetilde{P}_2^+ \widetilde{P}_2^-$ serve as pre-images of the midpoint P^{mid} of the segment $P^+ P^-$ lying in the intersection $\mathrm{Cyl}_1 \cap \mathrm{Cyl}_2$.

Evidently, Cyl_1 and Cyl_2 can be prolonged to realize homotopies between the pairs of closed curves (c_1^+, c_1^-) and (c_2^+, c_2^-) as it was announced in our plan, but it will not be needed further.

Let us study properties of the cylinders constructed alike Cyl_1 and Cyl_2.

3.2. Properties of the cylinders of the type Cyl

DEFINITION 3.2. — A cylinder Cyl_0 is said to be *of the type* Cyl if and only if Cyl_0 possesses

1) a fundamental domain $FD(\mathrm{Cyl}_0) \overset{\text{def}}{=} \triangle \widetilde{R}^+ \widetilde{R}^- \widetilde{Q}^+ \cup \triangle \widetilde{Q}^+ \widetilde{Q}^- \widetilde{R}^-$ constructed of two totally geodesic triangles in $\mathbb{H}^3$ such that $\mathrm{d}_{\mathbb{H}^3}(\widetilde{Q}^+, \widetilde{Q}^-) = \mathrm{d}_{\mathbb{H}^3}(\widetilde{R}^+, \widetilde{R}^-)$, and

2) the hyperbolic isometry $\tilde{\gamma} \in \mathcal{I}(\mathbb{H}^3)$ sending the geodesic segment $\widetilde{R}^+ \widetilde{R}^-$ to the geodesic segment $\widetilde{Q}^+ \widetilde{Q}^-$ and such that for every point $\widetilde{R}_\sharp^- \in \{\tilde{\gamma}_\sharp.\widetilde{R}^- | \tilde{\gamma}_\sharp \in \langle \tilde{\gamma} \rangle\}$ the inequality $\mathrm{d}_{\mathbb{H}^3}(\widetilde{R}^+, \widetilde{R}^-) \leq \mathrm{d}_{\mathbb{H}^3}(\widetilde{R}^+, \widetilde{R}_\sharp^-)$ holds true (here and below the symbol $\langle \tilde{\gamma} \rangle$ stands for the group generated by the element $\tilde{\gamma}$). Note that $\widetilde{Q}^- \in \{\tilde{\gamma}_\sharp.\widetilde{R}^- | \tilde{\gamma}_\sharp \in \langle \tilde{\gamma} \rangle\}$ by construction.
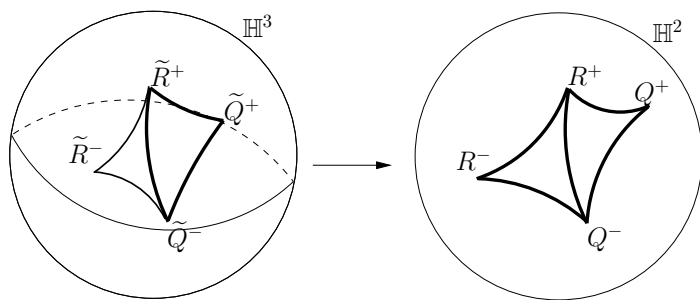

FIGURE 3.3. The quadrilaterals $\widetilde{R}^+ \widetilde{R}^- \widetilde{Q}^+ \widetilde{Q}^-$ in $\mathbb{H}^3$ and $R^+ R^- Q^+ Q^-$ in $\mathbb{H}^2$.

Remark that the metric of Cyl_0 induced from the ambient space is hyperbolic. Let us flatten $FD(\mathrm{Cyl}_0)$ and obtain a hyperbolic quadrilateral $R^+ R^- Q^+ Q^- \subset \mathbb{H}^2$ isometric to $FD(\mathrm{Cyl}_0)$ such that the vertices with tildes in $\mathbb{H}^3$ correspond to the vertices of the same name but without tildes in $\mathbb{H}^2$ (see Fig. 3.3).

The quadrilateral $R^+R^-Q^+Q^-$ serves as a fundamental domain of Cyl_0 in its universal covering in $\mathbb{H}^2$. Denote by χ_R and χ_Q the hyperbolic straight lines in $\mathbb{H}^2$ containing the segments R^+R^- and Q^+Q^- correspondingly. Remark that the connected domain of $\mathbb{H}^2$ between χ_R and χ_Q is actually a fundamental domain of the unbounded hyperbolic cylinder Cyl_0° containing Cyl_0. We will call it $FD(\mathrm{Cyl}_0^\circ)$. Indeed, the fundamental group $\pi_1(\mathrm{Cyl}_0^\circ) = \mathbb{Z}$. Hence, Cyl_0° possesses a closed geodesic χ° and there is a hyperbolic straight line χ in $\mathbb{H}^2$ serving as a lift of χ° and related to the isometry $\bar{\chi}$ of $\mathbb{H}^2$ such that $\mathrm{Cyl}_0^\circ = \mathbb{H}^2/\langle\bar{\chi}\rangle$. We show the existence of such geodesic χ in the following

LEMMA 3.3. — *Consider two nonintersecting geodesics χ_R and χ_Q in $\mathbb{H}^2$ which are not asymptotic, with marked points $R \in \chi_R$ and $Q \in \chi_Q$. There is a unique hyperbolic straight line χ in $\mathbb{H}^2$ such that the angles of intersection of χ with χ_R and χ_Q are equal, and moreover, if we denote $R' \overset{\mathrm{def}}{=} \chi_R \cap \chi$ and $Q' \overset{\mathrm{def}}{=} \chi_Q \cap \chi$, then $\mathrm{d}_{\mathbb{H}^2}(R, R') = \mathrm{d}_{\mathbb{H}^2}(Q, Q')$ and the points R and Q lie in the same half-plane with respect to χ.*

Proof. — Let us consider the Beltrami-Klein model $\mathbb{K}^2$ of the hyperbolic plane $\mathbb{H}^2$. Recall that $\mathbb{K}^2$ is a unit disk in the Euclidean plane $\mathbb{R}^2$ and all geodesics of $\mathbb{K}^2$ are restrictions of Euclidean straight lines on this disk. Without loss of generality the geodesics $\chi_R \subset \mathbb{K}^2$ and $\chi_Q \subset \mathbb{K}^2$ can be taken symmetric with respect to the axis Ox of the cartesian coordinate system on $\mathbb{R}^2$, both at an arbitrary distance ζ from Ox. Let χ_R lie in the upper half-space of $\mathbb{R}^2$ with respect to Ox and χ_Q lie in the lower half-space of $\mathbb{R}^2$ with respect to Ox. At last we fix arbitrary points $R \in \chi_R$ and $Q \in \chi_Q$.

