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THE GALOIS ACTION AND A SPIN INVARIANT FOR PRYM-TEICHMÜLLER CURVES IN GENUS 3

by Jonathan Zachhuber

Abstract. — Given a Prym-Teichmüller curve in $\mathcal{M}_3$, this note provides an invariant that sorts the cusp prototypes of Lanneau and Nguyen by component. This can be seen as an analog of McMullen's genus 2 spin invariant, although the source of this invariant is different. Moreover, we describe the Galois action on the cusps of these Teichmüller curves, extending the results of Bouw and Möller in genus 2. We use this to show that the components of the genus 3 Prym-Teichmüller curves are homeomorphic.

Résumé (*L'action galoisienne et un invariant spin pour les courbes de Prym-Teichmüller en genre 3*). — Étant donnée une courbe de Prym–Teichmüller dans $\mathcal{M}_3$, cette note introduit un invariant qui trie par composante les prototypes cusp de Lanneau et Nguyen. Il peut être vu comme l'analogue en genre 3 de l'invariant spin en genre 2 de McMullen, bien que la source de cet invariant soit différente. De plus, nous décrivons l'action de Galois sur les cusps des courbes de Teichmüller, étendant les résultats en genre 2 de Bouw et Möller. Cela nous permet de montrer que les composants des courbes de Prym–Teichmüller en genre 3 sont homéomorphes.

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1. Introduction

A *Teichmüller curve* is a curve inside the moduli space $\mathcal{M}_g$ of smooth projective genus g curves that is totally geodesic for the Teichmüller metric. Every Teichmüller curve arises as the projection of the $\mathrm{GL}_2^+(\mathbb{R})$ orbit of a flat surface (see section 2 and the references therein for background and definitions). Only a few infinite families of primitive Teichmüller curves are known. McMullen constructed several primitive families in low genera, among them, for every non-square discriminant D, the *Prym-Teichmüller* or *Prym-Weierstraß* curves W_D in genus 3 [4].

This family is fairly well understood. In particular, Möller calculated the Euler characteristic [8], Lanneau and Nguyen enumerated the cusps and connected components [2], and the number and type of orbifold points are determined in [12]. The aim of this note is to complete the classification of the topological components by showing that the connected components of W_D are always homeomorphic.

To be more precise, in [2], Lanneau and Nguyen show that W_D has at most two components for any D and has two components if and only if $D \equiv 1 \bmod 8$.

Theorem 1.1. — *Let $D \equiv 1 \bmod 8$, which is not a square. Then the two components of W_D are homeomorphic (as orbifolds). In particular, they have the same number of cusps and elliptic points.*

A similar result was obtained by Bouw and Möller [1] for Teichmüller curves in genus 2. Note that a Teichmüller curve is always defined over a number field but is never compact. Both approaches rely on determining the stable curves associated to the cusps of the Teichmüller curve and describing explicitly the Galois action on these cusps. At this point, it is crucial that we are able to determine of a pair of cusps if they lie on the same component or not. In genus 2, Bouw and Möller could use McMullen's spin invariant [3] to achieve this.

However, while Lanneau and Nguyen list prototypes corresponding to the cusps of Prym-Teichmüller curves [2], they do not provide an effective analog of the spin invariant. Here we give such an invariant, which is, moreover, easy to compute.

Theorem 1.2. — *Let $D \equiv 1 \bmod 8$, which is not a square. Given a cusp prototype $[w, h, t, e, \varepsilon]$ (see section 2), the associated cusp of W_D lies on the component W_D^i if and only if*

$$2i \equiv e + \varepsilon \bmod 4,$$

for $i = 1, 2$.

In section 3, we prove Theorem 1.2 essentially using topological arguments. More precisely, we analyze the intersection pairing on a certain intrinsic subspace of homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. This is similar to the approach of [3] where the Arf invariant of a quadratic form that was associated to the flat structure was analyzed on such a subspace, but the nature of these subspaces is different (cf. [2, Remark 2.9]). Note also that in genus 3 the two components lie on disjoint Hilbert Modular Surfaces (cf. [8, Proposition 4.6]) and that the $(1,2)$-polarization of the Prym variety plays a special role in this case, essentially yielding a much more compact formula (cf. [3, Theorem 5.3]).

In section 4, we proceed to give an explicit description of the Galois action on Lanneau and Nguyen's cusp prototypes (Proposition 4.7) and combine this with Theorem 1.2 to show that Galois-conjugate cusps always lie on different components of W_D, thus proving Theorem 1.1.

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2. Cusp Prototypes

A *flat surface* is a pair (X, ω) where X is a compact Riemann surface of genus g and $\omega \in \mathrm{H}^0(X, \omega_X)$ is a holomorphic 1-form on X. Note that X obtains a flat structure away from the zeros of ω via integrating ω and affine shearing of this flat structure gives an action of $\mathrm{GL}_2^+(\mathbb{R})$. A *Teichmüller curve* is a $\mathrm{GL}_2^+(\mathbb{R})$ orbit of a flat surface that projects to an algebraic curve inside the moduli space $\mathcal{M}_g$. See e.g., [7] for background on Teichmüller curves and flat surfaces. Not many families of primitive Teichmüller curves are known; McMullen constructed families in low genera by requiring a factor of the Jacobian of X to admit real multiplication, the (Prym-)Weierstraß curves. We briefly review the construction in genus 3, the case with which we are concerned.

Prym Varieties and Real Multiplication. — Let $D \equiv 0, 1 \bmod 4$ be a (positive) non-square discriminant and denote by $\mathcal{O}_D$ the corresponding order in the real quadratic number field $\mathbb{Q}(\sqrt{D})$. Let X be a genus 3 curve and ρ an involution with X/ρ of genus 1. Then we define the *Prym Variety* $\mathrm{Prym}(X, \rho)$ as the connected component of the identity of $\ker(\mathrm{Jac}(X) \to \mathrm{Jac}(X/\rho))$ and we say that (X, ρ) *admits real multiplication by* $\mathcal{O}_D$ if there exists an injective ring homomorphism $\iota \colon \mathcal{O}_D \to \mathrm{End}\,\mathrm{Prym}(X, \rho)$, such that

- every endomorphism $\iota(s)$ is self-adjoint with respect to the intersection pairing on H_1, and
- ι cannot be extended to any $\mathcal{O}_{D'} \supset \mathcal{O}_D$.

In other words, the ρ-anti-invariant part $\mathrm{H}_1(X, \mathbb{Z})^-$ of the homology admits a symplectic $\mathcal{O}_D$-module structure and $\mathcal{O}_D$ is maximal in this respect.

Prym-Weierstraß Curves. — Denote by W_D the space of genus 3 flat surfaces (X, ω, ρ, ι) with an involution ρ that admit real multiplication ι as above and where additionally ω has a single (4-fold) zero, is ρ-anti-invariant, and is an eigenform for the induced action of $\mathcal{O}_D$ on $\mathrm{H}^0(X, \omega_X)$. McMullen [4] showed that W_D is a union of Teichmüller curves, the *genus* 3 *Prym-Weierstraß* or *Prym-Teichmüller curves of discriminant* D. Prym-Weierstraß curves have been studied intensely, see e.g., [4], [8], [2] and [12]. Note, in particular, that W_D is empty for $D \equiv 5 \bmod 8$.

Again, we note that we explicitly exclude the case that $D = d^2$ is a square, see [2, Appendix B] for some results in this case.

Cusps. — Recall that a Teichmüller curve $\mathcal{C}$ is never compact. We describe the cusps first in the terminology of flat surfaces. Let (X, ω) be a flat surface generating $\mathcal{C}$ and consider a direction $v \in \mathbb{P}^1(\mathbb{R})$. Recall that a geodesic segment is said to be a *saddle connection* if its endpoints are (not necessarily distinct) zeros of ω and its interior contains no zeros of ω. The direction v is said to be *periodic* if all geodesics in direction v are either closed or saddle connections. We say that a *cylinder* is a maximal union of homotopic geodesics on (X, ω) and any closed geodesic inside a cylinder is a *core curve*. The length of a core curve is the *width* of the cylinder. A cylinder is called *simple* if each boundary consists of a saddle connection. The cusps of $\mathcal{C}$ are in one-to-one correspondence with the parabolic *cylinder decompositions* on (X, ω), see e.g., [3, §4], [13, §2] or [7, §5.4].

Prototypes. — To describe the cusps of W_D, Lanneau and Nguyen introduce prototypes that encode the cylinder decompositions [2, §3,4 and C]. We briefly summarize the results we need.

The following result is a slight refinement of [2, Proposition 3.2].

Lemma 2.1. — *Given D non-square and a point (X, ω) on W_D, any periodic direction decomposes (X, ω) into three cylinders.*

Proof. — By [2, Proposition 3.2], any periodic direction decomposes (X, ω) into either three cylinders, or two cylinders that are permuted by the Prym involution or one cylinder (that is fixed by the Prym involution). Obviously, in the last two cases, the ratio of cylinder circumferences is 1. However, [14, Theorem 1.9] asserts that adjoining the ratio of cylinder circumferences to $\mathbb{Q}$ gives the trace field of (X, ω), which is $\mathbb{Q}(\sqrt{D})$ (cf. [4, Corollary 3.6]), a contradiction. $\square$

Remark 2.2. — *Lemma 2.1 can be seen as a converse to* [2, Corollary 3.4].

Following [2], after rescaling, applying Dehn-twists, and normalizing so that the horizontal direction is periodic, this decomposition may be encoded in a

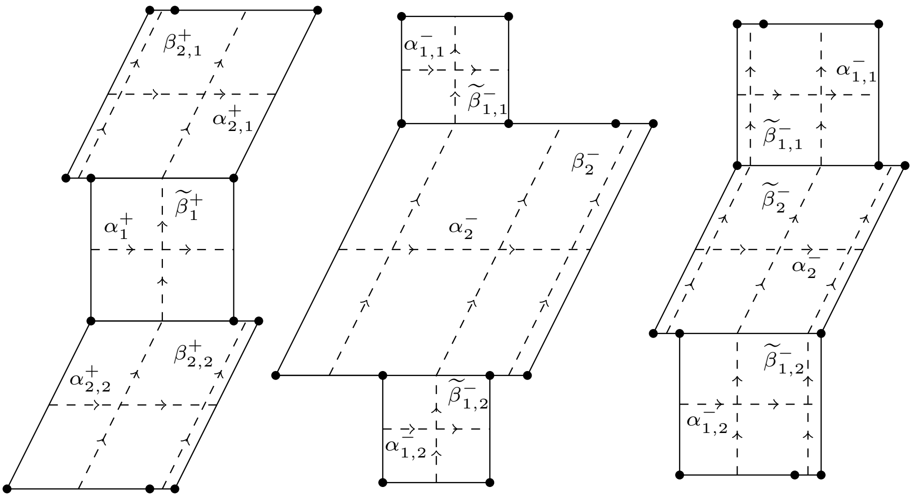

FIGURE 2.1. Prototypes of geometric type $A+$, $A-$ and B. Observe that all α_i are drawn in a horizontal direction, the β_i are drawn vertical. We set $\alpha_i = \alpha_{i,1} + \alpha_{i,2}$ and $\beta_i = \beta_{i,1} + \beta_{i,2}$ when appropriate, and furthermore, for the $A+$ prototype, $\beta_1^+ = \widetilde{\beta}_1^+ - \beta_2^+$, for the $A-$ prototype, $\beta_{1,i}^- = \widetilde{\beta}_{1,i}^- - \beta_2^-$, and, for the B prototype, $\beta_2^- = \widetilde{\beta}_{1,1}^- + \widetilde{\beta}_{1,2}^- - \widetilde{\beta}_2^-$ and $\beta_{1,i}^- = \widetilde{\beta}_{1,i}^- - \beta_2^-$. Thus the α_i and β_i give symplectic bases whose periods describe the cylinder heights and widths.

combinatorial prototype

$$P_D = [w, h, t, e, \varepsilon] \in \mathbb{Z}^5$$

subject to the following conditions:

$$\begin{cases} D = e^2 + 8wh, \ \varepsilon = \pm 1, \ w, h > 0, \\ w > \frac{\lambda}{2}, \ 0 \le t < \gcd(w, h), \ \gcd(w, h, t, e) = 1, \end{cases}$$

where we set

$$\lambda := \lambda_P := \frac{e + \sqrt{D}}{2}. \tag{1}$$

Moreover, if $\varepsilon = 1$, the stronger condition $w > \lambda$ is required.

Conversely, given a combinatorial prototype, we obtain a three-cylinder decomposition into one of the following three geometric types (see Figure 2.1):

- $A+$: If $\varepsilon = 1$ and $\lambda < w$, we obtain a cylinder decomposition with a single (short) simple cylinder of width and height λ and two cylinders of width w, height h and twist t.

- $A-$: If $\varepsilon = -1$ and $\lambda < w$, we obtain a cylinder decomposition with two (short) simple cylinders of width and height $\lambda/2$ and a third cylinder of width w, height h and twist t.
- B: If $\varepsilon = -1$ and $\lambda/2 < w < \lambda$, we obtain a cylinder decomposition with no simple cylinders but again two short cylinders of width and height $\lambda/2$ and a third cylinder of width w, height h and twist t.

Each geometric prototype corresponds to exactly one cusp of W_D.

3. Components and Spin

In analogy to the situtation in genus 2, Lanneau and Nguyen showed that, for any discriminant D, the locus W_D has at most two components [2, Theorem 2.8, 2.10]. More precisely, W_D has two components if and only if $D \equiv 1 \bmod 8$. In the following, we denote these components by W_D^1 and W_D^2.

The aim of this section is to provide an analog of McMullen's spin invariant in genus 2 [3], i.e., an invariant that determines if a cusp prototype is associated to a cusp on W_D^1 or W_D^2.

To each geometric prototype $P_D = [w, h, t, e, \varepsilon]$, Lanneau and Nguyen associate a basis $\mathfrak{b} = \mathfrak{b}(P_D)$ of $\mathrm{H}_1(X, \mathbb{Z})^-$ "spanning cylinders", cf. [2, §4]. We will see that, in fact, the behavior of the basis will depend only on ε, i.e., geometric type $A-$ and B will not be distinguished. Hence, we denote the bases by

$$\mathfrak{b}^\varepsilon = (\alpha_1^\varepsilon, \alpha_2^\varepsilon, \beta_1^\varepsilon, \beta_2^\varepsilon),$$

where α_i and β_i are as in Figure 2.1. In particular, the periods (with respect to ω) are

$$\int_{\alpha_1^+} \omega = \lambda, \quad \int_{\alpha_2^+} \omega = 2w, \quad \int_{\beta_1^+} \omega = \mathrm{i}\lambda, \quad \int_{\beta_2^+} \omega = 2t + 2\mathrm{i}h \tag{2}$$

if P_D is of geometric type $A+$ (i.e., $\varepsilon = 1$) and

$$\int_{\alpha_1^-} \omega = \lambda, \quad \int_{\alpha_2^-} \omega = w, \quad \int_{\beta_1^-} \omega = \mathrm{i}\lambda, \quad \int_{\beta_2^-} \omega = t + \mathrm{i}h \tag{3}$$

if P_D is of geometric type $A-$ or B (i.e., $\varepsilon = -1$).

Moreover, the intersection form on $\mathrm{H}_1(X, \mathbb{Z})^-$ is of type $(1, 2)$. Clearly, it is described by the matrices

$$\langle \cdot, \cdot \rangle_{\mathfrak{b}^+} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\mathfrak{b}^-} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{4}$$

In particular, $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0$ for any i, j and $\langle \alpha_i, \beta_j \rangle$ is nonzero iff $i = j$.

Recall that, for $D \equiv 1 \bmod 4$, the quadratic order is $\mathcal{O}_D = \mathbb{Z} \oplus T\mathbb{Z}$, where

$$T = \frac{1+\sqrt{D}}{2}.$$

As $(X,\omega) \in W_D$ admits real multiplication ι, $\mathrm{H}_1(X,\mathbb{Z})^-$ is an $\mathcal{O}_D$-module. In particular, for odd D, we may view T as an endomorphism $\iota(T)$ on $\mathrm{H}_1(X,\mathbb{Z})^-$. We now describe this endomorphism on the cusp prototypes. Note that this calculation essentially appears already in [2, §4], but due to differences in notation and for the convenience of the reader, we briefly restate the result.

Lemma 3.1. — *Let D be an odd discriminant. Given a prototype $P_D = [w,h,t,e,\varepsilon]$ associated to a flat surface (X,ω) the endomorphism $\iota(T)$ acts on $\mathrm{H}_1(X,\mathbb{Z})^-$ in the basis $\mathfrak{b}(P_D) = \mathfrak{b}^\varepsilon$ by $\iota(T)_{P_D} = \iota(T)^\varepsilon$, where*

$$\iota(T)^+ = \begin{pmatrix} \frac{e+1}{2} & 2w & 0 & 2t \\ h & -\frac{e-1}{2} & -t & 0 \\ 0 & 0 & \frac{e+1}{2} & 2h \\ 0 & 0 & w & -\frac{e-1}{2} \end{pmatrix} \text{ and } \iota(T)^- = \begin{pmatrix} \frac{e+1}{2} & w & 0 & t \\ 2h & -\frac{e-1}{2} & -2t & 0 \\ 0 & 0 & \frac{e+1}{2} & h \\ 0 & 0 & 2w & -\frac{e-1}{2} \end{pmatrix}.$$

Note that e is odd iff D is odd.

Proof. — Note first that $T = \lambda - \frac{e-1}{2}$ (cf. (1)) and that any $\gamma \in \mathrm{H}_1(X,\mathbb{Z})^-$ satisfies

$$\int_{\iota(T)\cdot\gamma} \omega = \int_\gamma \iota(T)^\vee \omega = T \cdot \int_\gamma \omega,$$

as ω is an eigenform. Now, using the periods of $\mathfrak{b}^\pm$ in (2) and (3), as well as the identities $\lambda^2 = e\lambda + 2wh$ and $T\cdot\lambda = 2wh + \lambda\frac{e+1}{2}$, the representations $\iota(T)^\pm$ are obtained by a straight-forward calculation. □

We are now in a position to describe the restriction of the intersection pairing $\langle\cdot,\cdot\rangle$ to the image of the endomorphism $\iota(T)$ in $\mathrm{H}_1(X,\mathbb{Z}/2\mathbb{Z})^-$ (for D odd). To ease notation, we will no longer distinguish T and $\iota(T)$, as no confusion can arise.

Proposition 3.2. — *Let D be an odd discriminant and T the endomorphism from above. Let (X,ω) be the geometric prototype associated to the cusp prototype $P_D = [w,h,t,e,\varepsilon]$. Then*

$$\langle\cdot,\cdot\rangle|_{\operatorname{Im} T} \equiv 0 \bmod 2 \iff e + \varepsilon \equiv 0 \bmod 4,$$

where $\langle\cdot,\cdot\rangle|_{\operatorname{Im} T}$ is the restriction of the intersection pairing on $\mathrm{H}_1(X,\mathbb{Z})^-$ to the image of T.

Proof. — We begin by observing that, as T is self-adjoint by the condition on real multiplication, we have $\langle T\gamma, T\delta\rangle = \langle T^2\gamma, \delta\rangle$ for any $\gamma,\delta \in \mathrm{H}_1(X,\mathbb{Z})^-$.

Moreover, by (4), any two elements $b_1, b_2 \in \mathfrak{b}^\varepsilon$ satisfy

$$\langle b_1, b_2\rangle \not\equiv 0 \bmod 2 \iff \{b_1, b_2\} = \begin{cases}\{\alpha_1^+, \beta_1^+\}, & \text{if } \varepsilon = 1,\\ \{\alpha_2^-, \beta_2^-\}, & \text{if } \varepsilon = -1.\end{cases}$$

Therefore, by checking mod 2 the $1,1$ entry of $(T^+)^2$ and the $2,2$ entry of $(T^-)^2$, we find (using $D = e^2 + 8wh$) that

$$\langle\cdot,\cdot\rangle_\pm|_{\operatorname{Im} T^\pm} \equiv 0 \bmod 2 \iff e \pm 1 = e + \varepsilon \equiv 0 \bmod 4,$$

as claimed. □

REMARK 3.3. — *Note that Lanneau and Nguyen use a similar idea (restriction of the intersection pairing to the image of an operator* mod 2*) to show that there are in fact two distinct components of W_D for $D \equiv 1 \bmod 8$* [2, Theorem 6.1]. *However, they use a different operator $T = T(P)$ for every prototype and this does not seem a feasible invariant.*

Proof of Theorem 1.2. — Let D be an odd discriminant. We denote by $\mathcal{X} \to W_D$ the universal family over the Teichmüller curve W_D, see [6, §1.4]. By definition of W_D, each fiber $\mathcal{X}_t$ has an involution ρ_t and the real multiplication gives endomorphisms T_t of $\mathrm{H}_1(\mathcal{X}_t, \mathbb{Z})^-$, allowing us to consider the restriction of the intersection form $\langle\cdot,\cdot\rangle_t$ to the image of T_t and take $\mathbb{Z}/2\mathbb{Z}$ coefficients. In particular, the map

$$t \mapsto \langle\cdot,\cdot\rangle|_{\operatorname{Im} T_t} \bmod 2$$

is continuous and as the range (the space of bilinear operators on an $\mathbb{F}_2$ vector space) is discrete, it is locally constant. Now, Proposition 3.2 asserts that two cusp prototypes P_D, P_D' are associated to cusps on the same component if and only if $e + \varepsilon \equiv e' + \varepsilon' \bmod 4$ and, as any such e must be odd, this yields the claim. □

4. The Galois Action on the Components

The aim of this section is to prove Theorem 1.1. The idea is to show that, for $D \equiv 1 \bmod 8$, the two components of W_D are in fact Galois-conjugate in analogy to the situation in genus 2 (cf. [1, Theorem 3.3]).

To achieve this, we first describe algebraic models of the stable curves associated to the cusps of W_D and then describe the Galois-action on these curves explicitly.

Stable Curves. — While a Teichmüller curve $\mathcal{C}$ is never compact, it admits a smooth completion $\overline{\mathcal{C}}$. Moreover, after passing to a finite cover, we may pull back the universal family over $\mathcal{M}_g$ to $\mathcal{C}$, thus obtaining a family of curves, which we – by abuse of notation – also denote by $\mathcal{X} \to \mathcal{C}$ and which extends to a family of stable curves $\overline{\mathcal{X}} \to \overline{\mathcal{C}}$, cf. [6, §1.4].

Much of the geometry of the stable fibers is given by the flat structure. By the above, given a flat surface (X, ω) on $\mathcal{X}$ together with a periodic direction v, we may associate a cusp $(X_\infty, \omega_\infty)$ to (X, ω, v), where X_∞ is a stable curve and ω_∞ is a stable differential on X_∞, see e.g., [7, §2.5 and §5.4]. In particular, X_∞ is obtained from X topologically by contracting the core curves of cylinders and ω_∞ has poles with residue equal to the cylinder widths at the nodes of X_∞.

LEMMA 4.1. — *Let $c \in \overline{W_D} \setminus W_D$ be a point such that the fiber $X_\infty = \overline{\mathcal{X}}_c$ is singular. Then X_∞ is a trinodal curve, i.e., X_∞ is a projective line with three pairs of points identified.*

Proof. — This follows immediately from [7, Corollary 5.11]: let $(X_\infty, \omega_\infty)$ be the stable flat surface associated to c. Then, as every component of X_∞ must contain a zero of ω_∞, the stable curve X_∞ is irreducible. Moreover, $(X_\infty, \omega_\infty)$ is obtained by contracting the core curves of a cylinder decomposition on some $(X, \omega) \in W_D$. But by Lemma 2.1, any such (X, ω) decomposes into three cylinders, hence X_∞ is obtained topologically by contracting three (homologically independent) curves on a genus 3 Riemann surface and therefore has geometric genus 0 and three nodes. □

Using the prototypes of [2] from section 2, we can describe the singular fibers of $\overline{W_D}$ more explicitly, in the spirit of [1, Proposition 3.2].

PROPOSITION 4.2. — *The stable curve above the cusp associated to the combinatorial prototype $[w, h, t, e, \varepsilon]$ may be normalized by a projective line with six marked points: $\pm 1, \pm x_1$, and $\pm x_3$, where*

$$x_1 = -s - \sqrt{\frac{1-s^2}{3}} \text{ and } x_3 = -s + \sqrt{\frac{1-s^2}{3}} \text{ for } s = \begin{cases} \dfrac{e+\sqrt{D}}{4w}, & \text{if } \varepsilon = 1, \\ \dfrac{2w}{e+\sqrt{D}}, & \text{if } \varepsilon = -1, \end{cases}$$

and the pairs of points $(+1, -1)$, $(x_1, -x_3)$, and $(x_3, -x_1)$ are identified in the stable model.

In particular, the absolute value of s uniquely determines the stable fibers.

Proof. — By Lemma 4.1, the normalization of the stable curve X_∞ associated to a cusp of W_D is a projective line with three pairs of marked points which we denote by $x_1, y_1, x_2, y_2, x_3, y_3$.

Now, the stable differential ω_∞ has poles at the nodes of X_∞ and the residues at each node must add up to zero, i.e., we have the crossratio equation

$$\omega_\infty = \left(\sum_{i=1}^{3} \frac{r_i}{z-x_i} - \frac{r_i}{z-y_i}\right) \mathrm{d}z = \frac{C\mathrm{d}z}{\prod_{i=1}^{3}(z-x_i)(z-y_i)}, \tag{5}$$

for the residues r_i, some constant C, and after choosing coordinates so that the unique zero of ω_∞ is at ∞.

Moreover, the Prym involution ρ acts on X_∞, hence also on the normalization, where we choose coordinates so that it acts as $z \mapsto -z$ (fixing the zero at ∞) and $x_2 = 1$. Recall that the stable fiber was obtained topologically by contracting the core curves of the three cylinders and that two cylinders are exchanged by the involution, one is fixed. We therefore find

$$y_1 = -x_3, \quad y_2 = -x_2 = -1, \quad y_3 = -x_1, \quad \text{and} \quad r_1 = r_3.$$

Comparing coefficients in (5), we obtain

$$x_1 = -x_3 - 2s \quad \text{and} \quad x_3 = -s \pm \sqrt{\frac{1-s^2}{3}} \quad \text{for} \quad s = \frac{r_2}{2r_1}.$$

Observe that the choice of sign in x_3 interchanges the values of x_1 and x_3 and that $-s$ gives the same set of points.

Now, consider the cusp associated to the prototype $[w, h, t, e, \varepsilon]$. If $\varepsilon = 1$, we have $r_1 = r_3 = w$ and $r_2 = \lambda$, while $\varepsilon = -1$ implies $r_1 = r_3 = \lambda/2$ and $r_2 = w$ (cf. Figure 2.1). This determines s.

Conversely, $|s|$ determines the points x_i. Identifying the points ± 1, x_1 and $-x_3$, and x_3 and $-x_1$, we obtain a stable curve with three nodes and an involution. □

Remark 4.3. — *Note that replacing s with $-s$ in Proposition 4.2 gives the same six points on $\mathbb{P}^1$, i.e., the same stable curve. This ambiguity corresponds to the action of the Prym involution on the stable curve.*

Remark 4.4. — *Observe that the stable curve does not "see" the twist parameter t, as it only depends on the cylinder widths. In particular, cusp prototypes that differ only in their twist parameter cannot be distinguished by the associated stable curves. This motivates the following definition.*

Definition 4.5. — *Given a prototype $P = [w, h, t, e, \varepsilon]$, we define the associated* algebraic cusp prototype *as* $[w, h, e, \varepsilon]$.

The Galois Action. — As Teichmüller curves are rigid, they are defined (as algebraic varieties) over a number field [5, 9] and one can show that the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of all Teichmüller curves (cf. [6, §5], [9, §6]), hence also on the set of cusps of Teichmüller curves.

Moreover, given $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, let $\mathcal{X} \to \mathcal{C}$ denote the universal family over a Teichmüller curve. Then the associated family of Jacobians splits with one factor admitting real multiplication, see e.g., [7, Corollary 5.7]. Now, σ acts on the universal family $\mathcal{X}$, as well as on the associated family of Jacobians, preserving the splitting and the real multiplication. In the case that $\mathcal{C}$ is a Prym-Weierstraß curve in $\mathcal{M}_3$, this implies that σ acts on the family of Prym-varieties over $\mathcal{C}$. In particular, a (fibrewise) ρ-anti-invariant eigenform for real multiplication by some $\mathcal{O}_D$ with a single zero is mapped again to a (fibrewise) ρ-anti-invariant eigenform for real multiplication by $\mathcal{O}_D$, as the splitting of the family, the real multiplication and the multiplicities of the zeros are all preserved by σ.

Note that Galois conjugation on curves in the moduli stack of curves preserves the number of cusps and number and type of orbifold points. In fact, the number of isomorphic fibers of the universal family over an orbifold chart near an orbifold point detects the orbifold order and is preserved by Galois conjugation.

Remark 4.6. — *While a Teichmüller curve $\mathcal{C}$ and its Galois conjugate $\mathcal{C}^\sigma$ are homeomorphic* as orbifolds*, they are in general, however, not isomorphic as complex curves. Indeed, by the calculations of the explicit equation of the Teichmüller curve W_{17}^{ε} in $\mathcal{M}_2$ in* [1, §7] *(the equations are also given in* [10, §6] *with a different normalization) it is not difficult to check that the two components are not isomorphic: using the notation of* [10, (35)]*, one can calculate the modular j function and clearly $j(\kappa_0) \neq j(\kappa_0^\sigma)$.*

Using the algebraic description of the stable curves, we may describe the Galois action on the cusps of W_D. As this is again independent of the twist parameter t, the action is given only on algebraic cusp prototypes.

Proposition 4.7. — *Let $P = [w,h,e,\varepsilon]$ be an algebraic cusp prototype, let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be a Galois automorphism that maps $\sqrt{D}$ to $-\sqrt{D}$, and denote by P^σ the prototype corresponding to the σ-conjugate cusp. Then, if $\varepsilon = 1$,*

$$P^\sigma = \begin{cases} [h,w,e,-\varepsilon], & \text{if } h > \lambda/2, \\ [w,h,-e,\varepsilon], & \text{if } h < \lambda/2, \end{cases}$$

and if $\varepsilon = -1$,

$$P^\sigma = \begin{cases} [h,w,e,-\varepsilon], & \text{if } h > \lambda, \\ [w,h,-e,\varepsilon], & \text{if } h < \lambda, \end{cases}$$

where $2\lambda = e + \sqrt{D}$, as above.

Proof. — Let $P = [w,h,e,\varepsilon]$ be an algebraic cusp prototype. By Proposition 4.2, the conjugate cusp will depend only on the action of σ on s. Recall that for $\varepsilon = 1$, we have

$$s = s(P) = s^+ = \frac{e+\sqrt{D}}{4w}, \quad \text{i.e.,} \quad (s^+)^\sigma = -\frac{-e+\sqrt{D}}{4w},$$

while for $\varepsilon = -1$

$$s = s(P) = s^- = \frac{2w}{e+\sqrt{D}} = \frac{-e+\sqrt{D}}{4h}, \quad \text{i.e.,} \quad (s^-)^\sigma = -\frac{e+\sqrt{D}}{4h},$$

as $D = e^2 + 8wh$.

Now, consider a prototype $P' = [w', h', e', \varepsilon']$ such that $|s(P)^\sigma| = |s(P')|$ (recall that by Remark 4.3, s is determined only up to sign, due to the action of the Prym involution). Comparing coefficients in $\mathbb{Q}(\sqrt{D})$ and as $w, h > 0$, it is clear that either $\varepsilon' = -\varepsilon$ and $e' = e$ or $e' = -e$ and $\varepsilon' = \varepsilon$. In the first case, $w' = h$ and $h' = w$, while in the second case $w' = w$ and $h' = h$.

Moreover, observe (using again that $D = e^2 + 8wh$) that

$$h < \frac{\lambda}{2} \iff e + \sqrt{D} > 4h = \frac{D - e^2}{2w} \iff \frac{\sqrt{D} - e}{2} < w,$$

and that any valid prototype $[w', h', e', 1]$ must satisfy $w' > \lambda'$. Hence, comparing h to λ (respectively $\lambda/2$), determines which of the above described choices for P' gives a valid prototype and thus yields the claim. □

We now combine Theorem 1.2 with Proposition 4.7 to show that, when D is odd any two conjugate cusps are on different components.

Proposition 4.8. — *Let $D \equiv 1 \bmod 8$ and $P_D = [w, h, e, \varepsilon]$ be an algebraic cusp prototype. Then P_D and P_D^σ are on different components of W_D.*

In particular, the cusps associated to $\left[\frac{D-1}{8}, 1, -1, -1\right]$ and $\left[\frac{D-1}{8}, 1, 1, -1\right]$ lie on W_D^1 and W_D^2, respectively, and are conjugate.

Proof. — Let $P_D = [w, h, e, \varepsilon]$ be an algebraic cusp prototype and denote by

$$c(P) = e + \varepsilon \bmod 4,$$

the component (see Theorem 1.2) of W_D that the associated cusp(s) of P lie on. Then, by Proposition 4.7, we have

$$c(P^\sigma) \equiv -e + \varepsilon \equiv e - \varepsilon \bmod 4,$$

as both e and ε are $\pm 1 \bmod 4$. In particular, $c(P) \not\equiv c(P^\sigma) \bmod 4$, hence the cusps lie on alternate components. □

Proof of Theorem 1.1. — Let $D \equiv 1 \bmod 8$, non-square, and W_D^i be a Teichmüller curve. Now, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on W_D^i and as this action extends to an action on the families of curves and their Jacobians, respects the (Prym) splitting, and maps eigenforms for real multiplication to eigenforms (for the same D), it preserves the locus W_D. Hence, any given element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts either trivially or interchanges the two components. But by Proposition 4.8, there exists an automorphism that *does not* fix W_D^i and therefore the components are Galois-conjugate. In particular, they are homeomorphic. □

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MEROMORPHIC QUOTIENTS FOR SOME HOLOMORPHIC G-ACTIONS

BY DANIEL BARLET

ABSTRACT. — Using mainly tools from previous articles we give necessary and sufficient conditions on the G-orbits' configuration in X in order that a holomorphic action of a connected complex Lie group G on a reduced complex space X admits a *strongly quasi-proper meromorphic quotient*. To show how these conditions can be used, we show, when $G = K.B$ with B a closed connected complex subgroup of G and K a real compact subgroup of G, the existence of a strongly quasi-proper meromorphic quotient for the G-action on X, assuming a slightly stronger condition than the existence of such a quotient for the B-action. We also give a similar result when the connected complex Lie group has the form $G = K.A.K$ where A is a closed connected complex subgroup and K is a compact (real) subgroup.

RÉSUMÉ (*Quotients méromorphes pour certaines G-actions holomorphes*). — En utilisant les résultats de précédents articles, nous donnons des conditions nécessaires et suffisantes sur la configuration des G-orbites dans X pour que l'action holomorphe d'un groupe de Lie complexe connexe sur un espace complexe réduit X admette un *quotient méromorphe fortement quasi-propre*. Pour illustrer l'intérêt de ces conditions, nous montrons, quand $G = K.B$ où B est un sous-groupe connexe complexe fermé et

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K un sous-groupe compact réel de G, l'existence d'un quotient méromorphe fortement quasi-propre pour l'action de G sur X sous une hypothèse légèrement plus forte que l'existence d'un tel quotient pour l'action de B sur X. Nous donnons également un résultat analogue quand $G = K.A.K$ où A est un sous-groupe complexe fermé et connexe et K un sous-groupe compact réel de G.

1. Introduction

In this article we explain how the tools developed in [9], [1], [2] and [3] can be applied to produce, in suitable cases, a meromorphic quotient of a holomorphic action of a connected complex Lie group G on a reduced complex space X. This uses the notion of *strongly quasi-proper map* introduced in *loc. cit.* and our first goal is to give three hypotheses, called [H.1], [H.2], [H.3], on the G-orbits' configuration in X which are *equivalent* to the existence of a *strongly quasi-proper meromorphic quotient*, notion defined in the Section 1.2.

The proof of this equivalence is the content of Proposition 2.7.1 and Theorem 2.8.1. Then we give a sufficient condition [H.1str], asking the existence of a G-invariant set $\Omega_1 \subset X$ which is dense, Zariski open and "good" for the action, to satisfy the condition [H.1].

Note that the conditions [H.1], [H.2], [H.3] introduced in Section 2.7 only depend on the G-orbits' configuration in X, but the condition [H.1str] depends on the action of G on X itself.

The existence theorem for a strongly quasi-proper meromorphic quotient under our three assumptions is applied to prove the following result:

THEOREM 1.0.1. — *Assume that we have a holomorphic action of a connected complex Lie group G on a reduced complex space X. Assume that $G = K.B$ where K is a compact (real) subgroup of G and B a connected complex closed subgroup of G. Assume that the action of B on X satisfies the condition* [H.1str] *on a G-invariant Zariski open dense subset Ω in $X^{(1)}$, and the conditions* [H.2] *and* [H.3]. *Then the G-action satisfies* [H.1str], [H.2] *and* [H.3]*; so it has a strongly quasi-proper meromorphic quotient.*

A first variant of this result is given by the following theorem.

THEOREM 1.0.2. — *Assume that we have a holomorphic action of a connected complex Lie group G on a reduced complex space X. Assume that $G = K.B$ where K is a compact (real) subgroup of G and B a connected complex closed subgroup of G. Assume that K normalizes B and that the B-action satisfies*

1. This precisely means that there exists a G-invariant dense Zariski open set in X which is a "good open set" for the B-action (see Section 2.5)

the conditions [H.1str]*,* [H.2] *and* [H.3]*. Then the G-action satisfies the conditions* [H.1str], [H.2] *and* [H.3] *and so has a strongly quasi-proper meromorphic quotient.*

Here is a second result obtained by a similar method.

THEOREM 1.0.3. — *Let G be a complex connected Lie group and assume that there exists a closed connected complex subgroup A and a compact (real) subgroup K such That $G = K.A.K$. Assume the we have a completely holomorphic action of G on an irreducible complex space X and that the action of A on X satisfies the following properties:*

i) *The hypothesis* [H.1str] *for the A-action is satisfied on a G-invariant (Zariski good) open set Ω_1 in X.*
ii) *The hypothesis* [H.2] *for the A-action is satisfied on a G-invariant open set $\Omega_0 \subset \Omega_1$ in X.*
iii) *The hypothesis* [H.3] *holds for the A-action.*

Then [H.1str]*,* [H.2] *and* [H.3] *hold for the action of G on X. So there exists a SQP meromorphic quotient of X for the G-action.*

Of course the hypothesis $G = K.A.K$ is more "general" than the case $G = K.B$. But the hypothesis of this last theorem is more restrictive for the action on X of the closed connected complex subgroup A of G: we ask also the G-invariance of the dense open subset Ω_0 of Ω_1 (the open set Ω_0 is defined in the condition [H.2]).

We conclude this article with two results (see Section 3.4) relating the SQP meromorphic quotients for the actions of B and G (resp. of A and G) when they exist:

1. The existence of a holomorphic map $h : Q_B \to Q_G$ (resp. $Q_A \to Q_G$) between the corresponding quotients.
2. The existence under the hypotheses of the Theorem 1.0.1 (resp. the Theorem 1.0.3) of a G-invariant dense Zariski open set Ω disjoint from the centers of the modifications, such that the corresponding map $h_\Omega : q_B(\Omega) \to q_G(\Omega)$ (resp. $h_\Omega : q_A(\Omega) \to q_G(\Omega)$) is proper.

2. Strongly quasi-proper meromorphic quotients

2.1. Preliminaries. — For the definition of the topology on the space $\mathcal{C}_n^f(X)$ of finite type n-cycles in X and its relationship with the topology of the space $\mathcal{C}_n^{\mathrm{loc}}(X)$ we refer to [4] ch. IV, [2] and [3].

For the convenience of the reader we recall shortly here the definitions of a geometrically f-flat map (f-GF map) and of a strongly quasi-proper map (SQP map) between irreducible complex spaces and we give a short summary on some properties of the SQP maps. For more details on these notions see [2] and [3].

Definition 2.1.1. — *Let $\pi : M \to N$ be a holomorphic map between two irreducible complex spaces and let $n := \dim M - \dim N$. We shall say that π is a* geometrically f-flat map (a f-GF-map *for short) if the following conditions are fulfilled:*

i) *The map is quasi-proper equidimensional and surjective.*
ii) *There exists a holomorphic map $\varphi : N \to \mathcal{C}_n^f(M)$[2] such that for y generic[3] in N the cycle $\varphi(y)$ is reduced and equal to the set-theoretic fiber $\pi^{-1}(y)$ of π at y.*

A holomorphic map $\pi : M \to N$ between two irreducible complex spaces will be strongly quasi-proper *(SQP map for short) if there exists a modification[4] $\tau : \tilde{N} \to N$ such that the strict transform[5] $\tilde{\pi} : \tilde{M} \to \tilde{N}$ of π by τ is a f-GF map.*

A meromorphic map $M \dashrightarrow N$ will be called strongly quasi-proper *when the projection on N of its graph is a SQP map.*

Note that a f-GF map has, by definition, a *holomorphic fiber map* and that a SQP holomorphic (or meromorphic) map has a *meromorphic fiber map* via the composition of the holomorphic fiber map of $\tilde{\pi}$ with the (holomorphic) direct image map for finite type n-cycles $\tau_* : \mathcal{C}_n^f(\tilde{M}) \to \mathcal{C}_n^f(M)$. Of course, a SQP holomorphic map is quasi-proper, but the converse is not true. The notion of strongly quasi-proper map is stable by modification of the target space, property which is not true in general for a quasi-proper map having "big fibers" (see [3] for such an example).

Let $\pi : M \to N$ be a SQP map between irreducible complex spaces and define $n := \dim M - \dim N$. By definition of a SQP map, we can find a Zariski open dense subset N_0 in N and a holomorphic map $\varphi_0 : N_0 \to \mathcal{C}_n^f(M)$ such that

i) For each y in N_0 we have the equality of subsets $|\varphi_0(y)| = \pi^{-1}(y)$.
ii) For y generic in N_0 the cycle $\varphi_0(y)$ is reduced.

Let $\Gamma \subset N_0 \times \mathcal{C}_n^f(M)$ be the graph of φ_0. Then, thanks to the Theorem 2.3.6 of [2], the closure $\bar{\Gamma}$ of Γ in $N \times \mathcal{C}_n^f(M)$ is proper over N. Then, using the semi-proper direct image Theorem 2.3.2 of [3], this implies that $\tilde{N} := \bar{\Gamma}$ is an irreducible complex space (locally of finite dimension) with the structure sheaf induced by the sheaf of holomorphic functions on $N \times \mathcal{C}_n^f(M)$. Moreover the natural projection $\tau : \tilde{N} \to N$ is a (proper) modification.

Let $\tilde{M} := M \times_{N,\mathrm{str}} \tilde{N}$ be the strict transform of M by τ, that is to say the irreducible component of $M \times_N \tilde{N}$ containing the graph of the restriction π_0

2. That is to say a f-analytic family of finite type n-cycles in M parametrized by N.
3. A dense subset in N of such y is enough here, thanks to the Proposition 3.2.2 of [3].
4. A modification is, by definition, always proper.
5. By definition $\tilde{M}$ is the irreducible component of $M \times_N \tilde{N}$ which dominates $\tilde{N}$ and $\tilde{\pi}$ is induced by the projection.

of π to the open set $\pi^{-1}(N_0)$[6]. Then let $\tilde{\pi} : \tilde{M} \to \tilde{N}$ the strict transform of π by the modification τ; it is induced on $\tilde{M}$ by the natural projection of $M \times_N \tilde{N}$ onto $\tilde{N}$. The set-theoretical fiber at $\tilde{y} := (y, C) \in \tilde{N}$ of $\tilde{\pi}$ is the subset $|C| \times \{\tilde{y}\}$ in $\tilde{M}$. The map $\psi : \tilde{N} \to \mathcal{C}_n^f(\tilde{M})$ given by $(y, C) \mapsto C \times \{\tilde{y}\}$ is holomorphic and satisfies $|\psi(\tilde{y})| = \tilde{\pi}^{-1}(\tilde{y})$ for all $\tilde{y}$ in $\tilde{N}$. Moreover $\psi(\tilde{y})$ is a reduced cycle for generic $\tilde{y}$ in $\tilde{N}$. So the map $\tilde{\pi}$ is geometrically f-flat. It is the *canonical f-GF-flattning* of π.

Then we have an isomorphism of $\tilde{N}$ on its image (see [3] th. 3.1.9) induced by ψ

$$\psi : \tilde{N} \longrightarrow \mathcal{C}_n^f(\tilde{\pi})$$

where $\mathcal{C}_n^f(\tilde{\pi}) := \{C \in \mathcal{C}_n^f(\tilde{M}) / \exists \tilde{y} \in \tilde{N} s.t. |C| \subset \tilde{\pi}^{-1}(\tilde{y})\}$ is a closed analytic subset of $\mathcal{C}_n^f(\tilde{M})$ (see [3] Proposition 2.1.7.); the inverse map is induced by the holomorphic map $\hat{\pi} : \mathcal{C}_n^f(\tilde{\pi}) \to \tilde{N}$ which associates to $\gamma \in \mathcal{C}_n^f(\tilde{\pi})$ the point in $\tilde{N}$ whose $\tilde{\pi}$-fiber contains γ (see loc. cit.).

The direct image of n-cycles by τ gives a holomorphic map $\tau_* : \mathcal{C}_n^f(\tilde{M}) \to \mathcal{C}_n^f(M)$ which sends $\tilde{N} \subset \mathcal{C}_n^f(\tilde{\pi})$ in $\mathcal{C}_n^f(\pi)$. Let us show that it is an isomorphism of $\tilde{N}$ onto its image in $\mathcal{C}_n^f(\pi)$:

We have an obvious holomorphic map $\tilde{N} \to \mathcal{C}_n^f(\pi)$ given by $(y, C) \mapsto C$. We have also a holomorphic map $\mathcal{C}_n^f(\pi) \to N \times \mathcal{C}_n^f(M)$ given by $C \mapsto (\hat{\pi}(C), C)$ where $\hat{\pi} : \mathcal{C}_n^f(\pi) \to N$ is the map associating to $C \in \mathcal{C}_n^f(\pi)$ the point $y \in N$ such that $|C| \subset \pi^{-1}(y)$. This proves our claim.

Remark that the closed analytic subset $\mathcal{C}_n^f(\pi)$ of $\mathcal{C}_n^f(M)$ is not, in general, even locally, a complex space of finite dimension.

2.2. Action of G on $\mathcal{C}_n^f(X)$. — Let G be a Lie group. We shall say that G acts *continuously holomorphically* on the reduced complex space X when the action $f : G \times X \to X$ is a continuous map such that for each $g \in G$ fixed, the map $x \mapsto f(g, x)$ is a (biholomorphic) automorphism of X. Then there is a natural action of G induced on the set $\mathcal{C}_n^f(X)$ of finite type n-cycles given by $(g, C) \mapsto g_*(C)$ where we denote $g_*(C)$ the direct image of the cycle C by the automorphism of X associated to $g \in G$. When G is a complex Lie group and the map f is holomorphic we shall say that the action is *completely holomorphic.*

PROPOSITION 2.2.1. — *Let G be a Lie group acting continuously holomorphically on a reduced complex space X. Then the action of G on $\mathcal{C}_n^f(X)$ is continuously holomorphic. This means that the map*

$$G \times \mathcal{C}_n^f(X) \to \mathcal{C}_n^f(X) \quad \textit{given by} \quad (g, C) \mapsto g_*(C)$$

is continuous and holomorphic for each fixed $g \in G$.

6. This graph is a Zariski open set in $M \times_N \tilde{N}$ which is irreducible as M an N are irreducible.

If G is complex Lie group and the action is completely holomorphic, the action of G on $\mathcal{C}_n^f(X)$ is completely holomorphic; so, for any f-analytic family of n-cycles $(C_s)_{s\in S}$ in X parametrized by a reduced complex space S, the family of n-cycles $g_*(C_s)_{(g,s)\in G\times S}$ parametrized by $G\times S$ is f-analytic.

Proof. — First we prove the continuity of the action of G on $\mathcal{C}_n^{\mathrm{loc}}(X)$. To apply the Theorem IV 2.5.6 of [4] it is enough to see that the map $F : G\times X \to G\times X$ given by $(g,x)\mapsto (g, g.x)$ is proper. But if $L\subset G$ and $K\subset X$ are compact sets, we have $F^{-1}(L\times K)\subset L\times (L^{-1}.K)$ which is a compact set in $G\times X$.

The only point left to prove the continuity statement for the topology of $\mathcal{C}_n^f(X)$, assuming that the continuity for the topology of $\mathcal{C}_n^{\mathrm{loc}}(X)$, is obtained as follows:

Let W be a relatively compact open set in X and $\mathcal{W}$ be the open set in $\mathcal{C}_n^f(X)$ of cycles C such any irreducible component of C meets W. Then we want to show that the set of $(g,s)\in G\times S$ such that $g_*(C_s)$ lies in $\mathcal{W}$ is an open set in $G\times S$. As the topology of $\mathcal{C}_n^f(X)$ has a countable basis(7) it is enough to show that if a sequence (g_ν, s_ν) converges to (g,s) with $g_*(C_s)\in\mathcal{W}$ then for $\nu\gg 1$ we have also $(g_\nu)_*(C_{s_\nu})\in\mathcal{W}$. If this not the case, we can choose, for infinitely many ν, an irreducible component Γ_ν of $(g_\nu)_*(C_{s_\nu})$ which does not meet W. Passing to a sub-sequence, we may assume that the sequence (Γ_ν) converges in $\mathcal{C}_n^{\mathrm{loc}}(X)$ to a cycle Γ which does not meet W and is contained in $g_*(C_s)$. This is a simple consequence of the continuity of the G-action on $\mathcal{C}_n^{\mathrm{loc}}(X)$ and the characterization of compact subsets in $\mathcal{C}_n^{\mathrm{loc}}(X)$ (see [4] ch. IV). As any irreducible component of $g_*(C_s)$ meets W this implies that Γ is the empty n-cycle. This means that for any compact K in X there exists an integer $\nu(K)$ such that for $\nu\geq\nu(K)$ we have $\Gamma_\nu\cap K=\emptyset$. Choose now a compact neighborhood L of $g(K)$. For ν large enough we shall have $K\subset g_\nu^{-1}(L)$. This comes from the fact that the automorphisms g_ν^{-1} converge to g^{-1} in the compact-open topology. Then this implies that for $\nu\geq\nu(L)$ the irreducible component $g_\nu^{-1}(\Gamma_\nu)$ of C_{s_ν} does not meet K. Then, when $s_\nu\to s$ the cycles C_{s_ν} does not converge to C_s for the topology of $\mathcal{C}_n^f(X)$ because we have some "escape at infinity" in a well choosen sub-sequence. Contradiction. $\square$

LEMMA 2.2.2. — *Let $\pi : M\to N$ be a SQP map between irreducible complex spaces. Assume now that a Lie group H acts continuously holomorphically on M and N and that π is H-equivariant for these actions. Let $\tau:\tilde{N}\to N$ be the canonical modification giving the canonical f-GF flattning of π and let $\tilde{\tau}:\tilde{M}\to M$ and $\tilde{\pi}:\tilde{M}\to\tilde{N}$ respectively be the strict transform τ and of π by the modification τ. Then $\tilde{N}$ and $\tilde{M}$ have natural continuous holomorphic actions of H such τ and $\tilde{\pi}$ are H-equivariant.*

7. This is a corollary of the fact that this is true for $\mathcal{C}_n^{\mathrm{loc}}(X)$ (see [4] ch. IV) as the topology of X has a countable basis of open sets; see the Lemma 2.1.1 in [3] for details.

Moreover, if H is a complex Lie group and if it acts completely holomorphically on M and N, it acts also completely holomorphically on $\tilde{M}$ and $\tilde{N}$.

Proof. — Let $N_0 \subset N$ the open dense subset of points y in N such that y is normal and $\pi^{-1}(y)$ is purely n-dimensional. Then N_0 is H-stable because H acts on M and N by bi-holomorphic equivariant automorphisms. As π is quasi-proper, the Theorem IV 3.4.1 of [4] gives a holomorphic map $\varphi_0 : N_0 \to \mathcal{C}_n^f(M)$ such that, for each $y \in N_0$, we have $|\varphi_0(y)| = \pi^{-1}(y)$, with $\varphi_0(y)$ reduced for y generic. Then the H-equivariance of π implies the H-equivariance of φ_0 for the action of H on $\mathcal{C}_n^f(M)$ defined in the Proposition 2.2.1. So the graph Γ of φ_0 is stable by the H-action and so is its closure $\tilde{N}$ in $N \times \mathcal{C}_n^f(M)$. Then $\tau : \tilde{N} \to N$ induced by the first projection is H-equivariant.

Now the strict transform of $\tilde{M}$ by τ is the closure in $M \times_N \tilde{N}$ of the subset $\pi^{-1}(N_0 \times_N N_0) \simeq \pi^{-1}(N_0)$. As it is stable by the action of H, so is its closure, and the map $\tilde{\pi} : \tilde{M} \to \tilde{N}$ induced by the second projection is then H-equivariant.

The case where H is complex and acts completely holomorphically on M and N follows from the previous proposition. □

Of course, the use of the action of H on cycles is crucial in order to have natural H-actions on $\tilde{M}$ and $\tilde{N}$ and H-equivariance of $\tilde{\tau}$ and $\tilde{\pi}$.

We shall also use the following simple tool from the cycle's space.

PROPOSITION 2.2.3. — *Let M be a reduced complex space and $(X_s)_{s\in S}$ the tautological f-continuous family of d-dimensional finite type cycles parametrized by a compact subset S in $\mathcal{C}_d^f(M)$. Let $(C_t)_{t\in T}$ be the tautological family of n-dimensional cycles in M parametrized by a compact subset $T \subset \mathcal{C}_n^{\mathrm{loc}}(M)$. We assume the following condition:*

- *There exists a dense subset T' in T such that each $C_t, t \in T'$, is non empty and equal to the union of some X_s.* (@@)

Then the property (@@) is satisfied for any $t \in T$ and T is in fact a compact subset of $\mathcal{C}_n^f(M)$.

Proof. — First remark that, as S is compact in $\mathcal{C}_d^f(M)$, there exists a compact set $L \subset M$ such that any irreducible component of any X_s meets L.

Let $(t_m)_{m\in\mathbb{N}}$ be a sequence of points in T' converging to a point $t \in T$ for the topology of $\mathcal{C}_n^{\mathrm{loc}}(M)$ and denote by C_m the cycle C_{t_m} for short and $C_t = C_\infty$. Now choose for each m an irreducible component Γ_m of some X_{s_m} contained in C_m. Remark that this is possible because $t_m \in T'$ implies that (@@) holds. Passing to a sub-sequence and using the compactness of S, we may assume that Γ_m converges in $\mathcal{C}_d^f(M)$ to a cycle Γ which is not empty (it contains at least a point in L) and is included in $|C_\infty|$. So C_∞ is not the empty cycle.

Let x be a generic point of an irreducible component D of C_∞. Then, passing to a sub-sequence, we may choose a sequence (x_m) of points respectively in C_m

which converges to x. Choose for each m an irreducible component Γ_m of some $X_{s_m} \subset |C_m|$ which contains x_m. This is possible again because the condition (@@) holds. Now, passing again to a sub-sequence, we may assume that the sequence $(\Gamma_m)_{m\in\mathbb{N}}$ converges in $\mathcal{C}_d^f(M)$ to a cycle Γ containing the point x and contained in $|C_\infty|$. Note that $|\Gamma|$ contains an irreducible component of some $|X_{s_\infty}|$ containing x, as we may assume, by compactness of S, that the sequence (s_m) converges to $s_\infty \in S$. Then we have $|X_{s_\infty}| \subset |C_\infty|$. As D is the only irreducible component of C_∞ containing x, it contains at least an irreducible component of $|X_{s_\infty}|$ containing x, and so D meets L. So we have proved that C_∞ is not the empty n-cycle and that any irreducible component of C_∞ meets the compact set L. This is enough to conclude thanks to the Corollary 2.1.3 in [3]. □

2.3. f-GF holomorphic quotient. — We shall consider a complex connected Lie group G and a completely holomorphic action of G on an irreducible complex space X.