By construction, every geodesic in $\mathbb{K}^2$ passing through the origin O of the cartesian coordinate system on $\mathbb{R}^2$ either intersects χ_R and χ_Q at the same angle or does not intersect them. Let us consider a family Φ_τ of such geodesics $R_\tau Q_\tau$ lying between the straight lines OR and OQ where $R_\tau \in \chi_R$, $Q_\tau \in \chi_Q$, τ stands for the hyperbolic distance between R and R_τ, and the line $OQ \in \Phi_\tau$ corresponds to the value $\hat{\tau}$ of the parameter τ.

Note that

- R and Q lie in the same half-plane with respect to any $R_\tau Q_\tau \in \Phi_\tau$.
- As τ grows up monotonically from 0 to $\hat{\tau}$, the distance $\mathrm{d}_{\mathbb{H}^2}(Q, Q_\tau)$ decreases monotonically from $\mathrm{d}_{\mathbb{H}^2}(Q, Q_{\hat{\tau}})$ to 0. Hence, there exists a unique $\tau_0 \in [0, \hat{\tau}]$ such that $\mathrm{d}_{\mathbb{H}^2}(R, R_{\tau_0}) = \mathrm{d}_{\mathbb{H}^2}(Q, Q_{\tau_0})$.

We choose χ to be $R_{\tau_0} Q_{\tau_0} \in \Phi_\tau$. χ is unique since τ_0 is unique. $\square$

REMARK 3.4. — Let $Set(R^-) \overset{\mathrm{def}}{=} \{\bar{\chi}_\sharp.R^- | \bar{\chi}_\sharp \in \langle\bar{\chi}\rangle\}$ (by construction, $Q^- \in Set(R^-)$). Then for every point $R_\sharp^- \in Set(R^-)$ the inequality $\mathrm{d}_{\mathbb{H}^2}(R^+, R^-) \leq \mathrm{d}_{\mathbb{H}^2}(R^+, R_\sharp^-)$ holds true.

Proof. — By construction, $d_{\mathbb{H}^3}(\widetilde{R}^+, \widetilde{R}^-) = d_{\mathbb{H}^2}(R^+, R^-)$, and the surfaces $\langle\bar{\chi}\rangle.R^+R^-Q^+Q^- \subset \mathbb{H}^2$ (which is the union $\bigcup_{\bar{\chi}_\sharp \in \langle\bar{\chi}\rangle} \bar{\chi}_\sharp.R^+R^-Q^+Q^-$ of the quadrilaterals $\bar{\chi}_\sharp.R^+R^-Q^+Q^-$ isometric to $R^+R^-Q^+Q^-$) and $\langle\bar{\chi}\rangle.FD(\mathrm{Cyl}_0) \subset \mathbb{H}^3$ are isometric in their intrinsic metrics. Evidently, for any points $\widetilde{T}_1$ and $\widetilde{T}_2$ in $\langle\bar{\chi}\rangle.FD(\mathrm{Cyl}_0)$ it is true that $d_{\mathbb{H}^3}(\widetilde{T}_1, \widetilde{T}_2) \leq d^{\mathrm{int}}_{\langle\bar{\chi}\rangle.FD(\mathrm{Cyl}_0)}(\widetilde{T}_1, \widetilde{T}_2)$, where $d^{\mathrm{int}}_{\langle\bar{\chi}\rangle.FD(\mathrm{Cyl}_0)}(\cdot, \cdot)$ stands for the intrinsic metric of $\langle\bar{\chi}\rangle.FD(\mathrm{Cyl}_0)$. At last, the part 2) of the definition of a cylinder Cyl_0 of the type Cyl allows us to conclude that Remark 3.4 is valid. $\qquad\square$

REMARK 3.5. — Let $R'Q'$ be a segment of the geodesic $\chi \subset \mathbb{H}^2$ between χ_R and χ_Q serving as a fundamental domain of $\chi^\circ \subset \mathrm{Cyl}_0^\circ$ on χ (here $R' \in \chi_R$ and $Q' \in \chi_Q$). Then either $R'Q' \subset R^+R^-Q^+Q^-$ or $R'Q' \cap R^+R^-Q^+Q^- = \emptyset$.

Proof. — Recall that the points R^+ and Q^+ are pre-images in $\mathbb{H}^2$ of the same point on Cyl_0, and one can be obtained from another by applying an isometry of $\mathbb{H}^2$ which is an element of the group $\langle\bar{\chi}\rangle$ preserving the straight hyperbolic line χ. Hence, R^+ and Q^+ lie in one half-plane of $\mathbb{H}^2$ with respect to χ and, by consequence, the segment R^+Q^+ does not intersect χ. Similarly, $R^-Q^- \cap \chi = \emptyset$.

We conclude that if R^+Q^+ and R^-Q^- lie in the same half-plane of $\mathbb{H}^2$ with respect to χ then $R'Q' \cap R^+R^-Q^+Q^- = \emptyset$. Otherwise, if R^+Q^+ and R^-Q^- lie in different half-planes with respect to χ, then $R'Q' \subset R^+R^-Q^+Q^-$. $\qquad\square$

3.3. h-neighborhood of a geodesic in $\mathbb{H}^2$. — In this section, we study hyperbolic quadrilaterals of one special type and half-neighborhoods of geodesics containing one of the sides of our quadrilaterals which are inscribed in and circumscribed about these quadrilaterals. Properties of these objects will be largely used in obtaining bounds on a possible size of cylinders of the type Cyl.

The object of our interest is a quadrilateral $O_RO_QRQ \subset \mathbb{H}^2$ with the sides $d_{\mathbb{H}^2}(O_RO_Q) = l$, $d_{\mathbb{H}^2}(R, Q) = l'$, and $d_{\mathbb{H}^2}(O_R, R) = d_{\mathbb{H}^2}(O_Q, Q) = h'$, such that the edges O_RR and O_QQ are perpendicular to O_RO_Q. Draw a curve γ_h at a distance $h < h'$ from the geodesic containing O_RO_Q such that γ_h intersects O_RR and O_QQ at points T and T' correspondingly. Denote a segment of γ_h between O_RR and O_QQ by $\widehat{TT'}$, and the hyperbolic length of $\widehat{TT'}$ by l_h.

A direct calculation shows that

REMARK 3.6. — The following relation holds true:

$$l_h = l\cosh h.$$

REMARK 3.7. — If $h = h'$ then T and T' coincide with R and Q, $\widehat{TT'}$ intersects O_RO_QRQ as a solid body only at its ends R and Q, and, evidently, $l_{h'} > l'$ (any path connecting two points can not be shorter then a geodesic segment between them).

REMARK 3.8. — Suppose that $h' > l'$. If $h \leq h' - l'$ then $\widehat{TT'} \subset O_R O_Q RQ$ and $l_h < l'$.