Definition 2.3.1. — *We shall say that the action of G on X has a* quasi-proper GF holomorphic quotient*, (a f-GF holomorphic quotient for short), when there exists a G-invariant quasi-proper geometrically flat holomorphic map $q : X \to Q$ onto a reduced complex space Q such that each fiber of q is set-theoretically equal to an orbit of G in X.*

Remark. — Assume that G is connected and that we have a G-invariant open set U and a GF surjective holomorphic map $q : U \to Q$ to a reduced complex space such that for each $x \in U$ we have the set-theoretic equality $q^{-1}(q(x)) = G.x$. Then the map q is quasi-proper: let $y := q(x)$ be any point in Q and let $V(x)$ be a relatively compact open neighborhood of x in U. As the map q is open, $q(V(x))$ is an open neighborhood of y and for any $y' \in q(V(x))$ there exists $x' \in V(x)$ such that $q^{-1}(y') = G.x'$. So the fiber $q^{-1}(y')$ meets the compact set $\bar{V}(x)$ of U, proving the quasi-properness of the map q.

Proposition 2.3.2. — *In the situation of the definition above there exists a holomorphic map, where we have defined $n := \dim X - \dim Q$,*

$$\Phi : X \longrightarrow \mathcal{C}_n^f(X)$$

with the following properties

i) *The subset $Q_u := \Phi(X)$ of $\mathcal{C}_n^f(X)$ is a closed analytic subset of finite dimension (so an irreducible complex space) with the complex structure sheaf induced by the structure sheaf of $\mathcal{C}_n^f(X)$[8].*

8. Recall that a continuous function $g : \mathcal{U} \to \mathbb{C}$ on an open set $\mathcal{U} \subset \mathcal{C}_n^f(X)$ is holomorphic if and only if for any holomorphic map $f : S \to \mathcal{U}$ of a reduced complex space S in $\mathcal{U}$ the composed function $f \circ g$ is holomorphic on S.

ii) *The map* $id_X \times \Phi : X \to X \times \mathcal{C}_n^f(X)$ *induces an isomorphism of* X *on the set-theoretic graph of the tautological family of finite type* n*-cycles in* X *parametrized by* Q_u*.*

iii) *There is a canonical isomorphism* $\theta : Q \to Q_u$ *such the diagram*

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "q_u"'] & Q \arrow[d, "\theta"] \arrow[dr, "\varphi"] & \\ & Q_u \arrow[r, "i"] & \mathcal{C}_n^f(X) \end{tikzcd}$$

commutes, where φ *is the holomorphic map classifying the fibers of the f-GF map* q *and where* $\Phi = q_u \circ i = q \circ \varphi$*.*

Conversely, if there exists a holomorphic map $\Phi : X \longrightarrow \mathcal{C}_n^f(X)$ *such that for each* $x \in X$ *we have* $|\Phi(x)| = G.x$ *and such* $\Phi(x)$ *is generically a reduced cycle then* $Q_u := \Phi(X)$ *is a closed analytic subset in* $\mathcal{C}_n^f(X)$ *of finite dimension and the map* $q_u : X \to Q_u$ *induced by* Φ *is a f-GF holomorphic quotient for the* G*-action on* X*.*

Proof. — The Theorem 3.1.9 of [3] gives that the classifying map $\varphi : Q \to \mathcal{C}_n^f(X)$ for the fibers of the f-GF map q is a proper holomorphic embedding. As $\Phi := q \circ \varphi$ and q is surjective, this gives the fact that Q_u is a closed locally finite dimensional subset of $\mathcal{C}_n^f(X)$ and that the map $\theta : Q \to Q_u$ induced by φ is an isomorphism.

Then the map $id_X \times \Phi$ sends X to the set-theoretic graph of the tautological family of finite type n-cycles in X parametrized by Q_u. The inverse map is given by the projection on X, so it is an isomorphism on this graph[9]. The point iii) is now clear.

The converse is consequence of the Theorem 2.3.2 of [3] as soon as we proved the following claim:

- the holomorphic map Φ is semi-proper.

Consider first the case of a non empty cycle $C_0 \in \mathcal{C}_n^f(X), C_0 \not\in \Phi(X)$. There exists two points $x, y \in |C_0|$ such that $y \not\in G.x$. Then choosing two adapted n-scales E_x, E_y to C_0 such that x and y are respectively in the domains of E_x and E_y with $\deg_{E_x}(C_0) \geq 1$ and $\deg_{E_y}(C_0) \geq 1$. Then any cycle which is near enough to C_0 in $\mathcal{C}_n^f(X)$ has the same degrees in these adapted scales. But as $\Phi(x) \neq \Phi(y)$ there exists two disjoint G-invariant open sets U and V in X containing respectively x and y. If we choose E_x to be a scale on U and E_y to be a scale on V we know that each cycle near enough to C_0 cannot be an orbit (set-theoretically) so is not in $\Phi(X)$.

Let now $C_0 = \Phi(x)$ and let W be an open relatively compact neighborhood of x in X. Let $\mathcal{W}$ be the open set in $\mathcal{C}_n^f(X)$ of cycles C such any irreducible of C

9. Note that, as the generic fiber of q_u is a reduced cycle, the cycle-graph is also reduced.

meets W. Then we have the equality $\Phi(X) \cap \mathcal{W} = \Phi(\bar{W}) \cap \mathcal{W}$: if $C = \Phi(y)$ is in $\mathcal{W}$ then $G.y$ has to meet W by definition of $\mathcal{W}$ and so there exists $z \in W$ such that $z \in G.y$. But then $G.z = G.y$ and we have $|\Phi(z)| = |\Phi(y)|$. But then $\Phi(y) = \Phi(z)$ as the cycles in $\Phi(X)$ are disjoint or equal; and we have find a $z \in \bar{W}$ with $\Phi(z) = \Phi(y)$, proving our claim. □

Remark. — It is easy to see that if, for x generic in X, the cycle $\Phi(x)$ is not reduced, there exists an integer $k \geq 2$ such that, for x generic in X we have $\Phi(x) = k.[G.x]$. Then, assuming that X is normal, there exists a holomorphic map $\Phi_1 : X \to \mathcal{C}_n^f(X)$ such for any $x \in X$ we have $\Phi(x) = k.\Phi_1(x)$ and then, for generic x in X, the cycle $\Phi_1(x)$ is reduced. So, up to replace X by its normalization $\tilde{X}$ (with its corresponding natural G-action) we obtain a f-GF holomorphic quotient for the G-action on $\tilde{X}$.

The following obvious corollary of the Proposition 2.3.2 will be useful.

COROLLARY 2.3.3. — *If X admits a f-GF holomorphic quotient $q : X \to Q$ for the G-action, the map q is unique up to an unique isomorphism of Q.*

LEMMA 2.3.4. — *Let $(\Omega_i)_{i\in I}$ be a collection of G-invariant open sets in X such that for any $i \in I$ and any $x \in \Omega_i$ the orbit $G.x$ is a closed set (resp. a closed analytic subset) in Ω_i. Then for each $x \in \Omega := \bigcup_{i\in I} \Omega_i$ the orbit $G.x$ is a closed set (resp. a closed analytic subset) in Ω. Moreover, if any orbit is a closed analytic subset in Ω, the subset*

$$Z := \{(x, y) \in \Omega \times \Omega / G.x = G.y\}$$

is a locally closed analytic subset in $\Omega \times \Omega$ as soon as its intersection with $\Omega_i \times \Omega_i$ is a closed analytic subset of $\Omega_i \times \Omega_i$ for each $i \in I$.

Proof. — Let $x \in \Omega$ and $y \in \overline{G.x} \cap \Omega$. Let $j \in I$ such that y belongs to Ω_j. Choose an open neighborhood V of y in Ω_j. Then $V \cap G.x$ is not empty. But if z is in $V \cap G.x$ we have $z \in \Omega_j$ and also $G.x = G.z$ lies in Ω_j. As $G.x$ is closed in Ω_j we conclude that y is in $G.x$ and so $G.x$ is closed in Ω. The assertion on analyticity is obvious.

Consider now the open set

$$B := \{(x, y) \in \Omega \times \Omega / \exists i \in I \text{ such that } x \in \Omega_i \text{ and } y \in \Omega_i\} = \bigcup_{i\in I} \Omega_i \times \Omega_i.$$

Then $Z \cap B$ is closed in B and is clearly analytic in this open set. □

COROLLARY 2.3.5. — *Assume that we can cover the G-space X by a family of G-invariant open sets $(\Omega_i)_{i\in I}$ such that for each $i \in I$ we have a f-GF holomorphic quotient $q_i : \Omega_i \to Q_i$. Then if the family of closed sets $(\partial\Omega_i)_{i\in I}$ is locally finite in X, the open G-invariant dense set $X' := X \setminus \bigcup_{ß\in I} \partial\Omega_i$ has a f-GF holomorphic quotient $q : X' \to Q'$.*

Proof. — First remark that if X admits a holomorphic f-GF quotient $q : X \to Q$ then for any G-invariant open set $\Omega \subset X$ the restriction of q to Ω induces a holomorphic f-GF quotient $q_\Omega : \Omega \to Q_\Omega := q(\Omega)$ because the map q is open and $\Omega = q^{-1}(q(\Omega))$ as Ω is G-invariant and so q-saturated.

Note also that, as X is countable at infinity, we may assume that I is countable.

For any $(i,j) \in I^2$ such that the open set $\Omega_i \cap \Omega_j$ is not empty, the G-invariant open set $\Omega_i \cap \Omega_j$ has two holomorphic f-GF quotients:

$$q_i|_{\Omega_i\cap\Omega_j} : \Omega_i \cap \Omega_j \to Q_{i,j} := q_i(\Omega_i \cap \Omega_j) \quad \text{and}$$
$$q_j|_{\Omega_i\cap\Omega_j} : \Omega_i \cap \Omega_j \to Q_{j,i} := q_j(\Omega_i \cap \Omega_j)$$

thanks to our first remark. Then we have a canonical isomorphism $\theta_{i,j} : Q_{i,j} \to Q_{j,i}$, and again by the uniqueness assertion of the previous corollary we have

$$\theta_{i,j} \circ \theta_{j,k} \circ \theta_{k,i} = \mathrm{id} \quad \text{on} \quad \Omega_i \cap \Omega_j \cap \Omega_k \quad \forall (i,j,k) \in I^3.$$

So there exists a, may be non Hausdorff, locally reduced complex space Q obtained by identifying $Q_{i,j}$ to $Q_{j,i}$ via $\theta_{i,j}$ in the disjoint union of the $Q_i, i \in I$ and a holomorphic map $q : X \to Q$ such that its restriction to Ω_i is equal to q_i for each $i \in I$. It is then easy to see that the open subset Q' of Q obtained by deleting the image in Q of $F := \bigcup_{ß\in I} \partial\Omega_i$ is a Hausdorff reduced complex space: to see this, it is enough to produce, for any distinct points $x \neq y$ in X' a pair of disjoint G-invariant neighborhoods of x and y. If there exists $i \in I$ such both are in Ω_i this is a consequence of the separation of the quotient Q_i. If this is not the case, there exists $i \neq j \in I$ such that $x \in \Omega_i \setminus \bar\Omega_j$ and $y \in \Omega_j \setminus \bar\Omega_i$. If $V(x)$ and $V(y)$ are open neighborhoods of x and y respectively in $\Omega_i \setminus \bar\Omega_j$ and $\Omega_j \setminus \bar\Omega_i$, then $G.V(x)$ and $G.V(y)$ answer the question. □

2.4. SQP meromorphic quotient

Definition 2.4.1. — *A* strongly quasi-proper meromorphic quotient, *we shall say a* SQP-meromorphic quotient *for short, for a completely holomorphic action $f : G \times X \to X$ of a complex Lie group G on an irreducible complex space X will be the following data:*

1. *a G-modification*[10] *$\tau : \tilde X \to X$ with center Σ;*
2. *a G-invariant holomorphic f-GF map $q : \tilde X \to Q$ where Q is a (irreducible) complex space;*
3. *an analytic G-invariant subset $Y \subset X$ containing Σ, with no interior point in X.*

10. This means that we have a completely holomorphic G-action on $\tilde X$ and that the modification τ is G-equivariant.

We shall denote $\tilde{Y} := \tau^{-1}(Y)$, $\Omega := X \setminus Y$ *and we shall identify* Ω *with its pull-back by* τ. *We shall also define* $Q' := q(\Omega)$*; note that, as* q *is an open surjective map,* Q' *is open and dense in* Q.

Now we ask that these data satisfy the following properties:

i) *There exists a* G*-invariant dense open set* $\Omega_1 \subset \Omega$ *which admits a f-GF holomorphic quotient for the* G*-action on it.*

ii) *There exists an open dense* G*-invariant subset* $\Omega_0 \subset \Omega_1$ *such that for each* x *in* Ω_0 *the* closure $\overline{G.x}$ *of* $G.x$ *in* $\tilde{X}$ *is exactly the reduced cycle* $q^{-1}(q(x))$.

Note that, in general, the G-invariant f-GF map $q : \tilde{X} \to Q$ is not a holomorphic f-GF quotient of $\tilde{X}$ as the generic G-orbits are not closed in $\tilde{X}$. Also, in general, the restriction of q to Ω is not a f-GF quotient (see the example below).

When we assume that Ω_1 is Zariski open, we could take $\Omega = \Omega_1$ and then Y is simply the complement of Ω_1.

In the other cases, we may always choose $Y = \Sigma$. So $\Omega_0 \subset \Omega_1$ are simply G-invariant open dense subsets in $X \setminus \Sigma$.

To illustrate this definition which seems a little complicate, let us look at the following very simple (algebraic) example.

Example. — [from many discussions in Bochum] Let $G := \mathbb{C}^*$ and $X := \mathbb{P}_2$ the action given by $t.(x_0, x_1, x_2) := (t.x_0, t^{-1}.x_1, x_2)$. Then there exist 3 fixed points $O := (0,0,1), P := (1,0,0), Q := (0,1,0)$ and 3 orbits which are the punctured lines OP, OQ, PQ which are copies of $\mathbb{C}^*$. The other orbits are the conics $\{x_0.x_1 = s.x_2^2\}$ for $s \in \mathbb{C}^*$.

The SQP meromorphic quotient for this action is given by the (quasi-proper) meromorphic map

$$q : \mathbb{P}_2 \dashrightarrow \mathbb{P}_1, \quad (x_0, x_1, x_2) \mapsto (x_0.x_1, x_2^2).$$

The graph of this meromorphic map is the blow-up $\tilde{X}$ of $\mathbb{P}_2$ in the 3 points O, P, Q. Then the G-invariant open dense subset $\Omega_1 := \mathbb{P}_2 \setminus \{(PQ) \cup (OP)\} \simeq \mathbb{C}^* \times \mathbb{C}$ admits a f-GF holomorphic quotient map $(x, y) \mapsto x.y$.

Note that we can make another choice: $\Omega'_1 := \mathbb{P}_2 \setminus \{(PQ) \cup (OQ)\} \simeq \mathbb{C} \times \mathbb{C}^*$ with the same map $(x, y) \mapsto x.y$ but now x may vanishes and $y \neq 0$.

Then we can choose the G-invariant open set

$$\Omega_0 := \mathbb{P}_2 \setminus \{(PQ) \cup (OP) \cup (OQ)\} = \Omega_1 \cap \Omega'_1.$$

Note that for $(x, 0) \in \Omega_1$ we have

$$\tau(q^{-1}(q(x,0)) = \overline{G.(x,0)} \cup (OQ) \quad \text{and} \quad \tau(\overline{G.(x,0)}) \cup \{O\} \cup \{P\}$$

so q does not induce a f-GF holomorphic quotient on $\mathbb{P}_2 \setminus \{0, P, Q\}$.

Remark. — In the situation of a connected complex Lie group acting completely holomorphically on an irreducible complex space X, the irreducibility of X gives that the subset of points in X for which the stabilizer has a dimension strictly bigger than the generic dimension is a closed analytic G-invariant subset Y_0 in X with no interior point. Then it is clear that any G-invariant open set Ω_1 for which there exists a f-GF holomorphic quotient has to be contained in $X \setminus Y_0$. In fact the G-invariant open dense subset $X \setminus Y_0$ is the first and best "candidate" for such an open set. But the example above shows that, even in the algebraic context, assuming moreover that each orbit in $X \setminus Y_0$ is a closed analytic subset in $X \setminus Y_0$, only some smaller open sets may have a f-GF holomorphic quotient.

PROPOSITION 2.4.2. — *Let G be a complex connected Lie group which acts completely holomorphically on an irreducible complex space X. Assume that we have a SQP meromorphic quotient for this action, given by a G-modification $\tau : \tilde{X} \to X$ and a G-invariant holomorphic f-GF map $q : \tilde{X} \to Q$.*

Let $\psi : Q \to \mathcal{C}_n^f(X)$ be the holomorphic map obtained by the composition of the fiber map of the f-GF map q and the direct image map for n-cycles by the modification τ. Define $Q_u := \psi(Q)$. Then we have the following properties:

1. *Q_u is a closed analytic subset in $\mathcal{C}_n^f(X)$ which is an irreducible complex space of finite dimension with the structure sheaf induced by the sheaf of holomorphic functions on $\mathcal{C}_n^f(X)$.*
2. *Let $\tilde{X}_u$ be the graph of the meromorphic map $q_u : X \dashrightarrow Q_u$ given by the holomorphic map $\psi \circ q : \tilde{X} \to Q_u$ and let $\tau_u : \tilde{X}_u \to X$ and $q_u : \tilde{X}_u \to Q_u$ be the projections on X and Q_u respectively of this graph. Then (τ_u, q_u) is also a SQP meromorphic quotient for the given G-action.*
3. *For any SQP meromorphic quotient (τ, q) there exists a unique holomorphic surjective map $\eta : Q \to Q_u$ such that the meromorphic maps $q : X \dashrightarrow Q$ and $q_u : X \dashrightarrow Q_u$ satisfies $\eta \circ q = q_u$.*

DEFINITION 2.4.3. — *In the situation of the previous proposition the SQP meromorphic quotient for the given G-action defined by (τ_u, q_u) will be called* the minimal SQP meromorphic quotient *of this G-action.*

So the proposition above says that the existence of a SQP meromorphic quotient for the given G-action implies the existence and uniqueness of a minimal meromorphic quotient for this G-action.

Proof of the Proposition 2.4.2. — To prove the point 1. we shall prove that the map

$$\psi \circ q : \tilde{X} \to \mathcal{C}_n^f(X)$$

is semi-proper. Let $C \neq \emptyset$ be in $\mathcal{C}_n^f(X)$ and fix a relatively compact open set W in X meeting all irreducible components of C. The subset $\mathcal{W}$ of $\mathcal{C}_n^f(X)$ of cycles C' such that any irreducible component of C' meets W is an open set containing C. Now $q(\tau^{-1}(\bar{W}))$ is a compact set in Q, as τ is proper. Take any $y \in Q$ such that $C' := \psi(y)$ is in $\mathcal{W}$. The point y is the limit in Q of points $y_\nu \in q(\tilde{\Omega}_0)$ such that the fiber of q at y is limit in $\mathcal{C}_n^f(\tilde{X})$ of the fibers $q^{-1}(y_\nu) = \overline{G.\tilde{x_\nu}}$ where, for $\nu \gg 1$, we can choose $\tilde{x}_\nu$ in $\tilde{\Omega}_0 \cap \tau^{-1}(W)$. Passing to a sub-sequence, we may assume that the sequence $(\tilde{x}_\nu)$ converges to a point $\tilde{x}$ in $\tau^{-1}(\bar{W})$. Then the continuity of q implies that $q(\tilde{x}) = y$ and C' is the limit of $\overline{G.\tilde{x}_\nu}$. So $|C'|$ is in the image by ψ of the compact set $q(\tau^{-1}(\bar{W}))$ and this gives the semi-properness of $\psi \circ q$.

Now the direct image Theorem 2.3.2 in [3] shows that Q_u is an irreducible complex space (locally of finite dimension) and the point 1. is proved.

Now, using the G-invariance of the map $\psi \circ q$ we see that Q_u is point by point invariant by the natural action of G on $\mathcal{C}_n^f(X)$ defined in the Proposition 2.2.1. Then the graph $\tilde{X}_u$ of the G-invariant meromorphic map $q_u : X \dashrightarrow Q_u$ is G-stable in $X \times Q_u$ and the G-action induced on it makes the projection $q_u : \tilde{X}_u \to Q_u$ G-invariant and the projection $\tau_u : \tilde{X}_u \to X$ G-equivariant.

To prove the second point we have to show that the map $q_u : \tilde{X}_u \to Q_u$ is a f-GF map. By definition $\tilde{X}_u$ is the closure in $X \times Q_u$ of the graph of the map $q|_{\Omega_0}$ where $\Omega_{0,u} := \Omega_0$ is an open dense set in X such that for any point $x \in \Omega_0$ we have $\psi(q(x)) = \overline{G.x}$ (as Ω_0 is disjoint from the center of the modification τ we identify via τ the open sets Ω_0 and $\tilde{\Omega}_0 := \tau^{-1}(\Omega_0)$). Then, by irreducibility of Q_u and X, the closed analytic subset $\tilde{X}_u \subset X \times Q_u$ is equal to the graph of the tautological family of cycles in X parametrized by $Q_u \subset \mathcal{C}_n^f(X)$. This will complete the proof of the point 2. when we shall be able to find

i) a closed G-invariant analytic subset $\tilde{Y}_u$ with no interieur point containing $\tau_u^{-1}(\Sigma_u)$ where Σ_u is the center of the modification τ_u; then let $Y_u := \tau(\tilde{Y}_u)$;
ii) a dense G-invariant open set $\Omega_{1,u} \subset X \setminus Y_u$ such that the restriction of q_u to $\Omega_{1,u}$ will give a f-GF holomorphic quotient for the action of G on $\Omega_{1,u}$.

These facts will be deduced from the point 3.

Now consider a SQP meromorphic quotient of the G-action given by the maps $\tau : \tilde{X} \to X$ and $q : \tilde{X} \to Q$. Then, by the construction in the proof of the point 1, we know that $\psi \circ q(\tilde{X}) = Q_u$ and then the map $\psi \circ q$ induces a surjective holomorphic map $\eta : Q \to Q_u$. Now $\tilde{X}$ is a G-equivariant modification of the graph $\tilde{X}_0$ of the G-invariant meromorphic map $q : X \dashrightarrow Q$. Then, as $\tilde{X}_u$ is the graph of the G-invariant meromorphic map $q_u : X \dashrightarrow Q_u$ the holomorphic map $\mathrm{id}_X \times \eta : X \times Q \to X \times Q_u$ sends $\tilde{X}_0$ to $\tilde{X}_u$ because this is true over the dense open set Ω_0 in X where the maps q and q_u are holomorphic and satisfy $q^{-1}(q(x)) = q_u^{-1}(q_u(x)) = \overline{G.x}$. This complete the proof of the point 3.

Now the composition of the holomorphic G-equivariant modifications $\tilde{X} \to \tilde{X}_0$ and $\tilde{X}_0 \to \tilde{X}_u$ allows to define $\tilde{Y}_u$ and $\Omega_{1,u}$ as the images of $\tilde{Y}$ and Ω_1 by this modification. $\square$

Note that the subsets $\Omega_{0,u}, \Omega_{1,u}$ and $\tilde{Y}_u, Y_u$ are not intrinsically defined in $\tilde{X}_u$.

Corollary 2.4.4. — *In the situation of the previous proposition, assume that another Lie group H acts continuously holomorphically on X. Assume that the action of H normalizes the action of G, that is to say that for any $(h,x) \in H\times X$ we have $h.[G.x] = G.(h.x)$. Then the minimal SQP meromorphic quotient of X for the action of G is H-equivariant in the following sense:*

- *There are natural continuously holomorphic H-actions on $\tilde{X}_u$ and on Q_u such that the maps $\tau_u : \tilde{X}_u \to X$ and $q_u : \tilde{X}_u \to Q_u$ are H-equivariant.*

Moreover, if H is a complex Lie group and the action of H on X is completely holomorphic, the natural actions of H on $\tilde{X}_u$ and Q_u are completely holomorphic.

Proof. — As H acts on $\mathcal{C}_n^f(X)$ via $(h,C) \mapsto h_*(C)$ the only thing to prove is the fact that $\tilde{X}_u \subset X \times \mathcal{C}_n^f(X)$ is stable by the action of H. But, by definition of the minimal SQP meromorphic quotient, $\tilde{X}_u$ is the closure in $X \times \mathcal{C}_n^f(X)$ of the set of couples $(x, \overline{G.x})$ for x in a dense G-invariant open set in X.

We have for such an x,

$$h.(x, \overline{G.x})) = (h.x, h_*(\overline{G.x}) = (h.x, \overline{h.[G.x]}) = (h.x, \overline{G.(h.x)})$$

as $h.[G.x] = G.(h.x)$ and the fact that H acts by complex automorphisms on X.

If H is a complex Lie group and the action of H on X is completely holomorphic, its action on $\mathcal{C}_n^f(X)$, X_u and Q_u are also completely holomorphic. $\square$

The next result shows that the minimal SQP quotient for a G-action on X factorizes any G-invariant meromorphic map defined on X.

Theorem 2.4.5. — *Let G be a complex Lie group acting on a irreducible complex space X with a minimal SQP meromorphic quotient $\tau : \tilde{X} \to X$ and $q : \tilde{X} \to Q$. Let $\gamma : \tilde{X} \to T$ be a G-invariant holomorphic map. Then there exists a holomorphic map $h : Q \to T$ such $\gamma = q \circ h$.*

More generally, for any G-invariant meromorphic map $\gamma : X \dashrightarrow T$[11] *there exists a holomorphic map $H : Q \to T$ such that $\tilde{\gamma} = \tilde{q} \circ H$ on $\tilde{X} \times_{X,\mathrm{str}} Z$, where $\theta : Z \to X$ is the graph of γ in $X \times T$, where $\tilde{\gamma}$ is the composition of*

11. by definition that means that the projection $\theta : Z \to X$ of the graph $Z \subset X \times T$ of γ is G-equivariant, where G acts trivially on T, and that the projection $Z \to T$ is G-invariant.

the projection of $\tilde{X} \times_{X,\mathrm{str}} Z$ on Z with the projection of Z on T and $\tilde{q}$ the composition of the projection of $\tilde{X} \times_{X,\mathrm{str}} Z$ on $\tilde{X}$ with q:

$$\begin{tikzcd}
Z \times_{X,\mathrm{str}} \tilde{X} \arrow[r] \arrow[d] \arrow[ddr, "\tilde{\gamma}"] \arrow[rr, bend left, "\tilde{q}"] & \tilde{X} \arrow[r, "q"] \arrow[d, "\tau"] & Q \arrow[ddl, "H"] \\
Z \arrow[dr] & X \arrow[d, "\gamma"] & \\
 & T. &
\end{tikzcd}$$

The main ingredient to prove this theorem is the following lemma (see Lemma 2.1.8 in [3] for a proof).

Lemma 2.4.6. — *Let U and B be open polydisks in $\mathbb{C}^n$ and $\mathbb{C}^p$ and let $F : U \times B \to \mathbb{C}^N$ be a holomorphic map. For any positive integer k the subset $S(F)$ of $H(\bar{U}, \mathrm{Sym}^k(B))$ of the multiform graphs contained in a fiber of F is a closed banach analytic subset of $H(\bar{U}, \mathrm{Sym}^k(B))$. Moreover, the map $\tilde{F} : S(F) \to \mathbb{C}^N$ given by the value of F on $X \in S(F)$, is a holomorphic map on $S(F)$.*

Proposition 2.4.7. — *Let X be a reduced complex space and let $(C_s)_{s\in S}$ be a f-analytic family of n-cycles in X parametrized by a reduced complex space S. Let $h : X \to T$ be a holomorphic map and assume that the restriction of h to the set $|C_s|$, for s in a dense set S' in S, is constant. Then there exists a holomorphic map $H : S \to T$ such that for each $s \in S$ we have $|C_s| \subset h^{-1}(H(s))$.*

Note that this means that the classifying map $\varphi : S \to \mathcal{C}_n^f(X)$ of the f-analytic family $(C_s)_{s\in S}$ takes its values in $\mathcal{C}_n^f(h)$; the holomorphic map H is then the composition of φ with the natural holomorphic map $\tilde{h} : \mathcal{C}_n^f(h) \to T$ (see [3] Prop. 2.1.7).

Proof. — A local embedding of T in an open set in $\mathbb{C}^N$ and the consideration of finitely many adapted scales[12] to a cycle C_{s_0} of the family allow to deduce the proposition from the lemma above. $\square$

The following corollary of the Proposition 2.4.7 is immediate, as, by definition, the fibers of a f-GF holomorphic map is given by a f-analytic family of cycles.

12. Here the quasi-properness over S of the graph of the family is used in a crucial way in order that a finite number of adapted scales are enough to determine, locally on the parameter space, the cycles of the family .

Corollary 2.4.8. — *Let $q : X \to Q$ a f-GF holomorphic map between irreducible complex spaces. Let $n := \dim X - \dim Q$. If $h : X \to T$ is a holomorphic map such that for y in a dense set in Q the restriction of h to the fiber $q^{-1}(y)$ is constant, then there exists a holomorphic map $H : Q \to T$ such that $h = q \circ H$.*

Proof of the Theorem 2.4.5. — Let $\sigma : Z \to \tilde{X}$ be the strict transform of the modification $\theta : Y \to X$ given by the projection of the graph of γ on X by the modification $\tau : \tilde{X} \to X$. As these two modifications are G-equivariant, this is also the case for σ, and the maps q and γ are holomorphic on Z. The generic fiber of q is the closure of a G-orbit in $\tilde{X}$, so it is contained in a fiber of γ as G is connected. Then the Corollary 2.4.8 applies and we have on Z the factorization $\gamma = q \circ H$ where $H : Q \to T$ is a holomorphic map. □

2.5. Good points, good open sets. — Let X be a reduced complex space and G be a connected complex Lie group. Let $f : G \times X \to X$ be a completely holomorphic action of G on X. We shall often note $g.x := f(g, x)$ for $g \in G$ and $x \in X$.

Definition 2.5.1. — *We shall say that a point $x \in X$ is a* good point *for the action f if the following condition is satisfied*

- *for each compact set K in X there exists an open neighborhood V of x and a compact set L in G such that if $y \in V$ and $g \in G$ are such that $g.y \in K$, there exists $\gamma \in L$ with $\gamma.y = g.y$.*

We shall say that the action of G on X is good *when each point in X is a good point. If Ω is a G-invariant open set in X, we shall say that Ω is a* a good open set *for the action f when all points in Ω are good points for the G-action given by f* restricted to Ω.

Remarks. —

1. If $x \in X$ is a good point, then, for any $g_0 \in G$, $g_0.x$ is also a good point: for K given, choose $g_0.V$ as neighborhood of $g_0.x$ and the compact set $L.g_0^{-1} \subset G$ to satisfy the needed conditions.
2. Consider $\Omega' \subset \Omega$ two G-invariant open sets in X and assume that $x \in \Omega'$ is a good point in Ω. Then x is a good point in Ω'.
   So, if Ω is a good open set, Ω' is also a good open set.
3. But conversely, if $x \in \Omega' \subset \Omega$ is a good point in Ω', it is not true, in general, that x is a good point in Ω. So if Ω is a good open set, points in Ω are not in general good points for the action on X.
4. If M is a compact set of good points in X for any compact set K in X we can find a neighborhood V of M in X and a compact set L in G such that for any point $y \in V$ and any $g \in G$ such that $g.y \in K$ there exists $\gamma \in L$ with $\gamma.y = g.y$. This is easily obtained by a standard compactness argument. We shall say that a compact set of good points is *uniformely good.*

LEMMA 2.5.2. — *Let x be a point in X. Then x is a good point for the G-action on X if and only if the map $F_X : G \times X \to X \times X$ given by $(g, x) \mapsto (x, g.x)$ is semi-proper at each point of $\{x\} \times X$. As a consequence a G-invariant open set Ω in X is a good open set for the G-action if and only if the map $F_\Omega : G \times \Omega \to \Omega \times \Omega$ given by $(g, x) \mapsto (x, g.x)$ is semi-proper.*

Proof. — Let $x \in X$ be a good point and fix any $z \in X$. To prove that the map F_X is semi-proper at (x, z) choose compact neighborhoods V_0 and K of x and z in X and apply the definition of a good point to the compact set K. So we can find a neighborhood V of x, that we may assume to be contained in V_0, and a compact set L in G such that for any $y \in V$ such that $g.y \in K$ we have a $\gamma \in L$ with $g.y = \gamma.y$. Then we have $F_X(G \times X) \cap (V \times K) = F_X(L \times V_0) \cap (V \times K)$ and $L \times V_0$ is a compact set in $G \times X$.

Conversely, assume that the map F_X is semi-proper at each point of $\{x\} \times X$. Take a compact set K in X and apply the semi-properness to each point (x, z) where z is in K. For each $z \in K$ we obtain open neighborhoods V_z and W_z of x and z in X and a compact set $L_z \times M_z$ in $G \times X$ such that

$$F_X(G \times X) \cap (V_z \times W_z) = F_X(L_z \times M_z) \cap (V_z \times W_z).$$

Extract a finite sub-cover $W_1, \dots, W_N$ of K by the open sets W_z and define the compact set $L := \bigcup_{i \in [1,N]} L_{z_i}$ and the neighborhood $V := \bigcap_{i \in [1,N]} V_{z_i}$ of x. Then if y is in V and $g.y$ is in K there exists $i \in [1, N]$ such that $g.y \in W_i$. As y is in V_i we can find a $\gamma \in L_{z_i} \subset L$ with $F_X(g, y) = F_X(\gamma, y)$ and this implies that x is a good point. The second assertion is an easy consequence of the first one. $\square$

LEMMA 2.5.3. — *Let G be a connected complex Lie group. Let $f : G \times X \to X$ be a completely holomorphic action of G on a reduced complex space X. Consider a countable family $(\Omega_i)_{i \in I}$ of good open sets for f and the family $(F_i)_{i \in I}$ of closed sets in $\Omega := \bigcup_{i \in I} \Omega_i$ defined by $F_i := \partial\Omega_i \cap \Omega$. Let $F := \bigcup_{i \in I} F_i$. Then any point in the dense set $\Omega \setminus F$ of Ω is a good point in Ω.*

Proof. — Of course Ω is a G-invariant open set in X as a good open set is G-invariant by definition. Then $F \cap \Omega$ is a G-invariant set in Ω which is a countable union of nowhere dense closed sets in Ω. So $\Omega \setminus F$ is a dense G_δ in Ω. Now let $x \in \Omega \setminus F$ and K be a compact subset in Ω. Choose a sub-covering $(\Omega)_i, i \in [1, N]$ of K by some open sets $\Omega_i, i \in [1, N]$ and let $\{i \in [1, p], p \in [0, N]\}$ be the subset of $i \in [1, N]$ such that x is in Ω_i. Choose now a compact neighborhood V of x such that V is contained in $\bigcap_{i=1}^{p} \Omega_i$ and such that $V \cap \Omega_j = \emptyset$ for each $j \in [p+1, N]$. This is possible because x is not in $\partial\Omega_j$ for $j \in [p+1, N]$. Remark that if y is in V and $g.y$ is in K, by the G-invariance of Ω_j we have $g.y \notin \Omega_j$ for each $j \in [p+1, N]$. So let $\tilde{K} := K \setminus \bigcup_{j=p+1}^{N} \Omega_j$. This is a compact set in $\bigcup_{i=0}^{N} \Omega_i$. Choose now compact sets $K_1, \dots, K_p$ such that $K_i \subset \Omega_i$ and such that $\tilde{K} \subset \bigcup_{i=1}^{p} K_i$. For each $i \in [1, p]$, as x is in Ω_i

and as K_i is a compact set in Ω_i which is a good open set, there exist an open neighborhood $W_i \subset V$ of x and a compact set L_i in G such that for any $y \in W_i$ and any $g \in G$ such that $g.y$ is in K_i there exists a $\gamma \in L_i$ with $\gamma.y = g.y$.

Let $W := \bigcap_{i=1}^{p} W_i$ and $L := \bigcup_{i=1}^{p} L_i$. If now $y \in W$ and $g \in G$ are such that $g.y$ is in K, then first we have $g.y$ which is in $\tilde{K}$. If $g.y$ is in K_{i_0} as y is in W_{i_0} there exists $\gamma \in L_{i_0} \subset L$ such that $g.y = \gamma.y$. This shows that any x in $\Omega \setminus F$ is a good point in the G-invariant open set Ω. □

Remarks. —

1. If the family $(F_i)_{i\in I}$ is locally finite in Ω then $\Omega \setminus F$ is a dense good open set in Ω.
2. As Ω is countable at infinity (it is an open set in a complex space), if the family $(\Omega_i)_{i\in I}$ is not countable, it is always possible to find a countable sub-family $(\Omega_i)_{i\in I'}$ such that $\Omega = \bigcup_{i\in I'} \Omega_i$.

PROPOSITION 2.5.4. — *Let G be a connected complex Lie group. Let $f : G \times X \to X$ be a completely holomorphic action of G on a reduced complex space X. Then we have the following properties:*

i) *If x is a good point for f the orbit $G.x$ is a closed analytic subset of X.*
ii) *If x is a good point for f and if $(G.x) \cap K = \emptyset$ where $K \subset X$ is a compact set, there exists a neighborhood V of x in X such that $(\overline{G.x'}) \cap K = \emptyset$ for any x' in V (x' is not assumed here to be a good point).*
iii) *If x is a good point for f there exists a neighborhood V of x such that any good point $x' \in V$ has an orbit which is a closed analytic subset of the same dimension than $G.x$.*
iv) *Let Ω be a good connected open set in X which is normal and let n be the dimension of $G.x$ for $x \in \Omega$. Then there exists a holomorphic map[(13)] $\varphi : \Omega \to \mathcal{C}_n^f(\Omega)$ given generically by $\varphi(x) := G.x$ as a reduced n-cycle in Ω.*
v) *When we have a good open set Ω in X which is normal, there exists a f-GF holomorphic quotient of Ω for the action restricted to Ω.*

Proof. — We already proved that x is a good point if the map $G \to X$ given by $g \mapsto g.x$ s semi-proper in Lemma 1.3.2. Now Kuhlmann's theorem [7], [8] gives that $f_x(G) = G.x$ is a closed analytic subset of X. This proved i).

Assume ii) is not true ; then we have a compact set K such that $(G.x) \cap K = \emptyset$ and a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ converging to x and such that $(\overline{G.x_\nu}) \cap K$ is not empty for each ν. Fix a compact neighborhood $\tilde{K}$ of K such that $(G.x) \cap \tilde{K} = \emptyset$. This is possible thanks to i). Pick a point $y_\nu = \lim_{\alpha\to\infty} g_{\nu,\alpha}.x_\nu$ in $(\overline{G.x_\nu}) \cap K$ for each ν. Passing to a subsequence we may assume that sequence (y_ν) converges

13. Recall that this means that we have an f-analytic family of n-cycles in Ω parametrized by Ω.

to $y \in K$ when $\nu \to +\infty$. So, for $\alpha \geq \alpha(\nu)$, we can assume that $g_{\nu,\alpha}.x_\nu$ is in $\tilde{K}$. But, as x is a good point, for the given compact set $\tilde{K}$ there exists a neighborhood V of x and a compact set $L \subset G$ as in the definition. We may assume that x_ν is in V for $\nu \geq \nu_0$ and so we may find, for $\nu \geq \nu_0, \alpha \geq \alpha(\nu)$, elements $\gamma_{\nu,\alpha} \in L$ such that $\lim_{\alpha\to\infty} \gamma_{\nu,\alpha}.x_\nu = y_\nu \quad \forall \nu \geq \nu_0$.

Passing to a sub-sequence for each given $\nu \geq \nu_0$, we may assume that the sequence $(\gamma_{\nu,\alpha})_\alpha$ converges to some $\gamma_\nu \in L$. And again, that the sequence (γ_ν) converges to some $\gamma \in L$. So the continuity of f gives $y_\nu = \gamma_\nu.x_\nu \to \gamma.x = y \in \tilde{K}$ giving a contradiction because we assume $(G.x) \cap \tilde{K} = \emptyset$. This proves ii).

Let $E := (U, B, j)$ be a n-scale on Ω adapted to the n-cycle $G.x$. Then the compact set $K := j^{-1}(\bar{U} \times \partial B)$ does not meet $G.x$, by definition of an adapted scale. Using ii), there exists a neighborhood V of x such that for any $x' \in V$ we have $(\overline{G.x'}) \cap K = \emptyset$. As for a good point $x' \in V$ we know that $G.x'$ is a closed analytic subset, the n-scale is then adapted to $G.x'$. This implies that the dimension of $G.x'$ is at most equal to n. But the semi-continuity of the dimension of the stabilizers implies that the dimension of $G.x' \simeq G\big/St(x')$ is at least equal to $n = \dim(G\big/St(x))$. This proves iii).

Remark that for any $x' \in V$ such that $\overline{G.x'} \cap \Omega$ is a (closed) analytic subset in Ω, the previous proof shows also that $\overline{G.x'} \cap \Omega$ is of dimension n.

To prove iv) fix a good connected open set Ω and define

$$Z := \{(g, x, y) \in G \times \Omega \times \Omega / y = g.x\}.$$

This is a closed analytic subset in $G \times \Omega \times \Omega$. It is isomorphic to $G \times \Omega$ by the projection $(g, x, y) \mapsto (g, x)$ and so the projection $p : Z \to \Omega \times \Omega$ is semi-proper, thanks to the Lemma 2.5.2. Its image $p(Z)$ is then a closed analytic subset of $\Omega \times \Omega$ by Kuhlmann's theorem [7], [8], which is a generalization of Remmert's theorem [10] to the semi-proper case. But now the projection

$$\pi : p(Z) \to \Omega$$

is n-equidimensional, thanks to iii), and has irreducible generic fibers on a normal basis Ω. So its fibers (with generic multiplicity equal to 1) define an analytic family of n-cycles of X parametrized by Ω. It is clearly f-analytic because each fiber is irreducible(14) and we have an holomorphic section because each x lies in $G.x$.

To prove v) let us prove that the holomorphic map $\varphi : \Omega \to \mathcal{C}_n^f(\Omega)$ classifying the fibers of $p(Z)$ is semi-proper. Fix a non empty cycle $C \in \mathcal{C}_n^f(\Omega)$ and choose a point $x_i, i \in [1, k]$, in each irreducible component of $|C|$. Let W a relatively compact open neighborhood of $\{x_1, \dots, x_k\}$ in Ω and let $\mathcal{W}$ be the open set in $\mathcal{C}_n^f(\Omega)$ of cycles such that each irreducible component meets W. Let C' be in $\mathcal{W} \cap \varphi(\Omega)$; we know that if $C' = \varphi(z)$ we have $|C'| = G.z$. So $G.z$ has to meet W and we can choose y in the compact set $\bar{W}$ such that $|C'| = |\varphi(y)|$; but the

14. But some multiplicities may occur.

equality of supports implies equality of cycles in this family. So $\varphi(z) = \varphi(y)$. This gives the semi-properness of φ. Now the semi-proper direct image Theorem 2.3.2 of [3] implies that the image Q of φ is a locally finite dimensional reduced complex space. Moreover, it parametrizes a f-analytic family of n-cycles in Ω which coïncides generically with the reduced G-orbits. Then the holomorphic map $q : \Omega \to Q$ is a quasi-proper GF holomorphic quotient for the action f on Ω as *each* fiber of q is set-theoretically a G-orbit. □

2.6. Nice points. — We consider a connected complex Lie group G and a completely holomorphic action of G on a reduced complex space X given by a holomorphic map

$$f : G \times X \to X.$$

Define the closed analytic subset

$$Z := \{(g, x, y) \in G \times X \times X / y = g.x\}$$

and let $p : Z \to X \times X$ the natural projection. Remark that Z is isomorphic to $G \times X$ and that the projection $p : Z \to X \times X$ is equivalent, via this isomorphism, to the map $f \times \mathrm{id} : G \times X \to X \times X$ given by $(g, x) \mapsto (g.x, x)$.

Definition 2.6.1. — *We shall say that a couple $(x, y) \in X \times X$ is a* good couple *when there exists open neighborhoods $V(x)$ and $V(y)$ respectively of x and y in X and a compact subset L in G such that for any $x' \in V(x)$, any $y' \in V(y)$ and any $g \in G$ such that $y' = g.x'$ there exists $\gamma \in L$ with $y' = \gamma.x'$.*

Remarks. —

1. A couple (x, y) is good if and only if the map $p : Z \to X \times X$ is semi-proper at the point (x, y), and then it is semi-proper at any point in $V(x) \times V(y)$. So the set of good couples in $X \times X$ is an open subset.
2. A couple (x, y) is good if and only the couple (y, x) is good.
3. Assume that the couple (x, y) is good ; then, with the notation of the previous definition, for any $x' \in V(x)$ the subset $G.x' \cap V(y)$ is a closed subset in $V(y)$. Also for any $y' \in V(y), G.y' \cap V(x)$ is closed in $V(x)$.
4. If a couple (x, y) is good, then for any $(g_1, g_2) \in G \times G$, the couple $(g_1.x, g_2.y)$ is a good couple. So the subset of good couple in $X \times X$ is $(G \times G)$-invariant.
5. For any point $x \in X$ the subset $\Omega(x) := \{y \in X/(x, y)$ is a good couple$\}$ is an open G-invariant subset in X. Moreover, the open set $\Omega(x)$ only depends on the orbit $G.x$ of x and not of the choice of x in its G-orbit.

In order to obtain a canonical G-invariant open set in X on which there exists (at least locally) a f-GF holomorphic quotient we shall introduce the following notion.

Definition 2.6.2. — *We shall say that a point x in X is a* nice point *when the couple (x, x) is a good couple.*

Note that nice points in X are points corresponding via the diagonal embedding $\delta : X \to X \times X$ to the intersection of the set of good couples with the diagonal.

So the subset of nice points is a G-invariant open set in X (may be empty!).

We have the following characterization of nice points in X.

LEMMA 2.6.3. — *A point $x \in X$ is a nice point if and only there exists a G-invariant open set U containing x such that x is a good point in U.*

Proof. — For any $x \in X$ the subset $\Omega(x)$ is a G-invariant open set, and x is a nice point in X if and only if x is in $\Omega(x)$ by definition. Let us show that in this case x is a good point in $\Omega(x)$. Then take any compact set K in $\Omega(x)$. For any $y \in K$ the couple (x, y) is a good couple, so there exist $V_y(x), V(y)$, respectively open neighborhoods of x and y in $\Omega(x)$ and a compact set L_y in G such that for any $x' \in V_y(x), y' \in V(y)$ such that $y' = g.x'$ for some $g \in G$, there exists $\gamma \in L_y$ with $\gamma.x' = y'$. Extract a finite sub-cover of the open cover of the compact K by the $V(y), y \in K$ corresponding to the points $y_1, \dots, y_N$. Let $W(x) := \bigcap_{i=1}^N V_{y_i}(x)$ and $L := \bigcup_{i=1}^N L_{y_i}$. Then for any $x' \in W(x)$ and any $g \in G$ such that $g.x' = z$ is in K, as there exists $i \in [1, N]$ such that $z \in V(y_i)$ and as x' is in $V_{y_i}(x)$ we can find $\gamma \in L_{y_i} \subset L$ with $\gamma.x' = z$. So x is a good point in $\Omega(x)$.

Conversely, if x is a good point in the G-invariant open set U in X, let V be a relatively compact open neighborhood of x in U. Let $K := \bar{V}$. As x is a good point in U, for the compact set K in U we may find $V(x)$ an open neighborhood of x in U and a compact set L in G such that for any $x' \in V(x)$ and any $g \in G$ such that $g.x'$ lies in K we can find $\gamma \in L$ such that $\gamma.x' = g.x'$. As we may assume that $V(x)$ is contained in V we see that the couple (x, x) is a good couple thanks to the choices $V(x), V(x)$ and L. □

COROLLARY 2.6.4. — *A point $x \in X$ is a nice point if and only it is contained in a good open set in X.*

Proof. — As the set of nice points in X is open, we can find a compact neighborhood V of x such that any couple $(x', y') \in V \times V$ is a good couple. Now define

$$\Omega(V) := \{y \in X / \forall x' \in V \text{ the couple } (x', y) \text{ is good}\}.$$

Then $\Omega(V)$ is a G-invariant open set in X and the interior V_0 of V is an open set of good points in $\Omega(V)$. Then the G-invariant open set $G.V_0$ is a good open set containing x. The converse is obvious thanks to the previous lemma. □

The next proposition summarizes the relations between the several definitions given above.

PROPOSITION 2.6.5. — *Let Ω be a G-invariant open set in X. Then the following properties holds:*

i) *A point $x \in \Omega$ is a good point in Ω if and only if $\Omega \subset \Omega(x)$.*

ii) *Ω is a good open set if and only if $\Omega \times \Omega$ is contained in the open set of good couples in $X \times X$. This implies that each point of Ω is a nice point but also (see* i)*) that $\Omega \subset \bigcap_{x\in\Omega} \Omega(x)$.*

iii) *A G-invariant open set Ω in X is a good open set if and only if the restriction map $(f \times \mathrm{id})|_{\Omega} : G \times \Omega \to \Omega \times \Omega$ is semi-proper.*

iv) *The set of nice points in X, denoted Ω_{nice}, is the union of all good open sets in X. But, in general, it is not true that Ω_{nice} is itself a good open set (see the Lemma 2.5.3).*

v) *On the G-invariant open set of normal points of Ω_{nice} we have* locally *a f-GF holomorphic quotient for the G-action.*

Proof. — As Ω is locally compact, it is clear that for any good point x in Ω and any point $y \in \Omega$ the couple (x, y) is a good couple. The converse is easy (see the begining of the proof of the Lemma 2.6.3). If Ω is a good open set it is again easy to see that any couple $(x, y) \in \Omega \times \Omega$ is a good couple. The converse is easily obtained following the same argument than in the beginning of the proof of the Lemma 2.6.3.

The assertion iii) is already in the Lemma 2.5.2. The inclusion of good open set in Ω_{nice} is obtained in Lemma 2.6.3 and the Corollary 2.6.4 implies the equality. The assertion v) is consequence of iv) and of the property v) in the Proposition 2.5.4. □

2.7. The conditions [H.1], [H.1str], [H.2] and [H.3]. — Now we shall consider the following conditions on the action f.

- *There exists a G-invariant dense open set Ω_1 in X which admits a quasi-proper GF holomorphic quotient.* [H.1]

Recall that this means that there exists a G-invariant geometrically f-flat holomorphic map $q : \Omega_1 \to Q_1$ onto a reduced complex space Q_1 such that each fiber of q over a point in Q_1 is set-theoretically an orbit in Ω_1. So the hypothesis [H.1] implies that all G-orbits in Ω_1 are closed analytic subsets in Ω_1 of the same dimension.

The following stronger form will be useful.

- *There exists a G-invariant good open set Ω_1 which is Zariski open and dense in X.* [H.1str]

This condition implies [H.1] thanks to the Proposition 2.5.4. But it is not true in general that [H.1] implies the existence of a dense good open set in X.

Remarks. —

1. The existence of a dense good open set in X is a natural hypothesis on the G-action to obtain [H.1], thanks to the Proposition 2.5.4. We add here the condition "Zariski open" for this dense good open set because

this assumption is crucial in our application in order to use the "sub-analytic lemma".

2. A good open set is always contained in Ω_{nice} which is a G-invariant open set canonically defined by the action. So a necessary condition for the existence of a dense good open set in X is the density of Ω_{nice} in X. Also to have a dense Zariski good open set in X it is necessary that the complement of Ω_{nice} is contained in a closed nowhere dense analytic subset in X.
3. The hypotheses [H.1] concerns only the structure of the orbits in a G-invariant dense open set Ω. The existence of a dense good open set in X involves the defining map $f : G \times X \to X$ for the action of G on X. So these hypotheses are at different levels.

Now assume [H.1] and define $\mathcal{R} := \{(x, y) \in \Omega_1 \times \Omega_1 / y \in G.x\}$. It is a closed analytic set in $\Omega_1 \times \Omega_1$: on the G-invariant open set Ω_1 on which there exists a f-GF holomorphic quotient $q_1 : \Omega_1 \to Q_1$, so the equality $G.x = G.y$ is equivalent to $q_1(x) = q_1(y)$.

Our second assumption will be:

- *The closure $\tilde{\mathcal{R}}$ of $\mathcal{R}$ in $X \times X$ is an analytic subset and there exists a G-invariant open dense subset $\Omega_0 \subset \Omega_1$ such that for each $x \in \Omega_0$*

$$\overline{\mathbf{G.x}} = \tilde{\mathcal{R}} \cap (\{\mathbf{x}\} \times \mathbf{X}). \qquad \text{[H.2]}$$

Remark. — For $x \in \Omega_0$ the orbit $G.x$ is a closed analytic subset in Ω_1, so, as we have $\tilde{\mathcal{R}} \cap (\{x\} \times \Omega_1) = (\{x\} \times G.x)$, $G.x$ is open in $\overline{G.x}$. Then $\overline{G.x}$ is irreducible of dimension n and when Ω_1 is Zariski open in X, $\overline{G.x} \setminus G.x$ is a closed analytic subset and has dimension at most $n - 1$. As this analytic set is G-invariant, it is contained in $Y_0 \subset X$, the closed analytic subset in X where the stabilizer has a bigger dimension than the generic one.

Now the first projection $p_1 : \tilde{\mathcal{R}} \cap (\Omega_0 \times X) \to \Omega_0$ is n-equidimensional surjective (with irreducible fibers, thanks to [H.2]) and quasi-proper because we have a holomorphic section of this map which is given by $x \mapsto (x, x)$.

Assuming that Ω_0 contains only normal points[15] in X, the equidimensionality and quasi-properness on Ω_0 of the projection of $\tilde{\mathcal{R}}$ imply that there exists a holomorphic map

$$\bar{\varphi}_0 : \Omega_0 \to \mathcal{C}_n^f(X)$$

where the supports are given by $x \mapsto \overline{G.x}$ and where the multiplicity is generically equal to 1. Our last hypothesis is:

- *The first projection $p_1 : \tilde{\mathcal{R}} \to X$ is strongly quasi-proper .* [H.3]

15. This not restrictive, as we may always assume that $X \setminus \Omega_0$ contains the non normal points in X. We shall always assume that Ω_0 is normal in the sequel, without any more comment.

Recall that this condition implies that there exists a modification $\tau : \tilde{X} \to X$ with center in the complement of Ω_0 such that the map $\bar{\varphi}_0$ extends holomorphically to $\tilde{X}$. Note that, thanks to [2] Theorem 2.4.4 (see also [3] Proposition 3.2.2) a sufficient condition for [H.3] is that the closure in $X \times \mathcal{C}_n^f(X)$ of the graph of $\bar{\varphi}_0$ is proper over X.

The following proposition shows that these conditions [H.1], [H.2] and [H.3] are necessary for the existence of a SQP-meromorphic quotient for a completely holomorphic action of G on X.

Proposition 2.7.1. — *Assuming that $f : G \times X \to X$ is a completely holomorphic action of the connected complex Lie group G on the irreducible complex space X which has a SQP-meromorphic quotient, then the conditions* [H.1], [H.2] *and* [H.3] *are satisfied.*

Proof. — Let $\tau : \tilde{X} \to X$ and $q : \tilde{X} \to Q$ be respectively the G-equivariant modification of X and the f-GF holomorphic map given by the existence of a SQP meromorphic quotient for the G-action on X. The conditions to be a SQP-meromorphic quotient gives an open set Ω_1 which is dense, G-stable and which admits a f-GF holomorphic quotient for the action of G on Ω_1. So [H.1] is clear.