Proof. — Consider hyperbolic balls $B_{l'}(R)$ and $B_{l'}(Q)$ of the radius l' with the centers R and Q. These balls contain the segment RQ. Also, $B_{l'}(R)$ and $B_{l'}(Q)$ are perpendicular to $O_R R$ and $O_Q Q$ correspondingly. By construction, $\widehat{TT'}$ is perpendicular to $O_R R$ and $O_Q Q$ as well. Moreover, $\widehat{TT'}$ is a convex curve. Hence, $\widehat{TT'}$ lies outside the interior of $B_{l'}(R)$ and $B_{l'}(Q)$ for $h \leq h' - l'$. It means that the geodesic segment RQ does not intersect $\widehat{TT'}$, and $\widehat{TT'} \subset O_R O_Q RQ$.

Denote by $O_R O_Q \widehat{TT'}$ the convex domain in $\mathbb{H}^2$ bounded by the segments $O_R T$, $O_R O_Q$, $O_Q T'$ and the curve $\widehat{TT'}$. By construction, the orthogonal projection of RQ onto $O_R O_Q \widehat{TT'}$ is $\widehat{TT'}$. Since the orthogonal projection on the boundary of a convex hyperbolic domain is contracting [4, p. 9] (see also [7, II.1.3.4, p. 124]), we get $l_h < l'$. □

Also, we need

LEMMA 3.9. — *Let us consider a quadrilateral $O_R O_Q RQ$ as in Section 3.3 with the fixed length l_{RQ} of the edge RQ. There is a constant*

$$h_{\mathrm{int}}^{\mathrm{ort}} = l_{RQ} + \mathrm{arcosh}\frac{e^{l_{RQ}} l_{RQ}^2}{\varepsilon_3^2}.$$

such that if the length h_{RQ} of the sides $O_R R$ and $O_Q Q$ is greater than $h_{\mathrm{int}}^{\mathrm{ort}}$ then the length of the path $\widehat{T_R T_Q}$ at distance $h_T \stackrel{\mathrm{def}}{=} h_{RQ}/2$ from $O_R O_Q$ connecting the midpoints T_R and T_Q of $O_R R$ and $O_Q Q$ is smaller than the Margulis constant ε_3.

Proof. — Denote by l_O the length of $O_R O_Q$. Once l_{RQ} is fixed, suppose that h_{RQ} can be arbitrarily big, in particular, bigger than l_{RQ}.

There are points $T_R' \in O_R R$ and $T_Q' \in O_Q Q$ at distance h_T' from O_R and O_Q correspondingly, such that the length of the path $\widehat{T_R' T_Q'}$ as in Section 3.3 is equal to ε_3. By Remark 3.6,

$$(3.1) \qquad\qquad l_O \cosh h_T' = \varepsilon_3.$$

Indeed, if T_R' and T_Q' do not exist then

$$(3.2) \qquad\qquad l_O > \varepsilon_3.$$

By Remarks 3.6 and 3.8 applied to the quadrilateral $O_R O_Q RQ$,

$$(3.3) \qquad\qquad l_O \cosh(h_{RQ} - l_{RQ}) < l_{RQ}.$$

Mixing (3.2) and (3.3), we get

$$\varepsilon_3 \cosh(h_{RQ} - l_{RQ}) < l_{RQ},$$

$$h_{RQ} < l_{RQ} + \operatorname{arcosh}\frac{l_{RQ}}{\varepsilon_3},$$

which leads us to a contradiction with the unboundedness of h_{RQ}.

The length of $\widehat{T_R T_Q}$ is less than the length ε_3 of $\widehat{T'_R T'_Q}$ when the inequality

$$(3.4) \qquad\qquad h'_T > h_T\left(= \frac{h_{RQ}}{2}\right)$$

is satisfied, which is equivalent to the validity of

$$\cosh h'_T > \cosh\frac{h_{RQ}}{2},$$

and, by (3.1), is also equivalent to

$$(3.5) \qquad\qquad \frac{\varepsilon_3}{l_O} > \cosh\frac{h_{RQ}}{2}.$$

Due to the following property of the hyperbolic cosine: $\cosh 2x = \cosh^2 x + \sinh^2 x$, we see that

$$\cosh^2\left(\frac{h_{RQ}}{2}\right) \leq \cosh h_{RQ}.$$

Hence, the validity of the formula

$$(3.6) \qquad\qquad \cosh h_{RQ} < \frac{\varepsilon_3^2}{l_O^2}$$

implies the validity of (3.5).

Let us exclude l_O from (3.6) using (3.3).

At first, we perform a series of modifications of (3.3). By the formula for the hyperbolic cosine of the sum of two angles, we get

$$\cosh h_{RQ}\cosh l_{RQ} - \sinh h_{RQ}\sinh l_{RQ} < \frac{l_{RQ}}{l_O}.$$

Then, as $\sinh x > 0$ for each $x > 0$, and because $\cosh x > \sinh x$ and $\cosh x > 0$ for all $x \in \mathbb{R}$, we obtain

$$\cosh h_{RQ}(\cosh l_{RQ} - \sinh l_{RQ}) < \frac{l_{RQ}}{l_O},$$

and the definitions of the hyperbolic sine and cosine,

$$(3.7) \qquad \sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2},$$

imply

$$\cosh h_{RQ} < \frac{e^{l_{RQ}} l_{RQ}}{l_O}.$$

It means that the validity of the formula

$$(3.8) \qquad\qquad \frac{e^{l_{RQ}} l_{RQ}}{l_O} < \frac{\varepsilon_3^2}{l_O^2}$$

implies the validity of (3.6). We rewrite the condition (3.8) in a more convenient form:

$$(3.9) \qquad\qquad l_O < \frac{\varepsilon_3^2}{e^{l_{RQ}} l_{RQ}}.$$

By (3.3), we know that

$$l_O < \frac{l_{RQ}}{\cosh(h_{RQ} - l_{RQ})}.$$

Hence, the validity of

$$(3.10) \qquad\qquad \frac{l_{RQ}}{\cosh(h_{RQ} - l_{RQ})} < \frac{\varepsilon_3^2}{e^{l_{RQ}} l_{RQ}}$$

implies the validity of (3.8).