Let S be the graph of the equivalence relation given by q on $\tilde{X}$. Then the proper direct image $(\tau \times \tau)(S)$ is a closed analytic subset in $X \times X$.

Let $\mathcal{R} := \{(x, y) \in \Omega_1 \times \Omega_1 / G.x = G.y\}$. Then we have $\mathcal{R} \subset (\tau \times \tau)(S) \cap (\Omega_1 \times \Omega_1)$. To see that $\mathcal{R}$ is dense in $(\tau \times \tau)(S)$, remark that on the dense open set $\Omega_0 \times \Omega_1$ we have the equality of $\mathcal{R}$ and $(\tau \times \tau)(S)$. So the closure $\tilde{\mathcal{R}}$ of $\mathcal{R}$ in $X \times X$ is analytic. Moreover, for $x \in \Omega_0$ the fiber of q at $q(x)$ is equal to $\overline{G.x}$, the closure in $\tilde{X}$ of $G.x$. So the fiber of $\tilde{\mathcal{R}}$ at x is $\tau(\overline{G.x})$ which is equal to the closure in X of $G.x$ because τ is proper and G-equivariant. So the condition [H.2] is satisfied.

The composition of q with the holomorphic classifying map $\varphi : Q \to \mathcal{C}_n^f(\tilde{X})$ for the fibers of q gives a holomorphic map $\tilde{\psi} : \tilde{X} \to \mathcal{C}_n^f(\tilde{X})$. Composed with the direct image map, which is holomorphic (see [4] ch. IV; the "quasi-proper" part of this result is easy, as τ is proper) $\tau_* : \mathcal{C}_n^f(\tilde{X}) \to \mathcal{C}_n^f(X)$, we obtain a holomorphic map

$$\Phi : \tilde{X} \to \mathcal{C}_n^f(X)$$

and the restriction of this map to $\tau^{-1}(\Omega_0) \simeq \Omega_0$ satisfies $|\Phi(\tilde{x})| = \overline{G.\tau(x)}$. So the map Φ is a holomorphic extension to $\tilde{X}$ of the map $\tilde{\varphi}_0 : \Omega_0 \to \mathcal{C}_n^f(X)$ classifying the fibers over Ω_0 of $\tilde{\mathcal{R}}$ via the first projection. This implies that the closure $\bar{\Gamma}$ in $X \times \mathcal{C}_n^f(X)$ of the graph Γ of $\tilde{\varphi}_0$ is contained in $(\tau \times \tau_*)(\Delta)$, where Δ is the graph of $\tilde{\psi}$. But Δ is proper on X via $\tau \circ p_1$ and the set $(\tau \times \tau_*)(\Delta)$ is closed in $X \times \mathcal{C}_n^f(X)$: if the sequence $(\tau(\tilde{x}_\nu), \tau_*(C_\nu))_\nu$ converges to $(x, C) \in X \times \mathcal{C}_n^f(X)$, passing to a subsequence, we may assume that the sequence $(\tilde{x}_\nu)_\nu$ converges to some $\tilde{x}$ in $\tau^{-1}(x)$, and then the sequence $(C_\nu)_\nu$ converges to $\psi(\tilde{x})$ so that $(x, C) = (\tau \times \tau_*)(\tilde{x}, \psi(\tilde{x}))$; this shows our claim. Then $\bar{\Gamma}$ is a closed subset in the X-proper set $(\tau \times \tau_*)(\Delta)$, so $\bar{\Gamma}$ is proper over X and the condition [H.3] is fulfilled. □

2.8. Existence theorem for a SQP-meromorphic quotient. — Now we shall prove that conditions [H.1], [H.2] and [H.3] on a completely holomorphic action of a connected complex Lie group G on an irreducible complex space X are sufficient for the existence of a SQP meromorphic quotient.

THEOREM 2.8.1. — *Under the hypotheses* [H.1], [H.2] *and* [H.3] *there exists a proper G-equivariant modification $\tau : \tilde{X} \to X$ with center contained in $X \setminus \Omega_0$*(16) *and a geometrically f-flat holomorphic map*

$$q : \tilde{X} \to Q$$

on a reduced complex space, which gives a strongly quasi-proper meromorphic quotient for the given G-action.

Let $\Gamma \subset \Omega_0 \times \mathcal{C}_n^f(X)$ be the graph of the holomorphic map $\tilde{\varphi}_0 : \Omega_0 \to \mathcal{C}_n^f(X)$ classifying the fibers of the projection of $\tilde{\mathcal{R}}$ on Ω_0. Of course the complex space $\tilde{X}$ is the topological space $\bar{\Gamma}$ with a structure of a reduced complex space such that the projection on X is a proper modification. Then the space Q is the image of $\tilde{X}$ in $\mathcal{C}_n^f(X)$. So we need some semi-proper direct image theorem for such a map to prove this result. Such a result is the content of the Theorem 2.3.2 of [3].

Proof. — The first remark is that the hypothesis [H.3] says that the projection $p : \bar{\Gamma} \to X$ is a proper topological modification of X. But to apply directly the part ii) of the Theorem 2.3.6 of [2] to the projection $p_1 : \tilde{\mathcal{R}} \to X$ we need quasi-properness of this map. This is given by the Proposition 3.2.2 of [3] as we have the condition [H.3].

Then we obtain a proper (holomorphic) modification $\tau : \tilde{X} \to X$, with center $\Sigma \subset X \setminus \Omega_0$, and a f-analytic family of cycles in X parametrized by $\tilde{X}$ extending the family $(\overline{G.x})_{x \in \Omega_0}$, corresponding to a "holomorphic" map extending $\bar{\varphi}_0$:

$$\tilde{\varphi} : \tilde{X} \to \mathcal{C}_n^f(X).$$

Now let us prove that this map $\tilde{\varphi}$ is quasi-proper(17). This will allow us to apply the Theorem 2.3.2 of *loc. cit.* and to define the reduced complex space Q as the image $\tilde{\varphi}(\tilde{X})$. Then it will be easy to check that the map $\tilde{\varphi} : \tilde{X} \to Q$ is a strongly quasi-proper meromorphic quotient for the G-action we consider.

If C_0 is in $\mathcal{C}_n^f(X)$ and is not the empty cycle, choose a relatively compact open sets W in X such that any irreducible component of $|C_0|$ meets W. Then

16. The dense open subset $\Omega_0 \subset \Omega_1$ is defined in the condition [H.2].
17. This makes sense as the fibers are closed analytic subsets of $\tilde{X}$.

let $\mathcal{W}$ be the open set in $\mathcal{C}_n^f(X)$ defined by the condition on the cycle C that any irreducible component of C meets W. Then we shall prove that there exists a compact set K in $\tilde{X}$ such that any irreducible component of the fiber of $\tilde{\varphi}$ at a point in $\mathcal{W} \cap \tilde{\varphi}(\tilde{X})$ meets K. Let $K := \tau^{-1}(\bar{W})$. If (y, C) is in $\tilde{X}$[18] with $C \in \mathcal{W}$, each irreducible component of C meets W. But the fiber of $\tilde{\varphi}$ at C is equal to $|C|$, so each irreducible component of $\tilde{\varphi}^{-1}(C)$ meets K and the quasi-properness is proved. □

We conclude by a simple sufficient condition in order to obtain the condition [H.1] and [H.2] (assuming already [H.2]) which can be useful because the G-invariant open set Ω_{nice} is canonically defined by the action and so its density in X is a condition which can be tested directly.

PROPOSITION 2.8.2. — *Consider a completely holomorphic action of the complex connected Lie group G on the complex space X. Assume that it has a G-invariant dense open set Ω_1 which admits* locally *a f-GF holomorphic quotient and that it satisfies also the hypothesis* [H.2]. *Then it satisfies* [H.1] *(and* [H.2]*).*

Proof. — The hypothesis [H.2] gives an open G-invariant dense subset $\Omega_0 \subset \Omega_1$ on which we have

$$\tilde{\mathcal{R}} \cap (\Omega_0 \times \Omega_0) = \mathcal{R} \cap (\Omega_0 \times \Omega_0)$$

showing that $\mathcal{R}_0 := \mathcal{R} \cap (\Omega_0 \times \Omega_0)$ is a closed analytic subset in $\Omega_0 \times \Omega_0$. Note also that we can assume Ω_0 normal, as the set of normal points in X is open dense and G-invariant.

Now remark that the projection $p_1 : \mathcal{R}_0 \to \Omega_0$ is quasi-proper and equidimensional. The quasi-properness is consequence of the existence of the holomorphic section $x \mapsto (x, x)$ of p_1 on Ω_0. So the map p_1 is a f-GF holomorphic map, and is the projection of the graph of the f-analytic family of cycles in Ω_0 given by the fibers of p_1 corresponding to a holomorphic map

$$\varphi_0 : \Omega_0 \to \mathcal{C}_n^f(\Omega_0).$$

We shall prove now that φ_0 is semi-proper and then, using the semi-proper direct image Theorem 2.3.6 in [3], we shall conclude that $\varphi_0(\Omega_0)$ is a reduced complex space and the map φ_0 induced a f-GF holomorphic quotient on Ω_0 proving [H.1].

So consider a non empty cycle $C \in \mathcal{C}_n^f(\Omega_0)$ and choose a point $x_i, i \in [1, k]$ in each irreducible component of C, where k is a positive integer. Let, for each i, V_i be an open relatively compact neighborhood of x_i in Ω_0 and let $\mathcal{V}$ be the open set in $\mathcal{C}_n^f(\Omega_0)$ of cycles C' such that each irreducible of C' meets $\bigcup_{i \in [1,k]} V_i$.

18. Recall that, as a topological space, $\tilde{X} = \bar{\Gamma}$.

Then the compact set $\bigcup_{i\in[1,k]} \bar{V}_i$ in Ω_0 satisfies:

$$\varphi_0(\Omega_0) \cap \mathcal{V} = \varphi_0(\bigcup_{i\in[1,k]} \bar{V}_i) \cap \mathcal{V}$$

which gives the semi-properness of φ_0: if $\varphi_0(y)$ lies in $\mathcal{V}$ then let y_1 be in $\varphi_0(y)\cap V_1$. Then we have $y_1 \in G.y$ and so $\varphi_0(y_1)$ and $\varphi_0(y)$ have the same support. But a cycle in $\varphi_0(\Omega_0)$ is determined by its support. Then we have $\varphi_0(y_1) = \varphi_0(y)$. □

The point v) of the in Proposition 2.6.5 shows that the G-invariant open set of normal points in Ω_{nice} admits locally a f-GF holomorphic quotient, we obtain the following corollary of the Theorem 2.8.1.

COROLLARY 2.8.3. — *Let G be a complex connected Lie group action completely holomorphically on an irreducible complex space X. Assume that the open set Ω_{nice} is dense in X and that the hypotheses* [H.2] *(with $\Omega_1 := \Omega_{\text{nice}}$) and* [H.3] *holds, then there exists a SQP meromorphic quotient.*

3. Application

3.1. The sub-analytic lemma. — We shall use the following lemma (see [6]) in our application.

LEMMA 3.1.1. — *Let M be a reduced complex space and $Y \subset M$ a closed analytic subset with no interior point in M. Let R be a closed (complex) analytic subset in $M \setminus Y$ such that $\bar{R}$ is a sub-analytic set in M. Then $\bar{R}$ is a (complex) analytic subset in M.*

This important lemma is a consequence of Bishop's theorem (see [5]) and of a classical result on sub-analytic subsets (see [6] for more references).

3.2. The $G = K.B$ case: proof of the Theorems 1.0.1 and 1.0.2. — Now we shall assume that G is a connected complex Lie group such that we have $G = K.B$ where B is a closed complex connected subgroup of G and K a compact real subgroup of G.

LEMMA 3.2.1. — *In the situation of the Theorem 1.0.1, a couple $(x, y) \in X \times X$ is a good couple for the G-action if for any $k \in K$ the couple $(x, k.y)$ is a good couple for the B-action.*

Moreover, if K normalizes B, for any good couple (x, y) for the B-action and any $k \in K$ the couple $(k.x, k.y)$ is again a good couple for the B-action. This implies that the open set $\Omega_{\text{nice}/B}$ of nice points for the B-action is stable by K, when K normalizes B.

Proof. — For each $k \in K$ there exist $V_k(x), V(k.y)$ respectively open neighborhoods of x and $k.y$ and L_k a compact subset in B such that for any $x' \in V_k(x)$, any $y' \in V(k.y)$ with $y' = b.x'$ for some $b \in B$ there exists $\beta \in L_k$ with $y' = \beta.x'$. Now choose $k_1, \dots, k_N$ in K such that the compact set $K.y$ is contained in the open set $U := \bigcup_{i=1}^N V(k_i.y)$. Then the subset $W(y) := \{y' \in X / K.y' \subset U\}$ is an open neighborhood of y in X[19]. Define also $W(x) := \bigcap_{i=1}^N V_{k_i}(x)$ and $L := \bigcup_{i=1}^N L_{k_i}$. Then $W(x)$ is an open neighborhood of x in X, L is a compact set in B and $\Lambda := K.L$ is a compact set in G.

Take now $x' \in W(x)$ and $y' \in W(y)$ such that $y' = g.x'$ for some $g \in G$. Write $g = k.b$ with $k \in K$ and $b \in B$. Then $k^{-1}.y$ is in $V(k_i.y)$ for some $i \in [1, N]$. As x' is in $W(x) \subset V_{k_i}(x)$, the equality $k^{-1}.y' = b.x'$ allows to find $\beta \in L$ such that $k^{-1}.y' = \beta.x'$ and then $\gamma := k.\beta$ is in $K.L$ and $y' = \gamma.x'$.

So the condition that for any $k \in K$ the couple $(x, k.y)$ is a good couple for the B-action implies that the couple (x, y) is a good couple for the G-action.

To prove the converse, it is enough to remark that any compact set Λ in G is contained in the compact set $K.L$ where L is the compact set in B defined as $L := (K.\Lambda) \cap B$.

If we assume now that K normalizes B then for any $k \in K$ the neighborhoods $k.V(x)$ and $k.V(y)$ and the compact $k.L.k^{-1}$ of B give the fact that $(k.x, k.y)$ is good for the B-action:

if $k.y'$ and $k.x'$ are in $k.V(x)$ and $k.V(y)$ respectively and satisfy $k.y' = b.k.x'$ for some $b \in B$, we have $y' = b_1.x'$ with $b_1 := k^{-1}.b.k$ and so there exists $\beta_1 \in L$ such that $y' = \beta_1.x'$ and this implies $k.y' = \beta.k.x'$ with $\beta := k.\beta_1.k^{-1} \in k.L.k^{-1}$. □

COROLLARY 3.2.2. — *In the situation of the Theorem 1.0.1, assume that we have a G-invariant open set Ω which is a good open set for the B-action, then Ω is a good open set for the G-action.*

Proof. — Consider a point $x \in \Omega$ and a compact set M in Ω. Then there exists a neighborhood V of x in Ω and a compact set L in B such that $b.y \in M$ for some $y \in V$ and some $b \in B$ implies that we can find $\beta \in L$ with $b.y = \beta.y$. Now assume that M is K-invariant (here we use the G-invariance of Ω) and that $g.y$ is in M for some $g \in G$ and some $y \in V$. Write $g = k.b$ for some $k \in K$ and $b \in B$. Then $b.y$ is again in M so we can find $\beta \in L$ with $\beta.y = b.y$ and then $g.y = k.\beta.y$ with $k.\beta \in K.L$ which is a compact set in G. So x is a good point for the G-action on Ω. □

The corollary of the next lemma will give the first part of [H.2] for the G-action assuming that we have a G-invariant dense, Zariski open, good open set Ω for the B-action with the condition [H.2] for the B-action.

19. We let this exercice on compactness to the reader.

LEMMA 3.2.3. — *Let Ω be an open G-invariant set. Define the map*

$$\chi : K \times X \times X \to X \times X \quad \text{by} \quad (k, x, y) \mapsto (k.x, y)$$

and let $p : K \times X \times X \to X \times X$ be the natural projection. Then we have: $p(\chi^{-1}(\overline{\mathcal{R}_B})) = \overline{\mathcal{R}_G}$ where we define

$$\mathcal{R}_B := \{(x, y) \in \Omega \times \Omega / B.x = B.y\} \quad \text{and} \quad \mathcal{R}_G := \{(x, y) \in \Omega \times \Omega / G.x = G.y\},$$

and where the closures are taken in $X \times X$.

Proof. — Remark first that

$$p(\chi^{-1}(\mathcal{R}_B)) = \{(x, y) \in \Omega \times \Omega / \exists k \in K \quad B.k.x = B.y\}.$$

So $(x, y) \in p(\chi^{-1}(\mathcal{R}_B))$ implies $y \in B.k.x \subset G.x$ and also $k.x \in B.y$; we conclude that x is in $K.B.y = G.y$. This gives the inclusion $p(\chi^{-1}(\mathcal{R}_B)) \subset \mathcal{R}_G$. The opposite inclusion is easy because $G.x = G.y$ implies that $x \in K.B.y$ so there exists $k \in K$ such that $k.x \in B.y$.This gives the equality

$$p(\chi^{-1}(\mathcal{R}_B)) = \mathcal{R}_G.$$

Now the maps χ and p are continuous and proper, so we obtain the inclusion

$$\overline{\mathcal{R}_G} \subset p(\chi^{-1}(\overline{\mathcal{R}_B})).$$

Now take $(x, y) \in p(\chi^{-1}(\overline{\mathcal{R}_B}))$; there exists a sequence (k_ν, x_ν, y_ν) in $\chi^{-1}(\mathcal{R}_B)$ such that $(k_\nu.x_\nu, y_\nu)$ is a sequence in $\mathcal{R}_B$ converging to (x, y), as χ is proper and surjective. So we have $B.k_\nu.x_\nu = B.y_\nu$ and then $G.k_\nu.x_\nu = G.y_\nu$, so $(k_\nu.x_\nu, y_\nu)$ are in $\mathcal{R}_G$. We conclude that (x, y) is in $\overline{\mathcal{R}_G}$. $\square$

COROLLARY 3.2.4. — *In the situation of the previous lemma, assume that $X \setminus \Omega$ is a (complex) analytic subset with no interior point in X. Assume also that $\mathcal{R}_G$ is a closed analytic subset in $\Omega \times \Omega$. Then if the subset $\overline{\mathcal{R}_B}$ is (complex) analytic in $X \times X$, the subset $\overline{\mathcal{R}_G}$ is also a (complex) analytic subset of $X \times X$.*

Proof. — Note first that the maps χ and p are real analytic, so assuming that $\overline{\mathcal{R}_B}$ is analytic implies that $p(\chi^{-1}(\overline{\mathcal{R}_B}))$ is sub-analytic. Then, as we know that $\mathcal{R}_G$ is an irreducible closed complex analytic subset, the conclusion follows from the Lemma 3.1.1, as our assumption that Ω is a Zariski (dense) open set in X implies that $\Omega \times \Omega$ is Zariski open (and dense) in $X \times X$. $\square$

A first step to prove the quasi-properness of $\overline{\mathcal{R}_G}$ is our next result.

LEMMA 3.2.5. — *Let assume that the B-action on X satisfies* [H.1] *and* [H.2]. *Let $\Omega_0 \subset \Omega_1$ be an open set on which the fiber at any $x \in \Omega_0$ of $\overline{\mathcal{R}_B}$ is equal to $\overline{B.x}$ (with some multiplicity). Then the fiber at any $x \in \Omega_0$ of $\overline{\mathcal{R}_G}$ is equal to $\overline{G.x}$ (with some multiplicity).*

Proof. — As we know that the map $x \mapsto \overline{B.x}$, with generic multiplicity 1, is a f-analytic family of cycles of X parametrized by Ω_0, for each sequence $(x_\nu)_{\nu\in\mathbb{N}}$ of points in Ω_0 converging to a point $x \in \Omega_0$ we have (with suitable multiplicity) $\overline{B.x} = \lim_{\nu\to\infty} \overline{B.x_\nu}$ in the topology of $\mathcal{C}_d^f(X)$. We shall show that this implies, also with suitable multiplicity, the equality $\overline{G.x} = \lim_{\nu\to\infty} \overline{G.x_\nu}$ in the topology of $\mathcal{C}_n^f(X)$. As we have $G = K.B$ with K compact, for any $y \in X$ we have $\overline{G.y} = K.\overline{B.y}$. So the inclusion of $\lim_{\nu\to\infty} \overline{G.x_\nu}$ in the fiber at x of $\overline{\mathcal{R}_G}$ is clear. The point is to prove the opposite inclusion. Let y be a point in the fiber at $x \in \Omega_0$ of $\overline{\mathcal{R}_G}$. It is a limit of a sequence $y_\nu \in G.x_\nu$ where $x_\nu \in \Omega_0$ converges to x. Write $y_\nu = k_\nu.b_\nu.x_\nu$ with $k_\nu \in K$ and $b_\nu \in B$. Up to pass to a subsequence, we may assume that the sequence (k_ν) converges to a point $k \in K$. So we have $k^{-1}.y$ which is the limit of the sequence $b_\nu.x_\nu$. We obtain that $k^{-1}.y$ is in the limit of $\overline{B.x_\nu}$ which has support equal to $\overline{B.x}$. Then y is in $K.\overline{B.x} = \overline{G.x}$, concluding the proof. □

Proof of the Theorem 1.0.1. — The hypothesis gives a G-invariant dense, Zariski open Ω_1 which is a good open set for the B-action. The Corollary 3.2.2 shows that it is also a good open set for the G-action.

The analyticity of $\overline{\mathcal{R}_G}$ in $X \times X$ is proved at Corollary 3.2.4 as the complement of Ω_1 is Zariski closed. The Lemma 3.2.5 gives a dense open set Ω_0 where the fiber of the projection p_1 of $\overline{\mathcal{R}_G}$ at each point $x \in \Omega_0$ is equal to $\overline{G.x}$ as a set. This implies the quasi-properness of p_1 over Ω_0, because x is in $G.x$ and G is connected; assuming (which is not restrictive) that Ω_0 is normal, we obtain a holomorphic map

$$\Phi : \Omega_0 \longrightarrow \mathcal{C}_n^f(X)$$

where the support of $\Phi(x)$ is equal to $\overline{G.x}$ for each $x \in \Omega_0$ and with generic multiplicity equal to 1. This complete the proof of [H.2] for the G-action.

Thanks to Proposition 3.2.2 of [3], to prove [H.3] it is enough to show that the closure of the graph Γ_G of Φ in $X \times \mathcal{C}_n^f(X)$ is proper on X.

The projection $p_B : \overline{\mathcal{R}_B} \to X$ is strongly quasi-proper so, for any compact $\bar{V}$ in X, the subset T in $\mathcal{C}_n^{\mathrm{loc}}(X)$ of limits of the generic fibers of the projection $p_G : \overline{\mathcal{R}_G} \to X$ for $x \in \bar{V}$ is a compact set of $\mathcal{C}_n^{\mathrm{loc}}(X)$ thanks to [2] Theorem 2.3.6 i).

Now choose a relatively compact open set W in X and let $W' := W \cap \Omega_0$. Then T' the subset of T corresponding to the cycles $\Phi(x), x \in W'$ is a dense open set in T. Note that for each $x \in W'$ we have $|\Phi(x)| = \overline{G.x} = \bigcup_{k\in K} k.\overline{B.x}$. This means that for each $x \in W'$ the n-cycle $\Phi(x)$ is union of d-cycles in the subset $S := K.q_B(\tau_B^{-1}(\bar{W}))$ where q_B is the minimal SQP meromorphic quotient of X for the B-action, and where K acts on $\mathcal{C}_d^f(X)$ by direct image of the cycles (note that Q_B is a closed analytic subset in $\mathcal{C}_d^f(X)$ by definition of the minimal SQP meromorphic quotient). Then S is a compact subset of $\mathcal{C}_d^f(X)$

and we may apply the Proposition 2.2.3. It gives that T is a compact subset of $\mathcal{C}_n^f(X)$ and this proves [H.3] for the G-action. $\square$

Proof of the Theorem 1.0.2. — We shall reduce the proof of this result to the Theorem 1.0.1 using the following proposition.

PROPOSITION 3.2.6. — *In the situation of the Theorem 1.0.2 there exists a G-invariant Zariski open set Ω_2' which is dense in X, disjoint from the center Σ_B of the modification $\tau_B : \tilde{X}_B \to X$ such that the map $q_B : \tilde{X}_B \to Q_B$ induces on the open set $\tau_B^{-1}(\Omega_2')$ a f-GF holomorphic quotient map on a open dense set Q_B' in Q_B.*

Proof. — As the argument is not so simple we shall divide this proof in several steps.

Step 1. — The Theorem 2.8.1 gives the existence of a SQP meromorphic quotient for the B-action and thanks to the Proposition 2.4.2 we may use the minimal SQP meromorphic quotient (see Definition 2.4.3). Now, using the Corollary 2.4.4 we can assume that K acts continuously and holomorphically on $\tilde{X}_B$ and Q_B and that the holomorphic maps τ_B and q_B are K-equivariant.

Step 2. — As the center Σ_B of τ_B is K and B-invariant (see Lemma 2.2.2) with no interior point in X, we can replace the Zariski open set Ω_1 by the Zariski open set $\Omega_1 \setminus \Sigma_B$ which is still dense and B-invariant and good for the B-action. To avoid too many change of notations , we shall simply assume now that Ω_1 is disjoint to Σ_B and also identify Ω_1 with the open set $\tau_B^{-1}(\Omega_1)$.

Let $Y := X \setminus \Omega_1$. It is a closed analytic subset with no interior point in X thanks to [H.1str] for B. Let $T \subset Q_B$ the subset of points t in Q_B such that the cycle t has an irreducible component in Y. Then T is a closed analytic subset in Q_B (see [3] Lemma 2.1.9) with no interior point in Q because the generic fiber of q_B is irreducible. Then $\Omega_2 := \Omega_1 \setminus q_B^{-1}(T)$ is a Zariski dense open set in X. Let us show now that, for $x \in \Omega_2 \cap \Omega_0$ which is a B-invariant and dense in Ω_2 we have $\overline{\varphi(x)} = \tau_{B*}(q_B^{-1}(q_B(x)))$ where $\varphi : \Omega_1 \to \mathcal{C}_n^f(\Omega_1)$ is the fiber map of the f-GF quotient of the B-action on Ω_1 (remember that Ω_1 is a good open set for the B-action and use Proposition 2.5.4 v)). But by definition of Ω_0 in [H.2] for the B-action for such an x we have $\overline{B.x} = q_B^{-1}(q_B(x))$ in $\tilde{X}$ and then we also have $\overline{B.x} = (\tau_B)_*(q_B^{-1}(q_B(x)))$ in X. This implies $\varphi(x) = B.x = (\tau_B)_*(q_B^{-1}(q_B(x))) \cap \Omega_1$ for such an x.

Then the f-analytic families of cycles in Ω_1, $x \mapsto B.x$ and $x \mapsto q_B^{-1}(q_B(x))$ parametrized by Ω_2 coincide on the dense open set $\Omega_2 \cap \Omega_0$ in Ω_2. So are equal on Ω_2. So for each $x \in \Omega_2$ the cycles $\overline{B.x}$ and $(\tau_B)_*(q_B^{-1}(q_B(x)))$ are equal because the inclusion $\overline{B.x} \subset (\tau_B)_*(q_B^{-1}(q_B(x)))$ is clear and $(\tau_B)_*(q_B^{-1}(q_B(x)))$ has no irreducible component in Y.

Step 3. — Let us show that the open set $\Omega_2' := K.\Omega_2 \subset \Omega_1$ satisfies the following properties:

1. It is a G-invariant open set (we have K.B = G = B.K).
2. It is Zariski open by construction (remember that Ω_2 is Zariski open).
3. It is dense in X (already Ω_2 is dense in X).
4. It is contained in Ω_1, so disjoint from the center of τ_B.
5. We have proved that, after intersection with Ω_2, the restriction of q_B to Ω_2 induces a f-GF holomorphic quotient for the B-action and this extends to Ω_2' by K-equivariance of q_B. This will be our last step.

Step 4. — It is given by the following lemma.

LEMMA 3.2.7. — *Let S and X be irreducible complex spaces and let $\varphi : S \to \mathcal{C}_n^f(X)$ be an holomorphic map, so the classifying map of a f-analytic family of n-cycles in X. Assume that all cycles have irreducible supports and that for generic s in S, the corresponding cycle is reduced. Let $\Gamma \subset S \times X$ the graph of this family. Let X' be a Zariski open subset in X and assume that there exists a holomorphic map $\sigma : S \to X'$ such that $\sigma(x) \in |\varphi(x)|$. Then the family $s \mapsto \varphi(x) \cap X'$ is a f-analytic family of cycles in X'.*

Proof. — The only point to prove is that $\Gamma' := \Gamma \cap (S \times X')$ is quasi-proper on S as the restriction to an open set of a analytic family of cycles is always an analytic family of cycles of this open set. But the existence of a continuous map as σ is enough for that purpose under our hypothesis: for any compact set K in S the compact set $(\mathrm{id}_S \times \sigma)(K)$ in Γ' meets any irreducible component of any cycle associated to some point in K. Remark that the fact that X' is Zariski open in X is used to insure that for any cycle $\varphi(s)$ the cycle $\varphi(s) \cap X'$ has at most one irreducible component. Then the existence of σ implies that it has exactly one irreducible component. □

This completes the proof of the Proposition 3.2.6. □

End of the proof of the Theorem 1.0.2. — The Proposition 3.2.6 gives the G-invariant dense, Zariski open subset Ω_2' satisfying [H.1str] for the B-action and we can apply the Theorem 1.0.1. □

3.3. The $G = K.A.K$ case: proof of the Theorem 1.0.3. — The proof of the Theorem 1.0.3 will use the next lemmata (analoguous to 3.2.2 and 3.2.3).

LEMMA 3.3.1. — *In the $G = K.A.K$ case, a G-invariant good open set for A is a good open set for G.*

Proof. — Let M be a K-invariant compact set in Ω and V be a K-invariant compact neighborhood in Ω of a point x in Ω. Then, as V is uniformly good in Ω for the action of A, there exists a compact set L in A such that for $y \in V$ and $a \in A$ satisfying $a.y \in M$ there exists $\alpha \in L$ with $\alpha.y = a.y$. Assume now that for some $g \in G$ and some $y \in V$ we have $g.y \in M$. Write $g = k_1.a.k_2$. Then we have $a.k_2.y \in M$ and also $k_2.y \in V$ by the K-invariance of M and V. So there exists $\alpha \in L$ with $\alpha.k_2.y = a.k_2.y$ and so $g.y = k_1.\alpha.k_2.y$ where $k_1.\alpha.k_2$ is in the compact set $K.L.K$ of G. This shows that any point x in Ω is a good point for the G-action. □

LEMMA 3.3.2. — *In the $G = K.A.K$ case, consider a G-invariant good open set Ω for the A-action. Let $\chi : K \times K \times X \times X \to X \times X$ be the map given by $\chi(k_1, k_2, x, y) = (k_1.x, k_2.y)$. Define $\mathcal{R}_A := \{(x, y) \in \Omega \times \Omega / A.x = A.y\}$ and $\mathcal{R}_G := \{(x, y) \in \Omega \times \Omega / G.x = G.y\}$. Then we have*

$$p(\chi^{-1}(\mathcal{R}_A)) = \mathcal{R}_G \quad \text{and} \quad p(\chi^{-1}(\overline{\mathcal{R}_A})) = \overline{\mathcal{R}_G}$$

where $p : K \times K \times X \times X \to X \times X$ is the projection.

Proof. — Remark first that for $(x, y) \in \Omega \times \Omega$, the condition $G.y = G.x$ is equivalent to the existence of $(k_1, k_2) \in K \times K$ such that $A.k_1.x = A.k_2.y$. So the inclusion $\mathcal{R}_G = p(\chi^{-1}(\mathcal{R}_A))$ is clear. As p is proper, this implies that $\overline{\mathcal{R}_G} \subset p(\chi^{-1}(\overline{\mathcal{R}_A}))$. Conversely, consider a sequence $(x_\nu, y_\nu) \in \mathcal{R}_A$ converging to $(x, y) \in X \times X$. As $p(\chi^{-1}(x, y)) = \{(k_1.x, k_2.y)/(k_1, k_2) \in K \times K\}$, we want to prove that for any fixed $(k_1, k_2) \in K \times K$ we have $(k_1.x, k_2.y) \in \overline{\mathcal{R}_G}$. There exists a sequence $((x_\nu, y_\nu))_\nu$ in $\mathcal{R}_A$ converging to (x, y). Then $(k_1^{-1}, k_2^{-1}, k_1.x_\nu, k_2.y_\nu)$ is in $\chi^{-1}((x_\nu, y_\nu))$. So $(k_1.x_\nu, k_2.y_\nu)$ is in $\mathcal{R}_G$ for each ν and this sequence converges to $(k_1.x, k_2.y)$ proving the inclusion $p(\chi^{-1}(\overline{\mathcal{R}_A})) \subset \overline{\mathcal{R}_G}$. □

LEMMA 3.3.3. — *In the $K.A.K$ case with the hypotheses of the Theorem 1.0.3, we have, for any $x \in \Omega_0$, the equality $\overline{G.x} = \bigcup_{(k_1,k_2) \in K \times K} k_1.\overline{A.k_2.x}$ in X.*

Proof. — Remember first that the open dense subset Ω_0 given by the hypothesis [H.2] for the A-action is assumed to be G-invariant.

The inclusion of $\bigcup_{(k_1,k_2) \in K \times K} k_1.\overline{A.k_2.x}$ in $\overline{G.x}$ is clear. To prove the opposite inclusion it is enough to prove that the right hand-side is a closed subset in X. But it is the image by τ_A of the subset $\bigcup_{(k_1,k_2) \in K \times K} q_A^{-1}(q_A(\overline{A.k_2.x})$ in $\tilde{X}_A$, where $\tau_A, \tilde{X}_A, q_A, Q_A$ define the minimal SQP quotient of X for the A-action which exists thanks to the Theorem 2.8.1 and the Proposition 2.4.2. But $q_A(\overline{A.k_2.x}) = q_A(k_2.x)$ by A-invariance and continuity of q_A. Now the set $q_A(K.x)$ is compact in Q_A and so $q_A^{-1}(q_A(K.x))$ is closed in $\tilde{X}_A$ and equal to $\bigcup_{(k_1,k_2) \in K \times K} q_A^{-1}(q_A(\overline{A.k_2.x})$ because for each $y \in \Omega_0$ we have $q_A^{-1}(q_A(y)) = \overline{A.y}$. As τ_A is proper, it is a closed map and our right hand-side is a closed set in X. □

Proof of the Theorem 1.0.3. — The Lemma 3.3.1 implies that the Zariski dense good open set Ω_1 for the A-action is good for the G-action, so [H.1str] is true for G. The Lemma 3.3.2 and the sub-analytic Lemma 3.1.1 shows that $\overline{\mathcal{R}_G}$ is a closed analytic subset in $X \times X$ and its projection on Ω_0 is a f-GF flat map, as we may assume Ω_0 normal. The last point is to prove that the closure in $X \times \mathcal{C}_n^f(X)$ of the graph of the holomorphic map $\bar{\varphi}_0 : \Omega_0 \to \mathcal{C}_n^f(X)$ given generically by $x \mapsto \overline{G.x}$ is proper over X. We shall apply the Proposition 2.2.3 using the following two facts:

1. For a compact set $\bar{V}$ in X the set $K.q_A(\tau_A^{-1}(\bar{V}))$ is compact in $\mathcal{C}_d^f(X)$ and parametrizes a f-continuous family of d-cycles in X.
2. If $\bar{V}$ is the compact closure of an open set V in X, then for each point x in $V' := V \cap \Omega_0$ the cycle $\bar{\varphi}_0(x)$ is an union of some d-cycles in the above family,

where we use the minimal SQP meromorphic quotient of X for the A-action. The first point uses the continuous action of K on $\mathcal{C}_d^f(X)$ by the direct image of cycles. The second point is consequence of the Lemma 3.3.3. Then the Proposition 2.2.3 gives the condition [H.3] for the G-action as in the proof of the Theorem 1.0.1 and we conclude the proof using the Theorem 2.8.1. □

3.4. Relation between the two quotients. — In this section we consider a connected complex Lie group G and we assume that we have $G = K.B$ where K is a compact real subgroup and B a complex connected closed subgroup of G. We shall also indicate some analogous results in the case $G = K.A.K$. We also assume that G acts completely holomorphically on a irreducible complex space X.

Proposition 3.4.1. — *Assume that there exists a SQP meromorphic quotient for this action but also for the corresponding action of B. Then there exists a holomorphic map*

$$h : Q_B \to Q_G$$

where Q_B and Q_G are the minimal SQP quotients of X respectively for the B-action and the G-action, such that we have the equality $q_B \circ h = q_G$ on the strict transform of $\tilde{X}_G$ by the modification τ_B.

Moreover, if K normalizes B, there are natural K-actions on $\tilde{X}_B$ and Q_B, the map q_B is K-equivariant and the holomorphic map h is K-invariant.

Proof. — Consider the two diagrams corresponding to the two SQP meromorphic quotients of X by the actions of B and G

$$\begin{tikzcd} \tilde{X}_B \arrow[r, "q_B"] \arrow[d, "\tau_B"'] & Q_B \\ X & \end{tikzcd} \qquad \begin{tikzcd} \tilde{X}_G \arrow[r, "q_G"] \arrow[d, "\tau_G"'] & Q_G \\ X & \end{tikzcd}$$

and let $\sigma : Y \to \tilde{X}_B$ be the strict transform of the modification τ_G by the modification τ_B. Then the first assertion is a consequence of the Theorem 2.4.5 as the meromorphic map $q_G : X \dashrightarrow Q_G$ is B-invariant.

The second assertion is consequence of the Corollary 2.4.4. □

In the $G = K.A.K$ case, assuming that the SQP meromorphic quotient exists for the actions of A and G, using the A-invariance of the meromorphic map $X \dashrightarrow Q_G$ and the Theorem 2.4.5, we obtain that there exists also a holomorphic map between the corresponding minimal quotients $h : Q_A \to Q_G$ which satisfies the equality $q_A \circ h = q_G$ on the strict transform of $\tilde{X}_G$ by the modification τ_A.

Of course a natural question about the holomorphic map $h : Q_B \to Q_G$ defined in the previous proposition is its properness. Our next result gives a sufficient condition to obtain a partial result.

Proposition 3.4.2. — *Assume that the B-action and the G-action on X admit a SQP meromorphic quotient. Assume that there exists a G-invariant open set Ω in X, disjoint from the centers of the modifications τ_B and τ_G associated to the minimal SQP quotients of X, on which we have a f-GF holomorphic quotient for the G-action, and such that for each $x \in \Omega$ we have $q_G^{-1}(q_G(x)) = \overline{G.x}$ in $\tilde{X}_G$. Then the map $h_\Omega : q_B(\Omega) \to q_G(\Omega)$, induced by the restriction of the holomorphic map $h : Q_B \to Q_G$, is proper.*

Proof. — We shall prove that if M is a compact set in Ω then we have the inclusion $h^{-1}(q_G(M)) \cap q_B(\Omega) \subset q_B(K.M)$. As $K.M$ is a compact set in Ω, this will prove the properness of the map h_Ω because the map q_G is open, so each compact set in $q_G(\Omega)$ can be cover by finitely many open sets $q_G(V_i)$ where V_i is a relatively compact open subset in Ω; then any compact set in $q_G(\Omega)$ is contained in $q_G(M)$ where M is the compact set $\bigcup_{i \in I} \bar{V}_i$ of Ω.

Consider a point $y \in h^{-1}(q_G(M)) \cap q_B(\Omega)$. So there exists a point $x \in M$ such that $h(y) = q_G(x)$. But from our hypothesis we know that $q_G^{-1}(q_G(x)) = \overline{G.x}$ in $\tilde{X}_G$. Also there exists also a point $x_0 \in \Omega$ such that $y = q_B(x_0)$. This implies the equality $h(q_B(x_0)) = q_G(x_0) = q_G(x)$ because $q_B \circ h = q_G$ on Ω. So x_0 is in $\overline{G.x} \cap \Omega = G.x$. We conclude that there exists $k \in K$ such that x_0 is in $B.k.x$ and then $q_B(x_0) = q_B(k.x)$ and $k.x$ is in $K.M$. □

Remark that the existence of a SQP meromorphic quotient for the G-action implies the existence of a G-invariant open dense subset Ω_0 satisfying all the hypotheses of the previous proposition *excepted* the fact that Ω_0 is disjoint from the center of τ_B. So under the hypothesis that the closed set $K.\Sigma_B$, where Σ_B is the center of the modification τ_B, has no interior point in X, the existence of the two SQP meromorphic quotients is enough to conclude that there exists a G-invariant open dense set Ω in X for which the map h_Ω is proper.

So the following corollary is immediate.

Corollary 3.4.3. — *Under the hypotheses of the Theorem 1.0.1 there exists a dense open set Ω in X, disjoint from the center of the modifications τ_B and τ_G, such that the map $h_\Omega : q_B(\Omega) \to q_G(\Omega)$ is proper.*

With analoguous argument we obtain also such a result in the $K.A.K$ case.

COROLLARY 3.4.4. — *Under the hypotheses of the Theorem 1.0.3 there exists a dense open set Ω in X, disjoint from the center of the modifications τ_A and τ_G, such that the map $h_\Omega : q_A(\Omega) \to q_G(\Omega)$ is proper.*

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TISSUS PLATS ET FEUILLETAGES HOMOGÈNES SUR LE PLAN PROJECTIF COMPLEXE

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RÉSUMÉ. — Le but de ce travail est d'étudier les feuilletages du plan projectif complexe ayant une transformée de Legendre (tissu dual) plate. Nous établissons quelques critères effectifs de la platitude du d-tissu dual d'un feuilletage homogène de degré d et nous décrivons quelques exemples explicites. Ces résultats nous permettent de montrer qu'à automorphisme de $\mathbb{P}^2_{\mathbb{C}}$ près il y a 11 feuilletages homogènes de degré 3 ayant cette propriété. Nous verrons aussi qu'il est possible, sous certaines hypothèses, de ramener l'étude de la platitude du tissu dual d'un feuilletage inhomogène au cadre homogène. Nous obtenons quelques résultats de classification de feuilletages à singularités non-dégénérées et de transformée de Legendre plate.

ABSTRACT (*Flat webs and homogeneous foliations on the complex projective plane*). — The aim of this work is to study the foliations on the complex projective plane with flat Legendre transform (dual web). We establish some effective criteria for the flatness of the dual d-web of a homogeneous foliation of degree d and we describe some explicit examples. These results allow us to show that up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ there are 11 homogeneous foliations of degree 3 with flat dual web. We will see also that it is possible, under certain assumptions, to bring the study of flatness of the dual web of a general foliation to the homogeneous framework. We get some classification results about foliations with non-degenerate singularities and flat Legendre transform.

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Introduction

Un d-tissu (régulier) $\mathcal{W}$ de $(\mathbb{C}^2,0)$ est la donnée d'une famille $\{\mathcal{F}_1,\mathcal{F}_2,\ldots,\mathcal{F}_d\}$ de feuilletages holomorphes réguliers de $(\mathbb{C}^2,0)$ deux à deux transverses en l'origine. Le premier résultat significatif dans l'étude des tissus a été obtenu par W. Blaschke et J. Dubourdieu autour des années 1920. Ils ont montré ([3]) que tout 3-tissu régulier $\mathcal{W}$ de $(\mathbb{C}^2,0)$ est conjugué, via un isomorphisme analytique de $(\mathbb{C}^2,0)$, au 3-tissu trivial défini par $\mathrm{d}x.\mathrm{d}y.\mathrm{d}(x+y)$, et cela sous l'hypothèse d'annulation d'une 2-forme différentielle $K(\mathcal{W})$ connue sous le nom de courbure de Blaschke de $\mathcal{W}$. La courbure d'un d-tissu $\mathcal{W}$ avec $d>3$ se définit comme la somme des courbures de Blaschke des sous-3-tissus de $\mathcal{W}$. Un tissu de courbure nulle est dit plat. Cette notion est utile pour la classification des tissus de rang maximal ; un résultat de N. Mihăileanu montre que la platitude est une condition nécessaire pour la maximalité du rang, voir par exemple [8, 14].

Depuis peu, l'étude des tissus globaux holomorphes définis sur les surfaces complexes a été réactualisée, voir par exemple [6, 12, 9]. Nous nous intéressons dans ce qui suit aux tissus du plan projectif complexe. Un d-tissu (global) sur $\mathbb{P}^2_{\mathbb{C}}$ est donné dans une carte affine (x,y) par une équation différentielle algébrique $F(x,y,y')=0$, où $F(x,y,p)=\sum_{i=0}^{d}a_i(x,y)p^{d-i}\in\mathbb{C}[x,y,p]$ est un polynôme réduit à coefficient a_0 non identiquement nul. Au voisinage de tout point $z_0=(x_0,y_0)$ tel que $a_0(x_0,y_0)\Delta(x_0,y_0)\neq 0$, où $\Delta(x,y)$ est le p-discriminant de F, les courbes intégrales de cette équation définissent un d-tissu régulier de $(\mathbb{C}^2,z_0)$.

La courbure d'un tissu $\mathcal{W}$ sur $\mathbb{P}^2_{\mathbb{C}}$ est une 2-forme méromorphe à pôles le long du discriminant $\Delta(\mathcal{W})$. La platitude d'un tissu $\mathcal{W}$ sur $\mathbb{P}^2_{\mathbb{C}}$ se caractérise par l'holomorphie de sa courbure $K(\mathcal{W})$ le long des points génériques de $\Delta(\mathcal{W})$, voir §1.2.

D. Marín et J. Pereira ont montré, dans [9], comment on peut associer à tout feuilletage $\mathcal{F}$ de degré d sur $\mathbb{P}^2_{\mathbb{C}}$, un d-tissu sur le plan projectif dual $\check{\mathbb{P}}^2_{\mathbb{C}}$, appelé *transformée de Legendre* de $\mathcal{F}$ et noté $\mathrm{Leg}\mathcal{F}$; les feuilles de $\mathrm{Leg}\mathcal{F}$ sont les droites tangentes aux feuilles de $\mathcal{F}$, voir §1.1.

L'ensemble des feuilletages de degré d sur $\mathbb{P}^2_{\mathbb{C}}$, noté $\mathbf{F}(d)$, s'identifie à un ouvert de Zariski dans un espace projectif de dimension $(d+2)^2-2$ sur lequel agit le groupe $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$. Le sous-ensemble $\mathbf{FP}(d)$ de $\mathbf{F}(d)$ formé des $\mathcal{F}\in\mathbf{F}(d)$ tels que $\mathrm{Leg}\mathcal{F}$ soit plat est un fermé de Zariski de $\mathbf{F}(d)$. La classification des feuilletages $\mathcal{F}\in\mathbf{FP}(d)$ modulo $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ reste entière. Le premier cas non trivial que l'on rencontre est celui où $d=3$; on dispose actuellement d'une caractérisation géométrique ([2, Théorème 4.5]) des éléments de $\mathbf{FP}(3)$, mais ce résultat reste insuffisant pour avancer dans leur classification. C'est dans cette optique que nous nous proposons d'étudier cette question de platitude au niveau des éléments de $\mathbf{F}(d)$ qui sont *homogènes*, i.e. qui sont invariants par homothétie. En fait nous établirons, pour des feuilletages homogènes $\mathcal{H}\in\mathbf{F}(d)$, quelques critères effectifs de l'holomorphie de la courbure de $\mathrm{Leg}\mathcal{H}$; de plus nous verrons (Proposition 6.4) que l'étude de la platitude de la transformée de

Legendre d'un feuilletage inhomogène se ramène, sous certaines hypothèses, au cadre homogène.

Un feuilletage homogène $\mathcal{H}$ de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ est donné, pour un bon choix de coordonnées affines (x,y), par une 1-forme homogène $\omega_d = A_d(x,y)\mathrm{d}x + B_d(x,y)\mathrm{d}y$, où $A_d, B_d \in \mathbb{C}[x,y]_d$ et $\mathrm{pgcd}(A_d, B_d) = 1$.

L'homogénéité de $\mathcal{H}$ implique (voir [9, page 177]) que le discriminant de $\mathrm{Leg}\mathcal{H}$ se décompose en produit de $(d-1)(d+2)$ droites comptées avec multiplicités ; certaines parmi elles sont invariantes par $\mathrm{Leg}\mathcal{H}$ et d'autres non, i.e. sont transverses. De plus la multiplicité de $\Delta(\mathrm{Leg}\mathcal{H})$ le long d'une droite transverse est comprise entre 1 et $d-1$; en degré 3 elle est donc soit minimale (égale à 1) soit maximale (égale à 2).

Le théorème 3.1 affirme que le d-tissu $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si sa courbure est holomorphe sur la partie transverse de $\Delta(\mathrm{Leg}\mathcal{H})$.

Le théorème 3.5 (resp. théorème 3.8) contrôle de façon effective l'holomorphie de la courbure $K(\mathrm{Leg}\mathcal{H})$ le long d'une droite $\ell \subset \Delta(\mathrm{Leg}\mathcal{H})$ non invariante par $\mathrm{Leg}\mathcal{H}$ de multiplicité minimale 1 (resp. maximale $d-1$).

Ces théorèmes nous permettront de décrire certains feuilletages homogènes appartenant à $\mathbf{FP}(d)$ pour d arbitraire (Propositions 4.1, 4.2 et 4.3).

En combinant les théorèmes 3.1, 3.5 et 3.8 nous obtenons une caractérisation complète de la platitude de la transformée de Legendre d'un feuilletage homogène de degré 3 (Corollaire 3.10). Ce résultat nous permettra de classifier les éléments de $\mathbf{FP}(3)$ qui sont homogènes : *à automorphisme de* $\mathbb{P}^2_{\mathbb{C}}$ *près, il y a* 11 *feuilletages homogènes de degré* 3 *ayant une transformée de Legendre plate*, voir théorème 5.1.

En se basant essentiellement sur cette classification, nous obtenons un résultat (Théorème 6.1) qui sort du cadre homogène : *tout feuilletage* $\mathcal{F} \in \mathbf{FP}(3)$ *à singularités non-dégénérées (i.e. ayant pour nombre de Milnor* 1*) est linéairement conjugué au feuilletage de Fermat défini par la* 1*-forme* $(x^3 - x)\mathrm{d}y - (y^3 - y)\mathrm{d}x$.

Comme application du théorème 6.1 nous donnons une réponse partielle (Corollaire 6.9) à [9, Problème 9.1].

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1. Préliminaires

1.1. Tissus. — Soit $k \geq 1$ un entier. Un *k-tissu (global)* $\mathcal{W}$ sur une surface complexe S est la donnée d'un recouvrement ouvert $(U_i)_{i\in I}$ de S et d'une collection de k-formes symétriques $\omega_i \in \mathrm{Sym}^k\Omega^1_S(U_i)$, à zéros isolés, satisfaisant :

($\mathfrak{a}$) il existe $g_{ij} \in \mathcal{O}^*_S(U_i \cap U_j)$ tel que ω_i coïncide avec $g_{ij}\omega_j$ sur $U_i \cap U_j$;

($\mathfrak{b}$) en tout point générique m de U_i, $\omega_i(m)$ se factorise en produit de k formes linéaires deux à deux non colinéaires.

L'ensemble des points de S qui ne vérifient pas la condition ($\mathfrak{b}$) est appelé le *discriminant* de $\mathcal{W}$ et est noté $\Delta(\mathcal{W})$. Lorsque $k = 1$ cette condition est toujours vérifiée et on retrouve la définition usuelle d'un feuilletage holomorphe $\mathcal{F}$ sur S. Le cocycle (g_{ij}) définit un fibré en droites N sur S, appelé le *fibré normal* de $\mathcal{W}$, et les ω_i se recollent pour définir une section globale $\omega \in \mathrm{H}^0(S, \mathrm{Sym}^k\Omega^1_S \otimes N)$.

Un k-tissu global $\mathcal{W}$ sur S sera dit *décomposable* s'il existe des tissus globaux $\mathcal{W}_1, \mathcal{W}_2$ sur S n'ayant pas de sous-tissus communs tels que $\mathcal{W}$ soit la superposition de $\mathcal{W}_1$ et $\mathcal{W}_2$; on écrira $\mathcal{W} = \mathcal{W}_1 \boxtimes \mathcal{W}_2$. Dans le cas contraire $\mathcal{W}$ sera dit *irréductible*. On dira que $\mathcal{W}$ est *complètement décomposable* s'il existe des feuilletages globaux $\mathcal{F}_1, \ldots, \mathcal{F}_k$ sur S tels que $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. Pour en savoir plus à ce sujet, nous renvoyons à [12].

On se restreindra dans ce travail au cas $S = \mathbb{P}^2_{\mathbb{C}}$. Se donner un k-tissu sur $\mathbb{P}^2_{\mathbb{C}}$ revient à se donner une k-forme symétrique polynomiale

$$\omega = \sum_{i+j=k} a_{ij}(x,y)\mathrm{d}x^i\mathrm{d}y^j,$$

à zéros isolés et de discriminant non identiquement nul. Ainsi tout k-tissu sur $\mathbb{P}^2_{\mathbb{C}}$ peut se lire dans une carte affine donnée (x,y) de $\mathbb{P}^2_{\mathbb{C}}$ par une équation différentielle polynomiale $F(x,y,y') = 0$ de degré k en y'. Un k-tissu $\mathcal{W}$ sur $\mathbb{P}^2_{\mathbb{C}}$ est dit de *degré* d si le nombre de points où une droite générique de $\mathbb{P}^2_{\mathbb{C}}$ est tangente à une feuille de $\mathcal{W}$ est égal à d ; c'est équivalent de dire que $\mathcal{W}$ est de fibré normal $N = \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(d+2k)$. Il est bien connu, *voir* par exemple [12, Proposition 1.4.2], que les tissus de degré 0 sont les tissus algébriques (leurs feuilles sont les droites tangentes à une courbe algébrique réduite).

Les auteurs dans [9] ont associé, à tout k-tissu $\mathcal{W}$ de degré $d \geq 1$ sur $\mathbb{P}^2_{\mathbb{C}}$, un d-tissu de degré k sur le plan projectif dual $\check{\mathbb{P}}^2_{\mathbb{C}}$, appelé *transformée de Legendre* de $\mathcal{W}$ et noté $\mathrm{Leg}\mathcal{W}$; les feuilles de $\mathrm{Leg}\mathcal{W}$ sont les droites tangentes aux feuilles de $\mathcal{W}$. Plus explicitement, soit (x,y) une carte affine de $\mathbb{P}^2_{\mathbb{C}}$ et considérons la carte affine (p,q) de $\check{\mathbb{P}}^2_{\mathbb{C}}$ associée à la droite $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$. Soit $F(x,y;p) = 0$, $p = \frac{\mathrm{d}y}{\mathrm{d}x}$, une équation différentielle implicite décrivant $\mathcal{W}$; alors $\mathrm{Leg}\mathcal{W}$ est donné par l'équation différentielle implicite

$$\check{F}(p,q;x) := F(x, px-q; p) = 0, \qquad \text{avec} \qquad x = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

En particulier, si $\mathcal{F}$ est un feuilletage de degré $d \geq 1$ sur $\mathbb{P}^2_{\mathbb{C}}$ défini par une 1-forme $\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$, où $A, B \in \mathbb{C}[x,y]$, $\mathrm{pgcd}(A,B) = 1$, alors $\mathrm{Leg}\mathcal{F}$ est le d-tissu irréductible de degré 1 de $\check{\mathbb{P}}^2_{\mathbb{C}}$ défini par

$$A(x, px-q) + pB(x, px-q) = 0, \qquad \text{avec} \qquad x = \frac{\mathrm{d}q}{\mathrm{d}p}.$$

Inversement, tout d-tissu irréductible de degré 1 sur $\check{\mathbb{P}}^2_{\mathbb{C}}$ est nécessairement la transformée de Legendre d'un certain feuilletage de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ (voir [9]).