We can now conclude that the condition

$$h_{RQ} > h_{\mathrm{int}}^{\mathrm{ort}}$$

obtained from (3.10) implies (3.4). □

3.4. Fundamental domains of Cyl_1 and Cyl_2 in $\mathbb{H}^2$. — Following the construction of a fundamental domain of a cylinder of the type Cyl in $\mathbb{H}^2$ from Section 3.2, we define for the cylinder Cyl_1 its fundamental domain $P_0^+ P_0^- P_1^+ P_1^- \subset \mathbb{H}_1^2$, where $\mathbb{H}_1^2$ is just a copy of the hyperbolic plane $\mathbb{H}^2$. We denote by χ_{P_0} and χ_{P_1} the hyperbolic straight lines in $\mathbb{H}_1^2$ containing the segments $P_0^+ P_0^-$ and $P_1^+ P_1^-$ correspondingly. Following the content of Section 3.3, we find the hyperbolic segment $O_0 O_1 \subset \mathbb{H}_1^2$ corresponding to the element γ_1 of the fundamental group $\pi_1(\mathcal{M}^\circ)$ (see Section 3.1) with the points $O_0 \in \chi_{P_0}$ and $O_1 \in \chi_{P_1}$.

Similarly, we define the quadrilateral $P_0^+ P_0^- P_2^+ P_2^- \subset \mathbb{H}_2^2$ to be a fundamental domain of the cylinder Cyl_2, where $\mathbb{H}_2^2$ is another copy of $\mathbb{H}^2$. Denote by χ_{P_0} and χ_{P_2} the geodesics in $\mathbb{H}_2^2$ containing $P_0^+ P_0^-$ and $P_2^+ P_2^-$ correspondingly. We also find the hyperbolic segment $O_0 O_2 \subset \mathbb{H}_2^2$ corresponding to $\gamma_2 \in \pi_1(\mathcal{M}^\circ)$ with the points $O_0 \in \chi_{P_0}$ and $O_2 \in \chi_{P_2}$.

An attentive reader has already remarked the following abuse of notation: the geodesic χ_{P_0} with the points P_0^+, P_0^-, and O_0 on it lie both in $\mathbb{H}_1^2$ and $\mathbb{H}_2^2$ as if these copies $\mathbb{H}_1^2$ and $\mathbb{H}_2^2$ of the hyperbolic plane intersect at χ_{P_0}. It is very logical since the segment $P_0^+ P_0^- \subset \chi_{P_0}$ corresponds to the segment $P^+ P^-$ in the intersection of the cylinders Cyl_1 and Cyl_2 related to $\mathbb{H}_1^2$ and $\mathbb{H}_2^2$.

We are now prepared to prove Theorem 3.2. In order to do this, according to Remark 3.5 we must consider two separate situations.

Situation 1. — If for both cylinders Cyl_1 and Cyl_2 their fundamental domains $P_0^+ P_0^- P_1^+ P_1^- \subset \mathbb{H}_1^2$ and $P_0^+ P_0^- P_2^+ P_2^- \subset \mathbb{H}_1^2$ contain the segments $O_0 O_1$ and $O_0 O_2$ correspondingly (see Fig. 3.4), then the distance between the surfaces $\mathcal{S}^+$ and $\mathcal{S}^-$ from the statement of Theorem 3.2 is bounded from above due to the Margulis lemma.

Indeed, recall that P^{mid} is the midpoint of the segment $P^+ P^- \subset \mathrm{Cyl}_1 \cap \mathrm{Cyl}_2$, then the midpoints P_0^{mid}, P_1^{mid}, and P_2^{mid} of the segments $P_0^+ P_0^- \subset \chi_{P_0}$, $P_1^+ P_1^- \subset \chi_{P_1}$, and $P_2^+ P_2^- \subset \chi_{P_2}$ are the pre-images of P^{mid} in $P_0^+ P_0^- P_1^+ P_1^- \subset \mathbb{H}_1^2$ or $P_0^+ P_0^- P_2^+ P_2^- \subset \mathbb{H}_2^2$. Following the content of Section 3.3, we construct the paths $\widehat{P_0^{\mathrm{mid}} P_1^{\mathrm{mid}}} \subset \mathbb{H}_1^2$ and $\widehat{P_0^{\mathrm{mid}} P_2^{\mathrm{mid}}} \subset \mathbb{H}_2^2$ connecting P_0^{mid} with P_1^{mid} and P_2^{mid}, and lying at distance $\mathrm{d}_{\mathbb{H}^2}(P_0^{\mathrm{mid}}, O_0)$ from $O_0 O_1$ and $O_0 O_2$. We will demonstrate that, once the distance between $\mathcal{S}^+$ and $\mathcal{S}^-$ (consequently, the hyperbolic length of $P^+ P^-$) is bigger then a constant depending on l_1^+, l_1^-, l_2^+, and l_2^- (see Section 3.1 for definitions), then two intersecting homotopically different curves in $\mathcal{M}$ with fundamental domains $\widehat{P_0^{\mathrm{mid}} P_1^{\mathrm{mid}}} \subset \mathbb{H}_1^2$ and $\widehat{P_0^{\mathrm{mid}} P_2^{\mathrm{mid}}} \subset \mathbb{H}_2^2$ have the lengths less than the Margulis constant ε_3, which is impossible.
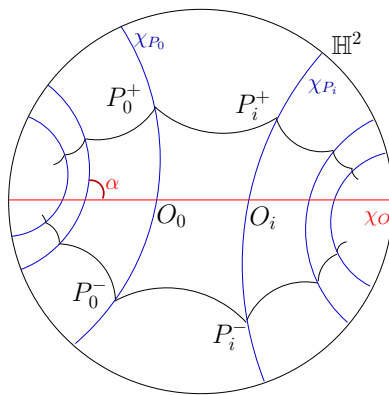

FIGURE 3.4. The quadrilateral $P_0^+ P_0^- P_i^+ P_i^-$, $i = 1, 2$, in Situation 1.

Situation 2. — If for at least one of the cylinders Cyl_1 or Cyl_2 the corresponding segment $O_0 O_1$ or $O_0 O_2$ does not intersect $P_0^+ P_0^- P_1^+ P_1^-$ or $P_0^+ P_0^- P_2^+ P_2^-$ (see Fig. 3.5), then we will prove that the hyperbolic length of the segment $P^+ P^- \subset \mathrm{Cyl}_1 \cap \mathrm{Cyl}_2$ (and, hence, the distance between $\mathcal{S}^+$ and $\mathcal{S}^-$) is necessarily bounded by a constant depending on either l_1^+ and l_1^-, or l_2^+ and l_2^-.