1.2. Courbure et platitude. — On rappelle ici la définition de la courbure d'un k-tissu $\mathcal{W}$. On suppose dans un premier temps que $\mathcal{W}$ est un germe de k-tissu de $(\mathbb{C}^2, 0)$ complètement décomposable, $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. Soit, pour tout $1 \leq i \leq k$, une 1-forme ω_i à singularité isolée en 0 définissant le feuilletage $\mathcal{F}_i$. D'après [13], pour tout triplet (r,s,t) avec $1 \leq r < s < t \leq k$, on définit $\eta_{rst} = \eta(\mathcal{F}_r \boxtimes \mathcal{F}_s \boxtimes \mathcal{F}_t)$ comme l'unique 1-forme méromorphe satisfaisant les égalités suivantes :

$$\begin{cases} \mathrm{d}(\delta_{st}\,\omega_r) = \eta_{rst} \wedge \delta_{st}\,\omega_r \\ \mathrm{d}(\delta_{tr}\,\omega_s) = \eta_{rst} \wedge \delta_{tr}\,\omega_s \\ \mathrm{d}(\delta_{rs}\,\omega_t) = \eta_{rst} \wedge \delta_{rs}\,\omega_t \end{cases} \tag{1.1}$$

où δ_{ij} désigne la fonction définie par $\omega_i \wedge \omega_j = \delta_{ij}\,\mathrm{d}x \wedge \mathrm{d}y$. Comme chacune des 1-formes ω_i n'est définie qu'à multiplication près par un inversible de $\mathcal{O}(\mathbb{C}^2, 0)$, il en résulte que chacune des 1-formes η_{rst} est bien déterminée à l'addition près d'une 1-forme holomorphe fermée. Ainsi la 1-forme

$$\eta(\mathcal{W}) = \eta(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k) = \sum_{1 \leq r<s<t \leq k} \eta_{rst} \tag{1.2}$$

est bien définie à l'addition près d'une 1-forme holomorphe fermée. La *courbure* du tissu $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ est par définition la 2-forme

$$K(\mathcal{W}) = K(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k) = \mathrm{d}\,\eta(\mathcal{W}).$$

On peut vérifier que $K(\mathcal{W})$ est une 2-forme méromorphe à pôles le long du discriminant $\Delta(\mathcal{W})$ de $\mathcal{W}$, canoniquement associée à $\mathcal{W}$; plus précisément, pour toute application holomorphe dominante φ, on a $K(\varphi^*\mathcal{W}) = \varphi^* K(\mathcal{W})$.

Si maintenant $\mathcal{W}$ est un k-tissu sur une surface complexe S (non forcément complètement décomposable), alors on peut le transformer en un k-tissu complètement décomposable au moyen d'un revêtement galoisien ramifié. L'invariance de la courbure de ce nouveau tissu par l'action du groupe de Galois permet de la redescendre en une 2-forme méromorphe globale sur S, à pôles le long du discriminant de $\mathcal{W}$ (voir [9]).

Un k-tissu $\mathcal{W}$ est dit *plat* si sa courbure $K(\mathcal{W})$ est identiquement nulle.

Signalons qu'un k-tissu $\mathcal{W}$ sur $\mathbb{P}^2_{\mathbb{C}}$ est plat si et seulement si sa courbure est holomorphe le long des points génériques des composantes irréductibles de $\Delta(\mathcal{W})$. Ceci résulte de la définition de $K(\mathcal{W})$ et du fait qu'il n'existe pas de 2-forme holomorphe sur $\mathbb{P}^2_{\mathbb{C}}$ autre que la 2-forme nulle.

1.3. Singularités et diviseur d'inflexion d'un feuilletage du plan projectif. — Un feuilletage holomorphe $\mathcal{F}$ de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ est défini par une 1-forme du type

$$\omega = a(x,y,z)\mathrm{d}x + b(x,y,z)\mathrm{d}y + c(x,y,z)\mathrm{d}z,$$

où a, b et c sont des polynômes homogènes de degré $d+1$ sans facteur commun satisfaisant la condition d'Euler $i_{\mathrm{R}}\omega = 0$, où $\mathrm{R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ désigne le champ radial et i_{R} le produit intérieur par R. Le *lieu singulier* $\mathrm{Sing}\mathcal{F}$ de $\mathcal{F}$ est le projectivisé du lieu singulier de ω

$$\mathrm{Sing}\omega = \{(x,y,z) \in \mathbb{C}^3 \,|\, a(x,y,z) = b(x,y,z) = c(x,y,z) = 0\}.$$

Rappelons quelques notions locales attachées au couple $(\mathcal{F}, s)$, où $s \in \mathrm{Sing}\mathcal{F}$. Le germe de $\mathcal{F}$ en s est défini, à multiplication près par une unité de l'anneau local $\mathcal{O}_s$ en s, par un champ de vecteurs
$\mathrm{X} = A(\mathrm{u},\mathrm{v})\frac{\partial}{\partial \mathrm{u}} + B(\mathrm{u},\mathrm{v})\frac{\partial}{\partial \mathrm{v}}$.
La *multiplicité algébrique* $\nu(\mathcal{F}, s)$ de $\mathcal{F}$ en s est donnée par

$$\nu(\mathcal{F}, s) = \min\{\nu(A, s), \nu(B, s)\},$$

où $\nu(g, s)$ désigne la multiplicité algébrique de la fonction g en s. L'*ordre de tangence* entre $\mathcal{F}$ et une droite générique passant par s est l'entier

$$\tau(\mathcal{F}, s) = \min\{k \geq \nu(\mathcal{F}, s) \,:\, \det(J^k_s\mathrm{X}, \mathrm{R}_s) \neq 0\},$$

où $J^k_s\mathrm{X}$ est le k-jet de X en s et R_s est le champ radial centré en s. Le *nombre de Milnor* de $\mathcal{F}$ en s est l'entier

$$\mu(\mathcal{F}, s) = \dim_{\mathbb{C}} \mathcal{O}_s/\langle A, B\rangle,$$

où $\langle A, B\rangle$ désigne l'idéal de $\mathcal{O}_s$ engendré par A et B.

La singularité s est dite *radiale d'ordre* $n-1$ si $\nu(\mathcal{F}, s) = 1$ et $\tau(\mathcal{F}, s) = n$.

La singularité s est dite *non-dégénérée* si $\mu(\mathcal{F}, s) = 1$, c'est équivalent de dire que la partie linéaire $J^1_s\mathrm{X}$ de X possède deux valeurs propres λ, μ non nulles. La quantité $\mathrm{BB}(\mathcal{F}, s) = \frac{\lambda}{\mu} + \frac{\mu}{\lambda} + 2$ est appelée l'*invariant de Baum-Bott* de $\mathcal{F}$ en s (voir [1]). D'après [5] il passe par s au moins un germe de courbe $\mathcal{C}$ invariante par $\mathcal{F}$; à isomorphisme local près, on peut se ramener à $s = (0,0)$, $\mathrm{T}_s\mathcal{C} = \{\mathrm{v} = 0\}$ et $J^1_s\mathrm{X} = \lambda\mathrm{u}\frac{\partial}{\partial \mathrm{u}} + (\varepsilon\mathrm{u} + \mu\mathrm{v})\frac{\partial}{\partial \mathrm{v}}$, où l'on peut prendre $\varepsilon = 0$ si $\lambda \neq \mu$. La quantité $\mathrm{CS}(\mathcal{F}, \mathcal{C}, s) = \frac{\lambda}{\mu}$ est appelée l'*indice de Camacho-Sad* de $\mathcal{F}$ en s par rapport à $\mathcal{C}$.

Rappelons la notion du diviseur d'inflexion de $\mathcal{F}$. Soit $\mathrm{Z} = E\frac{\partial}{\partial x} + F\frac{\partial}{\partial y} + G\frac{\partial}{\partial z}$ un champ de vecteurs homogène de degré d sur $\mathbb{C}^3$ non colinéaire au champ radial décrivant $\mathcal{F}$, i.e. tel que $\omega = i_{\mathrm{R}}i_{\mathrm{Z}}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. Le *diviseur d'inflexion* de $\mathcal{F}$, noté $\mathrm{I}_{\mathcal{F}}$, est le diviseur défini par l'équation

$$\begin{vmatrix} x & E & \mathrm{Z}(E) \\ y & F & \mathrm{Z}(F) \\ z & G & \mathrm{Z}(G) \end{vmatrix} = 0. \tag{1.3}$$

Ce diviseur a été étudié dans [11] dans un contexte plus général. En particulier, les propriétés suivantes ont été prouvées.

1. Sur $\mathbb{P}^2_{\mathbb{C}} \smallsetminus \mathrm{Sing}\mathcal{F}$, $\mathrm{I}_{\mathcal{F}}$ coïncide avec la courbe décrite par les points d'inflexion des feuilles de $\mathcal{F}$.
2. Si $\mathcal{C}$ est une courbe algébrique irréductible invariante par $\mathcal{F}$, alors $\mathcal{C} \subset \mathrm{I}_{\mathcal{F}}$ si et seulement si $\mathcal{C}$ est une droite invariante.
3. $\mathrm{I}_{\mathcal{F}}$ peut se décomposer en $\mathrm{I}_{\mathcal{F}} = \mathrm{I}_{\mathcal{F}}^{\mathrm{inv}} + \mathrm{I}_{\mathcal{F}}^{\mathrm{tr}}$, où le support de $\mathrm{I}_{\mathcal{F}}^{\mathrm{inv}}$ est constitué de l'ensemble des droites invariantes par $\mathcal{F}$ et où le support de $\mathrm{I}_{\mathcal{F}}^{\mathrm{tr}}$ est l'adhérence des points d'inflexion qui sont isolés le long des feuilles de $\mathcal{F}$.
4. Le degré du diviseur $\mathrm{I}_{\mathcal{F}}$ est $3d$.

Le feuilletage $\mathcal{F}$ sera dit *convexe* si son diviseur d'inflexion $\mathrm{I}_{\mathcal{F}}$ est totalement invariant par $\mathcal{F}$, i.e. si $\mathrm{I}_{\mathcal{F}}$ est le produit de droites invariantes.

L'application de Gauss est l'application rationnelle $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \check{\mathbb{P}}^2_{\mathbb{C}}$ qui à un point régulier m associe la droite tangente $\mathrm{T}_m\mathcal{F}$. Si $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{C}}$ est une courbe passant par certains points singuliers de $\mathcal{F}$, on définit $\mathcal{G}_{\mathcal{F}}(\mathcal{C})$ comme étant l'adhérence de $\mathcal{G}_{\mathcal{F}}(\mathcal{C} \setminus \mathrm{Sing}\mathcal{F})$. Il résulte de [2, Lemme 2.2] que

$$\Delta(\mathrm{Leg}\mathcal{F}) = \mathcal{G}_{\mathcal{F}}(\mathrm{I}_{\mathcal{F}}^{\mathrm{tr}}) \cup \check{\Sigma}_{\mathcal{F}}, \tag{1.4}$$

où $\check{\Sigma}_{\mathcal{F}}$ désigne l'ensemble des droites duales des points de $\Sigma_{\mathcal{F}} := \{s \in \mathrm{Sing}\mathcal{F} : \tau(\mathcal{F}, s) \geq 2\}$.

2. Géométrie des feuilletages homogènes

DÉFINITION 2.1. — Un feuilletage de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ est dit *homogène* s'il existe une carte affine (x, y) de $\mathbb{P}^2_{\mathbb{C}}$ dans laquelle il est invariant sous l'action du groupe des homothéties $(x, y) \longmapsto \lambda(x, y)$, $\lambda \in \mathbb{C}^*$.

Un tel feuilletage $\mathcal{H}$ est alors défini par une 1-forme

$$\omega = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y,$$

où A et B sont des polynômes homogènes de degré d sans facteur commun. Cette 1-forme s'écrit en coordonnées homogènes

$$zA(x, y)\mathrm{d}x + zB(x, y)\mathrm{d}y - (xA(x, y) + yB(x, y))\,\mathrm{d}z.$$

Ainsi le feuilletage $\mathcal{H}$ a au plus $d + 2$ singularités dont l'origine O de la carte affine $z = 1$ est le seul point singulier de $\mathcal{H}$ qui n'est pas situé sur la droite à l'infini $L_\infty = (z = 0)$; de plus $\nu(\mathcal{H}, O) = d$.

Dorénavant nous supposerons que d est supérieur ou égal à 2. Dans ce cas le point O est la seule singularité de $\mathcal{H}$ de multiplicité algébrique d.

Le champ de vecteurs homogène $-B(x,y)\frac{\partial}{\partial x}+A(x,y)\frac{\partial}{\partial y}+0\frac{\partial}{\partial z}$ définit aussi le feuilletage $\mathcal{H}$ car est dans le noyau de la 1-forme précédente. Notons $A_x=\frac{\partial A}{\partial x}$, $A_y=\frac{\partial A}{\partial y}$, $B_x=\frac{\partial B}{\partial x}$, $B_y=\frac{\partial B}{\partial y}$; d'après la formule (1.3), le diviseur d'inflexion $\mathrm{I}_{\mathcal{H}}$ de $\mathcal{H}$ est donné par

$$0=\begin{vmatrix} x & -B & BB_x-AB_y \\ y & A & AA_y-BA_x \\ z & 0 & 0 \end{vmatrix} = z\begin{vmatrix} -\frac{1}{d}(xB_x+yB_y) & BB_x-AB_y \\ \frac{1}{d}(xA_x+yA_y) & AA_y-BA_x \end{vmatrix}$$
$$=\frac{z}{d}(xA+yB)(A_xB_y-A_yB_x)=\frac{z}{d}\mathrm{C}_{\mathcal{H}}\mathrm{D}_{\mathcal{H}},$$

où $\mathrm{C}_{\mathcal{H}}=xA+yB\in\mathbb{C}[x,y]_{d+1}$ désigne le *cône tangent* de $\mathcal{H}$ en l'origine O et $\mathrm{D}_{\mathcal{H}}=A_xB_y-A_yB_x\in\mathbb{C}[x,y]_{2d-2}$.

Il en résulte que :

(i) le support du diviseur $\mathrm{I}_{\mathcal{H}}^{\mathrm{inv}}$ est constitué des droites du cône tangent $\mathrm{C}_{\mathcal{H}}=0$ et de la droite à l'infini L_∞ ;

(ii) le diviseur $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}}$ se décompose sous la forme $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}}=\prod_{i=1}^{n}T_i^{\rho_i-1}$ pour un certain nombre $n\leq\deg\mathrm{D}_{\mathcal{H}}=2d-2$ de droites T_i passant par O, ρ_i-1 étant l'ordre d'inflexion de la droite T_i. Lorsque $\rho_i=2$ on parle d'une droite d'inflexion simple pour $\mathcal{H}$, lorsque $\rho_i=3$ d'une droite d'inflexion double, etc.

Proposition 2.2. — *Avec les notations précédentes, pour tout point singulier* $s\in\mathrm{Sing}\mathcal{H}\cap L_\infty$, *nous avons*

1. *1.* $\nu(\mathcal{H},s)=1$;
2. *2. la droite L_s passant par l'origine O et le point s est invariante par $\mathcal{H}$ et elle apparaît avec multiplicité $\tau(\mathcal{H},s)-1$ dans le diviseur $\mathrm{D}_{\mathcal{H}}=0$, i.e.*

$$\mathrm{D}_{\mathcal{H}}=\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}}\prod_{s\in\mathrm{Sing}\mathcal{H}\cap L_\infty}L_s^{\tau(\mathcal{H},s)-1}.$$

Démonstration. — Soit s un point singulier de $\mathcal{H}$ sur $L_\infty=(z=0)$. Sans perte de généralité, nous pouvons supposer que les coordonnées homogènes de s sont de la forme $[x_0:1:0]$, $x_0\in\mathbb{C}$. Dans la carte affine $y=1$, $\mathcal{H}$ est décrit par la 1-forme

$$\theta=zA(x,1)\mathrm{d}x-(xA(x,1)+B(x,1))\,\mathrm{d}z;$$

la condition $s\in\mathrm{Sing}\mathcal{H}$ est équivalente à $B(x_0,1)=-x_0A(x_0,1)$. L'égalité $\mathrm{pgcd}(A,B)=1$ implique alors que $A(x_0,1)\neq0$; d'où $\nu(\mathcal{H},s)=1$.

Montrons la seconde assertion. Le fait que

$$\theta=A(x,1)\,(z\mathrm{d}(x-x_0)-(x-x_0)\mathrm{d}z)-(x_0A(x,1)+B(x,1))\,\mathrm{d}z$$

entraîne que

$$\tau := \tau(\mathcal{H}, s) = \min\{k \geq 1 : J^k_{x_0}(x_0\, A(x,1) + B(x,1)) \neq 0\},$$

cela permet d'écrire $x_0\, A(x,1) + B(x,1) = \sum_{k=\tau}^{d} c_k(x-x_0)^k$, avec $c_\tau \neq 0$. Par suite

$$B(x,y) = (x - x_0 y)^\tau P(x,y) - x_0\, A(x,y),$$

où $P(x,y) = \sum_{k=0}^{d-\tau} c_{k+\tau}(x-x_0y)^k y^{d-\tau-k}$.

Un calcul élémentaire montre que $\mathrm{D}_{\mathcal{H}} = A_xB_y - A_yB_x$ est de la forme $\mathrm{D}_{\mathcal{H}} = -(x-x_0y)^{\tau-1}Q(x,y)$, avec $Q \in \mathbb{C}[x,y]$ et

$$Q(x_0,1) = \tau P(x_0,1)\,(xA_x + yA_y)\Big|_{(x,y)=(x_0,1)}.$$

Comme $P(x_0,1) = c_\tau$ et $xA_x + yA_y = dA$, $Q(x_0,1) = \tau c_\tau dA(x_0,1) \neq 0$. □

Définition 2.3. — Soit $\mathcal{H}$ un feuilletage homogène de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ ayant un certain nombre $m \leq d+1$ de singularités radiales s_i d'ordre $\tau_i - 1$, $2 \leq \tau_i \leq d$ pour $i = 1,2,\ldots,m$. Le support du diviseur $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ est constitué d'un certain nombre $n \leq 2d-2$ de droites d'inflexion transverses T_j d'ordre $\rho_j - 1$, $2 \leq \rho_j \leq d$ pour $j = 1,2,\ldots,n$. On définit le *type du feuilletage* $\mathcal{H}$ par

$$\begin{aligned}\mathcal{T}_{\mathcal{H}} &= \sum_{i=1}^{m} \mathrm{R}_{\tau_i-1} + \sum_{j=1}^{n} \mathrm{T}_{\rho_j-1}\\ &= \sum_{k=1}^{d-1} (r_k \cdot \mathrm{R}_k + t_k \cdot \mathrm{T}_k) \in \mathbb{Z}\,[\mathrm{R}_1, \mathrm{R}_2, \ldots, \mathrm{R}_{d-1}, \mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_{d-1}]\end{aligned}$$

et le *degré du type* $\mathcal{T}_{\mathcal{H}}$ par $\deg \mathcal{T}_{\mathcal{H}} = \sum_{k=1}^{d-1}(r_k + t_k) \in \mathbb{N}\setminus\{0,1\}$; c'est le nombre de droites distinctes qui composent le diviseur $\mathrm{D}_{\mathcal{H}}$.

Exemple 2.4. — Considérons le feuilletage homogène $\mathcal{H}$ de degré 5 sur $\mathbb{P}^2_{\mathbb{C}}$ défini par

$$\omega = y^5 \mathrm{d}x + 2x^3(3x^2 - 5y^2)\mathrm{d}y.$$

Un calcul élémentaire conduit à

$$\mathrm{C}_{\mathcal{H}} = xy\left(6x^4 - 10x^2y^2 + y^4\right) \text{ et } \mathrm{D}_{\mathcal{H}} = 150x^2y^4(x-y)(x+y);$$

on constate que l'ensemble des singularités radiales de $\mathcal{H}$ est constitué des deux points $[0:1:0]$ et $[1:0:0]$; leurs ordres de radialité sont égaux respectivement à 2 et 4. De plus le support du diviseur $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ est formé des deux droites d'équations $x-y=0$ et $x+y=0$; ce sont des droites d'inflexion transverses simples. Donc le feuilletage $\mathcal{H}$ est du type $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_2 + 1\cdot\mathrm{R}_4 + 2\cdot\mathrm{T}_1$ et le degré de $\mathcal{T}_{\mathcal{H}}$ est $\deg \mathcal{T}_{\mathcal{H}} = 4$.

À tout feuilletage homogène $\mathcal{H}$ de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ on peut associer une application rationnelle $\underline{\mathcal{G}}_{\mathcal{H}} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ de la façon suivante : si $\mathcal{H}$ est décrit par $\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$, A et B désignant des polynômes homogènes de degré d sans facteur commun, on définit $\underline{\mathcal{G}}_{\mathcal{H}}$ par

$$\underline{\mathcal{G}}_{\mathcal{H}}([x:y]) = [-A(x,y) : B(x,y)];$$

il est clair que cette définition ne dépend pas du choix de la 1-forme homogène ω décrivant le feuilletage $\mathcal{H}$.

Dorénavant nous noterons l'application $\underline{\mathcal{G}}_{\mathcal{H}}$ simplement par $\underline{\mathcal{G}}$. Le feuilletage homogène $\mathcal{H}$ ainsi que son tissu dual $\mathrm{Leg}\mathcal{H}$ peuvent être décrits analytiquement en utilisant uniquement l'application $\underline{\mathcal{G}}$. En effet, la pente p de $\mathrm{T}_{(x,y)}\mathcal{H}$ est donnée par $\underline{\mathcal{G}}([x:y]) = [p:1]$ et les pentes x_i $(i = 1, \ldots, d)$ de $\mathrm{T}_{(p,q)}\mathrm{Leg}\mathcal{H}$ sont données par $x_i = \frac{q}{p - p_i(p)}$, avec $\underline{\mathcal{G}}^{-1}([p:1]) = \{[p_i(p):1]\}$.

En carte affine $\mathbb{C} \subset \mathbb{P}^1_{\mathbb{C}}$ cette application s'écrit $\underline{\mathcal{G}} : z \mapsto -\frac{A(1,z)}{B(1,z)}$. On a

$$\underline{\mathcal{G}}(z) - z = -\frac{A(1,z) + zB(1,z)}{B(1,z)} = -\frac{\mathrm{C}_{\mathcal{H}}(1,z)}{B(1,z)};$$

de plus, les identités $dA = xA_x + yA_y$ et $dB = xB_x + yB_y$ permettent de réécrire $\mathrm{D}_{\mathcal{H}}$ sous la forme $\mathrm{D}_{\mathcal{H}} = -\frac{d}{x}\left(BA_y - AB_y\right)$ de sorte que

$$\underline{\mathcal{G}}'(z) = -\left(\frac{BA_y - AB_y}{B^2}\right)\Big|_{(x,y)=(1,z)} = \frac{\mathrm{D}_{\mathcal{H}}(1,z)}{dB^2(1,z)}.$$

On en déduit immédiatement les propriétés suivantes :

1. les points fixes de $\underline{\mathcal{G}}$ correspondent au cône tangent de $\mathcal{H}$ en l'origine O (*i.e.* $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ est fixe par $\underline{\mathcal{G}}$ si et seulement si la droite d'équation $by - ax = 0$ est invariante par $\mathcal{H}$) ;
2. le point $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ est critique fixe par $\underline{\mathcal{G}}$ si et seulement si le point $[b:a:0] \in L_\infty$ est singulier radial de $\mathcal{H}$. La multiplicité du point critique $[a:b]$ de $\underline{\mathcal{G}}$ est exactement égale à l'ordre de radialité de la singularité à l'infini ;
3. le point $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ est critique non fixe par $\underline{\mathcal{G}}$ si et seulement si la droite d'équation $by - ax = 0$ est une droite d'inflexion transverse pour $\mathcal{H}$. La multiplicité du point critique $[a:b]$ de $\underline{\mathcal{G}}$ est précisément égale à l'ordre d'inflexion de cette droite.

Remarque 2.5. — Pour qu'un feuilletage homogène de degré $d \geq 2$ sur $\mathbb{P}^2_{\mathbb{C}}$ soit convexe de type $(2d-2)\cdot\mathrm{R}_1$ il faut que $d \in \{2,3\}$, car tout feuilletage homogène de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ a au plus $d+1$ points singuliers à l'infini. En fait, même en degré $d \in \{2,3\}$, le type $(2d-2)\cdot\mathrm{R}_1$ ne se produit pas. Ceci découle du fait bien connu qu'une application rationnelle de la sphère de Riemann dans elle-même a au moins un point fixe non critique (voir par exemple [10, Théorème 12.4]).

3. Étude de la platitude du tissu dual d'un feuilletage homogène

La Proposition 3.2 de [2] est un critère de la platitude de la transformée de Legendre d'un feuilletage homogène de degré 3. Notre premier résultat généralise ce critère en degré arbitraire.

Théorème 3.1. — *Soit $\mathcal{H}$ un feuilletage homogène de degré $d \geq 3$ sur $\mathbb{P}^2_{\mathbb{C}}$. Alors le d-tissu* $\mathrm{Leg}\mathcal{H}$ *est plat si et seulement si sa courbure* $K(\mathrm{Leg}\mathcal{H})$ *est holomorphe sur* $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$.

Dans tout ce qui suit, $\mathcal{H}$ désigne un feuilletage homogène de degré $d \geq 3$ sur $\mathbb{P}^2_{\mathbb{C}}$ défini, en carte affine (x,y), par la 1-forme

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A, B \in \mathbb{C}[x,y]_d, \; \mathrm{pgcd}(A,B) = 1.$$

La démonstration de ce théorème utilise les deux lemmes suivants.

Lemme 3.2. — *Le discriminant de* $\mathrm{Leg}\mathcal{H}$ *se décompose en*

$$\Delta(\mathrm{Leg}\mathcal{H}) = \mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}) \cup \check{\Sigma}^{\mathrm{rad}}_{\mathcal{H}} \cup \check{O},$$

où $\check{\Sigma}^{\mathrm{rad}}_{\mathcal{H}}$ *désigne l'ensemble des droites duales des points de* $\Sigma^{\mathrm{rad}}_{\mathcal{H}} = \{s \in \mathrm{Sing}\mathcal{H} : \nu(\mathcal{H},s) = 1, \tau(\mathcal{H},s) \geq 2\}$.

Démonstration. — La formule (1.4) nous donne $\Delta(\mathrm{Leg}\mathcal{H}) = \mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}) \cup \check{\Sigma}_{\mathcal{H}}$, où $\check{\Sigma}_{\mathcal{H}}$ est l'ensemble des droites duales des points de $\Sigma_{\mathcal{H}} = \{s \in \mathrm{Sing}\mathcal{H} : \tau(\mathcal{H},s) \geq 2\}$. D'après la première assertion de la proposition 2.2, l'origine O est le seul point singulier de $\mathcal{H}$ de multiplicité algébrique supérieure ou égale à 2 ; par conséquent $\Sigma_{\mathcal{H}} = \Sigma^{\mathrm{rad}}_{\mathcal{H}} \cup \{O\}$. □

Lemme 3.3 ([2], Lemme 3.1). — *Si la courbure de* $\mathrm{Leg}\mathcal{H}$ *est holomorphe sur* $\check{\mathbb{P}}^2_{\mathbb{C}} \setminus \check{O}$, *alors* $\mathrm{Leg}\mathcal{H}$ *est plat.*

Démonstration. — Soit (a,b) la carte affine de $\check{\mathbb{P}}^2_{\mathbb{C}}$ associée à la droite $\{ax - by + 1 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$; le d-tissu $\mathrm{Leg}\mathcal{H}$ est donné par la d-forme symétrique $\check{\omega} = bA(\mathrm{d}b, \mathrm{d}a) + aB(\mathrm{d}b, \mathrm{d}a)$. L'homogénéité de A et B implique alors que toute homothétie $h_\lambda : (a,b) \longmapsto \lambda(a,b)$ laisse invariant $\mathrm{Leg}\mathcal{H}$; par suite

$$h_\lambda^*(K(\mathrm{Leg}\mathcal{H})) = K(\mathrm{Leg}\mathcal{H}).$$

En combinant l'hypothèse de l'holomorphie de la courbure en dehors de $\check{O}$ avec le fait que $\check{O}$ est la droite à l'infini dans la carte (a,b), on constate que $K(\mathrm{Leg}\mathcal{H}) = P(a,b)\mathrm{d}a \wedge \mathrm{d}b$ pour un certain $P \in \mathbb{C}[a,b]$. On déduit de ce qui précède que $\lambda^2 P(\lambda a, \lambda b) = P(a,b)$, d'où l'énoncé. □

Démonstration du théorème 3.1. — L'implication directe est triviale. Montrons la réciproque ; supposons que $K(\mathrm{Leg}\mathcal{H})$ soit holomorphe sur $\mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$. D'après les lemmes 3.2 et 3.3, il suffit de prouver que $K(\mathrm{Leg}\mathcal{H})$ est holomorphe le long de $\Xi := \check{\Sigma}^{\mathrm{rad}}_{\mathcal{H}} \setminus \mathcal{G}_{\mathcal{H}}(\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}})$. Supposons donc Ξ non vide ; soit s une singularité radiale de $\mathcal{H}$ d'ordre $n-1$ telle que la droite $\check{s}$ duale de s ne soit pas contenue dans $\mathcal{G}_{\mathcal{H}}(\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}})$. D'après [9, Proposition 3.3], au voisinage de tout point générique m de $\check{s}$, le tissu $\mathrm{Leg}\mathcal{H}$ peut se décomposer comme le produit $\mathcal{W}_n \boxtimes \mathcal{W}_{d-n}$, où $\mathcal{W}_n$ est un n-tissu irréductible laissant $\check{s}$ invariante et $\mathcal{W}_{d-n}$ est un $(d-n)$-tissu transverse à $\check{s}$. De plus, la condition $\check{s} \not\subset \mathcal{G}_{\mathcal{H}}(\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}})$ assure que le tissu $\mathcal{W}_{d-n}$ est régulier au voisinage de m. Par conséquent $K(\mathrm{Leg}\mathcal{H})$ est holomorphe au voisinage de m, en vertu de [9, Proposition 2.6]. □

COROLLAIRE 3.4. — *Soit $\mathcal{H}$ un feuilletage homogène convexe de degré d sur le plan projectif. Alors le d-tissu* $\mathrm{Leg}\mathcal{H}$ *est plat.*

Le théorème suivant est un critère effectif d'holomorphie de la courbure (du tissu dual d'un feuilletage homogène) le long de l'image par l'application de Gauss d'une droite d'inflexion transverse simple, i.e. d'ordre d'inflexion minimal.

THÉORÈME 3.5. — *Soit $\mathcal{H}$ un feuilletage homogène de degré $d \geq 3$ sur $\mathbb{P}^2_{\mathbb{C}}$ défini par la* 1*-forme*

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A, B \in \mathbb{C}[x,y]_d, \ \ \mathrm{pgcd}(A,B) = 1.$$

Supposons que $\mathcal{H}$ possède une droite d'inflexion $T = (ax + by = 0)$ transverse et simple. Supposons en outre que $[-a : b] \in \mathbb{P}^1_{\mathbb{C}}$ soit le seul point critique de $\underline{\mathcal{G}}$ dans sa fibre $\underline{\mathcal{G}}^{-1}(\underline{\mathcal{G}}([-a : b]))$. Posons $T' = \mathcal{G}_{\mathcal{H}}(T)$ et considérons la courbe $\Gamma_{(a,b)}$ de $\mathbb{P}^2_{\mathbb{C}}$ définie par

$$Q(x,y;a,b) := \begin{vmatrix} \frac{\partial P}{\partial x} & A(b,-a) \\ \frac{\partial P}{\partial y} & B(b,-a) \end{vmatrix} = 0,$$

$$\textit{où} \quad P(x,y;a,b) := \frac{1}{(ax+by)^2} \begin{vmatrix} A(x,y) & A(b,-a) \\ B(x,y) & B(b,-a) \end{vmatrix}.$$

Alors la courbure de $\mathrm{Leg}\mathcal{H}$ *est holomorphe sur T' si et seulement si $T = \{ax + by = 0\} \subset \Gamma_{(a,b)}$, i.e. si et seulement si $Q(b,-a;a,b) = 0$.*

REMARQUE 3.6. — L'hypothèse que $T = (ax + by = 0)$ est une droite d'inflexion pour $\mathcal{H}$ implique que $P \in \mathbb{C}[x,y]_{d-2}$ et donc $Q \in \mathbb{C}[x,y]_{d-3}$. En particulier lorsque $d = 3$ on a

$$Q(b,-a;a,b) = \frac{\mathrm{C}_{\mathcal{H}}\left(B(b,-a), -A(b,-a)\right)}{\left(\mathrm{C}_{\mathcal{H}}(b,-a)\right)^2};$$

en effet si on pose $\tilde{a} = A(b,-a)$, $\tilde{b} = B(b,-a)$ et $P(x,y;a,b) = f(a,b)x + g(a,b)y$ on obtient

$$\begin{aligned} Q(b,-a;a,b) &= f(a,b)\tilde{b} - g(a,b)\tilde{a} = P(\tilde{b},-\tilde{a};a,b) \\ &= \frac{\tilde{b}A(\tilde{b},-\tilde{a}) - \tilde{a}B(\tilde{b},-\tilde{a})}{(a\tilde{b} - b\tilde{a})^2} = \frac{\mathrm{C}_{\mathcal{H}}\left(\tilde{b},-\tilde{a}\right)}{\left(\mathrm{C}_{\mathcal{H}}(b,-a)\right)^2}. \end{aligned}$$

Démonstration. — À isomorphisme linéaire près on peut se ramener à $T = (y = rx)$; si (p, q) est la carte affine de $\check{\mathbb{P}}^2_{\mathbb{C}}$ associée à la droite $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$, alors $T' = (p = \underline{\mathcal{G}}(r))$ avec $\underline{\mathcal{G}}(z) = -\frac{A(1,z)}{B(1,z)}$. Comme l'indice de ramification de $\underline{\mathcal{G}}$ en $z = r$ est égal à 2 et comme $z = r$ est l'unique point critique dans sa fibre $\underline{\mathcal{G}}^{-1}(\underline{\mathcal{G}}(r))$, cette fibre est formée de $d - 1$ points distincts, soit $\underline{\mathcal{G}}^{-1}(\underline{\mathcal{G}}(r)) = \{r, z_1, z_2, \ldots, z_{d-2}\}$. De plus, au voisinage de tout point générique de T', le tissu dual de $\mathcal{H}$ se décompose en $\mathrm{Leg}\mathcal{H} = \mathcal{W}_2 \boxtimes \mathcal{W}_{d-2}$ avec

$$\mathcal{W}_2\Big|_{T'} = (\mathrm{d}q - x_0(q)\mathrm{d}p)^2 \text{ et } \mathcal{W}_{d-2}\Big|_{T'} = \prod_{i=1}^{d-2} (\mathrm{d}q - x_i(q)\mathrm{d}p)\,,$$

où $x_0(q) = \frac{q}{\underline{\mathcal{G}}(r) - r}$ et $x_i(q) = \frac{q}{\underline{\mathcal{G}}(r) - z_i}$, $i = 1, 2, \ldots, d - 2$. D'après [9, Théorème 1], $K(\mathrm{Leg}\mathcal{H})$ est holomorphe le long de T' si et seulement si T' est invariante par le barycentre de $\mathcal{W}_{d-2}$ par rapport à $\mathcal{W}_2$. Or la restriction de $\beta_{\mathcal{W}_2}(\mathcal{W}_{d-2})$ à T' est donnée par $\mathrm{d}q - \beta(q)\mathrm{d}p = 0$ avec

$$\beta = x_0 + \frac{1}{\frac{1}{d-2} \sum\limits_{i=1}^{d-2} \frac{1}{x_i - x_0}}.$$

Ainsi la courbure de $\mathrm{Leg}\mathcal{H}$ est holomorphe sur T' si et seulement si $\beta = \infty$, i.e. si et seulement si $\sum_{i=1}^{d-2} \frac{1}{x_i - x_0} = 0$, car $x_0 \neq \infty$ ($z = r$ est non fixe par $\underline{\mathcal{G}}$). Cette dernière condition se réécrit

$$0 = \sum_{i=1}^{d-2} \frac{\underline{\mathcal{G}}(r) - z_i}{r - z_i} = d - 2 + (\underline{\mathcal{G}}(r) - r) \sum_{i=1}^{d-2} \frac{1}{r - z_i}. \tag{3.1}$$

D'autre part les z_i sont exactement les racines du polynôme

$$F(z) := \frac{P(1, z; -r, 1)}{B(1, r)} = \frac{A(1, z) + \underline{\mathcal{G}}(r)B(1, z)}{(z - r)^2}$$

et donc

$$\sum_{i=1}^{d-2} \frac{1}{r - z_i} = \sum_{i=1}^{d-2} \left(\frac{1}{F(r)} \prod_{\substack{j=1 \\ j \neq i}}^{d-2} (r - z_j) \right) = \frac{1}{F(r)} \sum_{i=1}^{d-2} \prod_{\substack{j=1 \\ j \neq i}}^{d-2} (r - z_j) = \frac{F'(r)}{F(r)}.$$

Ainsi l'équation (3.1) est équivalente à $(\underline{\mathcal{G}}(r) - r)F'(r) + (d - 2)F(r) = 0$, i.e. à

$$(d - 2)P(1, r; -r, 1) + (\underline{\mathcal{G}}(r) - r) \frac{\partial P}{\partial y}\Big|_{(x,y)=(1,r)} = 0; \tag{3.2}$$

comme $P \in \mathbb{C}[x, y]_{d-2}$ on peut réécrire (3.2) sous la forme

$$\left((d - 2)P(x, y; -r, 1) - y\frac{\partial P}{\partial y} + x\underline{\mathcal{G}}(r)\frac{\partial P}{\partial y} \right) \Big|_{y=rx} = 0;$$

celle-ci peut à son tour s'écrire

$$\left(\frac{\partial P}{\partial x}+\underline{\mathcal{G}}(r)\frac{\partial P}{\partial y}\right)\Big|_{y=rx}=0,$$

en vertu de l'identité d'Euler. Il en résulte que $K(\mathrm{Leg}\mathcal{H})$ est holomorphe le long de T' si et seulement si

$$\left(B(1,r)\frac{\partial P}{\partial x}-A(1,r)\frac{\partial P}{\partial y}\right)\Big|_{y=rx}=0. \qquad \square$$

REMARQUE 3.7. — En degré 3 l'équation (3.1) s'écrit $\frac{\underline{\mathcal{G}}(r)-z_1}{r-z_1}=0$; ainsi la courbure du 3-tissu $\mathrm{Leg}\mathcal{H}$ est holomorphe sur $T'=(p=\underline{\mathcal{G}}(r))$ si et seulement si $\underline{\mathcal{G}}(r)=z_1$, i.e. si et seulement si $\underline{\mathcal{G}}(\underline{\mathcal{G}}(r))=\underline{\mathcal{G}}(r)$.

Le théorème suivant est un critère effectif d'holomorphie de la courbure (du tissu dual d'un feuilletage homogène) le long de l'image par l'application de Gauss d'une droite d'inflexion transverse d'ordre maximal.

THÉORÈME 3.8. — *Soit $\mathcal{H}$ un feuilletage homogène de degré $d\geq 3$ sur $\mathbb{P}^2_{\mathbb{C}}$ défini par la 1-forme*

$$\omega=A(x,y)\mathrm{d}x+B(x,y)\mathrm{d}y,\quad A,B\in\mathbb{C}[x,y]_d,\;\; \mathrm{pgcd}(A,B)=1.$$

Supposons que $\mathcal{H}$ possède une droite d'inflexion transverse T d'ordre maximal $d-1$ et posons $T'=\mathcal{G}_{\mathcal{H}}(T)$. Alors la courbure de $\mathrm{Leg}\mathcal{H}$ est holomorphe le long de T' si et seulement si la 2-forme $\mathrm{d}\omega$ s'annule sur la droite T.

La démonstration de ce théorème utilise le lemme technique suivant, qui nous sera aussi utile ultérieurement.

LEMME 3.9. — *Soit $f:\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^1_{\mathbb{C}}$ une application rationnelle de degré d; $f(z)=\frac{a(z)}{b(z)}$ avec a et b des polynômes sans facteur commun et $\max(\deg a,\deg b)=d$. Soit $z_0\in\mathbb{C}$ tel que $f(z_0)\neq\infty$. Alors, z_0 est un point critique de f de multiplicité $m-1$ si et seulement s'il existe un polynôme $c\in\mathbb{C}[z]$ de degré $\leq d-m$ vérifiant $c(z_0)\neq 0$ et tel que $a(z)=f(z_0)b(z)+c(z)(z-z_0)^m$.*

Démonstration. — D'après la formule de Taylor, l'assertion $z=z_0$ est un point critique de f de multiplicité $m-1$ se traduit par $f(z)=f(z_0)+h(z)(z-z_0)^m$, avec $h(z_0)\neq 0$. Par suite

$$a(z)-f(z_0)b(z)=c(z)(z-z_0)^m$$

avec $c(z):=h(z)b(z)$, $c(z_0)\neq 0$; comme le membre de gauche est un polynôme en z de degré $\leq d$ celui de droite aussi. On constate alors que la fonction $c(z)$ est polynomiale en z de degré $\leq d-m$, d'où l'énoncé. $\square$

Démonstration du théorème 3.8. — On peut se ramener à $T = (y = rx)$; si (p,q) est la carte affine de $\check{\mathbb{P}}^2_{\mathbb{C}}$ associée à la droite $\{y = px - q\} \subset \mathbb{P}^2_{\mathbb{C}}$, alors $T' = (p = \underline{\mathcal{G}}(r))$ avec $\underline{\mathcal{G}}(z) = -\frac{A(1,z)}{B(1,z)}$. De plus, le d-tissu Leg$\mathcal{H}$ est décrit par $\prod_{i=1}^{d} \check{\omega}_i$, où $\check{\omega}_i = \frac{\mathrm{d}q}{q} - \lambda_i(p)\mathrm{d}p$, $\lambda_i(p) = \frac{1}{p - p_i(p)}$ et $\{p_i(p)\} = \underline{\mathcal{G}}^{-1}(p)$.

En appliquant les formules (1.1) et (1.2) à Leg$\mathcal{H} = \mathcal{W}(\check{\omega}_1, \check{\omega}_2, \ldots, \check{\omega}_d)$ on constate que $\eta(\mathrm{Leg}\mathcal{H})$ s'écrit sous la forme

$$\eta(\mathrm{Leg}\mathcal{H}) = \alpha(p)\mathrm{d}p + \frac{\mathrm{d}q}{q} \sum_{1 \le i < j < k \le d} \beta_{ijk}(p),$$

avec

$$\beta_{ijk}(p) = \frac{-\lambda_i'}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} + \frac{-\lambda_j'}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)} + \frac{-\lambda_k'}{(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)}.$$

Comme le point $z = r$ est critique non fixe pour $\underline{\mathcal{G}}$ de multiplicité $d-1$, il existe un isomorphisme analytique $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ tel qu'au voisinage de T' on ait

$$\lambda_i(p) = \frac{1}{\underline{\mathcal{G}}(r) - r} + \varphi\left(\zeta^i \left(p - \underline{\mathcal{G}}(r)\right)^{\frac{1}{d}}\right),$$

où ζ est une racine primitive d-ième de 1. Notons que

$$\lambda_i'(p) = \frac{1}{d}\left(p - \underline{\mathcal{G}}(r)\right)^{\frac{1-d}{d}} \left[\zeta^i \varphi'(0) + \zeta^{2i}\varphi''(0)\left(p - \underline{\mathcal{G}}(r)\right)^{\frac{1}{d}} + o\left(\left(p - \underline{\mathcal{G}}(r)\right)^{\frac{1}{d}}\right)\right],$$

et

$$\lambda_i(p) - \lambda_j(p) = \left(p - \underline{\mathcal{G}}(r)\right)^{\frac{1}{d}} \varphi'(0)(\zeta^i - \zeta^j) + o\left(\left(p - \underline{\mathcal{G}}(r)\right)^{\frac{1}{d}}\right).$$

Il s'en suit que

$$\beta_{ijk}(p) = \left(p - \underline{\mathcal{G}}(r)\right)^{-1-\frac{1}{d}} \tilde{\beta}_{ijk}\left(\left(p - \underline{\mathcal{G}}(r)\right)^{\frac{1}{d}}\right), \qquad \text{avec} \qquad \tilde{\beta}_{ijk}(z) \in \mathbb{C}\{z\}.$$

En fait, si $\langle i', j', k' \rangle$ désigne trois permutations circulaires de i, j et k, on a

$$\tilde{\beta}_{ijk}(0) = -\frac{1}{d\varphi'(0)} \underbrace{\sum_{\langle i', j', k' \rangle} \frac{\zeta^{i'}}{(\zeta^{i'} - \zeta^{j'})(\zeta^{i'} - \zeta^{k'})}}_{0} = 0,$$

et

$$\tilde{\beta}_{ijk}'(0) = \frac{\varphi''(0)}{2d\varphi'(0)^2} \underbrace{\sum_{\langle i', j', k' \rangle} \frac{\zeta^{i'}(\zeta^{j'} + \zeta^{k'})}{(\zeta^{i'} - \zeta^{j'})(\zeta^{i'} - \zeta^{k'})}}_{-1} = -\frac{\varphi''(0)}{2d\varphi'(0)^2}.$$

En posant $\beta(z) := \sum_{1 \le i < j < k \le d} \beta_{ijk}(z)$ et $\tilde{\beta}(z) := \sum_{1 \le i < j < k \le d} \tilde{\beta}_{ijk}(z)$, on obtient que

$$\beta(p) = \left(p - \underline{\mathcal{G}}(r)\right)^{-1-\frac{1}{d}} \tilde{\beta}\left(\left(p - \underline{\mathcal{G}}(r)\right)^{\frac{1}{d}}\right).$$

Comme $K(\mathrm{Leg}\mathcal{H}) = \mathrm{d}\eta(\mathrm{Leg}\mathcal{H}) = \frac{\beta'(p)}{q}\mathrm{d}p\wedge\mathrm{d}q$ et comme $\beta(p) \in \mathbb{C}\{p-\underline{\mathcal{G}}(r)\}\left[\frac{1}{p-\underline{\mathcal{G}}(r)}\right]$, on déduit que $K(\mathrm{Leg}\mathcal{H})$ est holomorphe le long de $T' = (p = \underline{\mathcal{G}}(r))$ si et seulement si $\tilde{\beta}(z) \in z\mathbb{C}\{z^d\}$ satisfait la condition

$$0 = \tilde{\beta}'(0) = \sum_{1\leq i<j<k\leq d} \tilde{\beta}'_{ijk}(0) = -\binom{d}{3}\frac{\varphi''(0)}{2d\varphi'(0)^2},$$

i.e. si et seulement si $\varphi''(0) = 0$.

D'après le lemme 3.9, le fait que $z = r$ est un point critique (non fixe) de $\underline{\mathcal{G}}$ de multiplicité $d-1$ se traduit par $-A(1,z) = \underline{\mathcal{G}}(r)B(1,z) + c(z-r)^d$, pour un certain $c \in \mathbb{C}^*$. Par suite

$$A(x,y) = -\underline{\mathcal{G}}(r)B(x,y) - c(y-rx)^d$$

$$\text{et} \qquad B(x,y) = b_0x^d + \sum_{i=1}^{d} b_i(y-rx)^i x^{d-i}.$$

Puisque $b_0 = B(1,r) \neq 0$, on peut supposer sans perte de généralité que $b_0 = 1$. Ainsi

$$\mathrm{d}\omega\Big|_{y=rx} = \left(d + b_1(\underline{\mathcal{G}}(r) - r)\right) x^{d-1}\mathrm{d}x \wedge \mathrm{d}y.$$

D'autre part, $\underline{\mathcal{G}}(z) = \underline{\mathcal{G}}(r) + \frac{c(z-r)^d}{1+b_1(z-r)+\cdots}$ et, pour tout $p \in \mathbb{P}^1_{\mathbb{C}}$ suffisamment voisin de $\underline{\mathcal{G}}(r)$, l'équation $\underline{\mathcal{G}}(z) = p$ est équivalente à

$$(p-\underline{\mathcal{G}}(r))^{\frac{1}{d}} = \frac{c^{\frac{1}{d}}(z-r)}{\sqrt[d]{1+b_1(z-r)+\cdots}} = c^{\frac{1}{d}}(z-r)\left[1 - \frac{1}{d}b_1(z-r) + \cdots\right].$$

Par suite les $p_i(p) \in \underline{\mathcal{G}}^{-1}(p)$ s'écrivent

$$p_i(p) = r + \frac{1}{c^{\frac{1}{d}}}\zeta^i\left(p-\underline{\mathcal{G}}(r)\right)^{\frac{1}{d}} + \frac{b_1}{dc^{\frac{2}{d}}}\zeta^{2i}\left(p-\underline{\mathcal{G}}(r)\right)^{\frac{2}{d}} + \cdots$$

et donc

$$\begin{aligned} p - p_i(p) = (\underline{\mathcal{G}}(r) - r) &- \frac{1}{c^{\frac{1}{d}}}\zeta^i\left(p-\underline{\mathcal{G}}(r)\right)^{\frac{1}{d}} \\ &- \frac{b_1}{dc^{\frac{2}{d}}}\zeta^{2i}\left(p-\underline{\mathcal{G}}(r)\right)^{\frac{2}{d}} + \cdots + (p-\underline{\mathcal{G}}(r)) + \cdots. \end{aligned}$$

Par conséquent

$$\lambda_i(p) = \frac{1}{p-p_i(p)} = \frac{1}{\underline{\mathcal{G}}(r)-r} + \varphi'(0)\zeta^i\left(p-\underline{\mathcal{G}}(r)\right)^{\frac{1}{d}} + \frac{\varphi''(0)}{2}\zeta^{2i}\left(p-\underline{\mathcal{G}}(r)\right)^{\frac{2}{d}} + \cdots,$$

avec

$$\varphi'(0) = \frac{1}{c^{\frac{1}{d}}(\underline{\mathcal{G}}(r)-r)^2} \neq 0 \text{ et } \varphi''(0) = \frac{2}{dc^{\frac{2}{d}}(\underline{\mathcal{G}}(r)-r)^3}\left[d + b_1(\underline{\mathcal{G}}(r)-r)\right],$$

ce qui termine la démonstration. □

Comme conséquence immédiate des théorèmes 3.1, 3.5, 3.8 et de la remarque 3.6 nous obtenons la caractérisation suivante de la platitude de la transformée de Legendre d'un feuilletage homogène de degré 3 sur le plan projectif.

COROLLAIRE 3.10. — *Soit $\mathcal{H}$ un feuilletage homogène de degré 3 sur $\mathbb{P}^2_{\mathbb{C}}$ défini par la 1-forme*

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A, B \in \mathbb{C}[x,y]_3, \ \mathrm{pgcd}(A,B) = 1.$$

Alors, le 3-tissu $\mathrm{Leg}\mathcal{H}$ *est plat si et seulement si les deux conditions suivantes sont satisfaites :*

(1) *pour toute droite d'inflexion de $\mathcal{H}$ transverse et simple $T_1 = (ax + by = 0)$, la droite d'équation $A(b,-a)x + B(b,-a)y = 0$ est invariante par $\mathcal{H}$;*
(2) *pour toute droite d'inflexion de $\mathcal{H}$ transverse et double T_2, la 2-forme $\mathrm{d}\omega$ s'annule sur T_2.*

En particulier, si le feuilletage $\mathcal{H}$ est convexe alors $\mathrm{Leg}\mathcal{H}$ *est plat.*

4. Platitude et feuilletages homogènes de type $\mathbb{Z}\,[\mathbf{R_1}, \mathbf{R_2}, \ldots, \mathbf{R}_{d-1}, \mathbf{T_1}, \mathbf{T}_{d-1}]$

Nous nous proposons dans ce paragraphe de décrire certaines feuilletages homogènes de degré $d \geq 3$ sur $\mathbb{P}^2_{\mathbb{C}}$, de type $\mathbb{Z}\,[\mathrm{R}_1, \mathrm{R}_2, \ldots, \mathrm{R}_{d-1}, \mathrm{T}_1, \mathrm{T}_{d-1}]$ et dont le d-tissu dual est plat. Nous considérons ici un feuilletage homogène $\mathcal{H}$ de degré $d \geq 3$ sur $\mathbb{P}^2_{\mathbb{C}}$ défini, en carte affine (x,y), par

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A, B \in \mathbb{C}[x,y]_d, \ \mathrm{pgcd}(A,B) = 1.$$

L'application rationnelle $\underline{\mathcal{G}} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, $\underline{\mathcal{G}}(z) = -\frac{A(1,z)}{B(1,z)}$, nous sera très utile pour établir les énoncés qui suivent.

PROPOSITION 4.1. — *Si $\deg \mathcal{T}_{\mathcal{H}} = 2$, alors le d-tissu* $\mathrm{Leg}\mathcal{H}$ *est plat si et seulement si $\mathcal{H}$ est linéairement conjugué à l'un des deux feuilletages $\mathcal{H}_1^d$ et $\mathcal{H}_2^d$ décrits respectivement par les 1-formes*

1. $\omega_1^d = y^d\mathrm{d}x - x^d\mathrm{d}y$;
2. $\omega_2^d = x^d\mathrm{d}x - y^d\mathrm{d}y$.

Démonstration. — L'égalité $\deg \mathcal{T}_{\mathcal{H}} = 2$ est réalisée si et seulement si nous sommes dans l'une des situations suivantes

(i) $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{R}_{d-1}$;
(ii) $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{T}_{d-1}$;
(iii) $\mathcal{T}_{\mathcal{H}} = 1 \cdot \mathrm{R}_{d-1} + 1 \cdot \mathrm{T}_{d-1}$.

Commençons par étudier l'éventualité (i). Nous pouvons supposer à conjugaison près que les deux singularités radiales de $\mathcal{H}$ sont $[0:1:0]$ et $[1:0:0]$, ce qui revient à supposer que les points $\infty = [1:0]$, $[0:1] \in \mathbb{P}^1_{\mathbb{C}}$ sont critiques fixes de $\underline{\mathcal{G}}$, de même multiplicité $d-1$. Cela se traduit par le fait que $A(x,y)=ay^d$ et $B(x,y)=bx^d$, avec $ab\neq 0$, en vertu du lemme 3.9. Par suite $\omega=ay^d\mathrm{d}x-(-b)x^d\mathrm{d}y$ et nous pouvons évidemment normaliser les coefficients a et $-b$ à 1. Ainsi $\mathcal{H}$ est conjugué au feuilletage $\mathcal{H}_1^d$ décrit par $\omega_1^d=y^d\mathrm{d}x-x^d\mathrm{d}y$; le d-tissu $\mathrm{Leg}\mathcal{H}_1^d$ est plat car $\mathcal{H}_1^d$ est convexe.

Intéressons-nous à la possibilité (ii). À isomorphisme linéaire près nous pouvons nous ramener à la situation suivante :

- les points $[0:1]$, $[1:1]\in\mathbb{P}^1_{\mathbb{C}}$ sont critiques non fixes de $\underline{\mathcal{G}}$, de même multiplicité $d-1$;
- $\underline{\mathcal{G}}(0)$ et $\underline{\mathcal{G}}(1)\neq\infty$.

Toujours d'après le lemme 3.9, il existe des constantes $\alpha,\beta\in\mathbb{C}^*$ telles que

$$-A(1,z)=\underline{\mathcal{G}}(0)B(1,z)+\alpha z^d=\underline{\mathcal{G}}(1)B(1,z)+\beta(z-1)^d$$

avec $\underline{\mathcal{G}}(0)\neq 0$, $\underline{\mathcal{G}}(1)\neq 1$ et $\underline{\mathcal{G}}(0)\neq\underline{\mathcal{G}}(1)$. L'homogénéité de A et B entraîne alors que

$$\omega=\big(\underline{\mathcal{G}}(0)s(y-x)^d-g(1)ry^d\big)\,\mathrm{d}x+\big(ry^d-s(y-x)^d\big)\,\mathrm{d}y$$

avec $r=\frac{\alpha}{\underline{\mathcal{G}}(1)-\underline{\mathcal{G}}(0)}\neq 0$ et $s=\frac{\beta}{\underline{\mathcal{G}}(1)-\underline{\mathcal{G}}(0)}\neq 0$. D'après les théorèmes 3.1 et 3.8, le d-tissu $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si $\mathrm{d}\omega$ s'annule sur les deux droites $y(y-x)=0$. Un calcul immédiat montre que

$$\mathrm{d}\omega\Big|_{y=0}=-sd(\underline{\mathcal{G}}(0)-1)x^{d-1}\mathrm{d}x\wedge\mathrm{d}y\quad\text{et}\quad\mathrm{d}\omega\Big|_{y=x}=rd\underline{\mathcal{G}}(1)x^{d-1}\mathrm{d}x\wedge\mathrm{d}y.$$

Ainsi $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si $\underline{\mathcal{G}}(0)=1$ et $\underline{\mathcal{G}}(1)=0$, auquel cas

$$\omega=s(y-x)^d\mathrm{d}x+\big(ry^d-s(y-x)^d\big)\,\mathrm{d}y;$$

quitte à remplacer ω par $\varphi^*\omega$, où $\varphi(x,y)=\Big(s^{\frac{-1}{d+1}}x-r^{\frac{-1}{d+1}}y,-r^{\frac{-1}{d+1}}y\Big)$, on se ramène à

$$\omega=\omega_2^d=x^d\mathrm{d}x-y^d\mathrm{d}y.$$

Considérons pour finir l'éventualité (iii). Nous pouvons supposer que la singularité radiale de $\mathcal{H}$ est le point $[0:1:0]$ et que la droite d'inflexion transverse de $\mathcal{H}$ est la droite $(y=0)$; $\underline{\mathcal{G}}(0)\neq\underline{\mathcal{G}}(\infty)=\infty$ car $\underline{\mathcal{G}}^{-1}(\underline{\mathcal{G}}(0))=\{0\}$. Un raisonnement analogue à celui du cas précédent conduit à

$$\omega=-\big(\underline{\mathcal{G}}(0)\beta x^d+\alpha y^d\big)\,\mathrm{d}x+\beta x^d\mathrm{d}y,\qquad\text{avec}\qquad\alpha\beta\underline{\mathcal{G}}(0)\neq 0.$$

La courbure du tissu associé à cette 1-forme ne peut pas être holomorphe sur $\mathcal{G}_{\mathcal{H}}(\{y=0\})$ car

$$\mathrm{d}\omega\Big|_{y=0}=d\beta x^{d-1}\mathrm{d}x\wedge\mathrm{d}y\not\equiv 0;$$

il en résulte que $\mathrm{Leg}\mathcal{H}$ ne peut pas être plat lorsque $\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_{d-1}+1\cdot\mathrm{T}_{d-1}$. □

PROPOSITION 4.2. — *Soit ν un entier compris entre* 1 *et* $d-2$. *Si le feuilletage* $\mathcal{H}$ *est de type*

$$\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_{\nu}+1\cdot\mathrm{R}_{d-\nu-1}+1\cdot\mathrm{R}_{d-1}, \qquad \textit{resp. } \mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_{\nu}+1\cdot\mathrm{R}_{d-\nu-1}+1\cdot\mathrm{T}_{d-1},$$

alors le d-tissu $\mathrm{Leg}\mathcal{H}$ *est plat si et seulement si* $\mathcal{H}$ *est linéairement conjugué au feuilletage* $\mathcal{H}_3^{d,\nu}$, *resp.* $\mathcal{H}_4^{d,\nu}$ *donné par*

$$\omega_3^{d,\nu} = \sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i\mathrm{d}x - \sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i\mathrm{d}y,$$
$$\textit{resp. } \omega_4^{d,\nu} = (d-\nu-1)\sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i\mathrm{d}x + \nu\sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i\mathrm{d}y.$$

Démonstration. — Dans les deux cas, nous pouvons supposer à conjugaison linéaire près que les points $[0:1]$, $[1:0]$, $[-1:1]\in\mathbb{P}^1_{\mathbb{C}}$ sont critiques de $\underline{\mathcal{G}}$, de multiplicité ν, $d-\nu-1$, $d-1$ respectivement. Les points $[0:1]$ et $[1:0]$ sont évidemment fixes par $\underline{\mathcal{G}}$; le feuilletage $\mathcal{H}$ est de type $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_{\nu}+1\cdot\mathrm{R}_{d-\nu-1}+1\cdot\mathrm{R}_{d-1}$ (resp. $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_{\nu}+1\cdot\mathrm{R}_{d-\nu-1}+1\cdot\mathrm{T}_{d-1}$) si et seulement si le point $[-1:1]$ est fixe (resp. non fixe) par $\underline{\mathcal{G}}$. Puisque $\underline{\mathcal{G}}^{-1}(\underline{\mathcal{G}}(-1)) = \{-1\}$ nous avons $\underline{\mathcal{G}}(-1)\neq\underline{\mathcal{G}}(\infty)=\infty$. Donc, d'après le lemme 3.9, il existe une constante $\alpha\in\mathbb{C}^*$ et un polynôme homogène $B_\nu\in\mathbb{C}[x,y]_\nu$ tels que

$$-A(x,y) = \underline{\mathcal{G}}(-1)B(x,y)+\alpha(y+x)^d,$$
$$B(x,y) = x^{d-\nu}B_\nu(x,y)$$

et $y^{\nu+1}$ divise $A(x,y)$. Il en résulte que

$$-A(x,y) = \underline{\mathcal{G}}(-1)x^{d-\nu}B_\nu(x,y)+\alpha\sum_{i=0}^{d}\binom{d}{i}x^{d-i}y^i$$
$$= \underline{\mathcal{G}}(-1)x^{d-\nu}B_\nu(x,y)+\alpha\sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i+\alpha\sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i;$$

par suite $A(x,y)$ est divisible par $y^{\nu+1}$ si et seulement si

$$-A(x,y) = \alpha\sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i \text{ et } \underline{\mathcal{G}}(-1)x^{d-\nu}B_\nu(x,y)+\alpha\sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i = 0.$$

Quitte à remplacer $\omega = A(x,y)\mathrm{d}x+B(x,y)\mathrm{d}y$ par $-\frac{1}{\alpha}\omega$ on se ramène à

$$\omega = \sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i\mathrm{d}x+\frac{1}{\underline{\mathcal{G}}(-1)}\sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i\mathrm{d}y, \quad \underline{\mathcal{G}}(-1)\neq\underline{\mathcal{G}}(0)=0.$$

- Si $\underline{\mathcal{G}}(-1)=-1$ nous obtenons le feuilletage $\mathcal{H}_3^{d,\nu}$ décrit par
$$\omega_3^{d,\nu}=\sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i\mathrm{d}x-\sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i\mathrm{d}y;$$
la transformée de Legendre $\mathrm{Leg}\mathcal{H}_3^{d,\nu}$ est plate car $\mathcal{H}_3^{d,\nu}$ est convexe.
- Si $\underline{\mathcal{G}}(-1)\neq-1$ alors, d'après les théorèmes 3.1 et 3.8, le d-tissu $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si
$$0\equiv \mathrm{d}\omega\Big|_{y=-x}=\binom{d}{\nu+1}\frac{(-1)^{\nu+1}(\nu+1)}{(d-1)\underline{\mathcal{G}}(-1)}\left[\underline{\mathcal{G}}(-1)\nu-d+\nu+1\right]x^{d-1}\mathrm{d}x\wedge\mathrm{d}y,$$
i.e. si et seulement si $\underline{\mathcal{G}}(-1)=\frac{d-\nu-1}{\nu}$, auquel cas
$$\begin{aligned}(d-\nu-1)\omega&=\omega_4^{d,\nu}\\&=(d-\nu-1)\sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i\mathrm{d}x+\nu\sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i\mathrm{d}y.\end{aligned}$$ □

PROPOSITION 4.3. — *Si le feuilletage $\mathcal{H}$ est de type*
$$\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_{d-2}+1\cdot\mathrm{T}_1+1\cdot\mathrm{R}_{d-1},\qquad \textit{resp. } \mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_{d-2}+1\cdot\mathrm{T}_1+1\cdot\mathrm{T}_{d-1},$$
alors le d-tissu $\mathrm{Leg}\mathcal{H}$ *est plat si et seulement si $\mathcal{H}$ est linéairement conjugué au feuilletage $\mathcal{H}_5^d$, resp. $\mathcal{H}_6^d$ décrit par*
$$\omega_5^d=2y^d\mathrm{d}x+x^{d-1}(yd-(d-1)x)\mathrm{d}y,$$
resp. $\omega_6^d=\left((d-1)^2x^d-d(d-1)x^{d-1}y+(d+1)y^d\right)\mathrm{d}x+x^{d-1}\left(yd-(d-1)x\right)\mathrm{d}y.$

Démonstration. — Nous allons traiter ces deux types simultanément. À isomorphisme linéaire près, nous pouvons nous ramener à la situation suivante : les points $[1:0]$, $[1:1]$, $[0:1]\in\mathbb{P}^1_{\mathbb{C}}$ sont critiques de $\underline{\mathcal{G}}$, de multiplicité $d-2$, 1, $d-1$ respectivement. Le point $[1:0]$ (resp. $[1:1]$) est fixe (resp. non fixe) par $\underline{\mathcal{G}}$; le feuilletage $\mathcal{H}$ est de type $\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_{d-2}+1\cdot\mathrm{T}_1+1\cdot\mathrm{R}_{d-1}$ (resp. $\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_{d-2}+1\cdot\mathrm{T}_1+1\cdot\mathrm{T}_{d-1}$) si et seulement si le point $[0:1]$ est fixe (resp. non fixe) par $\underline{\mathcal{G}}$. Puisque $\underline{\mathcal{G}}^{-1}(\underline{\mathcal{G}}(0))=\{0\}$ nous avons $\underline{\mathcal{G}}(0)\neq\underline{\mathcal{G}}(1)$ et $\underline{\mathcal{G}}(0)\neq\underline{\mathcal{G}}(\infty)=\infty$; de plus $\underline{\mathcal{G}}(1)\neq\underline{\mathcal{G}}(\infty)=\infty$ car $\underline{\mathcal{G}}^{-1}(\underline{\mathcal{G}}(\infty))=\{\infty,z_0\}$ pour un certain point $z_0\neq\infty$, non critique de $\underline{\mathcal{G}}$. Par suite, d'après le lemme 3.9, il existe une constante $\alpha\in\mathbb{C}^*$ telle que
$$-A(x,y)=\underline{\mathcal{G}}(0)B(x,y)+\alpha y^d\text{ et }B(x,y)=\frac{\alpha}{s}x^{d-1}\left(yd-(d-1)x\right),$$
avec $s=\underline{\mathcal{G}}(1)-\underline{\mathcal{G}}(0)\neq0$. Quitte à multiplier $\omega=A(x,y)\mathrm{d}x+B(x,y)\mathrm{d}y$ par $\frac{s}{\alpha}$ on se ramène à
$$\omega=-\left(\underline{\mathcal{G}}(0)\,x^{d-1}\left(yd-(d-1)x\right)+sy^d\right)\mathrm{d}x+x^{d-1}\left(yd-(d-1)x\right)\mathrm{d}y.$$

D'après ce qui précède le point $[1:1]$ est le seul point critique de $\underline{\mathcal{G}}$ dans sa fibre $\underline{\mathcal{G}}^{-1}(\underline{\mathcal{G}}(1))$. Donc, d'après le théorème 3.5, la courbure de $\mathrm{Leg}\mathcal{H}$ est holomorphe sur $\mathcal{G}_{\mathcal{H}}(\{y=x\})$ si et seulement si

$$0=Q(1,1;-1,1)=-\frac{1}{6}sd(d-1)(d-2)(\underline{\mathcal{G}}(0)+s+2),$$

i.e. si et seulement si $s=-\underline{\mathcal{G}}(0)-2$.

- Si $\underline{\mathcal{G}}(0)=0$ alors la condition $s=-\underline{\mathcal{G}}(0)-2=-2$ est suffisante pour que $\mathrm{Leg}\mathcal{H}$ soit plat, en vertu du théorème 3.1, et dans ce cas
$$\omega=\omega_5^d=2y^d\mathrm{d}x+x^{d-1}(yd-(d-1)x)\mathrm{d}y.$$
- Si $\underline{\mathcal{G}}(0)\neq 0$ alors, d'après les théorèmes 3.1 et 3.8, $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si
$$s=-\underline{\mathcal{G}}(0)-2\quad\text{et}\quad 0\equiv \mathrm{d}\omega\Big|_{y=0}=d(\underline{\mathcal{G}}(0)-d+1)x^{d-1}\mathrm{d}x\wedge\mathrm{d}y,$$
i.e. si et seulement si $\underline{\mathcal{G}}(0)=d-1$ et $s=-d-1$, auquel cas

$$\omega=\omega_6^d=\big((d-1)^2x^d-d(d-1)x^{d-1}y+(d+1)y^d\big)\,\mathrm{d}x+x^{d-1}\,(yd-(d-1)x)\,\mathrm{d}y.$$

□

5. Classification des feuilletages homogènes de degré trois à transformée de Legendre plate

Dans ce paragraphe nous allons classifier, à automorphisme de $\mathbb{P}^2_{\mathbb{C}}$ près, les feuilletages homogènes de degré 3 sur le plan projectif dont le 3-tissu dual est plat. Plus précisément nous allons démontrer l'énoncé suivant.

THÉORÈME 5.1. — *Soit $\mathcal{H}$ un feuilletage homogène de degré 3 sur le plan projectif $\mathbb{P}^2_{\mathbb{C}}$. Alors le 3-tissu dual $\mathrm{Leg}\mathcal{H}$ de $\mathcal{H}$ est plat si et seulement si $\mathcal{H}$ est linéairement conjugué à l'un des onze feuilletages $\mathcal{H}_1,\ldots,\mathcal{H}_{11}$ décrits respectivement en carte affine par les 1-formes*

1. $\omega_1=y^3\mathrm{d}x-x^3\mathrm{d}y$;
2. $\omega_2=x^3\mathrm{d}x-y^3\mathrm{d}y$;
3. $\omega_3=y^2(3x+y)\mathrm{d}x-x^2(x+3y)\mathrm{d}y$;
4. $\omega_4=y^2(3x+y)\mathrm{d}x+x^2(x+3y)\mathrm{d}y$;
5. $\omega_5=2y^3\mathrm{d}x+x^2(3y-2x)\mathrm{d}y$;
6. $\omega_6=(4x^3-6x^2y+4y^3)\mathrm{d}x+x^2(3y-2x)\mathrm{d}y$;
7. $\omega_7=y^3\mathrm{d}x+x(3y^2-x^2)\mathrm{d}y$;
8. $\omega_8=x(x^2-3y^2)\mathrm{d}x-4y^3\mathrm{d}y$;
9. $\omega_9=y^2\left((-3+\mathrm{i}\sqrt{3})x+2y\right)\mathrm{d}x+x^2\left((1+\mathrm{i}\sqrt{3})x-2\mathrm{i}\sqrt{3}y\right)\mathrm{d}y$;
10. $\omega_{10}=(3x+\sqrt{3}y)y^2\mathrm{d}x+(3y-\sqrt{3}x)x^2\mathrm{d}y$;
11. $\omega_{11}=(3x^3+3\sqrt{3}x^2y+3xy^2+\sqrt{3}y^3)\mathrm{d}x+(\sqrt{3}x^3+3x^2y+3\sqrt{3}xy^2+3y^3)\mathrm{d}y$.

Considérons un feuilletage homogène $\mathcal{H}$ de degré 3 sur $\mathbb{P}^2_{\mathbb{C}}$ défini, en carte affine (x, y), par

$$\omega = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y,$$

où A et B désignent des polynômes homogènes de degré 3 sans composante commune ; la classification menant au théorème 5.1 est établie au cas par cas suivant que $\deg \mathcal{T}_{\mathcal{H}} = 2, 3$ ou 4, i.e. suivant la nature du support du diviseur $\mathrm{D}_{\mathcal{H}}$ qui peut être deux droites, trois droites ou quatre droites. Pour ce faire commençons par établir les deux lemmes suivants.

Lemme 5.2. — *Si $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{R}_2$, resp. $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{T}_2$, alors, à conjugaison linéaire près, la 1-forme ω décrivant $\mathcal{H}$ est du type*

$$\omega = y^3\mathrm{d}x + \left(\beta\, x^3 - 3\beta\, xy^2 + \alpha\, y^3\right)\mathrm{d}y, \qquad \beta\left((2\beta-1)^2 - \alpha^2\right) \neq 0,$$

$$\textit{resp. } \omega = \left(x^3 - 3xy^2 + \alpha\, y^3\right)\mathrm{d}x + \left(\delta\, x^3 - 3\delta\, xy^2 + \beta\, y^3\right)\mathrm{d}y,$$

$(\beta - \alpha\delta)\left((\beta-2)^2 - (\alpha - 2\delta)^2\right) \neq 0.$

Démonstration. — À isomorphisme près nous pouvons nous ramener à $\mathrm{D}_{\mathcal{H}} = cy^2(y-x)(y+x)$ pour un certain $c \in \mathbb{C}^*$. Le produit $\mathrm{C}_{\mathcal{H}}(1,1)\mathrm{C}_{\mathcal{H}}(1,-1)$ est évidemment non nul ; $\mathcal{H}$ est de type $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{T}_1 + 1\cdot\mathrm{R}_2$ (resp. $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{T}_1 + 1\cdot\mathrm{T}_2$) si et seulement si $\mathrm{C}_{\mathcal{H}}(1,0) = 0$ (resp. $\mathrm{C}_{\mathcal{H}}(1,0) \neq 0$). Écrivons les coefficients A et B de ω sous la forme

$A(x,y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3$ et $B(x,y) = b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3$;

nous avons donc

$$\mathrm{C}_{\mathcal{H}} = a_0x^4 + (a_1+b_0)x^3y + (a_2+b_1)x^2y^2 + (a_3+b_2)xy^3 + b_3y^4$$

et

$$\begin{aligned}\mathrm{D}_{\mathcal{H}} = (a_0b_1 - a_1b_0)x^4 + 2(a_0b_2 - a_2b_0)x^3y + (3a_0b_3 + a_1b_2 - a_2b_1 - 3a_3b_0)x^2y^2 \\ + 2(a_1b_3 - a_3b_1)xy^3 + (a_2b_3 - a_3b_2)y^4.\end{aligned}$$

Ainsi $\mathrm{C}_{\mathcal{H}}(1,0) = a_0$ et

$$\mathrm{D}_{\mathcal{H}} = cy^2(y-x)(y+x) \quad \Leftrightarrow \quad \begin{cases} a_0b_1 = a_1b_0 \\ a_0b_2 = a_2b_0 \\ a_1b_3 = a_3b_1 \\ a_2b_3 - a_3b_2 = c \\ 3a_0b_3 + a_1b_2 - a_2b_1 - 3a_3b_0 = -c. \end{cases} \tag{5.1}$$

- Si $a_0 \neq 0$ alors le système (5.1) est équivalent à

$$a_1 = 0, \qquad a_2 = -3a_0, \qquad b_1 = 0, \qquad b_2 = -3b_0, \qquad c = -3(a_0b_3 - a_3b_0).$$

Posons $a_3 = a_0\alpha$, $\quad b_0 = a_0\delta$, $\quad b_3 = a_0\beta$; alors, quitte à diviser ω par a_0, cette forme s'écrit

$$\omega = \left(x^3 - 3xy^2 + \alpha\, y^3\right)\mathrm{d}x + \left(\delta\, x^3 - 3\delta\, xy^2 + \beta\, y^3\right)\mathrm{d}y;$$

un calcul direct montre que la condition $c\,\mathrm{C}_{\mathcal{H}}(1,1)\mathrm{C}_{\mathcal{H}}(1,-1)\neq 0$ est vérifiée si et seulement si $(\beta-\alpha\delta)\big((\beta-2)^2-(\alpha-2\delta)^2\big)\neq 0$.

- Si $a_0=0$ alors le système (5.1) conduit à
$$a_1=a_2=b_1=0,\quad b_2=-3b_0,\quad c=3a_3b_0\neq 0.$$
Écrivons $b_0=a_3\beta$ et $b_3=a_3\alpha$; alors, quitte à remplacer ω par $\frac{1}{a_3}\omega$, on se ramène à
$$\omega=y^3\mathrm{d}x+\big(\beta\,x^3-3\beta\,xy^2+\alpha\,y^3\big)\,\mathrm{d}y,$$
et la non nullité du produit $c\,\mathrm{C}_{\mathcal{H}}(1,1)\mathrm{C}_{\mathcal{H}}(1,-1)$ est équivalente à $\beta\big((2\beta-1)^2-\alpha^2\big)\neq 0$. □

Lemme 5.3. — *Si le diviseur $\mathrm{D}_{\mathcal{H}}$ est réduit, i.e. si $\deg\mathcal{T}_{\mathcal{H}}=4$, alors ω est, à conjugaison linéaire près, de l'une des formes suivantes*

1. $y^2\,((2r+3)x-(r+2)y)\,\mathrm{d}x-x^2(x+ry)\mathrm{d}y$,
   où $r(r+1)(r+2)(r+3)(2r+3)\neq 0$ *;*
2. $sy^2\,((2r+3)x-(r+2)y)\,\mathrm{d}x-x^2(x+ry)\mathrm{d}y$,
   où $rs(s-1)(r+1)(r+2)(r+3)(2r+3)\,\big(s(2r+3)^2-r^2\big)\neq 0$ *;*
3. $ty^2\,((2r+3)x-(r+2)y)\,\mathrm{d}x-x^2(x+ry)\mathrm{d}(sy-x)$,
   où $rst(r+1)(r+2)(r+3)(2r+3)(s-t-1)(tu^3-r^2su-r^2v)\neq 0$,
   $u=2r+3$ *et* $v=r(r+2)$ *;*
4. $uy^2\,((2r+3)x-(r+2)y)\,\mathrm{d}(y-sx)-x^2(x+ry)\mathrm{d}(ty-x)$,
   où $ur(r+1)(r+2)(r+3)(2r+3)(st-1)(su+t-u-1)(uv^4+suwv^3+r^2twv+r^2w^2)\neq 0$,
   $v=2r+3$ *et* $w=r(r+2)$.

Ces quatre modèles sont respectivement de types $3\cdot\mathrm{R}_1+1\cdot\mathrm{T}_1,\ 2\cdot\mathrm{R}_1+2\cdot\mathrm{T}_1,\ 1\cdot\mathrm{R}_1+3\cdot\mathrm{T}_1, 4\cdot\mathrm{T}_1$.

Démonstration. — D'après la remarque 2.5 le feuilletage $\mathcal{H}$ ne peut être de type $4\cdot\mathrm{R}_1$; nous sommes donc dans l'une des situations suivantes

(i) $\mathcal{T}_{\mathcal{H}}=3\cdot\mathrm{R}_1+1\cdot\mathrm{T}_1$;
(ii) $\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_1+2\cdot\mathrm{T}_1$;
(iii) $\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_1+3\cdot\mathrm{T}_1$;
(iv) $\mathcal{T}_{\mathcal{H}}=4\cdot\mathrm{T}_1$.

À conjugaison linéaire près nous pouvons nous ramener à $\mathrm{D}_{\mathcal{H}}=cxy(y-x)(y-\alpha x)$ pour certains $c,\alpha\in\mathbb{C}^*,\alpha\neq 1$. Dans la dernière éventualité nous avons
$$\mathrm{C}_{\mathcal{H}}(0,1)\mathrm{C}_{\mathcal{H}}(1,0)\mathrm{C}_{\mathcal{H}}(1,1)\mathrm{C}_{\mathcal{H}}(1,\alpha)\neq 0$$
et dans les cas (i), resp. (ii), resp. (iii) nous pouvons supposer que
$$\begin{cases}\mathrm{C}_{\mathcal{H}}(0,1)=0\\ \mathrm{C}_{\mathcal{H}}(1,0)=0\\ \mathrm{C}_{\mathcal{H}}(1,1)=0\\ \mathrm{C}_{\mathcal{H}}(1,\alpha)\neq 0\end{cases}\text{resp.}\quad\begin{cases}\mathrm{C}_{\mathcal{H}}(0,1)=0\\ \mathrm{C}_{\mathcal{H}}(1,0)=0\\ \mathrm{C}_{\mathcal{H}}(1,1)\neq 0\\ \mathrm{C}_{\mathcal{H}}(1,\alpha)\neq 0\end{cases}\text{resp.}\quad\begin{cases}\mathrm{C}_{\mathcal{H}}(0,1)=0\\ \mathrm{C}_{\mathcal{H}}(1,0)\neq 0\\ \mathrm{C}_{\mathcal{H}}(1,1)\neq 0\\ \mathrm{C}_{\mathcal{H}}(1,\alpha)\neq 0\end{cases}$$

Comme dans le lemme précédent, en écrivant

$A(x,y) = a_0x^3+a_1x^2y+a_2xy^2+a_3y^3$ et $B(x,y) = b_0x^3+b_1x^2y+b_2xy^2+b_3y^3$

nous obtenons que

$$\mathrm{D}_{\mathcal{H}} = cxy(y-x)(y-\alpha x) \quad \Leftrightarrow \quad \begin{cases} a_0b_1 = a_1b_0 \\ a_2b_3 = a_3b_2 \\ 2(a_1b_3 - a_3b_1) = c \\ 2(a_0b_2 - a_2b_0) = c\alpha \\ 3a_0b_3 + a_1b_2 - a_2b_1 - 3a_3b_0 = -c(\alpha+1). \end{cases} \tag{5.2}$$

Envisageons l'éventualité (iv). Comme $c \neq 0$, $a_0 = \mathrm{C}_{\mathcal{H}}(1,0) \neq 0$ et $b_3 = \mathrm{C}_{\mathcal{H}}(0,1) \neq 0$, le système (5.2) est équivalent à

$$\begin{cases} b_1 = \frac{a_1b_0}{a_0} \\ a_2 = \frac{a_3b_2}{b_3} \\ c = \frac{2a_1(a_0b_3-a_3b_0)}{a_0} \\ a_0b_2 - \alpha a_1b_3 = 0 \\ (3a_0+2\alpha a_1+2a_1)b_3 + a_1b_2 = 0 \end{cases} \Leftrightarrow \begin{cases} b_1 = \frac{a_1b_0}{a_0} \\ a_2 = \frac{a_3a_1\alpha}{a_0} \\ c = \frac{2a_1(a_0b_3-a_3b_0)}{a_0} \\ b_2 = \frac{a_1b_3\alpha}{a_0} \\ a_1(a_1+2a_0)\alpha + a_0(2a_1+3a_0) = 0. \end{cases}$$

Donc $a_1 \neq 0$ et puisque $\alpha \neq 0$, le produit $(a_1+2a_0)(2a_1+3a_0)$ est non nul. Il s'en suit que

$$a_2 = -\frac{a_3(2a_1+3a_0)}{a_1+2a_0}, \quad b_1 = \frac{a_1b_0}{a_0}, \quad b_2 = -\frac{b_3(2a_1+3a_0)}{a_1+2a_0},$$
$$\alpha = -\frac{a_0(2a_1+3a_0)}{a_1(a_1+2a_0)}, \quad c = \frac{2a_1(a_0b_3-a_3b_0)}{a_0}.$$

Posons $r = \frac{a_1}{a_0}$, $s = -\frac{a_3}{b_3}$, $t = -\frac{b_0}{a_0}$, $u = -\frac{b_3}{a_1+2a_0}$; alors

$$b_0 = -ta_0, \quad b_1 = -rta_0, \quad b_2 = (2r+3)ua_0, \quad b_3 = -u(r+2)a_0,$$
$$a_1 = ra_0, \quad a_2 = -su(2r+3)a_0, \; a_3 = su(r+2)a_0,$$
$$\alpha = -\frac{2r+3}{r(r+2)}, \qquad c = 2r(r+2)u(st-1)a_0^2.$$

Quitte à remplacer ω par $\frac{1}{a_0}\omega$, le coefficient a_0 vaut 1 et ω s'écrit

$$\begin{aligned} \omega &= \left(x^3+rx^2y-su(2r+3)xy^2+su(r+2)y^3\right)\mathrm{d}x \\ &\quad + \left(-tx^3-rtx^2y+u(2r+3)xy^2-u(r+2)y^3\right)\mathrm{d}y \\ &= uy^2\left((2r+3)x-(r+2)y\right)\mathrm{d}(y-sx) - x^2(x+ry)\mathrm{d}(ty-x); \end{aligned}$$

un calcul direct montre que la condition

$$c\alpha(\alpha-1)\mathrm{C}_{\mathcal{H}}(0,1)\mathrm{C}_{\mathcal{H}}(1,0)\mathrm{C}_{\mathcal{H}}(1,1)\mathrm{C}_{\mathcal{H}}(1,\alpha) \neq 0$$

est équivalente à

$$ur(r+1)(r+2)(r+3)(2r+3)(st-1) \\ \times (su+t-u-1)(uv^4+suwv^3+r^2twv+r^2w^2) \neq 0,$$

avec $v = 2r+3$ et $w = r(r+2)$.

Maintenant nous étudions la possibilité (iii). Dans ce cas nous avons $b_3 = \mathrm{C}_{\mathcal{H}}(0,1) = 0$ et $a_0 = \mathrm{C}_{\mathcal{H}}(1,0) \neq 0$; le système (5.2) conduit à

$$a_2 = -\frac{a_3(2a_1+3a_0)}{a_1+2a_0},\ b_1 = \frac{a_1b_0}{a_0},\ b_2 = 0,\ \alpha = -\frac{a_0(2a_1+3a_0)}{a_1(a_1+2a_0)},\ c = -\frac{2a_1a_3b_0}{a_0}.$$

En posant $r = \frac{a_1}{a_0}$, $s = -\frac{b_0}{a_0}$ et $t = -\frac{a_3}{a_1+2a_0}$, nous obtenons que

$$\begin{array}{llll} b_0 = -sa_0, & b_1 = -rsa_0, & b_2 = b_3 = 0, & c = -2rst(r+2)a_0^2, \\ a_1 = ra_0, & a_2 = t(2r+3)a_0, & a_3 = -t(r+2)a_0, & \alpha = -\dfrac{2r+3}{r(r+2)}. \end{array}$$

Quitte à diviser ω par a_0 on se ramène à

$$\begin{aligned} \omega &= \big(x^3+rx^2y+t(2r+3)xy^2-t(r+2)y^3\big)\,\mathrm{d}x - sx^2(x+ry)\mathrm{d}y \\ &= ty^2\left((2r+3)x-(r+2)y\right)\mathrm{d}x - x^2(x+ry)\mathrm{d}(sy-x), \end{aligned}$$

et la non nullité du produit $c\alpha(\alpha-1)\mathrm{C}_{\mathcal{H}}(1,0)\mathrm{C}_{\mathcal{H}}(1,1)\mathrm{C}_{\mathcal{H}}(1,\alpha)$ se traduit par

$$rst(r+1)(r+2)(r+3)(2r+3)(s-t-1)(tu^3-r^2su-r^2v) \neq 0,$$

avec $u = 2r+3$ et $v = r(r+2)$. Les deux premiers cas se traitent de façon analogue. □

Démonstration du théorème 5.1. — Nous allons considérer trois cas.

Premier cas : $\deg \mathcal{T}_{\mathcal{H}} = 2$. — Dans ce cas le 3-tissu $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si la 1-forme ω définissant $\mathcal{H}$ est linéairement conjuguée à l'une des deux 1-formes

$$\omega_1 = y^3\mathrm{d}x - x^3\mathrm{d}y \qquad \text{et} \qquad \omega_2 = x^3\mathrm{d}x - y^3\mathrm{d}y.$$

C'est une application directe de la proposition 4.1 pour $d = 3$.

Second cas : $\deg \mathcal{T}_{\mathcal{H}} = 3$

- Si $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{R}_2$, resp. $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{T}_2$, alors, d'après la proposition 4.2, $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si ω est conjuguée à

$$\begin{aligned} \omega_3^{3,1} &= \sum_{i=2}^{3}\binom{3}{i}x^{3-i}y^i\mathrm{d}x - \sum_{i=0}^{1}\binom{3}{i}x^{3-i}y^i\mathrm{d}y \\ &= y^2(3x+y)\mathrm{d}x - x^2(x+3y)\mathrm{d}y \;=\; \omega_3, \\ \text{resp. } \omega_4^{3,1} &= \sum_{i=2}^{3}\binom{3}{i}x^{3-i}y^i\mathrm{d}x + \sum_{i=0}^{1}\binom{3}{i}x^{3-i}y^i\mathrm{d}y \\ &= y^2(3x+y)\mathrm{d}x + x^2(x+3y)\mathrm{d}y \;=\; \omega_4. \end{aligned}$$

- Si $\mathcal{T}_{\mathcal{H}} = 1 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{R}_2$, resp. $\mathcal{T}_{\mathcal{H}} = 1 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{T}_2$, alors, d'après la proposition 4.3, $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si ω est conjuguée à

$$\begin{aligned} \omega_5^{3} &\;=\; 2y^3\mathrm{d}x + x^2(3y-2x)\mathrm{d}y \;=\; \omega_5, \\ \text{resp. } \omega_6^{3} &\;=\; (4x^3-6x^2y+4y^3)\mathrm{d}x + x^2(3y-2x)\mathrm{d}y \;=\; \omega_6. \end{aligned}$$

- Si $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{R}_2$, alors, d'après le lemme 5.2, la 1-forme ω est du type

$$\omega = y^3\mathrm{d}x + \left(\beta\, x^3 - 3\beta\, xy^2 + \alpha\, y^3\right)\mathrm{d}y, \qquad \beta\left((2\beta-1)^2-\alpha^2\right) \neq 0,$$

  et dans ce cas nous avons $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}} = (y-x)(y+x)$. D'après le corollaire 3.10, le 3-tissu $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si

$$0 = Q(1,1;-1,1) = (2\beta+2-\alpha)\beta \text{ et } 0 = Q(1,-1;1,1) = -(2\beta+2+\alpha)\beta,$$

  i.e. si et seulement si $\alpha = 0$ et $\beta = -1$, auquel cas $\omega = \omega_7 = y^3\mathrm{d}x + x(3y^2-x^2)\mathrm{d}y$.

- Dans ce deuxième cas, il ne nous reste plus qu'à traiter l'éventualité $\mathcal{T}_{\mathcal{H}} = 2 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{T}_2$. Toujours d'après le lemme 5.2, ω est, à conjugaison près, de la forme

$$\omega = \left(x^3 - 3xy^2 + \alpha\, y^3\right)\mathrm{d}x + \left(\delta\, x^3 - 3\delta\, xy^2 + \beta\, y^3\right)\mathrm{d}y,$$

  $(\beta-\alpha\delta)\left((\beta-2)^2-(\alpha-2\delta)^2\right) \neq 0$, comme $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}} = y^2(y-x)(y+x)$ le 3-tissu $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si

$$\begin{cases} 0 \equiv \mathrm{d}\omega\Big|_{y=0} = 3\delta\, x^2\mathrm{d}x\wedge\mathrm{d}y \\ 0 = Q(1,1;-1,1) = (4+\beta-2\alpha-2\delta)(\beta-\alpha\delta) \\ 0 = Q(1,-1;1,1) = (4+\beta+2\alpha+2\delta)(\beta-\alpha\delta), \end{cases}$$

  en vertu du corollaire 3.10. Il s'en suit que $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si $\alpha = \delta = 0$ et $\beta = -4$, auquel cas $\omega = \omega_8 = x(x^2-3y^2)\mathrm{d}x - 4y^3\mathrm{d}y$.

Troisième cas : $\deg \mathcal{T}_{\mathcal{H}} = 4$. — Pour examiner la platitude dans ce dernier cas, nous allons appliquer le corollaire 3.10 aux différents modèles du lemme 5.3.

- Si $\mathcal{T}_{\mathcal{H}} = 3\cdot \mathrm{R}_1 + 1\cdot \mathrm{T}_1$, alors ω est du type
$$\omega = y^2\left((2r+3)x-(r+2)y\right)\mathrm{d}x - x^2(x+ry)\mathrm{d}y$$
avec $r(r+1)(r+2)(r+3)(2r+3)\neq 0$. Nous avons $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}} = sx+ty$ où $s=2r+3$ et $t=r(r+2)$; par suite le 3-tissu $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si
$$0 = Q(t,-s;s,t) = r(r+1)^2(r+2)^2(r+3)(2r+3)\left[r^2+3r+3\right],$$
i.e. si et seulement si $r=-\frac{3}{2}\pm \mathrm{i}\frac{\sqrt{3}}{2}$. Dans les deux cas la 1-forme ω est linéairement conjuguée à
$$\omega_9 = y^2\left((-3+\mathrm{i}\sqrt{3})x+2y\right)\mathrm{d}x + x^2\left((1+\mathrm{i}\sqrt{3})x-2\mathrm{i}\sqrt{3}y\right)\mathrm{d}y;$$
en effet si $r=-\frac{3}{2}-\mathrm{i}\frac{\sqrt{3}}{2}$, resp. $r=-\frac{3}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$, alors
$$\omega_9 = -(1+\mathrm{i}\sqrt{3})\omega, \qquad \text{resp. } \omega_9 = -2\varphi^*\omega, \quad \text{où } \varphi(x,y)=(y,x).$$
- Si $\mathcal{T}_{\mathcal{H}} = 2\cdot \mathrm{R}_1 + 2\cdot \mathrm{T}_1$, alors ω est de la forme
$$\omega = sy^2\left((2r+3)x-(r+2)y\right)\mathrm{d}x - x^2(x+ry)\mathrm{d}y$$
avec $rs(s-1)(r+1)(r+2)(r+3)(2r+3)\left(s(2r+3)^2-r^2\right)\neq 0$. Posons $t=2r+3$ et $u=r(r+2)$; nous avons $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}} = (y-x)(tx+uy)$. Donc $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si
$$\begin{cases} 0 = Q(1,1;-1,1) = -s(r+1)^2\left[s(r+2)+1\right] \\ 0 = Q(u,-t;t,u) = rs(r+1)^2(r+2)^2(2r+3)\left[s(2r+3)^2+(r+2)r^2\right], \end{cases}$$
i.e. si et seulement si $r=\pm\sqrt{3}$ et $s=-2+r$, car $rs(r+1)(r+2)(2r+3)\neq 0$. Dans les deux cas ω est linéairement conjuguée à
$$\omega_{10} = (3x+\sqrt{3}y)y^2\mathrm{d}x + (3y-\sqrt{3}x)x^2\mathrm{d}y;$$
en effet si $(r,s)=(-\sqrt{3},-2-\sqrt{3})$, resp. $(r,s)=(\sqrt{3},-2+\sqrt{3})$, alors
$$\omega_{10} = \sqrt{3}\omega, \qquad \text{resp. } \omega_{10} = -\sqrt{3}\varphi^*\omega, \quad \text{où } \varphi(x,y)=(x,-y).$$
- Si $\mathcal{T}_{\mathcal{H}} = 1\cdot \mathrm{R}_1 + 3\cdot \mathrm{T}_1$, alors ω est du type
$$\omega = ty^2\left((2r+3)x-(r+2)y\right)\mathrm{d}x - x^2(x+ry)\mathrm{d}(sy-x)$$
avec $rst(r+1)(r+2)(r+3)(2r+3)(s-t-1)(tu^3-r^2su-r^2v)\neq 0$, $u=2r+3$ et $v=r(r+2)$. Puisque $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}} = y(y-x)(ux+vy)$ la courbure de $\mathrm{Leg}\mathcal{H}$ est holomorphe le long de $\mathcal{G}_{\mathcal{H}}(\{y(y-x)=0\})$ si et seulement si
$$\begin{cases} 0 = Q(1,0;0,1) = st\left[(2r+3)s-(r+2)\right] \\ 0 = Q(1,1;-1,1) = -st(r+1)^2\left[(r+2)(t+1)+s\right], \end{cases}$$

i.e. si et seulement si $s=\frac{r+2}{2r+3}$ et $t=-\frac{2(r+2)}{2r+3}$, auquel cas $K(\mathrm{Leg}\mathcal{H})$ ne peut être holomorphe sur $\mathcal{G}_{\mathcal{H}}(\{ux+vy=0\})$ car

$$Q(v,-u;u,v)=12\,r(r+1)^3(r+2)^5(r+3)(2r+3)^{-2}\neq 0.$$

Par conséquent la transformée de Legendre $\mathrm{Leg}\mathcal{H}$ de $\mathcal{H}$ ne peut être plate lorsque $\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_1+3\cdot\mathrm{T}_1$.

- Si $\mathcal{T}_{\mathcal{H}}=4\cdot\mathrm{T}_1$, alors ω est de la forme

$$\omega=uy^2\left((2r+3)x-(r+2)y\right)\mathrm{d}(y-sx)-x^2(x+ry)\mathrm{d}(ty-x),$$

où $ur(r+1)(r+2)(r+3)(2r+3)(st-1)(su+t-u-1)(uv^4+suwv^3+r^2twv+r^2w^2)\neq 0$, $v=2r+3$ et $w=r(r+2)$. Comme $\mathrm{I}_{\mathcal{H}}^{\mathrm{tr}}=xy(y-x)(vx+wy)$ la courbure de $\mathrm{Leg}\mathcal{H}$ est holomorphe le long de $\mathcal{G}_{\mathcal{H}}(\{xy(y-x)=0\})$ si et seulement si

$$\begin{cases}0=Q(0,-1;1,0)=-u^2(r+2)^2(st-1)\,[rs+1]\\ 0=Q(1,0;0,1)=-u(st-1)\,[(2r+3)t-r-2]\\ 0=Q(1,1;-1,1)=-u(r+1)^2(st-1)\,[(rs+2s+1)u-t-r-2],\end{cases}$$

i.e. si et seulement si $s=-\frac{1}{r}$, $t=\frac{r+2}{2r+3}$ et $u=-\frac{r(r+2)^2}{2r+3}$, auquel cas

$$Q(w,-v;v,w)=16r(r+1)^5(r+2)^5(r+3)(2r+3)^{-2}\left[r^2+3r+3\right].$$

Par suite $\mathrm{Leg}\mathcal{H}$ est plat si et seulement si nous sommes dans l'un des deux cas suivants

(i) $r=-\frac{3}{2}+\mathrm{i}\frac{\sqrt{3}}{2},\quad s=\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{6},\quad t=\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{6},\quad u=1;$

(ii) $r=-\frac{3}{2}-\mathrm{i}\frac{\sqrt{3}}{2},\quad s=\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{6},\quad t=\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{6},\quad u=1.$

Dans les deux cas la 1-forme ω est linéairement conjuguée à

$$\omega_{11}=(3x^3+3\sqrt{3}x^2y+3xy^2+\sqrt{3}y^3)\mathrm{d}x+(\sqrt{3}x^3+3x^2y+3\sqrt{3}xy^2+3y^3)\mathrm{d}y;$$

en effet dans les cas (i), resp. (ii) nous avons

$$\omega_{11}=3\varphi_1^*\omega,\ \text{où}\ \varphi_1=(x,\mathrm{e}^{-5\mathrm{i}\pi/6}\,y),\ \text{resp.}\ \omega_{11}=3\varphi_2^*\omega,\ \text{où}\ \varphi_2=(x,\mathrm{e}^{5\mathrm{i}\pi/6}\,y).$$

□

Une particularité remarquable de la classification obtenue est que toutes les singularités des feuilletages $\mathcal{H}_i$, $i=1,\ldots,11$, sur la droite à l'infini sont non-dégénérées. Nous aurons besoin dans le prochain paragraphe des valeurs des indices $\mathrm{CS}(\mathcal{H}_i,L_\infty,s)$, $s\in\mathrm{Sing}\mathcal{H}_i\cap L_\infty$. Pour cela, nous avons calculé, pour chaque $i=1,\ldots,11$, le polynôme suivant (dit *polynôme de Camacho-Sad du feuilletage homogène* $\mathcal{H}_i$)

$$\mathrm{CS}_{\mathcal{H}_i}(\lambda)=\prod_{s\in\mathrm{Sing}\mathcal{H}_i\cap L_\infty}(\lambda-\mathrm{CS}(\mathcal{H}_i,L_\infty,s)).$$

Le tableau suivant résume les types et les polynômes de Camacho-Sad des feuilletages $\mathcal{H}_i$, $i=1,\ldots,11$.

i	$\mathcal{T}_{\mathcal{H}_i}$	$\mathrm{CS}_{\mathcal{H}_i}(\lambda)$
1	$2 \cdot \mathrm{R}_2$	$(\lambda-1)^2(\lambda+\frac{1}{2})^2$
2	$2 \cdot \mathrm{T}_2$	$(\lambda-\frac{1}{4})^4$
3	$2 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{R}_2$	$(\lambda-1)^3(\lambda+2)$
4	$2 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{T}_2$	$(\lambda-1)^2(\lambda+\frac{1}{2})^2$
5	$1 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{R}_2$	$(\lambda-1)^2(\lambda+\frac{1}{5})(\lambda+\frac{4}{5})$
6	$1 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{T}_2$	$(\lambda-1)(\lambda+\frac{2}{7})(\lambda-\frac{1}{7})^2$
7	$2 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{R}_2$	$(\lambda-1)(\lambda-\frac{1}{4})(\lambda+\frac{1}{8})^2$
8	$2 \cdot \mathrm{T}_1 + 1 \cdot \mathrm{T}_2$	$(\lambda-\frac{1}{10})^2(\lambda-\frac{2}{5})^2$
9	$3 \cdot \mathrm{R}_1 + 1 \cdot \mathrm{T}_1$	$(\lambda-1)^3(\lambda+2)$
10	$2 \cdot \mathrm{R}_1 + 2 \cdot \mathrm{T}_1$	$(\lambda-1)^2(\lambda+\frac{1}{2})^2$
11	$4 \cdot \mathrm{T}_1$	$(\lambda-\frac{1}{4})^4$

TABLE 5.1. Types et polynômes de Camacho-Sad des feuilletages homogènes donnés par le théorème 5.1

6. Feuilletages à singularités non-dégénérées et de transformée de Legendre plate

L'ensemble $\mathbf{F}(d)$ des feuilletages de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ est un ouvert de Zariski dans l'espace projectif $\mathbb{P}^{(d+2)^2-2}$. Le groupe des automorphismes de $\mathbb{P}^2_{\mathbb{C}}$ agit sur $\mathbf{F}(d)$; l'orbite d'un élément $\mathcal{F} \in \mathbf{F}(d)$ sous l'action de $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{PGL}_3(\mathbb{C})$ est notée $\mathcal{O}(\mathcal{F})$, voir [7]. Le sous-ensemble $\mathbf{FP}(d)$ de $\mathbf{F}(d)$ formé des $\mathcal{F} \in \mathbf{F}(d)$ tels que $\mathrm{Leg}\mathcal{F}$ soit plat est un fermé de Zariski de $\mathbf{F}(d)$. Signalons aussi que si $\mathcal{F} \in \mathbf{FP}(d)$ alors l'adhérence $\overline{\mathcal{O}(\mathcal{F})}$ (dans $\mathbf{F}(d)$) de $\mathcal{O}(\mathcal{F})$ est contenue dans $\mathbf{FP}(d)$.

Parmi les éléments de $\mathbf{FP}(d)$ n'ayant que des singularités non-dégénérées, il y a le *feuilletage de Fermat* $\mathcal{F}^d$ de degré d défini en carte affine par la 1-forme

$$\omega_F^d = (x^d - x)\mathrm{d}y - (y^d - y)\mathrm{d}x;$$

en effet, d'une part $\mathrm{Leg}\mathcal{F}^d$ est plat car il est algébrisable d'après [9, Proposition 5.2] ; d'autre part, un calcul élémentaire montre que toutes les singularités du feuilletage $\mathcal{F}^d$ sont non-dégénérées. Nous savons aussi d'après [9, Théorème 3] que $\overline{\mathcal{O}(\mathcal{F}^d)}$ est une composante irréductible de $\mathbf{FP}(d)$ pour $d \neq 4$.

Le théorème suivant est le résultat principal de ce paragraphe.

THÉORÈME 6.1. — *Soit $\mathcal{F}$ un feuilletage de degré 3 sur $\mathbb{P}^2_{\mathbb{C}}$. Supposons que toutes ses singularités soient non-dégénérées et que son 3-tissu dual* $\mathrm{Leg}\mathcal{F}$ *soit* *plat. Alors $\mathcal{F}$ est linéairement conjugué au feuilletage de Fermat $\mathcal{F}^3$ défini par la 1-forme $\omega_F^3 = (x^3 - x)\mathrm{d}y - (y^3 - y)\mathrm{d}x$.*

Remarque 6.2. — L'ensemble $\mathbf{FP}(4)$ contient des feuilletages à singularités non-dégénérées et qui ne sont pas conjugués au feuilletage $\mathcal{F}^4$, *e.g.* la famille $(\mathcal{F}_\lambda^4)_{\lambda\in\mathbb{C}}$ de feuilletages définis par

$$\omega_F^4 + \lambda((x^3-1)y^2\mathrm{d}y - (y^3-1)x^2\mathrm{d}x).$$

En effet, d'après [9, Théorème 8.1], pour tout λ fixé dans $\mathbb{C}$, $\mathcal{F}_\lambda^4 \in \mathbf{FP}(4)$; de plus un calcul facile montre que $\mathcal{F}_\lambda^4$ est à singularités non-dégénérées. Mais, si λ est non nul alors $\mathcal{F}_\lambda^4$ n'est pas conjugué à $\mathcal{F}^4$ car $\mathcal{F}_\lambda^4$ n'est pas convexe.

La démonstration du théorème 6.1 repose sur le théorème 5.1 de classification des feuilletages homogènes appartenant à $\mathbf{FP}(3)$, et sur les trois résultats qui suivent, dont les deux premiers sont valables en degré quelconque.

Notons d'abord que le feuilletage $\mathcal{F}^d$ possède trois singularités radiales d'ordre maximal $d-1$, non alignées. L'énoncé suivant montre que cette propriété caractérise l'orbite $\mathcal{O}(\mathcal{F}^d)$.

Proposition 6.3. — *Soit $\mathcal{F}$ un feuilletage de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ ayant trois singularités radiales d'ordre maximal $d-1$, non alignées. Alors $\mathcal{F}$ est linéairement conjugué au feuilletage de Fermat $\mathcal{F}^d$.*

Démonstration. — Par hypothèse $\mathcal{F}$ possède trois points singuliers $m_j, j = 1,2,3$, non alignés vérifiant $\nu(\mathcal{F},m_j) = 1$ et $\tau(\mathcal{F},m_j) = d$. D'après [4, Proposition 2, page 23], les égalités $\tau(\mathcal{F},m_j) = \tau(\mathcal{F},m_l) = d$ avec $l \neq j$ impliquent que la droite (m_jm_l) est invariante par $\mathcal{F}$. Choisissons des coordonnées homogènes $[x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$ telles que $m_1 = [0:0:1]$, $m_2 = [0:1:0]$ et $m_3 = [1:0:0]$. Les égalités $\nu(\mathcal{F},m_1) = 1$ et $\tau(\mathcal{F},m_1) = d$, combinées avec le fait que $(m_2m_3) = (z=0)$ est $\mathcal{F}$-invariante, assurent que toute 1-forme ω décrivant $\mathcal{F}$ dans la carte affine $z=1$ est du type

$$\omega = (x\mathrm{d}y - y\mathrm{d}x)(\gamma + C_1(x,y) + \cdots + C_{d-2}(x,y)) + A_d(x,y)\mathrm{d}x + B_d(x,y)\mathrm{d}y$$

avec $\gamma \neq 0$, $\quad A_d, B_d \in \mathbb{C}[x,y]_d$, $\quad C_k \in \mathbb{C}[x,y]_k$ pour $k = 1,\ldots,d-2$.

Dans la carte affine $y=1$ le feuilletage $\mathcal{F}$ est donné par

$$\begin{aligned}\theta = -(\gamma z^d + C_1(x,1)z^{d-1} + \cdots + C_{d-2}(x,1)z^2)\mathrm{d}x&\\ + A_d(x,1)(z\mathrm{d}x - x\mathrm{d}z) - B_d(x,1)\mathrm{d}z;&\end{aligned}$$

nous avons $\theta \wedge (z\mathrm{d}x - x\mathrm{d}z) = zQ(x,z)\mathrm{d}x\wedge\mathrm{d}z$, avec

$$Q(x,z) = x\left[\gamma z^{d-1} + C_1(x,1)z^{d-2} + \cdots + C_{d-2}(x,1)z\right] + B_d(x,1).$$

L'égalité $\tau(\mathcal{F},m_2) = d$ entraîne alors que le polynôme $Q \in \mathbb{C}[x,z]$ est homogène de degré d, ce qui permet d'écrire $B_d(x,y) = \beta x^d$ et $C_k(x,y) = \delta_k x^k$, β,

$\delta_k \in \mathbb{C}$. Par suite nous avons $J^1_{(0,0)}\theta = A_d(0,1)(z\mathrm{d}x - x\mathrm{d}z)$; alors l'égalité $\nu(\mathcal{F}, m_2) = 1$ assure que $A_d(0,1) \neq 0$.

De la même manière, en se plaçant dans la carte affine $x = 1$ et en écrivant explicitement les égalités $\tau(\mathcal{F}, m_3) = d$ et $\nu(\mathcal{F}, m_3) = 1$, nous obtenons que $B_d(1,0) \neq 0$, $A_d(x,y) = \alpha y^d$ et $C_k(x,y) = \varepsilon_k y^k$, $\alpha, \varepsilon_k \in \mathbb{C}$. Donc $\alpha\beta \neq 0$, les C_k sont tous nuls et ω est du type

$$\omega = \gamma(x\mathrm{d}y - y\mathrm{d}x) + \alpha y^d \mathrm{d}x + \beta x^d \mathrm{d}y.$$

Écrivons $\alpha = \gamma\mu^{1-d}$ et $\beta = -\gamma\lambda^{1-d}$. Quitte à remplacer ω par $\varphi^*\omega$, où $\varphi(x,y) = (\lambda x, \mu y)$, le feuilletage $\mathcal{F}$ est défini, dans les coordonnées affines (x,y), par la 1-forme

$$\omega_F^d = (x^d - x)\mathrm{d}y - (y^d - y)\mathrm{d}x. \qquad \square$$

Le résultat suivant permet de ramener l'étude de la platitude au cadre homogène :

Proposition 6.4. — *Soit $\mathcal{F}$ un feuilletage de degré $d \geq 1$ sur $\mathbb{P}^2_{\mathbb{C}}$ ayant une droite invariante L. Supposons que toutes les singularités de $\mathcal{F}$ sur L soient non-dégénérées. Il existe un feuilletage homogène $\mathcal{H}$ de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ ayant les propriétés suivantes*

- $\mathcal{H} \in \overline{\mathcal{O}(\mathcal{F})}$;
- *L est invariante par $\mathcal{H}$;*
- $\mathrm{Sing}\mathcal{H} \cap L = \mathrm{Sing}\mathcal{F} \cap L$;
- $\forall\, s \in \mathrm{Sing}\mathcal{H} \cap L$, $\mu(\mathcal{H}, s) = 1$ *et* $\mathrm{CS}(\mathcal{H}, L, s) = \mathrm{CS}(\mathcal{F}, L, s)$.