It is now time to study

3.5. Distance between boundary components of a cylinder of the type Cyl. —

Let a quadrilateral $R_0^+ R_0^- R_1^+ R_1^- \subset \mathbb{H}^2$ with $h \overset{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^+, R_0^-) = \mathrm{d}_{\mathbb{H}^2}(R_1^+, R_1^-)$, $l^+ \overset{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^+, R_1^+)$, and $l^- \overset{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^-, R_1^-)$ be a fundamental domain in $\mathbb{H}^2$ of a cylinder Cyl_0 of the type Cyl. Denote by χ_{R_0} and χ_{R_1} the hyperbolic straight lines in $\mathbb{H}^2$ containing the segments $R_0^+ R_0^-$ and $R_1^+ R_1^-$ correspondingly. Then,
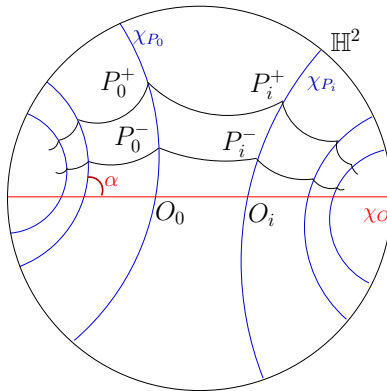
FIGURE 3.5. The quadrilateral $P_0^+ P_0^- P_i^+ P_i^-$, $i = 1, 2$, in Situation 2.

by Lemma 3.3 applied to the points $R_0^+ \in \chi_{R_0}$ and $R_1^+ \in \chi_{R_1}$ there is a a unique hyperbolic straight line $\chi_O \subset \mathbb{H}^2$ intersecting χ_{R_0} at a point O_0, χ_{R_1} at a point O_1, such that R_0^+ and R_1^+ lie in the same half-plane with respect to χ_O, $h^+ \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^+, O_0) = \mathrm{d}_{\mathbb{H}^2}(R_1^+, O_1)$, and the angles of intersection $\angle(\chi_O, \chi_{R_0})$ and $\angle(\chi_O, \chi_{R_1})$ are equal to some $\alpha \in (0, \pi/2)$. Denote also $h^- \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R_0^-, O_0) = \mathrm{d}_{\mathbb{H}^2}(R_1^-, O_1)$ and $l_O \stackrel{\text{def}}{=} \mathrm{d}_{\mathbb{H}^2}(O_0, O_1)$.

Let the hyperbolic isometry $\bar{\chi}_O$ of $\mathbb{H}^2$ send O_0 to O_1 leaving the geodesic χ_O invariant. Note that $\bar{\chi}_O$ sends also R_0^+ to R_1^+ and R_0^- to R_1^-. We define points $R_i^+ \stackrel{\text{def}}{=} \bar{\chi}_O^i . R_0^+$, $R_i^- \stackrel{\text{def}}{=} \bar{\chi}_O^i . R_0^-$, and $O_i \stackrel{\text{def}}{=} \bar{\chi}_O^i . O_0$ for $i \in \mathbb{Z}$, where the symbol $\bar{\chi}_O^i$ stands for the isometry $\bar{\chi}_O$ applied i times when i is a positive integer, and for the inverse isometry $\bar{\chi}_O^{-1}$ applied $-i$ times when $i < 0$. Denote by χ_{R_i} the hyperbolic straight line containing the segment $R_i^+ R_i^-$, $i \in \mathbb{Z}$. Construct the curves $\nu_+ \stackrel{\text{def}}{=} \bigcup_{i \in \mathbb{Z}} R_i^+ R_{i+1}^+$ and $\nu_- \stackrel{\text{def}}{=} \bigcup_{i \in \mathbb{Z}} R_i^- R_{i+1}^-$ of the geodesic segments $R_i^+ R_{i+1}^+$ and $R_i^- R_{i+1}^-$, $i \in \mathbb{Z}$. Remark that for each $i \in \mathbb{Z}$ the quadrilateral $R_i^+ R_i^- R_{i+1}^+ R_{i+1}^- \subset \mathbb{H}^2$ serves as a fundamental domain of the cylinder Cyl_0 in $\mathbb{H}^2$, and the connected domain between the curves ν_+ and ν_- of the hyperbolic plane is a universal covering of Cyl_0 in $\mathbb{H}^2$. By construction, $\mathrm{d}_{\mathbb{H}^2}(R_i^+, R_i^-) = h$, $\mathrm{d}_{\mathbb{H}^2}(R_i^+, O_i) = h^+$, $\mathrm{d}_{\mathbb{H}^2}(R_i^-, O_i) = h^-$, $\mathrm{d}_{\mathbb{H}^2}(R_i^+, R_{i+1}^+) = l^+$, $\mathrm{d}_{\mathbb{H}^2}(R_i^-, R_{i+1}^-) = l^-$, $\angle(\chi_O, \chi_{R_i}) = \alpha$, $i \in \mathbb{Z}$.

Let us construct a family of hyperbolic straight lines χ_i^+ passing through R_i^+ and orthogonal to χ_O, $i \in \mathbb{Z}$. Define the points of intersection $O_i^+ \stackrel{\text{def}}{=} \chi_i^+ \cap \chi_O$, $T_i^- \stackrel{\text{def}}{=} \chi_i^+ \cap \nu_-$, $i \in \mathbb{Z}$. Note that, by construction, the connected sets Ξ_i^+

bounded by χ^+_{i+1}, ν_+, χ^+_i, and ν_- are fundamental domains of the cylinder Cyl_0 in $\mathbb{H}^2$, $i \in \mathbb{Z}$.

REMARK 3.10. — The geodesic segment $R^+_{i+1}R^-_{i+1}$ lies inside the fundamental domain $\Xi^+_i \subset \mathbb{H}^2$ of a cylinder Cyl_0 of the type Cyl; on the other hand, the geodesic segment $R^+_i T^-_i$ lies inside the fundamental domain $R^+_i R^-_i R^+_{i+1} R^-_{i+1} \subset \mathbb{H}^2$ of the same cylinder Cyl_0, $i \in \mathbb{Z}$.

Proof. — Since for every integer i the hyperbolic straight lines χ^+_i are orthogonal to the geodesic χ_O corresponding to the closed geodesic χ° of the unbounded cylinder $\mathrm{Cyl}^\circ_0 = \mathbb{H}^2/\langle \bar{\chi}_O \rangle$ which contains Cyl_0 (see also Section 3.2), the projection on Cyl_0 of a path $\xi \subset \Xi^+_i$ connecting any point P^u of the upper boundary $\partial \Xi^+_i \cap \nu_+ (= R^+_i R^+_{i+1})$ of Ξ^+_i with any point P^l of its lower boundary $\partial \Xi^+_i \cap \nu_-$ does not make a full turn around Cyl_0.