Si de plus $\mathrm{Leg}\mathcal{F}$ *est plat, alors* $\mathrm{Leg}\mathcal{H}$ *l'est aussi.*

Démonstration. — Choisissons des coordonnées homogènes $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ telles que $L = (z = 0)$; comme L est $\mathcal{F}$-invariante, $\mathcal{F}$ est défini dans la carte affine $z = 1$ par une 1-forme ω du type

$$\omega = \sum_{i=0}^{d} (A_i(x,y)\mathrm{d}x + B_i(x,y)\mathrm{d}y),$$

où les A_i, B_i sont des polynômes homogènes de degré i.

Montrons par l'absurde que $\mathrm{pgcd}(A_d, B_d) = 1$; supposons donc que $\mathrm{pgcd}(A_d, B_d) \neq 1$. Quitte à conjuguer ω par une transformation linéaire de $\mathbb{C}^2 = (z = 1)$, nous pouvons nous ramener à

$$A_d(x,y) = x\widetilde{A}_{d-1}(x,y) \text{ et } B_d(x,y) = x\widetilde{B}_{d-1}(x,y)$$

pour certains $\widetilde{A}_{d-1}$, $\widetilde{B}_{d-1}$ dans $\mathbb{C}[x,y]_{d-1}$; alors $s_0 = [0:1:0] \in L$ est un point singulier de $\mathcal{F}$. Dans la carte affine $y=1$, le feuilletage $\mathcal{F}$ est donné par

$$\begin{aligned}\theta &= \sum_{i=0}^{d} z^{d-i}[A_i(x,1)(z\mathrm{d}x - x\mathrm{d}z) - B_i(x,1)\mathrm{d}z]\\ &= [A_d(x,1) + z\,A_{d-1}(x,1) + \cdots](z\mathrm{d}x - x\mathrm{d}z)\\ &\quad - [B_d(x,1) + zB_{d-1}(x,1) + \cdots]\mathrm{d}z.\end{aligned}$$

Le 1-jet de θ au point singulier $s_0 = (0,0)$ s'écrit $-[\widetilde{B}_{d-1}(0,1)x + B_{d-1}(0,1)z]\mathrm{d}z$; ce qui implique que $\mu(\mathcal{F}, s_0) > 1$: contradiction avec l'hypothèse que toute singularité de $\mathcal{F}$ située sur L est non-dégénérée.

Il s'en suit que la 1-forme $\omega_d = A_d(x,y)\mathrm{d}x + B_d(x,y)\mathrm{d}y$ définit bien un feuilletage homogène de degré d sur $\mathbb{P}^2_{\mathbb{C}}$, que nous notons $\mathcal{H}$. Il est évident que L est $\mathcal{H}$-invariante et que $\mathrm{Sing}\mathcal{F} \cap L = \mathrm{Sing}\mathcal{H} \cap L$. Considérons la famille d'homothéties $\varphi = \varphi_\varepsilon = (\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$. Nous avons

$$\varepsilon^{d+1}\varphi^*\omega = \sum_{i=0}^{d}(\varepsilon^{d-i}A_i(x,y)\mathrm{d}x + \varepsilon^{d-i}B_i(x,y)\mathrm{d}y)$$

qui tend vers ω_d lorsque ε tend vers 0 ; il en résulte que $\mathcal{H} \in \overline{\mathcal{O}(\mathcal{F})}$.

Montrons que $\mathcal{H}$ vérifie la quatrième propriété de l'énoncé. Soit $s \in \mathrm{Sing}\mathcal{H} \cap L$. Quitte à conjuguer ω par un isomorphisme linéaire de $\mathbb{C}^2 = (z=0)$, nous pouvons supposer que $s = [0:1:0]$; il existe donc un polynôme $\widehat{B}_{d-1} \in \mathbb{C}[x,y]_{d-1}$ tel que $B_d(x,y) = x\widehat{B}_{d-1}(x,y)$. Le feuilletage $\mathcal{H}$ est décrit dans la carte affine $y=1$ par

$$\theta_d = A_d(x,1)(z\mathrm{d}x - x\mathrm{d}z) - B_d(x,1)\mathrm{d}z.$$

Posons $\lambda = A_d(0,1)$ et $\nu = A_d(0,1) + \widehat{B}_{d-1}(0,1)$. Le 1-jet de θ_d en $s=(0,0)$ s'écrit $J^1_{(0,0)}\theta_d = \lambda z\mathrm{d}x - \nu x\mathrm{d}z$, et celui de θ est donné par $J^1_{(0,0)}\theta = \lambda z\mathrm{d}x - \nu x\mathrm{d}z - zB_{d-1}(0,1)\mathrm{d}z$. L'hypothèse $\mu(\mathcal{F}, s) = 1$ signifie que $\lambda\nu$ est non nul. Par suite $\mu(\mathcal{H}, s) = 1$ et $\mathrm{CS}(\mathcal{H}, L, s) = \mathrm{CS}(\mathcal{F}, L, s) = \frac{\lambda}{\nu}$.

L'implication $K(\mathrm{Leg}\mathcal{F}) \equiv 0 \Longrightarrow K(\mathrm{Leg}\mathcal{H}) \equiv 0$ découle du fait que $\mathcal{H} \in \overline{\mathcal{O}(\mathcal{F})}$. □

Nous illustrons le résultat précédent en l'appliquant au feuilletage $\mathcal{F}^d$.

Exemple 6.5. — Le feuilletage de Fermat $\mathcal{F}^d$ est donné en coordonnées homogènes par la 1-forme

$$x^d(y\mathrm{d}z - z\mathrm{d}y) + y^d(z\mathrm{d}x - x\mathrm{d}z) + z^d(x\mathrm{d}y - y\mathrm{d}x).$$

Il possède les $3d$ droites invariantes suivantes :

(a) $x=0,\ y=0,\ z=0$;
(b) $y=\zeta x,\ y=\zeta z,\ x=\zeta z$ avec $\zeta^{d-1}=1$.

Les droites de la famille (a) (resp. (b)) donnent lieu à 3 (resp. $3d-3$) feuilletages homogènes appartenant à $\overline{\mathcal{O}(\mathcal{F}^d)} \subset \mathbf{FP}(d)$ et de type $2\cdot\mathrm{R}_{d-1}$ (resp. $1\cdot\mathrm{R}_{d-1}+(d-1)\cdot\mathrm{R}_1$). Ceux qui sont de type $2\cdot\mathrm{R}_{d-1}$ sont tous conjugués à $\mathcal{H}_1^d$, d'après la proposition 4.1, et ceux qui sont de type $1\cdot\mathrm{R}_{d-1}+(d-1)\cdot\mathrm{R}_1$ sont tous conjugués au feuilletage défini par

$$(y^{d-1}-dx^{d-1})y\mathrm{d}x+(d-1)x^d\mathrm{d}y.$$

Pour $d=3$ ce dernier feuilletage est conjugué au feuilletage $\mathcal{H}_3^{d,1}$ donné par la proposition 4.2, mais ce n'est plus le cas pour $d\geq 4$.

Remarque 6.6. — Si $\mathcal{F}$ est un feuilletage de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ et si m est un point singulier de $\mathcal{F}$, nous avons l'encadrement $\sigma(\mathcal{F},m)\leq\tau(\mathcal{F},m)+1\leq d+1$, où $\sigma(\mathcal{F},m)$ désigne le nombre de droites (distinctes) invariantes par $\mathcal{F}$ et qui passent par m.

Le lemme technique suivant joue un rôle clé dans la démonstration du théorème 6.1.

Lemme 6.7. — *Soit $\mathcal{F}$ un feuilletage de degré 3 sur $\mathbb{P}^2_{\mathbb{C}}$. Si le 3-tissu $\mathrm{Leg}\mathcal{F}$ est plat et si $\mathcal{F}$ possède une singularité m non-dégénérée vérifiant $\mathrm{BB}(\mathcal{F},m)\notin\{4,\frac{16}{3}\}$, alors par le point m passent exactement deux droites invariantes par $\mathcal{F}$, i.e. $\sigma(\mathcal{F},m)=2$.*

Démonstration. — Les deux conditions $\mu(\mathcal{F},m)=1$ et $\mathrm{BB}(\mathcal{F},m)\neq 4$ assurent l'existence d'une carte affine (x,y) de $\mathbb{P}^2_{\mathbb{C}}$ dans laquelle $m=(0,0)$ et $\mathcal{F}$ est défini par une 1-forme du type $\theta_1+\theta_2+\theta_3+\theta_4$, où

$$\begin{aligned}
\theta_1 &= \lambda y\mathrm{d}x+\mu x\mathrm{d}y,\\
\theta_2 &= \left(\sum_{i=0}^{2}\alpha_i x^{2-i}y^i\right)\mathrm{d}x+\left(\sum_{i=0}^{2}\beta_i x^{2-i}y^i\right)\mathrm{d}y,\\
\theta_3 &= \left(\sum_{i=0}^{3}a_i x^{3-i}y^i\right)\mathrm{d}x+\left(\sum_{i=0}^{3}b_i x^{3-i}y^i\right)\mathrm{d}y,\\
\theta_4 &= \left(\sum_{i=0}^{3}c_i x^{3-i}y^i\right)(x\mathrm{d}y-y\mathrm{d}x),
\end{aligned}$$

avec $\lambda\mu(\lambda+\mu)\neq 0$; comme $\mathrm{BB}(\mathcal{F},m)\neq\frac{16}{3}$ nous avons $\lambda\mu(\lambda+\mu)(\lambda+3\mu)(3\lambda+\mu)\neq 0$.

Commençons par montrer que $\alpha_0=0$. Supposons par l'absurde que $\alpha_0\neq 0$. Soit (p,q) la carte affine de $\check{\mathbb{P}}^2_{\mathbb{C}}$ associée à la droite $\{px-qy=1\}\subset\mathbb{P}^2_{\mathbb{C}}$; le

3-tissu $\mathrm{Leg}\mathcal{F}$ est donné par la 3-forme symétrique

$$\begin{aligned}\check{\omega} = &\big[\big(\beta_2 p+\alpha_2 q-\lambda q^2\big)\mathrm{d}p^2+\big(\beta_1 p+\alpha_1 q+\lambda pq-\mu pq\big)\mathrm{d}p\mathrm{d}q\\ &+\big(\beta_0 p+\alpha_0 q+\mu p^2\big)\mathrm{d}q^2\big](p\mathrm{d}q-q\mathrm{d}p)\\ &+q\big(a_3\mathrm{d}p^3+a_2\mathrm{d}p^2\mathrm{d}q+a_1\mathrm{d}p\mathrm{d}q^2+a_0\mathrm{d}q^3\big)\\ &+p\big(b_3\mathrm{d}p^3+b_2\mathrm{d}p^2\mathrm{d}q+b_1\mathrm{d}p\mathrm{d}q^2+b_0\mathrm{d}q^3\big)\\ &+c_3\mathrm{d}p^3+c_2\mathrm{d}p^2\mathrm{d}q+c_1\mathrm{d}p\mathrm{d}q^2+c_0\mathrm{d}q^3.\end{aligned}$$

Considérons la famille d'automorphismes $\varphi=\varphi_\varepsilon=(\alpha_0\varepsilon^{-1}p,\alpha_0\varepsilon^{-2}q)$. Nous constatons que

$$\check{\omega}_0:=\lim_{\varepsilon\to0}\varepsilon^9\alpha_0^{-6}\varphi^*\check{\omega}=(p\mathrm{d}q-q\mathrm{d}p)\big(-\lambda q^2\mathrm{d}p^2+pq(\lambda-\mu)\mathrm{d}p\mathrm{d}q+(\mu p^2+q)\mathrm{d}q^2\big).$$

Puisque μ est non nul $\check{\omega}_0$ définit un 3-tissu $\mathcal{W}_0$, qui appartient évidemment à $\overline{\mathcal{O}(\mathrm{Leg}\mathcal{F})}$. L'image réciproque de $\mathcal{W}_0$ par l'application rationnelle $\psi(p,q)=\big(\lambda(p+q),-\lambda(\lambda+\mu)^2pq\big)$ s'écrit $\psi^*\mathcal{W}_0=\mathcal{F}_1\boxtimes\mathcal{F}_2\boxtimes\mathcal{F}_3$, où

$$\begin{aligned}&\mathcal{F}_1: q^2\mathrm{d}p+p^2\mathrm{d}q=0,\\ &\mathcal{F}_2: \mu q^2\mathrm{d}p+p(\lambda q+\mu q-\lambda p)\mathrm{d}q=0,\\ &\mathcal{F}_3: \mu p^2\mathrm{d}q+q(\lambda p+\mu p-\lambda q)\mathrm{d}p=0.\end{aligned}$$

Un calcul direct, utilisant la formule (1.1), conduit à

$$\begin{aligned}\eta(\psi^*\mathcal{W}_0)=&\frac{5(\lambda+\mu)p^2-(8\lambda+7\mu)pq+(3\lambda+4\mu)q^2}{(\lambda+\mu)p(p-q)^2}\mathrm{d}p\\ &+\frac{5(\lambda+\mu)q^2-(8\lambda+7\mu)pq+(3\lambda+4\mu)p^2}{(\lambda+\mu)q(p-q)^2}\mathrm{d}q\end{aligned}$$

de sorte que

$$K(\psi^*\mathcal{W}_0)=\mathrm{d}\eta(\psi^*\mathcal{W}_0)=-\frac{4\mu(p+q)}{(\lambda+\mu)(p-q)^3}\mathrm{d}p\wedge\mathrm{d}q\not\equiv0;$$

comme $\mathrm{Leg}\mathcal{F}$ est plat par hypothèse, il en est de même pour $\mathcal{W}_0$; par suite $K(\psi^*\mathcal{W}_0)=\psi^*K(\mathcal{W}_0)=0$, ce qui est absurde. D'où l'égalité $\alpha_0=0$.

Montrons maintenant que $a_0=0$. Raisonnons encore par l'absurde en supposant $a_0\neq0$. Le feuilletage $\mathcal{F}$ est décrit dans la carte affine (x,y) par $\theta=\theta_1+\theta_2+\theta_3+\theta_4$ avec $\alpha_0=0$. En faisant agir la transformation linéaire diagonale $(\varepsilon x,a_0\varepsilon^3y)$ sur θ puis en passant à la limite lorsque $\varepsilon\to0$ nous obtenons

$$\theta_0=\lambda y\mathrm{d}x+\mu x\mathrm{d}y+x^3\mathrm{d}x$$

qui définit un feuilletage de degré trois $\mathcal{F}_0\in\overline{O(\mathcal{F})}$. Notons $\mathrm{I}_0=\mathrm{I}_{\mathcal{F}_0}^{\mathrm{tr}}$, $\mathcal{G}_0=\mathcal{G}_{\mathcal{F}_0}$ et $\mathrm{I}_0^{\perp}=\overline{\mathcal{G}_0^{-1}(\mathcal{G}_0(\mathrm{I}_0))\setminus\mathrm{I}_0}$, où l'adhérence est prise au sens ordinaire. Un calcul élémentaire montre que

$$\mathcal{G}_0(x,y)=\left(\frac{x^3+\lambda y}{x(x^3+\lambda y+\mu y)},-\frac{\mu}{x^3+\lambda y+\mu y}\right),$$
$$\mathrm{I}_0=\{(x,y)\in\mathbb{C}^2\,:\,(\lambda-2\mu)x^3+\lambda(\lambda+\mu)y=0\}\subset\mathbb{P}^2_{\mathbb{C}}$$

et que la courbe $\mathrm{I}_0^{\perp}$ a pour équation affine $f(x,y)=y-\nu x^3=0$, où $\nu=-\frac{4\lambda+\mu}{4\lambda(\lambda+\mu)}$. Comme $\mathrm{Leg}\mathcal{F}$ est plat, $\mathrm{Leg}\mathcal{F}_0$ l'est aussi. Or, d'après [2, Corollaire 4.6], le 3-tissu $\mathrm{Leg}\mathcal{F}_0$ est plat si et seulement si $\mathrm{I}_0^{\perp}$ est invariante par $\mathcal{F}_0$, i.e. si et seulement si

$$0\equiv \mathrm{d}f\wedge\theta_0\Big|_{y=\nu x^3}=3(3\lambda+\mu)\mu x^3\mathrm{d}x\wedge\mathrm{d}y;$$

d'où $\mu(3\lambda+\mu)=0$: contradiction. Donc $a_0=\alpha_0=0$, ce qui signifie que la droite $(y=0)$ est $\mathcal{F}$-invariante.

Ce qui précède montre également que l'invariance de la droite $(y=0)$ par $\mathcal{F}$ découle uniquement du fait que $\lambda\mu(\lambda+\mu)(3\lambda+\mu)\neq 0$ et de l'hypothèse que $\mathrm{Leg}\mathcal{F}$ est plat. Ainsi en permutant les coordonnées x et y, la condition $\lambda\mu(\lambda+\mu)(\lambda+3\mu)\neq 0$ permet de déduire que $\beta_2=b_3=0$, i.e. que la droite $(x=0)$ est aussi invariante par $\mathcal{F}$.

La singularité m de $\mathcal{F}$ n'est pas radiale car $\mathrm{BB}(\mathcal{F},m)\neq 4$; de plus $\nu(\mathcal{F},m)=1$ car $\mu(\mathcal{F},m)=1$. Il s'en suit que $\tau(\mathcal{F},m)=1$; d'après la remarque 6.6, nous avons $\sigma(\mathcal{F},m)\leq\tau(\mathcal{F},m)+1=2$, d'où l'énoncé. □

Avant de commencer la démonstration du théorème 6.1, rappelons (voir [4]) que si $\mathcal{F}$ est un feuilletage de degré d sur $\mathbb{P}^2_{\mathbb{C}}$ alors

$$\text{(6.1)}\qquad \sum_{s\in\mathrm{Sing}\mathcal{F}}\mu(\mathcal{F},s)=d^2+d+1\qquad\text{et}\qquad\sum_{s\in\mathrm{Sing}\mathcal{F}}\mathrm{BB}(\mathcal{F},s)=(d+2)^2.$$

Démonstration du théorème 6.1. — Écrivons $\mathrm{Sing}\mathcal{F}=\Sigma^0\cup\Sigma^1\cup\Sigma^2$ avec

$$\Sigma^0=\{s\in\mathrm{Sing}\mathcal{F}\,:\,\mathrm{BB}(\mathcal{F},s)=\tfrac{16}{3}\},$$
$$\Sigma^1=\{s\in\mathrm{Sing}\mathcal{F}\,:\,\mathrm{BB}(\mathcal{F},s)=4\},$$
$$\Sigma^2=\mathrm{Sing}\mathcal{F}\setminus(\Sigma^0\cup\Sigma^1)$$

et notons $\kappa_i=\#\,\Sigma^i$, $i=0,1,2$. Par hypothèse, $\mathcal{F}$ est de degré 3 et toutes ses singularités ont leur nombre de Milnor 1. Les formules (6.1) impliquent alors que

$$\text{(6.2)}\quad \#\,\mathrm{Sing}\mathcal{F}=\kappa_0+\kappa_1+\kappa_2=13\quad\text{et}\quad\tfrac{16}{3}\kappa_0+4\kappa_1+\sum_{s\in\Sigma^2}\mathrm{BB}(\mathcal{F},s)=25;$$

il en résulte que Σ^2 est non vide. Soit m un point de Σ^2 ; d'après le lemme 6.7 il passe par m exactement deux droites $\ell_m^{(1)}$ et $\ell_m^{(2)}$ invariantes par $\mathcal{F}$. Alors, pour $i=1,2$, la proposition 6.4 assure l'existence d'un feuilletage homogène

$\mathcal{H}_m^{(i)}$ de degré 3 sur $\mathbb{P}^2_{\mathbb{C}}$ appartenant à $\overline{\mathcal{O}(\mathcal{F})}$ et tel que la droite $\ell_m^{(i)}$ soit $\mathcal{H}_m^{(i)}$-invariante. Comme $\mathrm{Leg}\mathcal{F}$ est plat par hypothèse, il en est de même pour $\mathrm{Leg}\mathcal{H}_m^{(1)}$ et $\mathrm{Leg}\mathcal{H}_m^{(2)}$. Donc chacun des $\mathcal{H}_m^{(i)}$ est linéairement conjugué à l'un des onze feuilletages homogènes donnés par le théorème 5.1. Pour $i = 1, 2$, la proposition 6.4 assure aussi que

($\mathfrak{a}$) $\mathrm{Sing}\mathcal{F} \cap \ell_m^{(i)} = \mathrm{Sing}\mathcal{H}_m^{(i)} \cap \ell_m^{(i)}$;

($\mathfrak{b}$) $\forall\, s \in \mathrm{Sing}\mathcal{H}_m^{(i)} \cap \ell_m^{(i)},\ \mu(\mathcal{H}_m^{(i)}, s) = 1$ et $\mathrm{CS}(\mathcal{H}_m^{(i)}, \ell_m^{(i)}, s) = \mathrm{CS}(\mathcal{F}, \ell_m^{(i)}, s)$.

Puisque $\mathrm{CS}(\mathcal{F}, \ell_m^{(1)}, m)\mathrm{CS}(\mathcal{F}, \ell_m^{(2)}, m) = 1$, nous avons

$$\mathrm{CS}(\mathcal{H}_m^{(1)}, \ell_m^{(1)}, m)\mathrm{CS}(\mathcal{H}_m^{(2)}, \ell_m^{(2)}, m) = 1.$$

Cette égalité et la Table 5.1 impliquent

$$\{\mathrm{CS}(\mathcal{H}_m^{(1)}, \ell_m^{(1)}, m),\ \mathrm{CS}(\mathcal{H}_m^{(2)}, \ell_m^{(2)}, m)\} = \{-2, -\tfrac{1}{2}\};$$

d'où $\mathrm{BB}(\mathcal{F}, m) = -\frac{1}{2}$. Le point $m \in \Sigma^2$ étant arbitraire, Σ^2 est formé des $s \in \mathrm{Sing}\mathcal{F}$ tels que $\mathrm{BB}(\mathcal{F}, s) = -\frac{1}{2}$. Par suite le système (6.2) se réécrit $\kappa_0 + \kappa_1 + \kappa_2 = 13$ et $\frac{16}{3}\kappa_0 + 4\kappa_1 - \frac{1}{2}\kappa_2 = 25$ dont l'unique solution est $(\kappa_0, \kappa_1, \kappa_2) = (0, 7, 6)$, c'est-à-dire que $\mathrm{Sing}\mathcal{F} = \Sigma^1 \cup \Sigma^2$, $\#\,\Sigma^1 = 7$ et $\#\,\Sigma^2 = 6$.

Pour fixer les idées, nous supposons que $\mathrm{CS}(\mathcal{H}_m^{(1)}, \ell_m^{(1)}, m) = -2$ pour n'importe quel choix de $m \in \Sigma_2$; donc $\mathrm{CS}(\mathcal{H}_m^{(2)}, \ell_m^{(2)}, m) = -\frac{1}{2}$. Dans ce cas, l'inspection de la Table 5.1 ainsi que les relations ($\mathfrak{a}$) et ($\mathfrak{b}$) conduisent à

$$\#\,(\Sigma^1 \cap \ell_m^{(1)}) = 3, \quad \#\,(\Sigma^1 \cap \ell_m^{(2)}) = 2, \quad \Sigma^2 \cap \ell_m^{(1)} = \{m\}, \quad \Sigma^2 \cap \ell_m^{(2)} = \{m, m'\}$$

pour un certain point $m' \in \Sigma^2 \backslash \{m\}$ vérifiant $\mathrm{CS}(\mathcal{F}, \ell_m^{(2)}, m') = -\frac{1}{2}$. Ce point m' satisfait à son tour l'égalité $\Sigma^2 \cap \ell_{m'}^{(1)} = \{m'\}$. Nous constatons que $\ell_{m'}^{(2)} = \ell_m^{(2)}$, $\ell_{m'}^{(1)} \neq \ell_m^{(1)}$, $\ell_{m'}^{(1)} \neq \ell_m^{(2)}$ et que ces trois droites distinctes satisfont $\Sigma^2 \cap (\ell_m^{(1)} \cup \ell_m^{(2)} \cup \ell_{m'}^{(1)}) = \{m, m'\}$. Comme $\#\,\Sigma^2 = 6 = 2 \cdot 3$, $\mathcal{F}$ possède $3 \cdot 3 = 9$ droites invariantes.

Posons $\Sigma^1 \cap \ell_m^{(2)} = \{m_1, m_2\}$. Notons $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_6$ les six droites $\mathcal{F}$-invariantes qui restent ; par construction chacune d'elles doit couper $\ell_m^{(1)}$ et $\ell_m^{(2)}$ en des points de Σ^1. Par ailleurs, d'après la remarque 6.6, pour tout $s \in \mathrm{Sing}\mathcal{F}$ nous avons $\sigma(\mathcal{F}, s) \leq \tau(\mathcal{F}, s) + 1 \leq 4$. Donc par chacun des points m_1 et m_2 passent exactement trois droites de la famille $\{\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_6\}$. Puisque $\#\,(\Sigma^1 \cap \ell_m^{(1)}) = 3$, $\Sigma^1 \cap \ell_m^{(1)}$ contient au moins un point, noté m_3, par lequel passent précisément trois droites de la famille $\{\ell_{m'}^{(1)}, \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_6\}$. Ainsi, pour $j = 1, 2, 3$ nous avons $\sigma(\mathcal{F}, m_j) = 4$, ce qui implique que $\tau(\mathcal{F}, m_j) = 3$. L'hypothèse sur les singularités de $\mathcal{F}$ assure que $\nu(\mathcal{F}, m_j) = 1$ pour $j = 1, 2, 3$. Il s'en suit que les singularités m_1, m_2 et m_3 sont radiales d'ordre 2 de $\mathcal{F}$.

Par construction ces trois points ne sont pas alignés. Nous concluons en appliquant la proposition 6.3. $\square$

Dans [9] les auteurs ont étudié les feuilletages de $\mathbf{F}(d)$ qui sont convexes à diviseur d'inflexion réduit ; ils ont montré que l'ensemble formé de tels feuilletages est contenu dans $\mathbf{FP}(d)$, voir [9, Théorème 2]. Ces feuilletages sont à singularités non-dégénérées comme le montre l'énoncé suivant qui est une légère généralisation de [9, Lemme 4.1].

LEMME 6.8. — *Tout feuilletage convexe sur $\mathbb{P}^2_{\mathbb{C}}$ à diviseur d'inflexion réduit est à singularités non-dégénérées.*

Démonstration. — Soit $\mathcal{F}$ un tel feuilletage et $s \in \mathrm{Sing}\mathcal{F}$ de multiplicité algébrique ν. Fixons une carte affine (x,y) telle que $s=(0,0)$; le germe $\mathcal{F}$ en s est défini par un champ de vecteurs X du type $\mathrm{X}=\mathrm{X}_\nu+\mathrm{X}_{\nu+1}+\cdots$, où les X_i sont homogènes de degré i. Le diviseur d'inflexion $\mathrm{I}_{\mathcal{F}}$ de $\mathcal{F}$ est donné par l'équation

$$0=\begin{vmatrix}\mathrm{X}(x) & \mathrm{X}(y)\\ \mathrm{X}^2(x) & \mathrm{X}^2(y)\end{vmatrix}=P_{3\nu-1}(x,y)+\cdots\,,$$

où $P_{3\nu-1}(x,y)=\mathrm{X}_\nu(x)\mathrm{X}_\nu^2(y)-\mathrm{X}_\nu(y)\mathrm{X}_\nu^2(x)$ est un polynôme homogène (éventuellement nul) de degré $3\nu-1$. Montrons d'abord que $\nu=1$. Les droites invariantes de $\mathcal{F}$ passant par l'origine sont contenues dans le cône tangent $yX_\nu(x)-xX_\nu(y)$ de X_ν qui est un polynôme homogène de degré $\nu+1$. L'hypothèse sur $\mathcal{F}$ implique alors que $\nu=1$. Il s'en suit aussi que le polynôme $P_{3\nu-1}$ n'est pas identiquement nul ; par suite la partie linéaire X_1 de X est saturée, ce qui implique que la singularité s est non-dégénérée. □

À notre connaissance les seuls feuilletages convexes à diviseur d'inflexion réduit connus dans la littérature sont ceux qui sont présentés dans [9, Table 1.1] : le feuilletage $\mathcal{F}^d$ en tout degré et les trois feuilletages donnés par les 1-formes

$$(2x^3-y^3-1)y\mathrm{d}x+(2y^3-x^3-1)x\mathrm{d}y\,,$$
$$(y^2-1)(y^2-(\sqrt{5}-2)^2)(y+\sqrt{5}x)\mathrm{d}x-(x^2-1)(x^2-(\sqrt{5}-2)^2)(x+\sqrt{5}y)\mathrm{d}y\,,$$
$$(y^3-1)(y^3+7x^3+1)y\mathrm{d}x-(x^3-1)(x^3+7y^3+1)x\mathrm{d}y\,,$$

qui sont de degré 4, 5 et 7 respectivement. Dans [9, Problème 9.1] les auteurs posent la question suivante : y a-t-il d'autres feuilletages convexes à diviseur d'inflexion réduit ? En combinant le théorème 6.1 avec le lemme 6.8 nous donnons une réponse négative en degré trois à ce problème.

COROLLAIRE 6.9. — *Tout feuilletage convexe de degré 3 sur $\mathbb{P}^2_{\mathbb{C}}$ à diviseur d'inflexion réduit est linéairement conjugué au feuilletage de Fermat $\mathcal{F}^3$.*

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DYNAMICS OF THE DOMINANT HAMILTONIAN

BY VADIM KALOSHIN & KE ZHANG

ABSTRACT. — It is well known that instabilities of nearly integrable Hamiltonian systems occur around resonances. Dynamics near resonances of these systems is well approximated by the associated averaged system, called *slow system*. Each resonance is defined by a basis (a collection of integer vectors). We introduce a class of resonances whose basis can be divided into two well separated groups and call them *dominant*. We prove that the associated slow system can be well approximated by a subsystem given by one of the groups, both in the sense of the vector field and weak KAM theory. As a corollary, we obtain perturbation results on normally hyperbolic invariant cylinders, and the Aubry/Mañe sets. This has applications in Arnold diffusion in arbitrary degrees of freedom.

RÉSUMÉ (*Dynamique de l'hamiltonien dominant*). — Il est bien connu que les instabilités des systèmes hamiltoniens presque intégrables interviennent au voisinage des résonances. La dynamique de ces systèmes près des résonances est bien approchée par les systèmes moyennés associés, appelés *systèmes lents*. Chaque résonance est définie par une base (une collection de vecteurs entiers). Nous introduisons une classe de résonances dont la base peut être divisée en deux groupes bien distincts, que nous appelons *dominantes*. Nous prouvons que le système lent associé peut être bien approché par un sous-système donné par l'un de ces deux groupes, à la fois comme champ de vecteurs et au sens de la théorie KAM faible. Comme corollaire, nous obtenons des résultats perturbatifs sur des cylindres invariants normalement hyperboliques, et sur

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les ensembles d'Aubry/Mañé. Cela a des applications en diffusion d'Arnold pour un nombre arbitraire de degrés de liberté.

1. Introduction

Consider a nearly integrable system with $n\frac{1}{2}$ degrees of freedom

$$H_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon H_1(\theta, p, t), \quad \theta \in \mathbb{T}^n, p \in \mathbb{R}^n, t \in \mathbb{T}. \tag{1.1}$$

We will restrict to the case where the integrable part H_0 is strictly convex, more precisely, we assume that there is $D > 1$ such that

$$D^{-1}\,\mathrm{Id} \leqslant \partial^2_{pp} H_0(p) \leqslant D\,\mathrm{Id}$$

as quadratic forms, where Id denotes the identity matrix.

The main motivation behind this work is the question of Arnold diffusion, that is, topological instability for the system H_ε. Arnold provided the first example in [3], and asks ([1, 2, 4]) whether topological instability is "typical" in nearly integrable systems with $n \geqslant 2$ (the system is stable when $n = 1$, due to low dimensionality).

It is well known that the instabilities of nearly integrable systems occurs along resonances. Given an integer vector $k = (\bar{k}, k^0) \in \mathbb{Z}^n \times \mathbb{Z}$ with $\bar{k} \neq 0$, we define the resonant submanifold to be $\Gamma_k = \{p \in \mathbb{R}^n : k \cdot (\omega(p), 1) = 0\}$, where $\omega(p) = \partial_p H_0(p)$. More generally, we consider a subgroup Λ of $\mathbb{Z}^{n+1}$ which does not contain vectors of the type $(0, \ldots, 0, k^0)$, called a *resonance lattice*. The *rank* of Λ is the dimension of the real subspace containing it. Then for a rank d resonance lattice Λ, we define

$$\Gamma_\Lambda = \bigcap \{\Gamma_k : k \in \Lambda\} = \bigcap_{i=1}^{d} \Gamma_{k_i},$$

where $\{k_1, \ldots, k_d\}$ is any linear independent set in Λ. We call such Γ_Λ a *d-resonance submanifold* (d-resonance for short), which is a co-dimension d submanifold of $\mathbb{R}^n$, and in particular, an n-resonant submanifold is a single point. We say that Λ is *irreducible* if it is not contained in any lattices of the same rank, or equivalently, $\mathrm{span}_{\mathbb{R}}\Lambda \cap \mathbb{Z}^{n+1} = \Lambda$.

We now consider the diffusion that occurs along a connected net Γ of $(n-1)$-resonances, which are curves in $\mathbb{R}^n$. The main difficulty in proving Arnold diffusion is in crossing the maximal (n-)resonances, which are intersections of Γ with a transveral 1-resonance manifold $\Gamma_{k'}$. A similar question is whether one can "switch" at the intersection of two resonant curves (see Figure 1.1).

For an n-resonance $\{p_0\} = \Gamma_\Lambda$, we assume that Λ is irreducible, and $\mathcal{B} = [k_1, \ldots, k_n]$ is a basis over $\mathbb{Z}$. The averaging theory of H_ϵ near p_0 reduces to

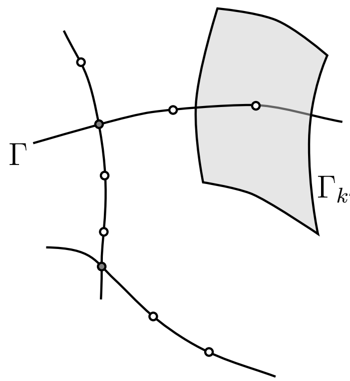

FIGURE 1.1. Diffusion path and essential resonances in $n = 3$. The hollow dots requires crossing, while the gray dots requires switching

the study of a particular *slow system* defined on $\mathbb{T}^n \times \mathbb{R}^n$, denoted $H^s_{p_0,\mathcal{B}}$. More precisely, in an $O(\sqrt{\varepsilon})$-neighborhood of p_0, the system H_ε admits the normal form (see [16], Appendix B)

$$H^s_{p_0,\mathcal{B}}(\varphi, I) + \sqrt{\varepsilon}P(\varphi, I, \tau), \quad \varphi \in \mathbb{T}^n, I \in \mathbb{T}^n, \tau \in \sqrt{\varepsilon}\mathbb{T},$$

where

$$\varphi_i = k_i \cdot (\theta, t),\ 1 \leqslant i \leqslant n, \quad (p - p_0)/\sqrt{\epsilon} = \bar{k}_1 I_1 + \cdots \bar{k}_n I_n.$$

Therefore, H_ϵ is conjugate to a fast periodic perturbation to $H^s_{p_0,\mathcal{B}}$. Note that our definition depend on the choice of basis $\mathcal{B}$. A basis free definition requires using a non-standard torus $\mathbb{T}^{n+1}/\omega(p_0)\mathbb{R}$ as the configuration space, and in this paper we choose to avoid this setting and fix a basis. Such averaged systems were studied in [24].

When $n = 2$, the slow system is a 2 degree of freedom mechanical system, the structure of its (minimal) orbits is well understood. This fact underlies the results on Arnold diffusion in two and half degrees of freedom (see [23], [24], [25], [10], [16], [14], [17], [20], [21]). This is no longer the case when $n > 2$, which is a serious obstacle to proving Arnold diffusion in higher degrees of freedom. In [15] it is proposed that we can sidestep this difficulty by using *dimension reduction*: using existence of normally hyperbolic invariant cylinders (NHICs) to restrict the system to a lower dimensional manifold. This approach only works when the slow system has a particular *dominant structure*, which is the topic of this paper.

In order to make this idea specific it is convenient to define the slow system for any p_0 and any d-resonance $d \leqslant n$. For $p_0 \in \mathbb{R}^n$, an irreducible rank d

resonance lattice Λ, and its basis $\mathcal{B} = [k_1, \dots, k_d]$, the slow system is

$$H^s_{p_0,\mathcal{B}}(\varphi, I) = K_{p_0,\mathcal{B}}(I) - U_{p_0,\mathcal{B}}(\varphi), \quad \varphi \in \mathbb{T}^d, I \in \mathbb{T}^d. \tag{1.2}$$

Suppose the Fourier expansion of H_1 is $\sum_{k\in\mathbb{Z}^{n+1}} h_k(p)e^{2\pi i k\cdot(\theta,t)}$, then

$$K_{p_0,\mathcal{B}}(I) = \frac{1}{2}\partial^2_{pp}H_0(p_0)(I_1\bar{k}_1 + \cdots + I_d\bar{k}_d)\cdot(I_1\bar{k}_1 + \cdots + I_d\bar{k}_d), \tag{1.3}$$

$$U_{p_0,\mathcal{B}}(\varphi_1, \dots, \varphi_d) = -\sum_{l\in\mathbb{Z}^d} h_{l_1k_1+\cdots l_dk_d}(p_0)\, e^{2\pi i(l_1\varphi_1+\cdots+l_d\varphi_d)}. \tag{1.4}$$

The system $H^s_{p_0,\mathcal{B}}$ is only dynamically meaningful when $p_0 \in \Gamma_\Lambda$. However, the more general set up allows us to embed the meaningful slow systems into a nice space.

In the sequel we fix a rank $m < n$ lattice, called the *strong lattice*, and its basis $\mathcal{B} = [k_1, \dots, k_m]$. We say an irreducible lattice $\Lambda \supset \Lambda^{\mathrm{st}}$ of rank d is *dominated* by Λ^{st} if

$$M(\Lambda|\Lambda^{\mathrm{st}}) := \min_{k\in\Lambda\setminus\Lambda^{\mathrm{st}}} |k| \gg \max_{k\in\mathcal{B}^{\mathrm{st}}} |k|, \tag{1.5}$$

where $|k| = \sup_i |k_i|$ is the sup-norm. Given the relation $\Lambda^{\mathrm{st}} \subset \Lambda$, we extend the basis $[k_1, \dots, k_m]$ of Λ^{st} to a basis $\mathcal{B} = [k_1, \dots, k_d]$ of Λ, such a basis is called *adapted*. Naturally, as $M(\Lambda|\Lambda^{\mathrm{st}}) \to \infty$, we have $|k_{m+1}|, \dots, |k_d| \to \infty$ for any adapted basis.

While we have fixed the basis $\mathcal{B}^{\mathrm{st}}$ of Λ^{st}, the system $H_{p_0,\mathcal{B}}$ strongly depends on the choice of the adapted basis. To get a meaningful result, we only consider particular bases that we call *κ-ordered*. Roughly speaking, given $\kappa > 1$, a basis $[k_1, \dots, k_d]$ is κ-ordered if k_i is, up to a factor of order κ, the vector of smallest norm in the set $\Lambda\setminus\mathrm{span}_{\mathbb{Z}}\{k_1, \dots, k_{i-1}\}$. The precise definition of this basis is given in Section 2.2. We will show that there exists κ depending only on $\mathcal{B}^{\mathrm{st}}$, such that any $\Lambda \supsetneq \Lambda^{\mathrm{st}}$ admits a κ-ordered basis.

After an ordered basis is chosen, we have two systems $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$ and $H^s_{p_0,\mathcal{B}}$, which we call the *strong system* and *slow system* respectively. When the lattices have a dominant structure (see (1.5)), the slow system $H^s_{p_0,\mathcal{B}}$ inherits considerable amount of information from the strong system. Indeed, let us denote

$$\begin{aligned} H^s_{p_0,\mathcal{B}^{\mathrm{st}}} &= K^{\mathrm{st}}(I_1, \dots, I_m) - U^{\mathrm{st}}(\varphi_1, \dots, \varphi_m), \\ H^s_{p_0,\mathcal{B}} &= K(I_1, \dots, I_d) - U(\varphi_1, \dots, \varphi_d), \end{aligned}$$

under (1.5) and we will show that

$$K^{\mathrm{st}}(I_1, \dots, I_m) = K(I_1, \dots, I_m, 0, \dots, 0), \quad \|U - U^{\mathrm{st}}\|_{C^2} \ll 1, \tag{1.6}$$

which indicates $H^s_{p_0,\mathcal{B}}$ can be approximated by an *extension* of $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$. The variables φ_i, I_i, $1 \leqslant i \leqslant m$ are called the *strong variables*, while φ_i, I_i, $m+1 \leqslant i \leqslant d$ are called the *weak variables*.

Recall that for each convex Hamiltonian H, we can associate a Lagrangian $L = L_H$, and the Euler-Lagrange flow is conjugate to the Hamiltonian flow. Denote by $X^{\mathrm{st}}_{\mathrm{Lag}}$ and X^{s}_{Lag} the Euler-Lagrange vector fields associated to the Hamiltonians $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$ and $H^s_{p_0,\mathcal{B}}$. Since the system for $X^{\mathrm{st}}_{\mathrm{Lag}}$ is only defined for the strong variables (φ_i, v_i), $1 \leqslant i \leqslant m$, we define a *trivial extension* of $X^{\mathrm{st}}_{\mathrm{Lag}}$ by setting $\dot\varphi_i = \dot v_i = 0$, $m+1 \leqslant i \leqslant d$.

We show that after performing a coordinate change[1] and rescaling transformation in the weak variables, the transformed vector field X^{s}_{Lag} converges to that of $X^{\mathrm{st}}_{\mathrm{Lag}}$ in some sense. In particular, if $X^{\mathrm{st}}_{\mathrm{Lag}}$ admits a normally hyperbolic invariant cylinder (NHIC), so does X^{s}_{Lag}. In a separate direction, we also obtain a limit theorem on the weak KAM solutions by variational arguments. We now formulate our main results in loose language, leaving the precise version for the next section.

Main Result. — *Assume that $r > n + 2(d-m) + 4$. Given a fixed lattice Λ^{st} of rank m with a fixed basis $\mathcal{B}^{\mathrm{st}}$, there exist $\kappa > 1$ depending only on $\mathcal{B}^{\mathrm{st}}$, and the following hold. Each rank $d, m \leqslant d \leqslant n$ irreducible lattice $\Lambda \supset \Lambda^{\mathrm{st}}$ admits a κ-ordered basis, under which we have:*

1. *(Geometrical) As $M(\Lambda|\Lambda^{\mathrm{st}}) \to \infty$, the projection of X^{s}_{Lag} to the strong variables (φ_i, v_i), $1 \leqslant i \leqslant m$ converges to $X^{\mathrm{st}}_{\mathrm{Lag}}$ uniformly. Moreover, by introducing a coordinate change and rescaling affecting only the weak variables (φ_i, v_i), $m+1 \leqslant i \leqslant d$, the transformed vector field of X^{s}_{Lag} converges to a trivial extension of the vector field of $X^{\mathrm{st}}_{\mathrm{Lag}}$. As a corollary, we obtain that if $X^{\mathrm{st}}_{\mathrm{Lag}}$ admits an NHIC, then so does X^{s}_{Lag} for sufficiently large $M(\Lambda|\Lambda^{\mathrm{st}})$.*
2. *(Variational) If, in addition, we have $r > n + 4(d-m) + 4$, then as $M(\Lambda|\Lambda^{\mathrm{st}}) \to \infty$, the weak KAM solution of $H^s_{p_0,\mathcal{B}}$ (of properly chosen cohomology classes) converges uniformly to a trivial extension of a weak KAM solution of $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$, considered as functions on $\mathbb{R}^d$. We also obtain corollaries concerning the limits of Mañe, Aubry sets, rotation vector of minimal measure, and Peierl's barrier function. The precise definitions of these objects will be given later.*

The statement that $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$ approximates $H^s_{p_0,\mathcal{B}}$ is related to the classical concept of partial averaging (see for example [5]). The statement $\min_{k\in\Lambda\setminus\Lambda^{\mathrm{st}}} |k| \gg \max_{k\in\mathcal{B}^{\mathrm{st}}} |k|$ says that the resonances in Λ^{st} is much *stronger* than the rest of the resonances in Λ. Partial averaging says that the weaker resonances contributes to smaller terms in a normal form.

However, our treatment of the partial averaging theory is quite different from the classical theory. By looking at the rescaling limit, we study the property

1. The coordinate change we perform is known in analytic mechanics as the Routhian coordinates, which is an half-Lagrangian, half-Hamiltonian setting, see (2.8).

of the averaging independent of the small parameter ε. The theory is far from a simple corollary of (1.6), with the main difficulty coming from the fact that *as* $M(\Lambda|\Lambda^{\mathrm{st}}) \to \infty$*, the quadratic part of the system* $H^s_{p_0,\mathcal{B}}$ *becomes unbounded.*

In [24], John Mather developed a theory of (partial) averaging for the nearly integrable Lagrangian system, which is dual to our setting. Quantitative estimates on the action of minimizing orbits of the original system versus the slow system are obtained. Our variational result is related to [24], but different in many ways. We work with the scaling limit system, and the small parameter ε does not show up in our analysis. We also avoid quantitative estimates (in the statement of the theorem) and obtain a limit theorem for weak KAM solutions.

The formulation of the limit theorem in weak KAM solution requires special care. A natural candidate is Tonelli convergence (convergence of Lagrangian within the Tonelli family, see [7]). In our setup, $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$ and $H^s_{p_0,\mathcal{B}}$ are defined on different spaces, we need to consider the trivial extension of $H^s_{p_0,\mathcal{B}^{\mathrm{st}}}$ to a higher dimensional space. The extended Lagrangian is then *degenerate* and obviously not Tonelli. Moreover, the standard C^2 norm of the Lagrangian becomes unbounded in the limit process. We nevertheless obtain the convergence of weak KAM solutions.

While this paper is mainly motivated by Arnold diffusion, the paper is self-contained and do not relate to the actual diffusion problem. We hope our treatment of partial averaging and its variational aspects is of independent interest.

The plan of the paper is as follows. The rigorous formulation of the results will be presented in Section 2. The choice of a basis is handled in Section 3, and the estimates of the vector fields, including the geometrical result is in Section 4. The variational aspect is more involved, and occupies Sections 5 and 6, with some technical estimates deferred to Section 7 . In Appendix A we prove Theorem 2.4 about existence of normally hyperbolic invariant cylinders stated in Section 2.4.

An earlier draft of the current paper is available on arxiv ([18]), which also includes a construction of the diffusion path such that the slow system at all strong resonances are dominant. We separated this construction from the current paper and it will appear in a future work.

2. Formulation of results

In order to state our results we establish an additional (filtrated) structure of the ambient lattice Λ relative to the strong lattice Λ^{st}:

$$\Lambda^{\mathrm{st}} = \Lambda_m \subset \Lambda_{m+1} \subset \cdots \subset \Lambda_d = \Lambda,$$

where each next lattice has rank by one exceeding the previous one, and associate to it a decomposition of the potential U into a filtrated sum (2.2).

2.1. The slow system and the choice of basis. — Recall that a slow system is

$$H^s_{p_0,\mathcal{B}}(\varphi, I) = K_{p_0,\mathcal{B}}(I) - U_{p_0,\mathcal{B}}(\varphi), \quad \varphi \in \mathbb{T}^d, I \in \mathbb{R}^d,$$

defined for a rank$-d$ irreducible lattice Λ with ordered basis $\mathcal{B} = [k_1, \dots, k_d]$, and a point $p_0 \in \mathbb{R}^n$. Let us denote

$$Z_{\mathcal{B}}(\varphi_1, \dots, \varphi_d, p) = \sum_{l \in \mathbb{Z}^d} h_{l_1 k_1 + \cdots l_d k_d}(p) e^{2\pi i (l_1\varphi_1 + \cdots + l_d \varphi_d)}, \tag{2.1}$$

where $H_1(\theta, p, t) = \sum_{k \in \mathbb{Z}^{d+1}} h_k(p) e^{2\pi i k\cdot(\theta,t)}$, then (1.4) becomes

$$U_{p_0,\mathcal{B}}(\varphi) = -Z_{\mathcal{B}}(\varphi, p_0).$$

Let $\Lambda^{\mathrm{st}}, \mathcal{B}^{\mathrm{st}}$ be the strong lattice and basis, and consider an adapted basis $\mathcal{B}$ of an irreducible lattice $\Lambda \supset \Lambda^{\mathrm{st}}$ of rank d. We define the *filtration* $\Lambda^{\mathrm{st}} = \Lambda_m \subset \Lambda_{m+1} \subset \cdots \subset \Lambda_d = \Lambda$ associated to $\mathcal{B}$ by $\Lambda_i = \mathrm{span}_{\mathbb{Z}}\{k_1, \dots, k_i\}$, $m \leqslant i \leqslant d$. Then each Λ_i is irreducible of rank i.

Given $\kappa > 1$, $\mathcal{B}$ is called κ-ordered if:

1. For $m+1 \leqslant i \leqslant n$, $|k_i| \leqslant \kappa M(\Lambda_i | \Lambda_{i-1})$.
2. For $m+1 \leqslant i < j \leqslant n$, $|k_i| \leqslant \kappa(1 + |k_j|)$.

The following proposition, proved in Section 3, ensures existence of ordered bases.

PROPOSITION 2.1. — *There is $\kappa = \kappa(\mathcal{B}^{\mathrm{st}}, d) > 0$ such that Λ admits a κ-ordered basis.*

We split the basis $\mathcal{B}$ into the strong and weak component, and introduce the following notations

$$\begin{aligned} \mathcal{B}^{\mathrm{st}} &= [k_1, \dots, k_m] = [k_1^{\mathrm{st}}, \dots, k_m^{\mathrm{st}}], \\ \mathcal{B}^{\mathrm{wk}} &= [k_{m+1}, \dots, k_d] = [k_1^{\mathrm{wk}}, \dots, k_{d-m}^{\mathrm{wk}}]. \end{aligned}$$

Denote also

$$\begin{aligned} \varphi^{\mathrm{st}} &= (\varphi_1, \dots, \varphi_m), \quad \varphi^{\mathrm{wk}} = (\varphi_{m+1}, \dots, \varphi_d), \\ I^{\mathrm{st}} &= (I_1, \dots, I_m), \quad I^{\mathrm{wk}} = (I_{m+1}, \dots, I_d), \end{aligned}$$

such naming convention will be kept for the whole paper.

Recall that a κ-ordered basis $\mathcal{B}$ comes with a filtration $\Lambda_m \subset \cdots \subset \Lambda_d$, with $\mathcal{B}_i = [k_1, \dots, k_i]$ being a basis of Λ_i. Let us define, for $m+1 \leqslant i < d$,

$$U_{p_0, \mathcal{B}_{i-1}, \mathcal{B}_i}(\varphi_1, \dots, \varphi_i) = U_{p_0,\mathcal{B}_i} - U_{p_0, \mathcal{B}_{i-1}},$$

As a result, we have

$$\begin{aligned} H^s_{p_0,\mathcal{B}} &= K_{p_0,\mathcal{B}} - U_{p_0,\mathcal{B}^{\mathrm{st}}} - U_{p_0,\mathcal{B}_m,\mathcal{B}_{m+1}} - \cdots - U_{\mathcal{B}_{d-1},\mathcal{B}_d} \\ &=: K_{p_0,\mathcal{B}} - U_{p_0,\mathcal{B}^{\mathrm{st}}} - U_{p_0,\mathcal{B}^{\mathrm{st}},\mathcal{B}^{\mathrm{wk}}}. \end{aligned} \tag{2.2}$$

Our first theorem gives estimates on $U_{p_0,\mathcal{B}_{i-1},\mathcal{B}_i}$ under a κ-ordered basis.

THEOREM 2.2. — *Let Λ^{st}, $\mathcal{B}^{\mathrm{st}}$ be the strong lattice and its basis, let $m < d \leqslant n$. Suppose H_1 is C^r with $r > n + 2(d-m) + 4$, and $\|H_1\|_{C^r} = 1$. Then there exists $\kappa = \kappa(\mathcal{B}^{\mathrm{st}}, n) > 1$ such that for each rank d irreducible lattice $\Lambda \supset \Lambda^{\mathrm{st}}$, there exists a κ-ordered basis $\mathcal{B}$ such that for $1 \leqslant i \leqslant d-m$, we have*

$$\|U^{\mathrm{wk}}_{p_0, \mathcal{B}_{i+m-1}, \mathcal{B}_{i+m}}\|_{C^2} \leqslant \kappa(1 + |k_i^{\mathrm{wk}}|)^{-r+n+2(d-m)+4}.$$

In particular, $\|U^{\mathrm{wk}}_{p_0, \mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}}\|_{C^2} \to 0$ as $M(\Lambda|\Lambda^{\mathrm{st}}) \to \infty$.

This theorem is proved in Section 3.

We will call any Hamiltonian that satisfy the conclusions of Theorem 2.2 a *dominant Hamiltonian.* In the next section, we define an abstract space of dominant Hamiltonians.

2.2. An abstract space of dominant Hamiltonians. — Define

$$\Omega^{m,d} := (\mathbb{Z}^{n+1})^d \times \mathbb{R}^n \times C^2(\mathbb{T}^m) \times C^2(\mathbb{T}^{m+1}) \times \cdots \times C^2(\mathbb{T}^d),$$
$$(\mathcal{B}^{\mathrm{st}} = [k_1^{\mathrm{st}}, \ldots, k_m^{\mathrm{st}}], \mathcal{B}^{\mathrm{wk}} = [k_1^{\mathrm{wk}}, \ldots, k_{d-m}^{\mathrm{wk}}], p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}} = (U_1^{\mathrm{wk}}, \ldots, U_{d-m}^{\mathrm{wk}}))$$

and a mapping with

$$\mathcal{H}^s : \Omega^{m,d} \to C^2(\mathbb{T}^d \times \mathbb{R}^d),$$
$$\mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) = K_{p_0,\mathcal{B}}(I) - U^{\mathrm{st}}(\varphi_1, \ldots, \varphi_m) - \sum_{i=1}^{d-m} U_i^{\mathrm{wk}}(\varphi_1, \ldots, \varphi_{i+m}),$$

where $\mathcal{B} = [\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}]$ and $K_{p_0,\mathcal{B}}$ is defined by (1.3).

We equip $\Omega^{m,d}$ with the product topology, with a discrete topology on k_i^{wk} and the standard norms on other components. The map $\mathcal{H}^s$ is smooth in p_0, $U^{\mathrm{st}}, U_1^{\mathrm{wk}}, \ldots, U_{d-m}^{\mathrm{wk}}$. Let $\Omega^{m,d}(\mathcal{B}^{\mathrm{st}})$ be the subset of $\Omega^{m,d}$ with fixed $\mathcal{B}^{\mathrm{st}}$.

We define $\Omega^{m,d}_{\kappa,q}(\mathcal{B}^{\mathrm{st}}) \subset \Omega^{m,d}(\mathcal{B}^{\mathrm{st}})$ to be the tuple $(\mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})$ satisfying the following conditions:

1. For any $1 \leqslant i < j \leqslant d-m$, $|k_i^{\mathrm{wk}}| \leqslant \kappa\ (1 + |k_j^{\mathrm{wk}}|)$.
2. For each $1 \leqslant i \leqslant d-m$, $\|U_i^{\mathrm{wk}}\|_{C^2} \leqslant \kappa\,(1 + |k_i^{\mathrm{wk}}|)^{-q}$.

Each element in $\mathcal{H}^s(\Omega^{m,d}_{\kappa,q})$ is called an (m,d)-dominant Hamiltonian with constants (κ, q). Define

$$\mu(\mathcal{B}^{\mathrm{wk}}) = \min_{1 \leqslant i \leqslant d-m} |k_i^{\mathrm{wk}}|,$$

then in $\Omega^{m,d}_{\kappa,q}(\mathcal{B}^{\mathrm{st}})$, we have $\|U_i^{\mathrm{wk}}\| \leqslant \kappa\mu(\mathcal{B}^{\mathrm{wk}})^{-q}$, i.e., the weak potential $U^{\mathrm{wk}} := \sum_{i=1}^{d-m} U_i^{\mathrm{wk}} \to 0$ as $\mu(\mathcal{B}^{\mathrm{wk}}) \to \infty$.

We restate Theorem 2.2 using the new language:

THEOREM (Theorem 2.2 restated). — *Under the assumptions of Theorem 2.2, there exists a constant $\kappa = \kappa(\mathcal{B}^{\mathrm{st}}, n) > 1$, integer vectors $\mathcal{B}^{\mathrm{wk}} = [k_1^{\mathrm{wk}}, \dots, k_{d-m}^{\mathrm{wk}}]$ with $\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}$ forming an adapted basis, such that for $q = r-n-2(d-m)-4 > 0$ we have*

$$(\mathcal{B}^{\mathrm{wk}}, p, U_{p_0,\mathcal{B}^{\mathrm{st}}}, (U_{p_0,\mathcal{B}_m,\mathcal{B}_{m+1}}, \dots, U_{p_0,\mathcal{B}_{d-1},\mathcal{B}_d})) \in \Omega_{\kappa,q}^{m,d}(\mathcal{B}^{\mathrm{st}}).$$

The strong Hamiltonian is defined by the mapping

$$\mathcal{H}^{\mathrm{st}} : (\mathbb{Z}^{n+1})^m \times \mathbb{R}^n \times C^2(\mathbb{T}^m) \to C^2(\mathbb{T}^m \times \mathbb{R}^m),$$
$$\mathcal{H}^{\mathrm{st}}(\mathcal{B}^{\mathrm{st}}, p_0, U^{\mathrm{st}}) = K_{p_0,\mathcal{B}^{\mathrm{st}}}(I^{\mathrm{st}}) - U^{\mathrm{st}}(\varphi^{\mathrm{st}}).$$

We extend the definition to $\Omega^{m,d}$ by writing

$$\mathcal{H}^{\mathrm{st}}(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) = \mathcal{H}^{\mathrm{st}}(\mathcal{B}^{\mathrm{st}}, p_0, U^{\mathrm{st}}).$$

We will prove all our limit theorems in the space $\Omega_{\kappa,q}^{m,d}(\mathcal{B}^{\mathrm{st}})$.

2.3. The rescaling limit. — We fix $\mathcal{B}^{\mathrm{st}}, \kappa > 1$ and $(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega_{\kappa,q}^{m,d}(\mathcal{B}^{\mathrm{st}})$. Denote

$$H^s = \mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}), \quad H^{\mathrm{st}} = \mathcal{H}^{\mathrm{st}}(p, U^{\mathrm{st}}).$$

We write

$$\begin{aligned} H^s(\varphi, I) &= K(I) - U^{\mathrm{st}}(\varphi^{\mathrm{st}}) - U^{\mathrm{wk}}(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}) \\ H^{\mathrm{st}}(\varphi^{\mathrm{st}}, I^{\mathrm{st}}) &= K^{\mathrm{st}}(I^{\mathrm{st}}) - U^{\mathrm{st}}(\varphi^{\mathrm{st}}) = K(I^{\mathrm{st}}, 0) - U^{\mathrm{st}}(\varphi^{\mathrm{st}}), \end{aligned} \tag{2.3}$$

where $U^{\mathrm{wk}} = \sum_{i=1}^{d-m} U_i^{\mathrm{wk}}$. We will keep using this notation throughout the paper.

As $\mu(\mathcal{B}^{\mathrm{wk}}) \to \infty$, we have $\|U^{\mathrm{wk}}\|_{C^2} \to 0$. However, $K(I^{\mathrm{st}}, I^{\mathrm{wk}})$ is not a small perturbation of $K(I^{\mathrm{st}}, 0)$, in fact, as $\mu(\mathcal{B}^{\mathrm{wk}}) \to \infty$, $K(I^{\mathrm{st}}, I^{\mathrm{wk}})$ becomes unbounded (see (2.5) below).

Let $Q_0(p) = \partial_{pp}^2 H_0(p)$ and $Q(p)$ be the $(n+1) \times (n+1)$ matrix $\begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix}$, then

$$K(I) = Q(p_0)(k_1 I_1 + \cdots + k_d I_d) \cdot (k_1 I_1 + \cdots + k_d I_d).$$

We write

$$\partial_{II}^2 K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \quad A = \partial_{I^{\mathrm{st}} I^{\mathrm{st}}}^2 K, B = \partial_{I^{\mathrm{st}} I^{\mathrm{wk}}}^2 K, C = \partial_{I^{\mathrm{wk}} I^{\mathrm{wk}}}^2 K, \tag{2.4}$$

then

$$(A)_{ij} = (k_i^{\mathrm{st}})^T Q k_j^{\mathrm{st}}, \quad (B)_{ij} = (k_i^{\mathrm{st}})^T Q k_j^{\mathrm{wk}}, \quad (C)_{ij} = (k_i^{\mathrm{wk}})^T Q k_j^{\mathrm{wk}}. \tag{2.5}$$

Note in particular that $A = \partial_{I^{\mathrm{st}} I^{\mathrm{st}}}^2 H^{\mathrm{st}}$. The Hamiltonian equation for H^s reads

$$\begin{cases} \dot{\varphi}^{\mathrm{st}} = A I^{\mathrm{st}} + B I^{\mathrm{wk}}, & \dot{I}^{\mathrm{st}} = \partial_{\varphi^{\mathrm{st}}} U, \\ \dot{\varphi}^{\mathrm{wk}} = B^T I^{\mathrm{st}} + C I^{\mathrm{wk}}, & \dot{I}^{\mathrm{wk}} = \partial_{\varphi^{\mathrm{wk}}} U, \end{cases}$$

where $U = U^{\mathrm{st}} + U^{\mathrm{wk}}$. Then the Euler-Lagrangian vector field X^s_{Lag} is

$$\begin{cases} \dot\varphi^{\mathrm{st}} = v^{\mathrm{st}}, & \dot v^{\mathrm{st}} = A\partial_{\varphi^{\mathrm{st}}}U + B\partial_{\varphi^{\mathrm{wk}}}U, \\ \dot\varphi^{\mathrm{wk}} = v^{\mathrm{wk}}, & \dot v^{\mathrm{wk}} = B^T\partial_{\varphi^{\mathrm{st}}}U + C\partial_{\varphi^{\mathrm{wk}}}U, \end{cases} \tag{2.6}$$

which will be compared to the Euler-Lagrange vector field of H^{st}

$$\dot\varphi^{\mathrm{st}} = v^{\mathrm{st}}, \quad \dot v^{\mathrm{st}} = A\partial_{\varphi^{\mathrm{st}}}U^{\mathrm{st}}, \tag{2.7}$$

denoted X^{st}. To show that the projection of (2.6) converges to (2.7), we only need to show $\|B\partial_{\varphi^{\mathrm{wk}}}U\| \to 0$.

For convergence of weak variables, we will need a rescaling. It turns out that it is better to rescale the I^{wk} variable. Introduce the coordinate change

$$\Phi : (\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}}) \mapsto (\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, v^{\mathrm{wk}}), \quad v^{\mathrm{wk}} = B^TA^{-1}v^{\mathrm{st}} - \tilde C I^{\mathrm{wk}}, \tag{2.8}$$

where $\tilde C = C - B^TA^{-1}B$ which is symmetric and invertible.

This is a "half Lagrangian" setting in the sense that $(\varphi^{\mathrm{st}}, v^{\mathrm{st}})$ is remain the Lagrangian setup, while $(\varphi^{\mathrm{wk}}, I^{\mathrm{wk}})$ is in the Hamiltonian format. This is known in analytic mechanics as the *Routhian coordinates.* Since the coordinate change is identity in $\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}$ let us compute the Jacobi matrix in the other two variables:

$$\frac{\partial(v^{\mathrm{st}}, v^{\mathrm{wk}})}{\partial(v^{\mathrm{st}}, I^{\mathrm{wk}})} = \begin{bmatrix} \mathrm{Id} & 0 \\ B^TA^{-1} & \tilde C \end{bmatrix}, \quad \frac{\partial(v^{\mathrm{st}}, I^{\mathrm{wk}})}{\partial(v^{\mathrm{st}}, v^{\mathrm{wk}})} = \begin{bmatrix} \mathrm{Id} & 0 \\ -\tilde C^{-1}B^TA^{-1} & \tilde C^{-1} \end{bmatrix}.$$

Thus, it is a diffeomorphism and the transformed vector field is

$$\begin{aligned} X^s &= (\Phi^{-1})_* X^s_{\mathrm{Lag}} = (D\Phi)^{-1} X^s_{\mathrm{Lag}} \circ \Phi \\ &= \begin{bmatrix} \mathrm{Id} & & & \\ & \mathrm{Id} & & \\ & & \mathrm{Id} & \\ & -\tilde C^{-1}B^TA^{-1} & & \tilde C^{-1} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} v^{\mathrm{st}} \\ A\partial_{\varphi^{\mathrm{st}}}U + B\partial_{\varphi^{\mathrm{wk}}}U \\ B^TA^{-1}v^{\mathrm{st}} - \tilde C I^{\mathrm{wk}} \\ B^T\partial_{\varphi^{\mathrm{st}}}U + C\partial_{\varphi^{\mathrm{wk}}}U \end{bmatrix} \\ &= \begin{bmatrix} v^{\mathrm{st}} \\ A\partial_{\varphi^{\mathrm{st}}}U + B\partial_{\varphi^{\mathrm{wk}}}U \\ B^TA^{-1}v^{\mathrm{st}} - \tilde C I^{\mathrm{wk}} \\ \partial_{\varphi^{\mathrm{wk}}}U \end{bmatrix} \end{aligned} \tag{2.9}$$

where the last line uses

$$\begin{aligned} &-\tilde C^{-1}B^TA^{-1}\left(A\partial_{\varphi^{\mathrm{st}}}U + B\partial_{\varphi^{\mathrm{wk}}}U\right) + \tilde C^{-1}\left(B^T\partial_{\varphi^{\mathrm{st}}}U + C\partial_{\varphi^{\mathrm{wk}}}U\right) \\ &= \tilde C^{-1}\left(-B^TA^{-1}B\partial_{\varphi^{\mathrm{wk}}}U + C\partial_{\varphi^{\mathrm{wk}}}U\right) = \partial_{\varphi^{\mathrm{wk}}}U \end{aligned}$$

by definition of $\tilde C$. We denote by $X^s(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}})$ the vector field of $(\Phi^{-1})_* X^s_{\mathrm{Lag}}$, lifted to the universal cover $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R}^{d-m}$.

Consider the trivial lift of the strong Lagrangian vector field X^{st}, defined on the universal cover

$$\begin{cases} \dot\varphi^{\mathrm{st}} = v^{\mathrm{st}}, & \dot v^{\mathrm{st}} = A\partial_{\varphi^{\mathrm{st}}}U, \\ \dot\varphi^{\mathrm{wk}} = 0, & \dot I^{\mathrm{wk}} = 0, \end{cases} \tag{2.10}$$

whose vector field we denote by $X_L^{\mathrm{st}}(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}})$. We show that X_L^{st} is a rescaling limit of X^s.

Given $1 \geqslant \sigma_1 \geqslant \cdots \geqslant \sigma_{d-m} > 0$, let $\Sigma = \mathrm{diag}\{\sigma_1, \ldots, \sigma_{d-m}\}$. We define a rescaling coordinate change $\Phi_\Sigma : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ by

$$\Phi_\Sigma : (\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}}) \mapsto (\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \Sigma^{-1}\varphi^{\mathrm{wk}}, \Sigma I^{\mathrm{wk}}). \tag{2.11}$$

The rescaled vector field for X^s is

$$\begin{aligned} \tilde X^s &:= (\Phi_\Sigma^{-1})_* X^s = (D\Phi_\Sigma)^{-1} X^s \circ \Phi_\Sigma, \\ \tilde X^s(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}}) &= (D\Phi_\Sigma)^{-1} X^s(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \Sigma^{-1}\varphi^{\mathrm{wk}}, \Sigma I^{\mathrm{wk}}), \end{aligned} \tag{2.12}$$

while X_L^{st} is unchanged under the rescaling.

Theorem 2.3. — *Fix $\mathcal{B}^{\mathrm{st}}$ and $\kappa > 1$. Assume that $q > 2$. Then there exists a constant $M = M(\mathcal{B}^{\mathrm{st}}, D, \kappa, q, d-m) > 1$, such that for $(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega_{\kappa,q}^{m,d}(\mathcal{B}^{\mathrm{st}})$, $H^s = \mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})$ and $H^{\mathrm{st}} = \mathcal{H}^{\mathrm{st}}(\mathcal{B}^{\mathrm{st}}, p, U^{\mathrm{st}})$, the following hold.*

For the rescaling parameter $\sigma_j = |k_j^{\mathrm{wk}}|^{-\frac{q+1}{3}}$, uniformly on $\mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R}^m \times \mathbb{R}^{d-m}$ we have

$$\begin{aligned} \|\Pi_{(\varphi^{\mathrm{st}}, v^{\mathrm{st}})}(\tilde X^s - X_L^{\mathrm{st}})\|_{C^0} &\leqslant M\,\mu(\mathcal{B}^{\mathrm{wk}})^{-(q-1)}, \\ \|D\tilde X^s - DX_L^{\mathrm{st}}\|_{C^0} &\leqslant M\,\mu(\mathcal{B}^{\mathrm{wk}})^{-\frac{q-2}{3}}. \end{aligned}$$

In particular, Theorem 2.3 implies that as $\mu(\mathcal{B}^{\mathrm{wk}}) \to \infty$, the vector field $\tilde X^s$ converges to X_L^{st} in the C^1 topology over compact sets. For applications, our version is more flexible as it is uniform over the whole space. Theorem 2.3 is proved in Section 4.1.

2.4. Persistence of normally hyperbolic invariant cylinders. — Our main application for Theorem 2.3 is to prove persistence of normally hyperbolic invariant cylinders (NHICs).

Let W be a manifold. For $R > 0$, let $B_R^l \subset \mathbb{R}^l$ denote the ball of radius R at the origin. A $2l$-cylinder Λ_R is defined by $\Lambda_R^l := \chi(\mathbb{T}^l \times B_R^l)$, where $\chi : \mathbb{T}^l \times B_R^l \to W$ is an embedding.

Let ϕ_t be a C^2 flow on W, and $\Lambda_R \subset W$ be a cylinder for some $R > 0$. We say that Λ_R^l is *normally hyperbolic (weakly) invariant cylinder* (NHWIC) if there exists $t_0 > 0$ such that the following hold.

- The vector field of ϕ_t is tangent to Λ_R^l at every $z \in \Lambda_R$.

- For each $z \in \Lambda^l_R$, there exists a splitting
$$T_zM = E^c(z) \oplus E^s(z) \oplus E^u(z), \quad \text{where } E^c(z) = T_z\Lambda,$$
weakly invariant in the sense that
$$D\phi_{t_0}(z)E^\sigma(z) = E^\sigma(\phi_{t_0}z), \quad \text{if } z,\ \phi_{t_0}z \in \Lambda \quad \text{and} \quad \sigma = c, s, u.$$
- There exist $0 < \alpha < \beta^{-1} < 1$ and a C^1 Riemannian metric g, called *an adapted metric, on a neighborhood of Λ^l_R such that whenever $z, \phi_{t_0}z \in \Lambda^l_R$,*
$$\|D\phi_{t_0}(z)|E^s\|, \quad \|(D\phi_{t_0}(z)|E^u)^{-1}\| < \alpha,$$
$$\|(D\phi_{t_0}(z)|E^c)^{-1}\|, \quad \|D\phi_{t_0}(z)|E^c\| < \beta,$$
where the norms are taken with respect to the metric g (see e.g., [13]*).*

The cylinder is called *normally hyperbolic (fully) invariant* (NHIC) if it satisfies the above conditions, and both Λ^l_R and $\partial\Lambda^l_R$ are invariant under ϕ_{t_0}. A more common definition of normally hyperbolic (fully) invariant cylinders assumes a spectral radius condition, but our definition is equivalent, see e.g., [9] Prop.5.2.2.

Moreover:

- If the parameters α, β satisfies the bunching condition $\alpha < \beta^2$, then the bundles E^s, E^u are C^1 smooth.
- When E^s, E^u are smooth, we can always choose the adapted metric g such that E^s, E^u and E^c are orthogonal.

Recall that $X^{\mathrm{st}}_{\mathrm{Lag}}$, X^s_{Lag} denotes the Lagrangian vector fields. Suppose $X^{\mathrm{st}}_{\mathrm{Lag}}$ admits a normally hyperbolic (fully) invariant cylinder Λ^{st}, we claim that X^s_{Lag} admits a normally hyperbolic weakly invariant cylinder diffeomorphic to $\Lambda^{\mathrm{st}} \times (\mathbb{T}^{d-m} \times \mathbb{R}^{d-m})$.

Theorem 2.4. — *Consider a strong lattice $\mathcal{B}^{\mathrm{st}}$, a strong potential $U^{\mathrm{st}} \in C^2(\mathbb{T}^m)$, $\kappa > 0, a > 0$, and $q > 2$. Assume that for some $1 \leqslant l \leqslant m-1$, the Euler-Lagrange vector field $X^{\mathrm{st}}_{\mathrm{Lag}}$ of $H^{\mathrm{st}} = \mathcal{H}^{\mathrm{st}}(\mathcal{B}^{\mathrm{st}}, p_0, U^{\mathrm{st}})$ admits a $2l$-dimensional NHIC $\Lambda^{\mathrm{st}}_{1+a} = \chi^{\mathrm{st}}(\mathbb{T}^l \times B^l_{1+a})$, given by the embedding $\mathbb{T}^l \times B^l_{1+a} \to \mathbb{T}^m \times \mathbb{R}^m$, with the parameters $0 < \alpha < \beta^2 < 1$.*

Then there exists an open set $V \supset \Lambda^{\mathrm{st}}_1$ such that for any $\delta > 0$, there exists $M > 0$, such that for any $(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}) \in \Omega^{\kappa,q}_{m,d}(\mathcal{B}^{\mathrm{st}})$ with $\mu(\mathcal{B}^{\mathrm{wk}}) > M$, $H^s = \mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})$, the following hold.

There exists a C^1 embedding
$$\eta^s = (\eta^{\mathrm{st}}, \eta^{\mathrm{wk}}) : (\mathbb{T}^l \times B^l) \times (\mathbb{T}^{d-m} \times \mathbb{R}^{d-m}) \to (\mathbb{T}^m \times B^m) \times (\mathbb{T}^{d-m} \times \mathbb{R}^{d-m}),$$
such that $\Lambda^s = \eta^s((\mathbb{T}^l \times B^l) \times (\mathbb{T}^{d-m} \times \mathbb{R}^{d-m}))$ is a $2(l+d-m)$-dimensional NHWIC under X^s_{Lag}. Moreover,
$$\|\eta^{\mathrm{st}}(z^{\mathrm{st}}, z^{\mathrm{wk}}) - \chi^{\mathrm{st}}(z^{\mathrm{st}})\| < \delta, \quad \forall z^{\mathrm{st}} \in \mathbb{T}^l \times B^l,\ z^{\mathrm{wk}} \in \mathbb{T}^{d-m} \times \mathbb{R}^{d-m},$$

and any X^s_{Lag}-invariant set contained in $V \times (\mathbb{T}^{d-m} \times \mathbb{R}^{d-m})$ is contained in Λ^s.

The assumption $\alpha < \beta^2$ is not necessary, and is assumed for simplicity of the proof. Nevertheless, the assumption is satisfied in our intended application and in most perturbative settings. The proof is presented in Appendix A.

2.5. The variational aspect of dominant Hamiltonians. — We will develop a similar perturbation theory for the weak KAM solutions of the dominant Hamiltonian. The weak KAM solution is closely related to some important invariant sets of the Hamiltonian system, known as the Mather, Aubry and Mañe sets.

Preliminaries in weak KAM solutions. In this section we give only enough concepts to formulate our theorem. A more detailed exposition will be given in Section 5.1. Let

$$H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$$

be a C^3 Hamiltonian satisfying the condition $D^{-1}\mathrm{Id} \leqslant \partial^2_{II} H(\varphi, I) \leqslant D\,\mathrm{Id}$. The associated Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is given by

$$L_H(\varphi, v) = \sup_{I \in \mathbb{R}^n} \{I \cdot v - H(\varphi, I)\}.$$

Let $c \in \mathbb{R}^d \simeq H^1(\mathbb{T}^d, \mathbb{R})$, we define Mather's alpha function to be

$$\alpha_H(c) = -\inf_{\mu} \left\{ \int (L_H - c \cdot v) d\mu \right\},$$

where the infimum is taken over all Borel probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ that is invariant under the Euler-Lagrange flow of L_H.

A continuous function $u : \mathbb{T}^d \to \mathbb{R}$ is called a (backward) weak KAM solution to $L_H - c \cdot v$ if for any $t > 0$, we have

$$u(x) = \inf_{y \in \mathbb{T}^d, \gamma(0)=y, \gamma(t)=x} \left(u(y) + \int_0^t (L_H(\gamma(t), \dot\gamma(t)) - c \cdot \dot\gamma(t) + \alpha_H(c)) dt \right),$$

where $\gamma : [0, t] \to \mathbb{T}^d$ is absolutely continuous. Weak KAM solutions exist and are Lipschitz (see [12], [7]).

The relation between cohomologies. We now turn to the weak KAM solutions of dominant Hamiltonians. Given

$$(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega^{m,d}(\mathcal{B}^{\mathrm{st}}),$$

we write as before $H^s = \mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})$, $H^{\mathrm{st}} = \mathcal{H}^{\mathrm{st}}(\mathcal{B}^{\mathrm{st}}, p, U^{\mathrm{st}})$ and recall the notations (2.3).

Denote $L^s = L_{H^s}$ and $L^{\mathrm{st}} = L_{H^{\mathrm{st}}}$, we have

$$L^s(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}, v^{\mathrm{st}}, v^{\mathrm{wk}}) = \check{K}(v^{\mathrm{st}}, v^{\mathrm{wk}}) + U^{\mathrm{st}}(\varphi^{\mathrm{st}}) + U^{\mathrm{wk}}(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}),$$
$$L^{\mathrm{st}}(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) = \check{K}^{\mathrm{st}}(v^{\mathrm{st}}) + U^{\mathrm{st}}(\varphi^{\mathrm{st}}),$$

where $\check{K}$, $\check{K}^{\mathrm{st}}$ are quadratic functions with $(\partial^2_{vv}\check{K}) = (\partial^2_{II}K)^{-1}$ and $(\partial^2_{v^{\mathrm{st}}v^{\mathrm{st}}}\check{K}^{\mathrm{st}}) = (\partial^2_{I^{\mathrm{st}}I^{\mathrm{st}}}K)^{-1}$ as matrices.

Given $c = (c^{\mathrm{st}}, c^{\mathrm{wk}}) \in \mathbb{R}^m \times \mathbb{R}^{d-m} = \mathbb{R}^d$, we show that the weak KAM solution of $L^s - c\cdot v$ is related to the weak KAM solution of $L^{\mathrm{st}} - \bar{c}\cdot v^{\mathrm{st}}$, where $\bar{c}$ is defined as

$$\bar{c} = c^{\mathrm{st}} + A^{-1}Bc^{\mathrm{wk}},$$

with $\partial^2_{II}K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ as in (2.5). To understand this definition, note that $\bar{c}$ uniquely satisfies

$$\partial_{I^{\mathrm{st}}}K^{\mathrm{st}}(\bar{c}) = \partial_{I^{\mathrm{st}}}K(c^{\mathrm{st}}, c^{\mathrm{wk}}).$$

If we view $c, \bar{c}$ as the momentum variable, then their corresponding velocities have the same strong component.

Semi-continuity of weak KAM solutions. We now state our main variational results. For $\nu \in \mathbb{N}$, we consider a sequence of dominant Hamiltonians with $\mu(\mathcal{B}^{\mathrm{wk}}_\nu) \to \infty$, and cohomology classes c_ν such that the corresponding $\bar{c}_\nu$ converge. Then the weak KAM solutions has a converging subsequence, and the limit point is the weak KAM solution of the strong Hamiltonian. This is sometimes referred to as upper semi-continuity.

Fix $\mathcal{B}^{\mathrm{st}}$ and $\kappa > 1$. For $\nu \in \mathbb{N}$, consider

$$(\mathcal{B}^{\mathrm{wk}}_\nu, p_\nu, U^{\mathrm{st}}_\nu, \mathcal{U}^{\mathrm{wk}}_\nu) \in \Omega^{m,d}_{\kappa,q}(\mathcal{B}^{\mathrm{st}}), \quad c_\nu = (c^{\mathrm{st}}_\nu, c^{\mathrm{wk}}_\nu) \in \mathbb{R}^m \times \mathbb{R}^{d-m},$$

write

$$H^s_\nu = \mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}_\nu, p_\nu, U^{\mathrm{st}}_\nu, \mathcal{U}^{\mathrm{wk}}_\nu), \quad L^s_\nu = L_{H^s_\nu},$$

and let u_ν be a weak KAM solution of $L^s_\nu - c_\nu \cdot v$.

Denote $\mathcal{B}_\nu = [\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}_\nu]$, $K_\nu = K_{p_\nu, \mathcal{B}_\nu}$, and

$$A_\nu = \partial^2_{I^{\mathrm{st}}I^{\mathrm{st}}}K_\nu, B_\nu = \partial^2_{I^{\mathrm{st}}I^{\mathrm{wk}}}K_\nu, C_\nu = \partial^2_{I^{\mathrm{wk}}I^{\mathrm{wk}}}K_\nu.$$

Theorem 2.5. — *Given $p_0 \in \mathbb{R}^n$, $U^{\mathrm{st}}_0 \in C^2(\mathbb{T}^m)$ and $\bar{c} \in \mathbb{R}^m$, assume as $\nu \to \infty$:*

- $\mu(B^{\mathrm{wk}}_\nu) \to \infty$, $p_\nu \to p_0$, $U^{\mathrm{st}}_\nu \to U^{\mathrm{st}}_0$.
- $c^{\mathrm{st}}_\nu + A_\nu^{-1}B_\nu c^{\mathrm{wk}}_\nu \to \bar{c}$.

Then:

1. *The sequence $\{u_\nu\}$ is equi-continuous. In particular, the sequence $\{u_\nu(\cdot) - u_\nu(0)\}$ is pre-compact in the C^0 topology.*
2. *Let u be any accumulation point of the sequence $u_\nu(\cdot) - u_\nu(0)$. Then there exists $u^{\mathrm{st}} : \mathbb{T}^m \to \mathbb{R}$ such that $u(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}) = u^{\mathrm{st}}(\varphi^{\mathrm{st}})$, i.e, u is independent of φ^{wk}.*
3. *u^{st} is a weak KAM solution of*

$$L_{\mathcal{H}^{\mathrm{st}}(\mathcal{B}^{\mathrm{st}}, p_0, U^{\mathrm{st}}_0)} - \bar{c}\cdot v^{\mathrm{st}}.$$

The proof of Theorem 2.5 occupies Sections 4 and 5, with some technical statements deferred to Section 7.

REMARK. — *Theorem 2.2 implies that under an ordered basis, we can express a slow system as a dominant system with parameters* κ, q, *where* $q = r - n - 2(d-m) - 4$. *For Theorem 2.3 we need* $q > 2$, *and for Theorem 2.5 we need* $q > 2(d-m)$. *Notice that to apply our theorems to the slow system, we need* $q > 2(d-m)$ *(or, equiv.,* $r > n + 4(d-m) + 4$*) and as stated in our main result. The additional requirement for Theorem 2.5 is due to the higher requirement of Proposition 5.3 (see also* (7.16)*).*

Using the point of view in [7], the semi-continuity of the weak KAM solution is closely related to the semi-continuity of the Aubry and Mañe sets. These properties have important applications to Arnold diffusion. In Section 6 we develop an analog of these results for the dominant Hamiltonians.

3. The choice of a basis and averaging

In this section we prove Proposition 2.1 and Theorem 2.2.

3.1. The choice of a basis. — Recall that we have a fixed irreducible lattice $\Lambda^{\mathrm{st}} \subset \mathbb{Z}^{n+1}$ of rank $m < n$, and a fixed basis $\mathcal{B}^{\mathrm{st}} = \{k_1, \dots, k_m\}$ for Λ^{st}. For $\Lambda^{\mathrm{st}} \subset \Lambda$ irreducible of rank d, we first construct a filtration $\Lambda^{\mathrm{st}} = \Lambda_m \subset \cdots \subset \Lambda_d = \Lambda$, with each Λ_i containing the vector with the smallest norm in $\Lambda \backslash \Lambda_{i-1}$, $m+1 \leqslant i \leqslant n$.

Explicitly, we define $l_i = k_i$ for $1 \leqslant i \leqslant m$, and l_i with $i > m$ inductively using the following procedure. Suppose $l_1, \dots, l_i$ are defined, let

$$\Lambda_i = \operatorname{span}_{\mathbb{R}}\{l_1, \dots, l_i\} \cap \Lambda, \quad M_{i+1} = \min\{|k| : k \in \Lambda \backslash \Lambda_i\}.$$

We define l_{i+1} to be a vector reaching the minimum in the definition of M_{i+1}, i.e $|l_{i+1}| = M_{i+1}$. Then $\Lambda_{i+1} = \operatorname{span}_{\mathbb{R}}(\Lambda_i \cup l_{i+1}) \cap \Lambda$. This provides the filtration as needed.

We have

$$|l_i| = M_i,\ m < i \leqslant d, \quad |l_j| \leqslant |l_i|,\ m < j < i \leqslant d,$$

but $l_1, \dots, l_d$ may not form a basis. We turn them into a basis using the following *explicit* procedure (see [28]).

For each $i = 1, \dots, d$, define

$$c_i = \min\{s_i : \quad s_{i,1} l_1 + \cdots + s_{i,i-1} l_{i-1} + s_i l_i \in \Lambda,\ s_i \in \mathbb{R}^+,\ s_{i,j} \in \mathbb{R}^+ \cup \{0\}\}. \tag{3.1}$$

We define $c_{i,i-1}$ using a similar minimization given the value c_i:

$$c_{i,i-1} = \min\{s_{i,i-1} : \quad s_{i,1} l_1 + \cdots + s_{i,i-1} l_{i-1} + c_i l_i,\ s_{i,j} \in \mathbb{R}^+ \cup \{0\}\}.$$

We now define $c_{i,j}$ for $1 \leqslant j \leqslant i-2$ inductively as j decreases. Assume that $c_{i,j}, \dots, c_{i,i-1}$ are all defined, then

$$\begin{aligned} c_{i,j-1} = \min\{s_{i,j-1} :&\\ &s_{i,1}l_1 + \cdots + s_{i,j-1}l_{i-1} + c_{i,j}l_i + \cdots + c_{i,i-1}l_{i-1} + c_i l_i \in \Lambda,\\ &\qquad\qquad\qquad\qquad\qquad s_{i,1}, \dots, s_{i,j-1} \in \mathbb{R}^+ \cup \{0\}\}. \end{aligned}$$

Finally,

$$k_i = c_{i,1}l_1 + \cdots + c_{i,i-1}l_{i-1} + c_i l_i.$$

We have the following lemma from the geometry of numbers.

LEMMA 3.1 (see [28]). — *Let $\Lambda \subset \mathbb{Z}^{n+1}$ be a lattice of rank $d \leqslant n$ and let $l_1, \dots, l_d$ be any linearly independent set in Λ. Let*

$$k_i = c_{i,1}l_1 + \cdots + c_{i,i-1}l_{i-1} + c_i l_i, \quad i \leqslant d.$$

be defined using the procedure above. Then

1. *For each $1 \leqslant i \leqslant d$, $k_1, \dots, k_i$ form a basis of* $\mathrm{span}_{\mathbb{R}}\{l_1, \dots, l_i\} \cap \Lambda$ *over $\mathbb{Z}$. In particular, $k_1, \dots, k_d$ form a basis of Λ.*
2. *For $1 \leqslant i < d$ and $1 \leqslant j \leqslant i-1$, we have*
$$0 \leqslant c_{i,j} < 1, \quad 0 < c_i \leqslant 1.$$
3. *If for some m such that $1 \leqslant m \leqslant d$, $l_1, \dots, l_m$ already form a basis of* $\mathrm{span}_{\mathbb{R}}\{l_1, \dots, l_m\} \cap \Lambda$ *over $\mathbb{Z}$, then $k_1 = l_1, \dots, k_m = l_m$.*

Proof. — For the proof of item 1, we refer to [28], Theorem 18. Item 2 and 3 follow from definition and item 1 as we explain below.

For item 2, note that for any $k_i = c_{i,1}l_1 + \cdots + c_{i,i-1}l_{i-1} + c_i l_i \in \Lambda$, we can always subtract an integer from any $c_{i,j}$ or c_i and remain in Λ. If the estimates do not hold, we can get a contradiction by reducing $c_{i,j}$ or c_i.

For item 3, if $l_1, \dots, l_m$ is a basis (over $\mathbb{Z}$) of $\mathrm{span}_{\mathbb{R}}\{l_1, \dots, l_m\} \cap \Lambda$, then all coefficients of $k_i = c_{i,1}l_1 + \cdots + c_{i,i-1}l_{i-1} + c_i l_i \in \Lambda$ for $i \leqslant m$ must be integers. Then the constraints of item 2 implies $c_{i,j} = 0$ and $c_i = 1$, namely $k_i = l_i$. □

Proof of Proposition 2.1. — We choose the basis $k_1, \dots, k_d$ as described. Lemma 3.1 implies $k_i = l_i$ for $1 \leqslant i \leqslant m$. Using

$$0 < c_{i+1} \leqslant 1, \quad 0 \leqslant c_{i+1,j} < 1,$$

we get

$$|k_i| \leqslant |l_1| + \cdots + |l_i| \leqslant |k_1| + \cdots + |k_m| + M_{m+1} + \cdots + M_i.$$

Since $M_{m+1} \leqslant \cdots \leqslant M_d$, and $\bar{M} = |k_1| + \cdots + |k_m|$, we get

$$|k_i| \leqslant \bar{M} + (i-m)M_i \leqslant \bar{M} + (d-m)M_i.$$

Moreover, for $i < j$, we have

$$|k_i| \leqslant \bar{M} + (d-m)M_i < \bar{M} + (d-m)M_j \leqslant \bar{M} + (d-m)|k_j|.$$

The proposition follows by taking $\chi = \bar{M} + (d-m)$. □

3.2. Estimating the weak potential. — In this section we prove Theorem 2.2. Assume that $H_1 \in C^r(\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T})$ with $r > n + 2d - 2m + 4$. Let the basis $k_1, \dots, k_d$ be chosen as in Proposition 2.1. Recall that

$$[H]_\Lambda(\theta, p, t) = \sum_{k \in \Lambda} h_k(p) e^{2\pi i k \cdot (\theta, t)},$$

then we have

$$(Z_{\mathcal{B}_i} - Z_{\mathcal{B}_{i-1}})(k_1 \cdot (\theta, t), \dots, k_i \cdot (\theta, t), p) = ([H_1]_{\Lambda_i} - [H_1]_{\Lambda_{i-1}})(\theta, p, t),$$

and the norm of $[H_1]_{\Lambda_i} - [H_1]_{\Lambda_{i-1}}$ can be estimated using standard estimates of the Fourier series.

LEMMA 3.2 (c.f. [8], Lemma 2.1, item 3). — *Let*

$$H_1(\theta, p, t) = \sum_{k \in \mathbb{Z}^{n+1}} h_k(p) e^{2\pi i k \cdot (\theta, t)}$$

satisfy $\|H_1\|_{C^r} = 1$, *with* $r \geqslant n + 4$. *There exists a constant* C_n *depending only on* n, *such that for any subset* $\tilde{\Lambda} \subset \mathbb{Z}^{n+1}$ *with* $\min_{k \in \tilde{\Lambda}} |k| = M > 0$, *we have*

$$\| \sum_{k \in \tilde{\Lambda}} h_k(p) e^{2\pi i k \cdot (\theta, t)} \|_{C^2} \leqslant C_n M^{-r+n+4}.$$

Since $\min_{k \in \Lambda_i \setminus \Lambda_{i-1}} |k| = M_i$, we apply Lemma 3.2 to $\Lambda_i \setminus \Lambda_{i-1}$ to get

$$\|[H_1]_{\Lambda_i} - [H_1]_{\Lambda_{i-1}}\|_{C^2} \leqslant C_n M_i^{-r+n+3}. \tag{3.2}$$

To estimate $Z_{\mathcal{B}_i} - Z_{\mathcal{B}_{i-1}}$, we apply a linear coordinate change. Given $k_1, \dots, k_i$, we choose $\hat{k}_{i+1}, \dots, \hat{k}_{n+1} \in \mathbb{Z}^{n+1}$ to be coordinate vectors (unit integer vectors) so that

$$P_i := \begin{bmatrix} k_1 & \cdots & k_i & \hat{k}_{i+1} & \cdots & \hat{k}_{n+1} \end{bmatrix}$$

is invertible. We extend $(Z_{\mathcal{B}_i} - Z_{\mathcal{B}_{i-1}})(\varphi_1, \dots, \varphi_i)$ trivially to a function of $(\varphi_1, \dots, \varphi_{n+1})$, then

$$(Z_{\mathcal{B}_i} - Z_{\mathcal{B}_{i-1}}) \left(P_i^T \begin{bmatrix} \theta \\ t \end{bmatrix} \right) = ([H_1]_{\Lambda_i} - [H_1]_{\Lambda_{i-1}})(\theta, t),$$

and as a result $Z_{\mathcal{B}_i} - Z_{\mathcal{B}_{i-1}} = ([H_1]_{\Lambda_i} - [H_1]_{\Lambda_{i-1}}) \circ (P_i^T)^{-1}$. Using the Faa-di Bruno formula, we have

$$\|Z_{\mathcal{B}_i} - Z_{\mathcal{B}_{i-1}}\|_{C^2} \leqslant c_n \|(P_i^T)^{-1}\|^2 \|[H_1]_{\Lambda_i} - [H_1]_{\Lambda_{i-1}}\|_{C^2}$$

for some $c_n > 1$ depending on n. We apply the following lemma in linear algebra:

LEMMA 3.3. — *Given $1 \leqslant s \leqslant n+1$, let $P = \begin{bmatrix} k_1 & \cdots & k_s \end{bmatrix}$ be an integer matrix with linearly independent columns. Then there exists $c_n > 1$ depending only on n such that*

$$\min_{\|v\|=1} \|Pv\| = \min_{\|v\|=1} (v^T P^T P v)^{\frac{1}{2}} = \|(P^T P)^{-1}\|^{-\frac{1}{2}} \geqslant c_n^{-1} |k_1|^{-1} \cdots |k_s|^{-1}.$$

In particular, if $s = n+1$, then $\|P^{-1}\| = \|(P^T)^{-1}\| \leqslant c_n |k_1| \cdots |k_{n+1}|$.

Proof. — We only estimate $\|(P^T P)^{-1}\|$. Let $a_{ij} = (P^T P)_{ij}$ and $b_{ij} = (P^T P)^{-1}_{ij}$, then using Cramer's rule and the definition of the cofactor, we have

$$|b_{ij}| \leqslant \frac{1}{\det(P^T P)} \sum_\sigma \prod_{s \neq i} a_{s\sigma(s)},$$

where σ ranges over all one-to-one mappings from $\{1, \dots, m\} \backslash \{i\}$ to $\{1, \dots, m\} \backslash \{j\}$. Since P is a nonsingular integer matrix, we have $\det(P^T P) \geqslant 1$. Moreover, $a_{ij} = k_i^T k_j \leqslant n |k_i||k_j|$. Therefore,

$$|b_{ij}| \leqslant \sum_\sigma \prod_{s \neq i} |k_s||k_{\sigma(s)}| \leqslant c_n \,(\prod_{s \neq i} |k_s|)\,(\prod_{s \neq j} |k_s|),$$

where c_n is a constant depending only on n. Using the fact that the norm of a matrix is bounded by its largest entry, up to a factor depending only on dimension, by changing to a different c_n, we have

$$\|(P^T P)^{-1}\| \leqslant c_n \sup_{i,j} |B_{ij}| \leqslant c_n \sup_{i,j} (\prod_{s \neq i} |k_s|)\,(\prod_{s \neq j} |k_s|) \leqslant c_n (\prod_{s=1}^m |k_s|)^2.$$

If $s = n+1$, then $\|P^{-1}\| = \|(P^T P)^{-1}\|^{\frac{1}{2}} = \|(P P^T)^{-1}\|^{\frac{1}{2}} = \|(P^T)^{-1}\|$. □

Proof of Theorem 2.2. — Using Lemma 3.3, there exists a constant $c_n > 0$ depending only on n such that

$$\|P_i^{-1}\| = \|(P_i^T)^{-1}\| \leqslant c_n |k_1| \cdots |k_i||\hat{k}_{i+1}| \cdots |\hat{k}_{n+1}|.$$

We have $|k_1|, \dots, |k_m| \leqslant \bar{M}$, $|\hat{k}_{i+1}| = \cdots = |\hat{k}_{n+1}| = 1$, and from Lemma 3.1, $|k_{m+1}|, \dots, |k_i| \leqslant \chi M_i$. Therefore,

$$\|P_i^{-1}\| = \|(P_i^T)^{-1}\| \leqslant c_n \bar{M}^m \chi^{i-m} M_i^{i-m}.$$

Combine with (3.2), we get for $\kappa = \kappa(n, \bar{M}, \chi)$,

$$\begin{aligned} \|Z_{\mathcal{B}_i} - Z_{\mathcal{B}_{i-1}}\|_{C^2} &\leqslant \kappa M_i^{-r+n+4+2(i-m)} \\ &\leqslant \kappa M_i^{-r+n+4+2(d-m)} \leqslant \kappa |k_i|^{-r+n+2d-2m+4}. \end{aligned}$$ □

4. Strong and slow systems of dominant Hamiltonians

In this section we study the relation between Hamiltonians and the corresponding Lagrangians for dominant systems. We start by comparing the Hamiltonian vector fields and then compare their Lagrangians.

4.1. Vector fields of dominant Hamiltonians. — In this section we expand on Section 2.3 and prove Theorem 2.3. Fix $\mathcal{B}^{\mathrm{st}}, \kappa > 1$ and let $(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega^{m,d}_{\kappa,q}(\mathcal{B}^{\mathrm{st}})$, we define H^{st}, H^s as before (see (2.3)). Recall from (2.4) that $\partial^2_{II} K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$, with

$$(A)_{ij} = (k_i^{\mathrm{st}})^T Q k_j^{\mathrm{st}}, \quad (B)_{ij} = (k_i^{\mathrm{st}})^T Q k_j^{\mathrm{wk}}, \quad (C)_{ij} = (k_i^{\mathrm{wk}})^T Q k_j^{\mathrm{wk}}.$$

The vector field $X^s(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}})$, defined on the universal cover $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R}^{d-m}$, is obtained from the Euler-Lagrange vector field via the non-degenerate coordinate change $\tilde{C} I^{\mathrm{wk}} = B^T A^{-1} v^{\mathrm{st}} - v^{\mathrm{wk}}$ (see (2.8)). The vector field $X_L^{\mathrm{st}}(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}})$ is defined as a trivial extension of the Euler-Lagrange vector field of H^{st}, also defined on the universal cover. More explicitly (see (2.9), (2.10))

$$X^s = \begin{bmatrix} v^{\mathrm{st}} \\ A\partial_{\varphi^{\mathrm{st}}} U + B \partial_{\varphi^{\mathrm{wk}}} U \\ B^T A^{-1} v^{\mathrm{st}} - \tilde{C} I^{\mathrm{wk}} \\ \partial_{\varphi^{\mathrm{wk}}} U \end{bmatrix}, \quad X_L^{\mathrm{st}} = \begin{bmatrix} v^{\mathrm{st}} \\ A \partial_{\varphi^{\mathrm{st}}} U^{\mathrm{st}} \\ 0 \\ 0 \end{bmatrix}. \tag{4.1}$$

Given $1 \geqslant \sigma_1 \geqslant \cdots \geqslant \sigma_{d-m} > 0$, let $\Sigma = \mathrm{diag}\{\sigma_1, \ldots, \sigma_{d-m}\}$. The rescaling is $\Phi_\Sigma : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, given by (2.11).

We denote by $\tilde{X}^s(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}})$ the rescaled X^s. Using (4.1), we have

$$\tilde{X}^s - X_L^{\mathrm{st}} = (\Phi_\Sigma)^{-1} X^s \circ \Phi_\Sigma - X_L^{\mathrm{st}} = \begin{bmatrix} 0 \\ (A\partial_{\varphi^{\mathrm{st}}} U^{\mathrm{wk}} + B \partial_{\varphi^{\mathrm{wk}}} U^{\mathrm{wk}})(\varphi^{\mathrm{st}}, \Sigma^{-1}\varphi^{\mathrm{wk}}) \\ \Sigma B^T A^{-1} v^{\mathrm{st}} - \Sigma \tilde{C} \Sigma I^{\mathrm{wk}} \\ \Sigma^{-1} \partial_{\varphi^{\mathrm{wk}}} U^{\mathrm{wk}}(\varphi^{\mathrm{st}}, \Sigma^{-1}\varphi^{\mathrm{wk}}) \end{bmatrix} \tag{4.2}$$

noting that U^{st} is independent of φ^{wk}, so $\partial_{\varphi^{\mathrm{wk}}} U = \partial_{\varphi^{\mathrm{wk}}} U^{\mathrm{wk}}$. Furthermore,

$$D(\tilde{X}^s - X_L^{\mathrm{st}}) = (\Phi_\Sigma)^{-1} D X^s \circ \Phi_\Sigma - X_L^{\mathrm{st}} = \tag{4.3}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ A\partial^2_{\varphi^{\mathrm{st}}\varphi^{\mathrm{st}}} U^{\mathrm{wk}} + B \partial^2_{\varphi^{\mathrm{st}}\varphi^{\mathrm{wk}}} U^{\mathrm{wk}} & 0 & A\partial^2_{\varphi^{\mathrm{st}}\varphi^{\mathrm{wk}}} U^{\mathrm{wk}} + B\partial^2_{\varphi^{\mathrm{wk}}\varphi^{\mathrm{wk}}} U^{\mathrm{wk}} \Sigma^{-1} & 0 \\ 0 & \Sigma B^T A^{-1} & 0 & -\Sigma \tilde{C} \Sigma \\ \Sigma^{-1} \partial^2_{\varphi^{\mathrm{st}}\varphi^{\mathrm{wk}}} U^{\mathrm{wk}} & 0 & \Sigma^{-1} \partial^2_{\varphi^{\mathrm{wk}}\varphi^{\mathrm{wk}}} U^{\mathrm{wk}} \Sigma^{-1} & 0 \end{bmatrix}.$$

The quantities in (4.2) and (4.3) are estimated as follows.

Lemma 4.1. — *Fix $\mathcal{B}^{\mathrm{st}}, \kappa > 1$. Assume $q > 2$. Then there exists a constant $M_1 = M_1(\mathcal{B}^{\mathrm{st}}, D, \kappa, q, d-m)$ such that for the parameters $\sigma_i = |k_i^{\mathrm{wk}}|^{-\frac{q+1}{3}}$, uniformly over $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R}^{d-m}$, the following hold.*

1. *For any $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant d-m$, $\|\partial_{\varphi_j^{\mathrm{wk}}} U^{\mathrm{wk}}\|_{C^0}, \|\partial^2_{\varphi_i^{\mathrm{st}}\varphi_j^{\mathrm{wk}}} U^{\mathrm{wk}}\|_{C^0} \leqslant M_1 |k_j^{\mathrm{wk}}|^{-q}$ for any $1 \leqslant i, j \leqslant d-m$, $\|\partial^2_{\varphi_i^{\mathrm{wk}}\varphi_j^{\mathrm{wk}}} U^{\mathrm{wk}}\|_{C^0} \leqslant M_1 \sup\{|k_i^{\mathrm{wk}}|^{-q}, |k_j^{\mathrm{wk}}|^{-q}\}$.*

2. $\|A\partial_{\varphi^{\rm wk}}U^{\rm wk}\|_{C^0} \leqslant M_1 \sup_i\{|k_i^{\rm wk}|^{-q}\}$, $\|A\partial^2_{\varphi^{\rm st}\varphi^{\rm st}}U^{\rm wk}\|_{C^0} \leqslant M_1 \sup_i\{|k_i^{\rm wk}|^{-q}\}$.
3. $\|B\partial_{\varphi^{\rm wk}}U^{\rm wk}\|_{C^0} \leqslant M_1 \sup_i\{|k_i^{\rm wk}|^{-(q-1)}\}$, $\|B\partial^2_{\varphi^{\rm st}\varphi^{\rm wk}}U\|_{C^0}$ $\leqslant M_1 \sup_i\{|k_i^{\rm wk}|^{-(q-1)}\}$.
4. $\|B\partial^2_{\varphi^{\rm st}\varphi^{\rm wk}}U\Sigma^{-1}\|_{C^0} \leqslant M_1 \sup_i\{|k_i^{\rm wk}|^{-\frac{2q-4}{3}}\}$.
5. $\|\Sigma^{-1}\partial^2_{\varphi^{\rm wk}\varphi^{\rm wk}}U\Sigma^{-1}\|_{C^0} \leqslant M_1 \sup_i\{|k_i^{\rm wk}|^{-\frac{q-2}{3}}\}$.
6. $\|\Sigma B^T A^{-1}\|_{C^0} \leqslant M_1 \sup_i\{|k_i^{\rm wk}|^{-\frac{q-2}{3}}\}$.
7. $\|\Sigma\tilde{C}\Sigma\|_{C^0} \leqslant M_1 \sup_i\{|k_i^{\rm wk}|^{-\frac{2q-4}{3}}\}$.

We first prove Theorem 2.3 using our lemma.

Proof of Theorem 2.3. — Noting that $\Pi_{(\varphi^{\rm st},v^{\rm st})}(\tilde{X}^s - X_L^{\rm st})$ is the first and second line of (4.2), using item 2 and 3 of Lemma 4.1 we get

$$\|\Pi_{(\varphi^{\rm st},v^{\rm st})}(\tilde{X}^s - X_L^{\rm st})\| \leqslant M \sup_j\{|k_j^{\rm wk}|^{-(q-1)}\} \leqslant 2M_1\mu(\mathcal{B}^{\rm wk})^{-(q-1)},$$

where M_1 is from Lemma 4.1.

Since $D(\tilde{X}^s - X_L^{\rm st})$ is bounded, up to a universal constant, the sum of the norms of all the non-zero blocks in (4.3), using Lemma 4.1 items 4-8, we get

$$\|D\tilde{X}^s - DX_L^{\rm st}\| \leqslant M \sup_j\{|k_j^{\rm wk}|^{-\frac{q-2}{3}}\} \leqslant 2M_1\mu(\mathcal{B}^{\rm wk})^{-\frac{q-2}{3}}. \qquad \square$$

The rest of the section is dedicated to proving Lemma 4.1.

Proof of Lemma 4.1. — Denote $\bar{M} = |k_1^{\rm st}| + \cdots + |k_m^{\rm st}|$, which depends only on $\mathcal{B}^{\rm st}$.

Item 1. — We have

$$\begin{aligned}\|\partial_{\varphi_i^{\rm wk}}U^{\rm wk}\|_{C^0} &\leqslant \sum_{l=1}^{d-m}\|\partial_{\varphi_i^{\rm wk}}U_l^{\rm wk}\|_{C^0} \leqslant \sum_{l\geqslant i}\|\partial_{\varphi_i^{\rm wk}}U_l^{\rm wk}\|_{C^0}\\ &\leqslant \kappa\sum_{l\geqslant i}|k_l^{\rm wk}|^{-q} \leqslant (d-m)\kappa^{q+1}|k_i^{\rm wk}|^{-q},\end{aligned}$$

where the second inequality is due to $U_l^{\rm wk}$ depending only on $(\varphi_1^{\rm wk},\dots,\varphi_l^{\rm wk})$, and the last two inequalities uses the definition of $\Omega^{m,d}_{\kappa,q}$, see Section 2.2. By the same reasoning, we have

$$\|\partial^2_{\varphi_i^{\rm st}\varphi_j^{\rm wk}}U^{\rm wk}\| \leqslant \sum_{l\geqslant j}\|U_l^{\rm wk}\|_{C^2} \leqslant (d-m)\kappa^{q+1}|k_j^{\rm wk}|^{-q},$$

$$\|\partial^2_{\varphi_i^{\rm wk}\varphi_j^{\rm wk}}U^{\rm wk}\| \leqslant \sum_{l\geqslant\sup\{i,j\}}\|U_l^{\rm wk}\|_{C^2} \leqslant (d-m)\kappa^{q+1}\sup\{|k_i^{\rm wk}|^{-q},|k_j^{\rm wk}|^{-q}\}$$

the second and third estimate follows.

Item 2. — We have

$$|(A\partial_{\varphi^{\rm st}}U^{\rm wk})_i| = |\sum_l (k_l^{\rm st})^T Q k_i^{\rm st}\partial_{\varphi_i^{\rm st}}U^{\rm wk}| \leqslant m\bar M^2\|Q\|\|\partial_{\varphi_i^{\rm st}}U\|$$
$$\leqslant m(d-m)\bar M^2\|Q\|\kappa^{q+1}|k_i^{\rm wk}|^{-q},$$

where the last line is due to item 1. Similarly,

$$|(A\partial^2_{\varphi^{\rm st}\varphi^{\rm st}}U^{\rm wk})_{ij}| \leqslant \bar M^2\|Q\|\|\partial_{\varphi_i^{\rm st}\varphi_j^{\rm st}}U\| \leqslant (d-m)\bar M^2\|Q\|\kappa^{q+1}|k_j^{\rm wk}|^{-q}.$$

Since the vector or matrix norm is bounded by the supremum of all matrix entries, up to a constant depending only on dimension, item 2 follows. In the sequel, we apply the same reasoning and only estimate the supremum of matrix/vector entries.

Item 3. — Similar to item 2,

$$|(B\partial_{\varphi^{\rm wk}}U)_i| = |\sum_l (k_l^{\rm st})^T Q k_i^{\rm wk}\partial_{\varphi_i^{\rm wk}}U^{\rm wk}| \leqslant (d-m)\bar M\|Q\||k_i^{\rm wk}|\|\partial_{\varphi_i^{\rm wk}}U^{\rm wk}\|$$
$$\leqslant (d-m)\bar M\|Q\|(d-m)\kappa^{q+1}|k_i^{\rm wk}|^{-(q-1)},$$

while

$$|(B\partial^2_{\varphi^{\rm st}\varphi^{\rm wk}}U^{\rm wk})_{ij}| = |(k_i^{\rm st})^T Q k_j^{\rm wk}\partial^2_{\varphi_i^{\rm st}\varphi_j^{\rm wk}}U^{\rm wk}| \leqslant (d-m)\kappa^{q+1}\bar M\|Q\||k_j^{\rm wk}|^{-(q-1)}.$$

Item 4. — We have

$$\begin{aligned}|(B\partial^2_{\varphi^{\rm wk}\varphi^{\rm wk}}U\Sigma^{-1})_{ij}| &= |\sum_l (k_i^{\rm st})^T Q k_l^{\rm wk}\partial^2_{\varphi_l^{\rm wk}\varphi_j^{\rm wk}}U\sigma_j^{-1}|\\ &\leqslant \bar M\|Q\|\sum_{l\geqslant j}|k_l^{\rm wk}|\sigma_j^{-1}|\partial^2_{\varphi_l^{\rm wk}\varphi_j^{\rm wk}}U|\\ &\leqslant \bar M\|Q\|(d-m)^2\kappa^{q+2}|k_j^{\rm wk}||k_j^{\rm wk}|^{-q}|k_j^{\rm wk}|^{\frac{q+1}{3}}\\ &= \bar M\|Q\|(d-m)\kappa^{q+2}|k_j^{\rm wk}|^{-\frac{2q-4}{3}},\end{aligned}$$

where the inequality of the second line uses $|k_l^{\rm wk}| \leqslant \kappa|k_j^{\rm wk}|$, item 1 and the choice of σ_j.