Let us fix $i \in \mathbb{Z}$. As $\Xi^+_i \subset \mathbb{H}^2$ is a fundamental domain of Cyl_0, the lower boundary $\partial \Xi^+_i \cap \nu_-$ of Ξ^+_i must contain at least one and at most two points of the family $\{R^-_j \in \mathbb{H}^2 | j \in \mathbb{Z}\}$ corresponding to one point on Cyl_0. Consider the point R^-_{i+1} of this family. By Remark 3.4, the length of the segment $R^+_{i+1}R^-_{i+1}$ is the smallest one among the lengths of all the segments $R^+_{i+1}R^-_j$, $j \in \mathbb{Z}$. Hence, the projection on Cyl_0 of $R^+_{i+1}R^-_{i+1}$ does not make a full turn around Cyl_0 (otherwise, there would be a path shorter than $R^+_{i+1}R^-_{i+1}$ among the segments $R^+_{i+1}R^-_j$, $j \in \mathbb{Z}$). Since $\alpha \in (0, \pi/2)$, we conclude that $R^+_{i+1}R^-_{i+1} \subset \Xi^+_i$. Similarly, $R^+_i R^-_i \subset \Xi^+_{i-1}$. Hence, $R^+_i T^-_i \subset R^+_i R^-_i R^+_{i+1} R^-_{i+1}$. $\square$

Similarly, we construct a family of hyperbolic straight lines χ^-_i passing through R^-_i and orthogonal to χ_O, $i \in \mathbb{Z}$, and define the points of intersection $O^-_i \overset{\mathrm{def}}{=} \chi^-_i \cap \chi_O$, $T^+_i \overset{\mathrm{def}}{=} \chi^-_i \cap \nu_+$, $i \in \mathbb{Z}$. By construction, the connected sets Ξ^-_i bounded by χ^-_{i+1}, ν_+, χ^-_i, and ν_- are fundamental domains of the cylinder Cyl_0 in $\mathbb{H}^2$ and, by analogy with Remark 3.10, the following statement holds true.

REMARK 3.11. — The geodesic segment $R^+_i R^-_i$ lies inside the fundamental domain $\Xi^-_i \subset \mathbb{H}^2$ of a cylinder Cyl_0 of the type Cyl; on the other hand, the geodesic segment $R^-_{i+1}T^+_{i+1}$ lies inside the fundamental domain $R^+_i R^-_i R^+_{i+1} R^-_{i+1} \subset \mathbb{H}^2$ of the same cylinder Cyl_0, $i \in \mathbb{Z}$.

Also, define $h^+_O \overset{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R^+_i, O^+_i)$, $h^-_O \overset{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(R^-_i, O^-_i)$, and note that $\mathrm{d}_{\mathbb{H}^2}(O_i, O_{i+1}) = \mathrm{d}_{\mathbb{H}^2}(O^+_i, O^+_{i+1}) = \mathrm{d}_{\mathbb{H}^2}(O^-_i, O^-_{i+1}) = l_O$, $i \in \mathbb{Z}$.

3.5.1. *Consideration of Situation 1.* — In this section, we demonstrate

LEMMA 3.12. — *Let a cylinder of the type* Cyl *contain a closed geodesic and possess a fundamental domain* $R^+_0 R^+_1 R^-_0 R^-_0 \subset \mathbb{H}^2$. *Define by* l^+ *and* l^- *the*

lengths of the sides $R_0^+ R_1^+$ *and* $R_0^- R_1^-$, *and by* h *the length of* $R_0^+ R_0^-$ *and* $R_1^+ R_1^-$. *Then the condition*

(3.11)
$$h \geq 2\max\left\{ \operatorname{arcosh}\left[\cosh l^+ \cosh\left(l^+ + \operatorname{arcosh}\frac{e^{l^+}(l^+)^2}{\varepsilon_3^2}\right)\right], \right.$$
$$\left. \operatorname{arcosh}\left[\cosh l^- \cosh\left(l^- + \operatorname{arcosh}\frac{e^{l^-}(l^-)^2}{\varepsilon_3^2}\right)\right]\right\}.$$

guarantees that there is a path in $R_0^+ R_1^+ R_0^- R_0^-$ *connecting the midpoints of* $R_0^+ R_0^-$ *and* $R_1^+ R_1^-$, *and such that its length is smaller than the Margulis constant* ε_3.

As we consider Situation 1, we suppose that $O_i \in R_i^- R_i^+$ for $i \in \mathbb{Z}$ and, consequently,

(3.12)
$$h = h^- + h^+.$$

For all $i \in \mathbb{Z}$, let us denote the midpoint of the segment $R_i^+ R_i^-$ by R_i^{mid}, the midpoints of $R_i^+ O_i$ and $R_i^- O_i$ by $R_i^{\mathrm{mid}+}$ and $R_i^{\mathrm{mid}-}$, the midpoints of $R_i^+ O_i^+$ and $R_i^- O_i^-$ by $O_i^{\mathrm{mid}+}$ and $O_i^{\mathrm{mid}-}$. Denote the distances from the points R_i^{mid} to the straight hyperbolic line χ_O by d, from $R_i^{\mathrm{mid}+}$ to χ_O by d^+, from $R_i^{\mathrm{mid}-}$ to χ_O by d^- and note that, by construction, the distances from the points $O_i^{\mathrm{mid}+}$ to χ_O are equal to $h_O^+/2$ and from the points $O_i^{\mathrm{mid}-}$ to χ_O are equal to $h_O^-/2$, $i \in \mathbb{Z}$.

Denote by $\hat{\chi}$ a curve in $\mathbb{H}^2$ at distance d from χ_O and passing through the points R_i^{mid} for all i integers; by $\hat{\chi}_R^+$ a curve in $\mathbb{H}^2$ at distance d^+ from χ_O and passing through the points $R_i^{\mathrm{mid}+}$; by $\hat{\chi}_R^-$ a curve in $\mathbb{H}^2$ at distance d^- from χ_O and passing through the points $R_i^{\mathrm{mid}-}$; by $\hat{\chi}_O^+$ a curve in $\mathbb{H}^2$ at distance $h_O^+/2$ from χ_O and passing through the points $O_i^{\mathrm{mid}+}$; by $\hat{\chi}_O^-$ a curve in $\mathbb{H}^2$ at distance $h_O^-/2$ from χ_O and passing through the points $O_i^{\mathrm{mid}-}$, $i \in \mathbb{Z}$.

REMARK 3.13. — In the notation defined above, the inequalities

(3.13)
$$d^+ \leq \frac{h_O^+}{2} \quad \text{and} \quad d^- \leq \frac{h_O^-}{2}$$

hold true.

Proof. — Define by $\hat{R}_0^{\mathrm{mid}+}$ the orthogonal projection of the point $R_0^{\mathrm{mid}+}$ on $\chi_O \subset \mathbb{H}^2$ and consider the hyperbolic triangles $\triangle O_0 O_0^+ R_0^+$ and $\triangle O_0 \hat{R}_0^{\mathrm{mid}+} R_0^{\mathrm{mid}+}$. Recall that $\mathrm{d}_{\mathbb{H}^2}(R_0^+, O_0^+) = h_O^+$, $\mathrm{d}_{\mathbb{H}^2}(R_0^{\mathrm{mid}+}, \hat{R}_0^{\mathrm{mid}+}) = d^+$, $\mathrm{d}_{\mathbb{H}^2}(R_0^+, O_0) = h^+$, $\mathrm{d}_{\mathbb{H}^2}(R_0^{\mathrm{mid}+}, O_0) = h^+/2$, $\angle R_0^+ O_0 O_0^+ = \angle R_0^{\mathrm{mid}+} O_0 \hat{R}_0^{\mathrm{mid}+} = \alpha$, and $\angle O_0 O_0^+ R_0^+ = \angle O_0 \hat{R}_0^{\mathrm{mid}+} R_0^{\mathrm{mid}+} = \pi/2$.

Applying Hyperbolic Law of Sines to $\triangle O_0 O_0^+ R_0^+$ and $\triangle O_0 \hat{R}_0^{\mathrm{mid}+} R_0^{\mathrm{mid}+}$, we obtain the formulas

$$\frac{\sin \alpha}{\sinh h_O^+} = \frac{\sin \frac{\pi}{2}}{\sinh h^+}$$

and

$$\frac{\sin\alpha}{\sinh d^+} = \frac{\sin\frac{\pi}{2}}{\sinh\frac{h^+}{2}},$$

or, after simplification,

$$(3.14) \qquad\qquad \sinh h_O^+ = \sin\alpha\sinh h^+$$

and

$$(3.15) \qquad\qquad \sinh d^+ = \sin\alpha\sinh\frac{h^+}{2}.$$

Note that when the formula

$$(3.16) \qquad\qquad \sinh d^+ \le \sinh\frac{h_O^+}{2}$$

holds true, the first relation in (3.13) is satisfied.

By (3.15), (3.16) is equivalent to

$$(3.17) \qquad\qquad \sin\alpha\sinh\frac{h^+}{2} \le \sinh\frac{h_O^+}{2}.$$

Due to the following property of the hyperbolic sine: $\sinh 2x = 2\sinh x\cosh x$, from (3.14) we get

$$(3.18) \qquad 2\sinh\frac{h_O^+}{2}\cosh\frac{h_O^+}{2} = 2\sin\alpha\sinh\frac{h^+}{2}\cosh\frac{h^+}{2}$$

As $h_O^+ \le h^+$ by construction and the function $\cosh x$ is monotonically increasing for $x \ge 0$, then it is true that $\cosh(h_O^+/2) \le \cosh(h^+/2)$ and, by (3.14), we obtain

$$(3.19) \qquad \sinh\frac{h_O^+}{2}\cosh\frac{h^+}{2} \ge \sin\alpha\sinh\frac{h^+}{2}\cosh\frac{h^+}{2}.$$

Simplifying (3.19), we see that the condition (3.17) is satisfied. Hence, the first inequality in (3.13) holds true.

The validity of the second relation in (3.13) we prove by the same method.

$\square$

Together with constructions made above, Remark 3.13 means geometrically that the curve $\hat{\chi}$ lies inside the connected domain of the hyperbolic plane bounded by the curves $\hat{\chi}_R^+$ and $\hat{\chi}_R^-$ which is embedded into the connected domain bounded by $\hat{\chi}_O^+$ and $\hat{\chi}_O^-$ which is embedded, in its turn, into the connected domain bounded by ν_+ and ν_-.

By Remark 3.6, the length of the path $\overparen{R_i^{\mathrm{mid}}R_{i+1}^{\mathrm{mid}}}$ connecting the points R_i^{mid} and R_{i+1}^{mid} on the curve $\hat{\chi}$ is $\hat{l} = l_O\cosh d$, the length of the path $\overparen{R_i^{\mathrm{mid}+}R_{i+1}^{\mathrm{mid}+}} \subset \hat{\chi}_R^+$ connecting the points $R_i^{\mathrm{mid}+}$ and $R_{i+1}^{\mathrm{mid}+}$ is $\hat{l}_R^+ = l_O\cosh d^+$, the length of the path $\overparen{R_i^{\mathrm{mid}-}R_{i+1}^{\mathrm{mid}-}} \subset \hat{\chi}_R^-$ connecting the points $R_i^{\mathrm{mid}-}$ and $R_{i+1}^{\mathrm{mid}-}$ is

$\hat{l}_R^- = l_O \cosh d^-$, the length of the path $O_i^{\mathrm{mid}+} O_{i+1}^{\mathrm{mid}+} \subset \hat{\chi}_O^+$ connecting the points $O_i^{\mathrm{mid}+}$ and $O_{i+1}^{\mathrm{mid}+}$ is $\hat{l}_O^+ = l_O \cosh(h_O^+/2)$, and the length of the path $O_i^{\mathrm{mid}-} O_{i+1}^{\mathrm{mid}-} \subset \hat{\chi}_O^-$ connecting the points $O_i^{\mathrm{mid}-}$ and $O_{i+1}^{\mathrm{mid}-}$ is $\hat{l}_O^- = l_O \cosh(h_O^-/2)$, $i \in \mathbb{Z}$.

Assume that $R_i^{\mathrm{mid}} \in R_i^+ O_i$, $i \in \mathbb{Z}$. According to Remark 3.13, we have

$$(3.20) \qquad\qquad l_O \leq \hat{l} \leq \hat{l}_R^+ \leq \hat{l}_O^+ \leq l^+.$$

Otherwise $R_i^{\mathrm{mid}} \in R_i^- O_i$, $i \in \mathbb{Z}$ and

$$(3.21) \qquad\qquad l_O \leq \hat{l} \leq \hat{l}_R^- \leq \hat{l}_O^- \leq l^-$$

(remind that we consider Situation 1). Hence, if we prove that for h big enough $\hat{l}_O^+ < \varepsilon_3$ and $\hat{l}_O^- < \varepsilon_3$, then $\hat{l} < \varepsilon_3$ and the projection of the path $R_i^{\mathrm{mid}} R_{i+1}^{\mathrm{mid}} \subset \mathbb{H}^2$ on the cylinder Cyl_0 is a closed curve which is shorter than the Margulis constant ε_3 and which passes through the midpoint R^{mid} of the segment $R^+ R^- \subset \mathrm{Cyl}_0$ corresponding to $R_i^+ R_i^- \subset \mathbb{H}^2$, $i \in \mathbb{Z}$.

First, fixing l^+ let us find a condition on h^+ which will guarantee $\hat{l}_O^+$ to be less than ε_3.

By Remark 3.10, the geodesic segment $R_0^+ T_0^-$ lies inside the fundamental domain $R_0^+ R_0^- R_1^+ R_1^- \subset \mathbb{H}^2$. Hence, the point O_0^+ of intersection of $R_0^+ T_0^-$ with χ_O belongs to the geodesic segment $O_0 O_1$.

Denote $l_{O_0^+ O_0} \overset{\mathrm{def}}{=} \mathrm{d}_{\mathbb{H}^2}(O_0^+, O_0)$ and consider the right-angled triangle $\triangle O_0 O_0^+ R_0^+$. Hyperbolic Pythagorean Theorem implies:

$$(3.22) \qquad\qquad \cosh h^+ = \cosh h_O^+ \cosh l_{O_0^+ O_0}.$$

Since $O_0 O_0^+ \subset O_0 O_1$, the inequality $l_{O_0^+ O_0} \leq l_O$ holds true and, together with (3.22) gives us

$$\cosh h^+ \leq \cosh h_O^+ \cosh l_O,$$

and, by (3.20),

$$\cosh h^+ \leq \cosh h_O^+ \cosh l^+,$$

or, in other form,

$$(3.23) \qquad\qquad \cosh h_O^+ \geq \frac{\cosh h^+}{\cosh l^+}.$$

It means that, once we take h^+ to satisfy the condition

$$(3.24) \qquad \cosh h^+ \geq \cosh l^+ \cosh\left(l^+ + \mathrm{arcosh}\frac{e^{l^+}(l^+)^2}{\varepsilon_3^2}\right),$$

then, according to (3.23),

$$h_O^+ \geq l^+ + \mathrm{arcosh}\frac{e^{l^+}(l^+)^2}{\varepsilon_3^2},$$

and, by Lemma 3.9 applied to the quadrilateral $O_0^+ O_1^+ R_0^+ R_1^+$, we conclude that

$$(3.25) \qquad\qquad\qquad\qquad \hat{l}_O^+ \leq \varepsilon_3.$$

Similarly, if we take h^- to verify the inequality

$$(3.26) \qquad\qquad \cosh h^- \geq \cosh l^- \cosh\left(l^- + \operatorname{arcosh} \frac{e^{l^-}(l^-)^2}{\varepsilon_3^2} \right),$$

then

$$(3.27) \qquad\qquad\qquad\qquad \hat{l}_O^- \leq \varepsilon_3.$$

Finally, let the condition (3.11) be satisfied. Supposing $h^+ \geq h^-$, we have $\widehat{R_0^{\mathrm{mid}} R_1^{\mathrm{mid}}} \subset O_0^+ O_1^+ R_0^+ R_1^+$ and, by (3.12), the inequality (3.24) holds true, which implies (3.25) and, due to (3.20), leads as to the validity of the condition

$$(3.28) \qquad\qquad\qquad\qquad \hat{l} \leq \varepsilon_3.$$

On the other hand, if $h^+ < h^-$ then $\widehat{R_0^{\mathrm{mid}} R_1^{\mathrm{mid}}} \subset O_0^- O_1^- R_0^- R_1^-$ and, by (3.12), the inequality (3.26) holds true, which implies (3.27) and, due to (3.21), leads as to the validity of (3.28).

Lemma 3.12 is proved.

3.5.2. *Consideration of Situation 2.* —

LEMMA 3.14. — *Let a cylinder of the type* Cyl *do not contain a closed geodesic and possess a fundamental domain* $R_0^+ R_1^+ R_0^- R_0^- \subset \mathbb{H}^2$. *Define by* l^+ *and* l^- *the lengths of the sides* $R_0^+ R_1^+$ *and* $R_0^- R_1^-$, *and by* h *the length of* $R_0^+ R_0^-$ *and* $R_1^+ R_1^-$. *Then*

$$h < \max\left\{ \left(l^+ + l^- + \ln \frac{2l^+}{l^-} \right), \left(l^+ + l^- + \ln \frac{2l^-}{l^+} \right) \right\}.$$

Lemma 3.14 can be proved in the same spirit as Lemma 3.12. For the detailed arguments see [23] (and also [24]).

3.6. Finalizing the proof of Theorem 3.2. —

Consider some points $P^+ \in c_1^+ \cap c_2^+$ and $P^- \in c_1^- \cap c_2^-$. As in Section 3.1, construct the cylinders Cyl_1 and Cyl_2 of the type Cyl homotopically equivalent to the pairs of curves (c_1^+, c_1^-) and (c_2^+, c_2^-), with the upper boundaries of the lengths l_1^+ and l_2^+, with the lower boundaries of the lengths l_1^- and l_2^-, and such that the hyperbolic geodesic segment $P^+ P^- \subset \mathcal{M}^\circ$ lies in the intersection $\mathrm{Cyl}_1 \cap \mathrm{Cyl}_2$.

If Situation 2 is realized for at least one of the cylinders Cyl_1 and Cyl_2, than Lemma 3.14 implies that

$$d(\mathcal{S}^+, \mathcal{S}^-) < \max \left\{ \left(l_1^+ + l_1^- + \ln \frac{2l_1^+}{l_1^-} \right), \left(l_1^+ + l_1^- + \ln \frac{2l_1^-}{l_1^+} \right), \right.$$
$$\left. \left(l_2^+ + l_2^- + \ln \frac{2l_2^+}{l_2^-} \right), \left(l_2^+ + l_2^- + \ln \frac{2l_2^-}{l_2^+} \right) \right\}.$$

Otherwise, Situation 1 is realized for both cylinders Cyl_1 and Cyl_2 and, once we suppose

$$d(\mathcal{S}^+, \mathcal{S}^-) < 2 \max \left\{ \operatorname{arcosh} \left[\cosh l_1^+ \cosh \left(l_1^+ + \operatorname{arcosh} \frac{e^{l_1^+} (l_1^+)^2}{\varepsilon_3^2} \right) \right], \right.$$
$$\operatorname{arcosh} \left[\cosh l_1^- \cosh \left(l_1^- + \operatorname{arcosh} \frac{e^{l_1^-} (l_1^-)^2}{\varepsilon_3^2} \right) \right],$$
$$\operatorname{arcosh} \left[\cosh l_2^+ \cosh \left(l_2^+ + \operatorname{arcosh} \frac{e^{l_2^+} (l_2^+)^2}{\varepsilon_3^2} \right) \right],$$
$$\left. \operatorname{arcosh} \left[\cosh l_2^- \cosh \left(l_2^- + \operatorname{arcosh} \frac{e^{l_2^-} (l_2^-)^2}{\varepsilon_3^2} \right) \right] \right\},$$

by Lemma 3.12, there are curves $cur_1 \subset \mathrm{Cyl}_1$ and $cur_2 \subset \mathrm{Cyl}_2$ with the lengths less than the Margulis constant ε_3, both passing through the midpoint of the segment $P^+ P^-$. Thus, we come to a contradiction with Margulis Lemma.

Theorem 3.2 is proved. $\square$

Acknowledgements

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