Item 5. — Using item 1 and choice of σ_j, we have

$$\begin{aligned}|(\Sigma^{-1}\partial^2_{\varphi^{\rm wk}\varphi^{\rm wk}}U^{\rm wk}\Sigma^{-1})_{ij}| &= |\sigma_i^{-1}\partial^2_{\varphi_i^{\rm wk}\varphi_j^{\rm wk}}U^{\rm wk}\sigma_j^{-1}|\\ &\leqslant (d-m)\kappa^{q+1}\sigma_i^{-1}\sigma_j^{-1}\sup\{|k_i^{\rm wk}|^{-q},|k_j^{\rm wk}|^{-q}\}\\ &\leqslant (d-m)\kappa^{q+1}\sup\{|k_i^{\rm wk}|^{-\frac{q-2}{3}},|k_j^{\rm wk}|^{-\frac{q-2}{3}}\}.\end{aligned}$$

Item 6. — We have

$$|(\Sigma B^T)_{ij}| = |\sigma_i(k_i^{\rm wk})^T Q k_j^{\rm st}| \leqslant \bar M\|Q\|\sup_j\{|k_j^{\rm wk}|\sigma_j\} = \bar M\|Q\|\sup_j\{|k_j^{\rm wk}|^{-\frac{q-2}{3}}\}$$

and uses $\|\Sigma B^T A^{-1}\| \leqslant \|\Sigma B^T\|\|A^{-1}\|$, noting that $\|A^{-1}\|$ depends only on Q and $\mathcal{B}^{\rm st}$.

Item 7. — Recall $\tilde{C} = C - B^T A^{-1} B$. We have

$$|(\Sigma C \Sigma)_{ij}| = |\sigma_i (k_i^{\mathrm{wk}})^T Q k_j^{\mathrm{wk}} \sigma_j| \leqslant (\sup_j \sigma_j |k_j^{\mathrm{wk}}|)^2 \|Q\| \leqslant \|Q\| \sup_j \{|k_j^{\mathrm{wk}}|^{-\frac{2q-4}{3}}\}.$$

Suppose S_1, S_2 are positive definite symmetric matrices with $S_1 \geqslant S_2$, for any $v \in \mathbb{R}^{d-m}$,

$$v^T S_1 v = v^T (S_1 - S_2 + S_2) v \geqslant v^T S_2 v,$$

we obtain $\|S_1\| \geqslant \|S_2\|$. Since $C - B^T A^{-1} B \geqslant 0$, we have $\Sigma C \Sigma - \Sigma B^T A^{-1} B \Sigma \geqslant 0$. Apply the observation to the matrices $\Sigma C \Sigma$ and $\Sigma B^T A^{-1} B \Sigma$ we get

$$\|\Sigma \tilde{C} \Sigma\| = \|\Sigma (C - B^T A^{-1} B) \Sigma\| \leqslant \|\Sigma C \Sigma\| + \|\Sigma B^T A^{-1} B \Sigma\| \leqslant 2 \|\Sigma C \Sigma\|.$$

Item 7 follows the previously obtained bound of $\|\Sigma C \Sigma\|$. □

4.2. The slow Lagrangian. — In this section, we derive the form of the slow Lagrangian in preparation for the variational part. We fix $\mathcal{B}^{\mathrm{st}}, \kappa > 1$ and $(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega_{\kappa,q}^{m,d}(\mathcal{B}^{\mathrm{st}})$. Recall the notations of H^{st}, H^s from (2.3).

We have

$$L^s(\varphi, v) = \check{K}(v) + U^{\mathrm{st}}(\varphi^{\mathrm{st}}) + U^{\mathrm{wk}}(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}), \quad L^{\mathrm{st}}(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) = \check{K}^{\mathrm{st}}(v^{\mathrm{st}}) + U^{\mathrm{st}}(\varphi^{\mathrm{st}}),$$

where $\partial^2_{vv} \check{K} = (\partial^2_{II} K)^{-1}$, $\partial^2_{v^{\mathrm{st}} v^{\mathrm{st}}} \check{K}^{\mathrm{st}} = (\partial^2_{I^{\mathrm{st}} I^{\mathrm{st}}} K)^{-1}$. Recall the notation

$$\partial^2_{II} K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \quad A = \partial^2_{I^{\mathrm{st}} I^{\mathrm{st}}} K, B = \partial^2_{I^{\mathrm{st}} I^{\mathrm{wk}}} K, C = \partial^2_{I^{\mathrm{wk}} I^{\mathrm{wk}}} K.$$

Lemma 4.2. — *With the above notations we have*

1.

$$L^s(v, \varphi) = L^{\mathrm{st}}(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) + \frac{1}{2}(v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}) \cdot \tilde{C}^{-1}(v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}) + U^{\mathrm{wk}}(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}), \tag{4.4}$$

where

$$\tilde{C} = C - B^T A^{-1} B.$$

2. *Let* $c = (c^{\mathrm{st}}, c^{\mathrm{wk}}) \in \mathbb{R}^m \times \mathbb{R}^{d-m}$. *We denote*[2]

$$\bar{c} = c^{\mathrm{st}} + A^{-1} B c^{\mathrm{wk}}, \quad w^{\mathrm{wk}} = v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}, \tag{4.5}$$

then

$$\begin{aligned} L^s(v, \varphi) - c \cdot v = L^{\mathrm{st}}(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) - \bar{c} \cdot v^{\mathrm{st}} \\ + \frac{1}{2}(w^{\mathrm{wk}} - \tilde{C} c^{\mathrm{wk}}) \cdot \tilde{C}^{-1}(w^{\mathrm{wk}} - \tilde{C} c^{\mathrm{wk}}) \\ - \frac{1}{2} c^{\mathrm{wk}} \cdot \tilde{C} c^{\mathrm{wk}} + U^{\mathrm{wk}}(\varphi^{\mathrm{wk}}, \varphi^{\mathrm{st}}). \end{aligned} \tag{4.6}$$

2. We stress here that no coordinate change is performed: w^{wk} is simply an abbreviation for $v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}$.

Proof. — We have the following identity in the block matrix inverse, which can be verified by a direct computation.

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -A^{-1}B \\ \mathrm{Id} \end{bmatrix} \tilde{C}^{-1} \begin{bmatrix} -B^T A^{-1} & \mathrm{Id} \end{bmatrix}.$$

Then

$$\begin{aligned} &\check{K}(v^{\mathrm{st}}, v^{\mathrm{wk}}) \\ &= \frac{1}{2} \begin{bmatrix} (v^{\mathrm{st}})^T (v^{\mathrm{wk}})^T \end{bmatrix} \left(\begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -A^{-1}B \\ \mathrm{Id} \end{bmatrix} \tilde{C}^{-1} \begin{bmatrix} -B^T A^{-1} & \mathrm{Id} \end{bmatrix} \right) \begin{bmatrix} v^{\mathrm{st}} \\ v^{\mathrm{wk}} \end{bmatrix} \\ &= \frac{1}{2} v^{\mathrm{st}} \cdot A^{-1} v^{\mathrm{st}} + \frac{1}{2}(v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}) \cdot \tilde{C}^{-1}(v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}) \\ &= \check{K}^{\mathrm{st}}(v^{\mathrm{st}}) + \frac{1}{2}(v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}) \cdot \tilde{C}^{-1}(v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}), \end{aligned}$$

and (4.4) follows.

Moreover,

$$\begin{aligned} &\check{K}(v^{\mathrm{st}}) - (c^{\mathrm{st}}, c^{\mathrm{wk}}) \cdot (v^{\mathrm{st}}, v^{\mathrm{wk}}) \\ &= \check{K}^{\mathrm{st}}(v^{\mathrm{st}}) - (c^{\mathrm{st}} + A^{-1}Bc^{\mathrm{wk}}) \cdot v^{\mathrm{st}} + \frac{1}{2} w^{\mathrm{wk}} \cdot \tilde{C}^{-1} w^{\mathrm{wk}} - c^{\mathrm{wk}} \cdot v^{\mathrm{wk}} + A^{-1}Bc^{\mathrm{wk}} \cdot v^{\mathrm{st}} \\ &= \check{K}^{\mathrm{st}}(v^{\mathrm{st}}) - \bar{c} \cdot v^{\mathrm{st}} + \frac{1}{2} w^{\mathrm{wk}} \cdot \tilde{C}^{-1} w^{\mathrm{wk}} - c^{\mathrm{wk}} \cdot (v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}}) \\ &= \check{K}^{\mathrm{st}}(v^{\mathrm{st}}) - \bar{c} \cdot v^{\mathrm{st}} + \frac{1}{2} w^{\mathrm{wk}} \cdot \tilde{C}^{-1} w^{\mathrm{wk}} - (\tilde{C} c^{\mathrm{wk}}) \cdot \tilde{C}^{-1} w^{\mathrm{wk}} \\ &= \check{K}^{\mathrm{st}}(v^{\mathrm{st}}) - \bar{c} \cdot v^{\mathrm{st}} + \frac{1}{2}(w^{\mathrm{wk}} - \tilde{C} c^{\mathrm{wk}}) \cdot \tilde{C}^{-1}(w^{\mathrm{wk}} - \tilde{C} c^{\mathrm{wk}}) - \frac{1}{2} c^{\mathrm{wk}} \cdot \tilde{C} c^{\mathrm{wk}}. \end{aligned}$$

We obtain (4.6). □

The Euler-Lagrange flow of L^s satisfies the following estimates expressed in notations of Section 2.3.

LEMMA 4.3. — *Fix $\mathcal{B}^{\mathrm{st}}, \kappa > 1$. Assume that $q > 1$, $L^s = L_{\mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})}$, with $(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega^{m,d}_{\kappa,q}(\mathcal{B}^{\mathrm{st}})$. Let $\gamma = (\gamma^{\mathrm{st}}, \gamma^{\mathrm{wk}}) : [0, T] \to \mathbb{T}^d$ satisfy the Euler-Lagrange equation of L^s.*

1. *There exists a constant $M_1 = M_1(\mathcal{B}^{\mathrm{st}}, Q, \kappa, q)$ such that*
$$\|\ddot{\gamma}^{\mathrm{st}} - A\partial_{\varphi^{\mathrm{st}}} U^{\mathrm{st}}(\gamma^{\mathrm{st}})\|_{C^0} \leqslant M_1(\mu(\mathcal{B}^{\mathrm{wk}}))^{-(q-1)}.$$
2. *There exists a constant $M_2 = M_2(\mathcal{B}^{\mathrm{st}}, Q, \kappa, q, \|U^{\mathrm{st}}\|)$ such that*
$$\|\ddot{\gamma}^{\mathrm{st}}\|_{C^0} \leqslant M_2.$$

Proof. — We note that for L^s, we have

$$\ddot{\gamma}^{\mathrm{st}} = A\partial_{\varphi^{\mathrm{st}}} U + B\partial_{\varphi^{\mathrm{wk}}} U = A\partial_{\varphi^{\mathrm{st}}} U^{\mathrm{st}} + \left(A\partial_{\varphi^{\mathrm{st}}} U^{\mathrm{wk}} + B\partial_{\varphi^{\mathrm{wk}}} U^{\mathrm{wk}}\right)$$

using that U^{st} is independent of φ^{wk}. We now use item 2, 3 of Lemma 4.1, to get item 1.

Since $\|A\partial_{\varphi^{\mathrm{st}}}U^{\mathrm{st}}\| \leqslant \|A\|\|U^{\mathrm{st}}\|$, and $\|A\|$ depends only on $\mathcal{B}^{\mathrm{st}}$ and Q, item 2 follows directly from item 1. □

5. Weak KAM solutions of dominant Hamiltonians and convergence

In this section, we provide some basic information about the weak KAM solution of the dominant system.

In Section 5.1, we give an overview on the relevant weak KAM theory. Recall that in Section 4.2, we derive the relation between the slow Lagrangian and the strong Lagrangian. In Section 5.2, we obtain a compactness result for the strong component of a minimizing curve. In Sections 5.3–5.5, we prove Theorem 2.5 with some technical statements deferred to Section 7.

5.1. Weak KAM solutions of Tonelli Lagrangian.

— For an extensive exposition of the topic, we refer to [12].

Tonelli Lagrangian. — The Lagrangian function $L = L(\varphi, v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is called Tonelli if it satisfies the following conditions:

1. (smoothness) L is C^r with $r \geqslant 2$;
2. (fiber convexity) $\partial^2_{vv}L$ is strictly positive definite;
3. (superlinearity) $\lim_{\|v\|\to\infty} |L(x,v)|/\|v\| = \infty$.

The Lagrangians considered in this paper are Tonelli.

Minimizers. — An absolutely continuous curve $\gamma : [a,b] \to \mathbb{T}^d$ is called *minimizing* for the Tonelli Lagrangian L if

$$\int_a^b L(\gamma,\dot\gamma)dt = \min_{\xi} \int_a^b L(\xi,\dot\xi)dt,$$

where the minimization is over all absolutely continuous curves $\xi : [a,b] \to \mathbb{T}^d$ with $b > a$, such that $\xi(a) = \gamma(a)$, $\xi(b) = \gamma(b)$. The functional

$$\mathbb{A}(\gamma) = \int_a^b L(\gamma,\dot\gamma)dt$$

is called *the action functional.* The curve γ is called an *extremal* if it is a critical point of the action functional. A minimizer is extremal, and it satisfies the Euler-Lagrange equation

$$\frac{d}{dt}(\partial_v L(\gamma,\dot\gamma)) = \partial_\varphi L(\gamma,\dot\gamma).$$

Tonelli Theorem and a priori compactness. — By the Tonelli Theorem (cf. [12], Corollary 3.3.1), for any $[a,b] \subset \mathbb{R}$ with $b > a$, $\varphi, \psi \in \mathbb{T}^d$, there always exists a C^r minimizer. Moreover, there exists $D > 0$ depending only on a lower bound of $b-a$ such that $\|\dot\gamma\| \leqslant D$ ([12] Corollary 4.3.2). This property is called the a priori compactness.

The alpha function and minimal measures. — A measure μ on $\mathbb{T}^d \times \mathbb{R}^d$ is called a closed measure (see [29], Remark 4.40) if for all $f \in C^1(\mathbb{T}^d)$,

$$\int df(\varphi) \cdot v \, d\mu(\varphi, v) = 0.$$

This notion is equivalent to the more well known notion of holonomic measure defined by Mañe ([19]).

For $c \in H^1(\mathbb{T}^d, \mathbb{R}) \simeq \mathbb{R}^d$, the alpha function

$$\alpha_L(c) = -\inf_{\mu} \int (L(\varphi, v) - c \cdot v) d\mu(\varphi, v),$$

where the minimization is over all closed Borel probability measures. When $L = L_H$ we also use the notation $\alpha_H(c)$. A measure μ is called a *c-minimizing* if it reaches the infimum above. A minimizing measure always exists, and is invariant under the Euler-Lagrange flow (c.f [19, 6]). Hence this definition of the alpha function is equivalent to the one given in Section 2.5, where the minimization is over invariant probability measures.

Rotation vector and the beta function. — The rotation vector ρ of a closed measure μ is defined by the relation

$$\int (c \cdot v) d\mu(\varphi, v) = c \cdot \rho, \qquad \text{for all } c \in H^1(\mathbb{T}^d, \mathbb{R}).$$

For $h \in H_1(\mathbb{T}^d, \mathbb{R}) \simeq \mathbb{R}^d$, the beta function is

$$\beta_L(h) = \inf_{\rho(\chi) = h} \int L(\varphi, v) d\chi(\varphi, v).$$

When $L = L_H$ we use the notation $\beta_H(h)$. The alpha function and beta function are Legendre duals:

$$\beta_L(h) = \sup_{c \in \mathbb{R}^d} \{c \cdot h - \alpha_L(c)\}.$$

The Legendre-Fenchel transform. — Define the Legendre-Fenchel transform associated to the beta function

$$\mathcal{LF}_\beta : H_1(\mathbb{T}^d, \mathbb{R}) \to \begin{array}{l}\text{the collection of nonempty,} \\ \text{compact convex subsets of } H^1(\mathbb{T}^d, \mathbb{R}),\end{array} \tag{5.1}$$

defined by

$$\mathcal{LF}_\beta(h) = \{c \in H^1(\mathbb{T}^n, \mathbb{R}) : \beta_L(h) + \alpha_L(c) = c \cdot h\}.$$

Domination and calibration. — For $\alpha \in \mathbb{R}$, a function $u : \mathbb{T}^d \to \mathbb{R}$ is dominated by $L + \alpha$ if for all $[a, b] \subset \mathbb{R}$ and piecewise C^1 curves $\gamma : [0, T] \to \mathbb{T}^d$, we have

$$u(\gamma(b)) - u(\gamma(a)) \leqslant \int_a^b L(\gamma, \dot\gamma)dt + \alpha(b - a).$$

A piecewise C^1 curve $\gamma : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is called (u, L, α)-calibrated if for any $[a, b] \subset I$,

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma, \dot\gamma)dt + \alpha(b - a).$$

Weak KAM solutions. — A function $u : \mathbb{T}^d \to \mathbb{R}$ is called a weak KAM solution of L if there exists $\alpha \in \mathbb{R}$ such that the following hold:

1. u is dominated by $L + \alpha$;
2. for all $\varphi \in \mathbb{T}^d$, there exists a (u, L, α)-calibrated curve $\gamma : (-\infty, 0] \to \mathbb{T}^d$ with $\gamma(0) = \varphi$.

This definition of the weak KAM solution is equivalent to the one given in Section 2.5 (see [12], Proposition 4.4.8), and the constant $\alpha = \alpha_L(0)$, where α_L is the alpha function.

Peierls' barrier. — For $T > 0$, we define the function $h_L^T : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$ by

$$h_L^T(\varphi, \psi) = \min_{\gamma(0)=\varphi, \gamma(T)=\psi} \int_0^T (L(\gamma, \dot\gamma) + \alpha_L)dt.$$

Peierls' barrier is $h_L(\varphi, \psi) = \lim_{T\to\infty} h_L^T(\varphi, \psi)$. The limit exists, and the function h_L is Lipschitz in both variables. Denote $h_{L,c} = h_{L-c\cdot v}$.

Mather, Aubry and Mañe sets. — These sets are defined by Mather (see [22]). Here we only introduce the projected version. Define the projected Aubry and Mañe sets as

$$\mathcal{A}_L(c) = \{x \in \mathbb{T}^d : \quad h_{L,c}(x, x) = 0\},$$
$$\mathcal{N}_L(c) = \left\{y \in \mathbb{T}^d : \quad \min_{x,z\in\mathcal{A}_L(c)} (h_{L,c}(x, y) + h_{L,c}(y, z) - h_{L,c}(x, z)) = 0\right\}.$$

The Mather set is $\tilde{\mathcal{M}}_L(c) = \overline{\bigcup_\mu \operatorname{supp}(\mu)}$ is the closure of the support of all c-minimal measures. Its projection $\pi\tilde{\mathcal{M}}(c) = \mathcal{M}(c)$ onto $\mathbb{T}^d$ is called the projected Mather set. Then

$$\mathcal{M}_L(c) \subset \mathcal{A}_L(c) \subset \mathcal{N}_L(c).$$

When $L = L_H$ we also use the subscript H to identify these sets.

Static classes. — For any $\varphi, \psi \in \mathcal{A}_L(c)$, Mather defined the following equivalence relation:

$$\varphi \sim \psi \text{ if } h_{L,c}(\varphi, \psi) + h_{L,c}(\psi, \varphi) = 0.$$

The equivalence classes, defined by this equivalence relation, are called *the static classes.* The static classes are linked to the family of weak KAM solutions, in particular, if there is only one static class, then the weak KAM solution is unique up to a constant.

In this section, we provide a few useful estimates in weak KAM theory, and prove Theorem 2.5. In Section 5.2, we prove a projected version of the a priori compactness property. We then introduce an approximate version of Lipschitz property and use it to prove Theorem 2.5.

5.2. Minimizers of strong and slow Lagrangians, their a priori compactness. — We prove a version of the a priori compactness theorem for the strong component.

PROPOSITION 5.1. — *Fix $\mathcal{B}^{\mathrm{st}}, \kappa > 1$. For any $R > 0, q > 2$, there exists $M = M(\mathcal{B}^{\mathrm{st}}, D, R, \kappa, n)$ such that the following hold. For any*

$$(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega^{m,d}_{\kappa,q}(\mathcal{B}^{\mathrm{st}}) \cap \{\|U^{\mathrm{st}}\|_{C^2} \leqslant R\},$$

the Lagrangian $L^s = L_{\mathcal{H}^s(\mathcal{B}^{\mathrm{st}},\mathcal{B}^{\mathrm{wk}},p,U^{\mathrm{st}},\mathcal{U}^{\mathrm{wk}})}$, let $T \geqslant \frac{1}{2}$, $c = (c^{\mathrm{st}}, c^{\mathrm{wk}}) \in \mathbb{R}^m \times \mathbb{R}^{d-m}$ and $\gamma = (\gamma^{\mathrm{st}}, \gamma^{\mathrm{wk}}) : [0,T] \to \mathbb{T}^d$ be a minimizer of $L^s - c \cdot v$. Then for $\bar{c} = c^{\mathrm{st}} + A^{-1}Bc^{\mathrm{wk}}$, we have

$$\|\dot{\gamma}^{\mathrm{st}} - A\bar{c}\| \leqslant M.$$

We first state a lemma on the strong component of the action and relate minimizers of the slow system with those of the strong one.

LEMMA 5.2. — *In the notations of Proposition 5.1 for $T \geqslant \frac{1}{2}$ and $c \in \mathbb{R}^d$, let $\gamma = (\gamma^{\mathrm{st}}, \gamma^{\mathrm{wk}}) : [0,T] \to \mathbb{T}^d$ be a minimizer for the Lagrangian $L^s - c \cdot v$. Then*

$$\int_0^T (L^{\mathrm{st}} - \bar{c} \cdot v^{\mathrm{st}})(\gamma^{\mathrm{st}}, \dot{\gamma}^{\mathrm{st}})dt \leqslant \min_{\zeta} \int_0^T (L^{\mathrm{st}} - \bar{c} \cdot v^{\mathrm{st}})(\zeta, \dot{\zeta})dt + 2T\|U^{\mathrm{wk}}\|_{C^0},$$

where the minimization is over all absolutely continuous $\zeta : [0,T] \to \mathbb{T}^m$ with $\zeta(0) = \gamma^{\mathrm{st}}(0)$, $\zeta(T) = \gamma^{\mathrm{st}}(T)$.

Proof. — Let $\gamma_0^{\mathrm{st}} : [0,T] \to \mathbb{T}^m$ be such that

$$\int_0^T (L^{\mathrm{st}} - \bar{c} \cdot v^{\mathrm{st}})(\gamma_0^{\mathrm{st}}, \dot{\gamma}_0^{\mathrm{st}})dt = \min_{\zeta} \int_0^T (L^{\mathrm{st}} - \bar{c} \cdot v^{\mathrm{st}})(\zeta, \dot{\zeta})dt$$

with $\zeta(0) = \gamma^{\mathrm{st}}(0)$, $\zeta(T) = \gamma^{\mathrm{st}}(T)$. Define $\gamma_0 = (\gamma_0^{\mathrm{st}}, \gamma_0^{\mathrm{wk}}) : [0,T] \to \mathbb{T}^d$, by

$$\gamma_0^{\mathrm{wk}}(t) = \gamma^{\mathrm{wk}}(t) - A^{-1}B\gamma^{\mathrm{st}}(t) + A^{-1}B\gamma_0^{\mathrm{st}}(t).$$

Note that

(5.2)
$$\gamma_0^{\mathrm{wk}}(0)=\gamma^{\mathrm{wk}}(0),\quad \gamma_0^{\mathrm{wk}}(T)=\gamma^{\mathrm{wk}}(T),\quad \dot\gamma_0^{\mathrm{wk}}-A^{-1}B\dot\gamma_0^{\mathrm{st}}=\dot\gamma^{\mathrm{wk}}-A^{-1}B\dot\gamma^{\mathrm{st}}.$$

Using (4.6) and (5.4), we have

$$\begin{aligned}(5.3)\quad L^s-c\cdot v+\frac{1}{2}c^{\mathrm{wk}}\cdot\tilde C^{-1}c^{\mathrm{wk}}&=L^{\mathrm{st}}-\bar c\cdot v^{\mathrm{st}}\\ &+\frac{1}{2}(v^{\mathrm{wk}}-B^TA^{-1}v^{\mathrm{st}}-\tilde C^{\mathrm{wk}})\cdot\tilde C(v^{\mathrm{wk}}-B^TA^{-1}v^{\mathrm{st}}-\tilde C^{\mathrm{wk}})+U^{\mathrm{wk}}\end{aligned}$$

Since γ is a minimizer for $L^s-c\cdot v$,

$$\int_0^T(L^s-c\cdot v)(\gamma,\dot\gamma)dt\leqslant\int_0^T(L^s-c\cdot v)(\gamma_0,\dot\gamma_0)dt.$$

By (5.3), we have

$$\begin{aligned}&\int_0^T(L^{\mathrm{st}}-\bar c\cdot v^{\mathrm{st}})(\gamma^{\mathrm{st}},\dot\gamma^{\mathrm{st}})dt+\int_0^T U^{\mathrm{wk}}(\gamma(t))dt\\ &\qquad+\int_0^T\frac{1}{2}(\dot\gamma^{\mathrm{wk}}-B^TA^{-1}\dot\gamma^{\mathrm{st}}-\tilde C^{\mathrm{wk}})\cdot\tilde C(\dot\gamma^{\mathrm{wk}}-B^TA^{-1}\dot\gamma^{\mathrm{st}}-\tilde C^{\mathrm{wk}})dt\\ &\quad\leqslant\int_0^T(L^{\mathrm{st}}-\bar c\cdot v^{\mathrm{st}})(\gamma_0^{\mathrm{st}},\dot\gamma_0^{\mathrm{st}})dt+\int_0^T U^{\mathrm{wk}}(\gamma_0(t))dt\\ &\qquad+\int_0^T\frac{1}{2}(\dot\gamma_0^{\mathrm{wk}}-B^TA^{-1}\dot\gamma_0^{\mathrm{st}}-\tilde C^{\mathrm{wk}})\cdot\tilde C(\dot\gamma_0^{\mathrm{wk}}-B^TA^{-1}\dot\gamma_0^{\mathrm{st}}-\tilde C^{\mathrm{wk}})\,dt.\end{aligned}$$

By (5.2), the second and fourth line of the above inequality cancel, therefore,

$$\int_0^T(L^{\mathrm{st}}-\bar c\cdot v^{\mathrm{st}})(\gamma^{\mathrm{st}},\dot\gamma^{\mathrm{st}})dt\leqslant\int_0^T(L^{\mathrm{st}}-\bar c\cdot v^{\mathrm{st}})(\gamma_0^{\mathrm{st}},\dot\gamma_0^{\mathrm{st}})dt+2T\|U^{\mathrm{wk}}\|_{C^0}.\quad\square$$

Proof of Proposition 5.1. — First, observe that any segments of a minimizer is still a minimizer. By dividing the interval $[0,T]$ into subintervals, it suffice to prove our proposition for $T\in[\frac{1}{2},1)$.

We first produce an upper bound for

$$\min_\zeta\int_0^T(L^{\mathrm{st}}-\bar c\cdot v^{\mathrm{st}}+\frac{1}{2}\bar c\cdot A\bar c)(\zeta,\dot\zeta)dt.$$

By completing the squares as in Lemma 4.2, we have

$$(5.4)\qquad L^{\mathrm{st}}-\bar c\cdot v^{\mathrm{st}}+\frac{1}{2}\bar c\cdot A\bar c=\frac{1}{2}(v^{\mathrm{st}}-A\bar c)\cdot A^{-1}(v^{\mathrm{st}}-A\bar c)+U^{\mathrm{st}}(\varphi^{\mathrm{st}}).$$

We then take

$$\zeta_0(t)=\gamma^{\mathrm{st}}(0)+tA\bar c+\frac{t}{T}y$$

where $y \in [0,1)^d$ is such that $\zeta_0(0) + TA\bar{c} + y = \gamma^{\mathrm{st}}(T) \mod \mathbb{Z}^m$. We then have $\dot{\zeta}_0 - A\bar{c} = \frac{1}{T}y$, so

$$\int_0^T (L^{\mathrm{st}} - \bar{c}\cdot v^{\mathrm{st}} + \frac{1}{2}\bar{c}\cdot A\bar{c})(\zeta_0, \dot{\zeta}_0)dt \leqslant \frac{1}{2T}\|A^{-1}\|\|y\|^2 + T\|U^{\mathrm{st}}\|_{C^0} \leqslant d\|A^{-1}\| + \|U^{\mathrm{st}}\|_{C^0}$$

using $T \in [0,1)$ and $\|y\|^2 \leqslant d$.

Using Lemma 5.2, and adding $\frac{1}{2}\bar{c}\cdot A\bar{c}$ to the Lagrangian to both sides, we obtain

$$\begin{aligned}\int_0^T (L^{\mathrm{st}} - \bar{c}\cdot v^{\mathrm{st}} + \frac{1}{2}\bar{c}\cdot A\bar{c})(\gamma^{\mathrm{st}}, \dot{\gamma}^{\mathrm{st}})dt &\leqslant 2T\|U^{\mathrm{wk}}\| + d\|A^{-1}\| + \|U^{\mathrm{st}}\|_{C^0} \\ &\leqslant d\|A^{-1}\| + \|U^{\mathrm{st}}\|_{C^0} + 2\|U^{\mathrm{wk}}\|_{C^0}\end{aligned}$$

since $T \in [\frac{1}{2}, 1)$.

We now use the above formula to get an L^2 estimate on $(\dot{\gamma}^{\mathrm{st}} - A\bar{c})$ and use the Poincaré estimate to conclude. Using the above formula and (5.4), we have

$$\int_0^T (\dot{\gamma}^{\mathrm{st}} - A\bar{c})\cdot A^{-1}(\dot{\gamma}^{\mathrm{st}} - A\bar{c})dt \leqslant d\|A^{-1}\| + 2\|U^{\mathrm{st}}\|_{C^0} + 2\|U^{\mathrm{wk}}\|_{C^0}.$$

Using the fact that A^{-1} is strictly positive definite, we get

$$\|\dot{\gamma}^{\mathrm{st}} - A\bar{c}\|_{L^2} \leqslant \|A\|\left(d\|A^{-1}\| + 2\|U^{\mathrm{st}}\|_{C^0} + 2\|U^{\mathrm{wk}}\|_{C^0}\right) =: M_1.$$

Then

$$\left\|\frac{1}{T}\int_0^T (\dot{\gamma}^{\mathrm{st}} - A\bar{c})\,dt\right\|^2 \leqslant \frac{1}{T^2}\int_0^T \|\dot{\gamma}^{\mathrm{st}} - A\bar{c}\|^2\,dt \leqslant 4M_1. \tag{5.5}$$

Moreover, from Lemma 4.3,

$$\|\ddot{\gamma}^{\mathrm{st}}\| \leqslant M_2(\mathcal{B}^{\mathrm{st}}, Q, \kappa, q, R).$$

The Poincaré estimate gives, for some uniform constant $Q > 0$,

$$\left\|(\dot{\gamma}^{\mathrm{st}} - A\bar{c}) - \frac{1}{T}\int_0^T (\dot{\gamma}^{\mathrm{st}} - A\bar{c})dt\right\|_{L^\infty} \leqslant \|\ddot{\gamma}^{\mathrm{st}}\|_{L^\infty} \leqslant QM_2.$$

Combine with (5.5) and we conclude the proof. □

5.3. Approximate Lipschitz property of weak KAM solutions. — The weak KAM solutions of the slow Hamiltonian is Lipschitz, however, it is not clear if the Lipschitz constant is bounded as $\mu(\mathcal{B}^{\mathrm{wk}}) \to \infty$. To get uniform estimates, we consider the following weaker notion.

Definition. — *For $C, \delta > 0$, a function $u : \mathbb{R}^d \to \mathbb{R}$ is called (C, δ) approximately Lipschitz if*

$$|u(x) - u(y)| \leqslant C\|x - y\| + \delta, \quad x, y \in \mathbb{R}^d.$$

For $u : \mathbb{T}^d \to \mathbb{R}$, the approximate Lipschitz property is defined by its lift to $\mathbb{R}^d$.

In Proposition 5.3 and 5.4 we state the approximate Lipschitz property of a weak KAM solution in weak and strong angles.

Proposition 5.3. — *Fix $\mathcal{B}^{\mathrm{st}}, \kappa > 1$. Assume that $q > 2(d-m)$. For $R > 0$, there exists a constant $M = M(\mathcal{B}^{\mathrm{st}}, D, \kappa, q, R) > 0$, such that for all*

$$(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega_{\kappa,q}^{m,d}(\mathcal{B}^{\mathrm{st}}) \cap \{\|U^{\mathrm{st}}\| \leqslant R\},$$

and

$$\delta(\mathcal{B}^{\mathrm{wk}}) = M\mu(\mathcal{B}^{\mathrm{wk}})^{-(\frac{q}{2}-d+m)},$$

let $u = u(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}) : \mathbb{T}^m \times \mathbb{T}^{d-m} \to \mathbb{R}$ be a weak KAM solution of

$$L_{\mathcal{H}^s}(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) - c \cdot v.$$

Then for all $\varphi^{\mathrm{st}} \in \mathbb{T}^m$, the function $u(\varphi^{\mathrm{st}}, \cdot)$ is (δ, δ) approximately Lipschitz.

Proposition 5.4. — *There exists a constant $M' = M'(\mathcal{B}^{\mathrm{st}}, D, \kappa, q, R) > 0$, let $\delta'(\mathcal{B}^{\mathrm{wk}}) = M'(\mu(\mathcal{B}^{\mathrm{wk}}))^{-(\frac{q}{2}-d+m)}$, and u be the weak KAM solution described in Proposition 5.3. Then for all $\varphi^{\mathrm{wk}} \in \mathbb{T}^{d-m}$, the function $u(\cdot, \varphi^{\mathrm{wk}})$ is (M', δ') approximately Lipschitz.*

The proof of these statements are deferred to Section 7.

5.4. The alpha function and rotation vector estimate. — In this section we provide a few useful estimates in weak KAM theory and prove Theorem 2.5 using Propositions 5.3 and 5.4. Recall that the notations $c = (c^{\mathrm{st}}, c^{\mathrm{wk}}), \bar{c} = c^{\mathrm{st}} + A^{-1}Bc^{\mathrm{wk}}$.

Proposition 5.5. — *With these notations we have the following estimate:*

$$\left|\alpha_{H^s}(c) - \alpha_{H^{\mathrm{st}}}(\bar{c}) + \frac{1}{2}(\tilde{C}c^{\mathrm{wk}}) \cdot c^{\mathrm{wk}}\right| \leqslant \|U^{\mathrm{wk}}\|_{C^0}.$$

Proof. — Let μ be a minimal measure for $L^s - c \cdot v$. Let π denote the natural projection from $(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}, v^{\mathrm{st}}, v^{\mathrm{wk}})$ to $(\varphi^{\mathrm{st}}, v^{\mathrm{st}})$. By Lemma 4.2 we have

$$\begin{aligned} -\alpha_{H^s}(c) &= \int (L^s - c \cdot v)d\mu \\ &= \int (L^{\mathrm{st}} - \bar{c} \cdot v^{\mathrm{st}})d\mu \circ \pi - \frac{1}{2}c^{\mathrm{wk}} \cdot \tilde{C}c^{\mathrm{wk}} \\ &\quad + \int \left(\frac{1}{2}(w^{\mathrm{wk}} - \tilde{C}c^{\mathrm{wk}}) \cdot \tilde{C}^{-1}(w^{\mathrm{wk}} - \tilde{C}c^{\mathrm{wk}}) + U^{\mathrm{wk}}\right) d\mu \\ &\geqslant -\alpha_{H^{\mathrm{st}}}(\bar{c}) - \|U^{\mathrm{wk}}\|_{C^0} - \frac{1}{2}c^{\mathrm{wk}} \cdot \tilde{C}c^{\mathrm{wk}}. \end{aligned} \tag{5.6}$$

On the other hand, let μ^{st} be an ergodic minimal measure for $L^{\mathrm{st}} - \bar{c} \cdot v^{\mathrm{st}}$. For an L^{st}-Euler-Lagrange orbit $\varphi^{\mathrm{st}}(t)$ in the support of μ^{st}, and any $\varphi_0^{\mathrm{wk}} \in \mathbb{T}^{d-m}$, define

$$\varphi^{\mathrm{wk}}(t) = \varphi_0^{\mathrm{wk}} + B^T A^{-1}\varphi^{\mathrm{st}}(t) + \tilde{C}c^{\mathrm{wk}}t, \quad t \in \mathbb{R} \tag{5.7}$$

and write $\gamma = (\gamma^{\mathrm{st}}, \gamma^{\mathrm{wk}})$. We take a weak$-*$ limit point μ^s of the probability measures $\frac{1}{T}(\gamma, \dot\gamma)|_{[0,T]}$ as $T \to +\infty$. Then μ^s is a closed measure (see Section 5.1).

Since on the support of μ^s, $v^{\mathrm{wk}} - B^T A^{-1} v^{\mathrm{st}} - \tilde{C} c^{\mathrm{wk}} = 0$, we have

$$\begin{aligned}
-\alpha_{H^s}(c) &\leqslant \int (L^s - c\cdot v) d\mu^s \\
&= \int (L^{\mathrm{st}} - \bar c \cdot v^{\mathrm{st}}) d\mu^{\mathrm{st}} + \int U^{\mathrm{wk}} d\mu - \frac{1}{2}\tilde C c^{\mathrm{wk}}\cdot c^{\mathrm{wk}} \\
&\leqslant -\alpha_{H^{\mathrm{st}}}(\bar c) + \|U\|_{C^0} - \frac{1}{2}\tilde C c^{\mathrm{wk}} \cdot c^{\mathrm{wk}}. \qquad \square
\end{aligned}$$

The following proposition establishes relations between rotation vectors of minimal measures of the slow and strong systems.

Proposition 5.6. — *Let μ^s be an ergodic minimal measure of $L^s - c\cdot v$, and let $(\rho^{\mathrm{st}}, \rho^{\mathrm{wk}})$ denote its rotation vector. Then*

$$0 \leqslant \frac{1}{2}(\tilde C(\rho^{\mathrm{wk}} - B^T A^{-1}\rho^{\mathrm{st}} - \tilde C c^{\mathrm{wk}}))\cdot(\rho^{\mathrm{wk}} - B^T A^{-1}\rho^{\mathrm{st}} - \tilde C c^{\mathrm{wk}}) \leqslant \|U^{\mathrm{wk}}\|_{C^0}$$

and

$$0 \leqslant \alpha_{H^{\mathrm{st}}}(\bar c) + \beta_{H^{\mathrm{st}}}(\rho^{\mathrm{st}}) - \bar c\cdot \rho^{\mathrm{st}} \leqslant \|U^{\mathrm{wk}}\|_{C^0}.$$

Proof. — Using (5.6) and the conclusion of Proposition 5.5, we have

$$\begin{aligned}
\|U^{\mathrm{wk}}\|_{C^0} &\geqslant \int (L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}} + \alpha_{H^{\mathrm{st}}}(\bar c)) d\mu^s \circ \pi \\
&\quad + \int \frac{1}{2}(\tilde C^{-1}(w - \tilde C c^{\mathrm{wk}}))\cdot (w - \tilde C c^{\mathrm{wk}}) d\mu^s\circ\pi.
\end{aligned} \tag{5.8}$$

Note the first of the two integrals is non-negative by definition, we obtain

$$0 \leqslant \int \frac{1}{2}(w^{\mathrm{wk}} - \tilde C c^{\mathrm{wk}})\cdot \tilde C^{-1}(w^{\mathrm{wk}} - \tilde C c^{\mathrm{wk}}) d\mu^s \leqslant \|U^{\mathrm{wk}}\|_{C^0}.$$

Denote $\bar w^{\mathrm{wk}} := \int w^{\mathrm{wk}} d\mu^s = \rho^{\mathrm{wk}} - B^T A^{-1}\rho^{\mathrm{st}}$, and rewrite the left hand side of the last formula as

$$\begin{aligned}
\frac{1}{2}(\tilde C^{-1}(\bar w^{\mathrm{wk}} - \tilde C c^{\mathrm{wk}}))\cdot(\bar w^{\mathrm{wk}} - \tilde C c^{\mathrm{wk}}) &+ \int \tilde C^{-1}(\bar w^{\mathrm{wk}} - \tilde C c^{\mathrm{wk}})\cdot (w^{\mathrm{wk}} - \bar w) d\mu^s \\
&+ \frac{1}{2}\int (\tilde C^{-1}(w^{\mathrm{wk}} - \bar w^{\mathrm{wk}}))\cdot (w^{\mathrm{wk}} - \bar w^{\mathrm{wk}}) d\mu^s.
\end{aligned}$$

Note that the second term vanishes and the third term is non-negative. Therefore

$$\frac{1}{2}(\tilde C^{-1}(\bar w^{\mathrm{wk}} - \tilde C c^{\mathrm{wk}}))\cdot(\bar w^{\mathrm{wk}} - \tilde C c^{\mathrm{wk}}) \leqslant \|U^{\mathrm{wk}}\|_{C^0}$$

which is the first conclusion.

For the second conclusion, using (5.8), we get

$$\|U^{\mathrm{wk}}\|_{C^0} \geqslant \int (L^{\mathrm{st}} - \bar{c}\cdot v^{\mathrm{st}} + \alpha_{H^{\mathrm{st}}}(\bar{c}))d\mu\circ\pi = \int L^{\mathrm{st}} d\mu\circ\pi - \bar{c}\cdot\rho^{\mathrm{st}} + \alpha_{H^{\mathrm{st}}}(\bar{c}).$$

Using $\int L^{\mathrm{st}} d\mu\circ\pi \geqslant \beta_{H^{\mathrm{st}}}(\rho^{\mathrm{st}})$ we get the upper bound of the second conclusion. The lower bound holds by definition. □

5.5. Convergence of weak KAM solutions. — We now prove Theorem 2.5. Fix $\mathcal{B}^{\mathrm{st}}$ and $\kappa > 1$.

For $\nu \in \mathbb{N}$, let $(\mathcal{B}^{\mathrm{wk}}_\nu, p_\nu, U^{\mathrm{st}}_\nu, \mathcal{U}^{\mathrm{wk}}_\nu) \in \Omega^{m,d}_{\kappa,q}(\mathcal{B}^{\mathrm{st}})$ and $c_\nu = (c^{\mathrm{st}}_\nu, c^{\mathrm{wk}}_\nu)$ be a sequence satisfying the assumption of the theorem, namely, $\mu(\mathcal{B}^{\mathrm{wk}}_\nu) \to \infty$, $p_\nu \to p_0$, $U^{\mathrm{st}}_\nu \to U^{\mathrm{st}}_0$ in C^2, and $c^{\mathrm{st}}_\nu + A_\nu^{-1}B_\nu c^{\mathrm{wk}}_\nu \to \bar{c}$. Let us fix the notations

$$(5.9)\qquad \begin{aligned} H^s_\nu &= \mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}_\nu, p_\nu, U^{\mathrm{st}}_\nu, \mathcal{U}^{\mathrm{wk}}_\nu), \quad L^s_\nu = L_{H^s_\nu}, \quad \alpha_\nu = \alpha_{H^s_\nu}(c_\nu),\\ H^{\mathrm{st}}_\nu &= \mathcal{H}^{\mathrm{st}}(\mathcal{B}^{\mathrm{st}}, p_\nu, U^{\mathrm{st}}_\nu), \quad L^{\mathrm{st}}_\nu = L_{H^{\mathrm{st}}_\nu}.\end{aligned}$$

Item 1. — Let u_ν be the weak KAM solution to $L^s_\nu - c_\nu\cdot v$. We first show the sequence $\{u_\nu\}$ is equi-continuous.

Let M^* be a constant larger than the constants in both Propositions 5.3 and 5.4. Using both propositions, for any $\varphi = (\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}})$, $\psi = (\psi^{\mathrm{st}}, \psi^{\mathrm{wk}})$,

$$|u_\nu(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}) - u_\nu(\psi^{\mathrm{st}}, \psi^{\mathrm{wk}})| \leqslant M^*\|\varphi^{\mathrm{st}} - \psi^{\mathrm{st}}\| + \delta_\nu\|\varphi^{\mathrm{wk}} - \psi^{\mathrm{wk}}\| + 2\delta_\nu,$$

where $\delta_\nu = M^*(\mu(\mathcal{B}^{\mathrm{wk}}_\nu))^{-\frac{q}{2}-d+m}$.

Since $\delta_\nu \to 0$ as $\nu\to\infty$, for any $0<\varepsilon<1$ there exists $M>0$ such that for all $\nu > M$, $3\delta_\nu < \frac{\varepsilon}{2}$. It follows that if $\|\varphi-\psi\| < \frac{\varepsilon}{2M^*} < 1$, then

$$|u_\nu(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}) - u_\nu(\psi^{\mathrm{st}}, \psi^{\mathrm{wk}})| < \varepsilon.$$

Since $\{u_\nu\}_{\nu\leqslant M}$ is a finite family, it is equi-continuous. In particular, there exist $\sigma > 0$ such that

$$|u_\nu(\varphi) - u(\psi)| < \varepsilon, \qquad \text{if } 1\leqslant\nu\leqslant M, \|\varphi-\psi\| < \sigma.$$

This proves equi-continuity. Moreover, since u_ν are all periodic, $u_\nu - u_\nu(0)$ are equi-bounded, therefore, Ascoli's theorem applies and the sequence is pre-compact in the uniform norm.

Item 2. — Let u be any accumulation point of $u_\nu - u_\nu(0)$, without loss of generality, we assume $u_\nu - u_\nu(0)$ converges to u uniformly. Proposition 5.3 implies that

$$\lim_{\nu\to\infty}\sup_{\varphi^{\mathrm{st}}}(\max_\nu u_\nu(\varphi^{\mathrm{st}},\cdot) - \min_\nu u_\nu(\varphi^{\mathrm{st}},\cdot)) \leqslant 2\lim_{\nu\to\infty}\delta_\nu = 0,$$

therefore, u is independent of φ^{wk}.

Item 3. — From item 2, there exists $u^{\mathrm{st}}(\varphi^{\mathrm{st}}) = \lim_{\nu\to\infty} u_\nu(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}})$. We show u^{st} is a weak KAM solution of $L_0^{\mathrm{st}} - \bar{c}\cdot v^{\mathrm{st}}$. Recall the notations in (5.9), we have $L_\nu^{\mathrm{st}} \to L_0^{\mathrm{st}}$ in C^2.

We first show that u^{st} is dominated by $L_0^{\mathrm{st}} - \bar{c}\cdot v^{\mathrm{st}}$. Let $\xi^{\mathrm{st}} : [0,T] \to \mathbb{T}^m$ be an extremal curve of L_0^{st}. In the same way as (5.7) in the proof of Proposition 5.5, we define $\xi_\nu = (\xi_\nu^{\mathrm{st}}, \xi_\nu^{\mathrm{wk}}) : [a,b] \to \mathbb{T}^m$ such that $\xi_\nu^{\mathrm{st}}(a) = \xi^{\mathrm{st}}(a)$, $\xi_\nu^{\mathrm{st}}(b) = \xi^{\mathrm{st}}(b)$ and $\dot{\xi}_\nu^{\mathrm{st}} - B_\nu^T A_\nu^{-1}\dot{\xi} - \tilde{C}_\nu c_\nu^{\mathrm{wk}} = 0$. Since u_ν are dominated by $L_\nu^s - c_\nu\cdot v + \alpha_\nu$ (see (5.9)), we have

$$\begin{aligned} u_\nu(\xi_\nu(b)) - u_\nu(\xi_\nu(a)) &\leqslant \int_a^b (L_\nu^s - c_\nu\cdot v^s + \alpha_\nu(\xi_\nu, \dot{\xi}_\nu)dt \\ &= \int_a^b (L_\nu^{\mathrm{st}} - \bar{c}_\nu\cdot v^{\mathrm{st}})(\xi_\nu^{\mathrm{st}}, \dot{\xi}_\nu^{\mathrm{st}})dt \\ &\quad + \int_a^b (U_\nu^{\mathrm{wk}}(\xi_\nu) + \alpha_\nu - \frac{1}{2}\tilde{C}_\nu c_\nu^{\mathrm{wk}}\cdot c_\nu^{\mathrm{wk}})dt, \end{aligned}$$

where the equality is due to $\dot{\xi}_\nu^{\mathrm{st}} - B_\nu^T A_\nu^{-1}\dot{\xi}_\nu - \tilde{C} c_\nu^{\mathrm{wk}} = 0$. Using the fact that $\|U_\nu^{\mathrm{wk}}\|_{C^0} \to 0$, $L_\nu^{\mathrm{st}} \to L_0^{\mathrm{st}}$, and from Proposition 5.5, $\alpha_\nu - \frac{1}{2}\tilde{C}_\nu c_\nu^{\mathrm{wk}}\cdot c_\nu^{\mathrm{wk}} \to \alpha_{H^{\mathrm{st}}}(\bar{c})$ as $\nu\to\infty$, we get

$$u^{\mathrm{st}}(\xi^{\mathrm{st}}(b)) - u^{\mathrm{st}}(\xi^{\mathrm{st}}(a)) \leqslant \int_a^b (L_0^{\mathrm{st}} - \bar{c}\cdot v^{\mathrm{st}} + \alpha_{H^{\mathrm{st}}}(\bar{c}))dt. \tag{5.10}$$

Therefore u^{st} is dominated by $L_0^{\mathrm{st}} - \bar{c}\cdot v^{\mathrm{st}}$.

Secondly, we show that for any $\varphi^{\mathrm{st}} \in \mathbb{T}^m$, there exists a $(u^{\mathrm{st}}, L_0^{\mathrm{st}}, \bar{c})$-calibrated curve $\gamma^{\mathrm{st}} : (-\infty, 0] \to \mathbb{T}^m$ with $\gamma^{\mathrm{st}}(0) = \varphi^{\mathrm{st}}$.

Because u_ν are weak KAM solutions of $L_\nu^s - c_\nu\cdot v$, for each ν there exists a $(u_\nu, L_\nu^s - c_\nu\cdot v, \alpha_\nu)$-calibrated curve $\gamma_\nu = (\gamma_\nu^{\mathrm{st}}, \gamma_\nu^{\mathrm{wk}}) : (-\infty, 0] \to \mathbb{T}^d$. By Proposition 5.1, all γ_ν^{st} are uniformly Lipschitz, so there exists a subsequence that converges in $C^1_{loc}((-\infty,0], \mathbb{T}^d)$. Assume without loss of generality that $\gamma_\nu^{\mathrm{st}} \to \gamma^{\mathrm{st}}$, since $\gamma_\nu = (\gamma_\nu^{\mathrm{st}}, \gamma_\nu^{\mathrm{wk}})$ is extremal for L_ν^s, we have

$$\ddot{\gamma}_\nu^{\mathrm{st}} = \frac{d}{dt}(A_\nu I^{\mathrm{st}} + B_\nu I^{\mathrm{wk}}) = A_\nu \partial_{\varphi^{\mathrm{st}}} U_\nu^{\mathrm{st}} + B_\nu \partial_{\varphi^{\mathrm{st}}} U_\nu^{\mathrm{wk}}.$$

By our assumption, as $i\to\infty$, $A_\nu \to A := \partial^2_{v^{\mathrm{st}}v^{\mathrm{st}}} L_0^{\mathrm{st}}$, and by Lemma 4.3 $\|B_\nu\|\|U_\nu^{\mathrm{wk}}\|_{C^2} \to 0$, we have

$$\ddot{\gamma}^{\mathrm{st}} = A\partial_{\varphi^{\mathrm{st}}} U^{\mathrm{st}}(\gamma^{\mathrm{st}}), \tag{5.11}$$

which is the Euler-Lagrange equation for L_0^{st}.

On the other hand, since γ_ν are $(u_\nu, L^s_\nu - c_\nu \cdot v, \alpha_\nu)$ calibrated, for any $[a,b] \subset (-\infty, 0]$,

$$u_\nu(\gamma_\nu(b)) - u_\nu(\gamma_\nu(a)) = \int_a^b (L^s_\nu - c_\nu \cdot v^s + \alpha_\nu(\gamma_\nu, \dot\gamma_\nu))dt$$

$$\geqslant \int_a^b (L^{\mathrm{st}} - \bar c_\nu \cdot v^{\mathrm{st}})(\gamma^{\mathrm{st}}_\nu, \dot\gamma^{\mathrm{st}}_\nu)dt + \int_a^b (U^{\mathrm{wk}}_\nu + \alpha_{H^s_\nu}(c_\nu) - \frac{1}{2}\tilde C_\nu c^{\mathrm{wk}}_\nu \cdot c^{\mathrm{wk}}_\nu)(\gamma_\nu, \dot\gamma_\nu)dt.$$

Take the limit again to get

$$u^{\mathrm{st}}(\gamma^{\mathrm{st}}(b)) - u^{\mathrm{st}}(\gamma^{\mathrm{st}}(a)) \geqslant \int_a^b (L^{\mathrm{st}}_0 - \bar c \cdot v^{\mathrm{st}} + \alpha_{H^{\mathrm{st}}}(\bar c))(\gamma^{\mathrm{st}}, \dot\gamma^{\mathrm{st}})dt.$$

Because γ^{st} is an L^{st} extremal curve (see (5.11)), (5.10) hold for γ^{st}. Combining with the last displayed formula, (5.10) becomes an equality. Then γ^{st} is a calibrated curve for $L^{\mathrm{st}} - \bar c \cdot v^{\mathrm{st}} + \alpha_{H^{\mathrm{st}}}(\bar c)$, and u^{st} is a weak KAM solution.

6. The Mañe and the Aubry sets and the barrier function

We prove the following result.

Proposition 6.1. — *Fix $\mathcal{B}^{\mathrm{st}}$ and $\kappa > 1$. Assume that $(\mathcal{B}^{\mathrm{wk}}_\nu, p_\nu, U^{\mathrm{st}}_\nu, \mathcal{U}^{\mathrm{wk}}_\nu)$ satisfies the assumptions of Theorem 2.5. Denote H^s_ν, H^{st}_0, L^s_ν and L^{st}_0 as in the previous section (see (5.9)).*

1. *Any limit point of $\varphi_\nu \in \mathcal{N}_{H^s_\nu}(c_\nu)$ is contained in $\mathcal{N}_{H^{\mathrm{st}}_0}(\bar c) \times \mathbb{T}^{d-m}$.*
2. *If $\mathcal{A}_{H^{\mathrm{st}}_0}(\bar c)$ contains only finitely many static classes, then any limit point of $\varphi_\nu \in \mathcal{A}_{H^s_\nu}(c_\nu)$ is contained in $\mathcal{A}_{H^{\mathrm{st}}_0}(\bar c) \times \mathbb{T}^{d-m}$.*
3. *Assume that $\mathcal{A}_{H^{\mathrm{st}}}(\bar c)$ contains only one static class. Let $\varphi_\nu = (\varphi^{\mathrm{st}}_\nu, \varphi^{\mathrm{wk}}_\nu) \in \mathcal{A}_{H^s_\nu}(c_\nu)$ be such that $\varphi^{\mathrm{st}}_\nu \to \varphi^{\mathrm{st}} \in \mathcal{A}_{H^{\mathrm{st}}_0}(\bar c)$. Then for any $\psi = (\psi^{\mathrm{st}}, \psi^{\mathrm{wk}}) \in \mathbb{T}^d$,*
$$\lim_{\nu\to\infty} h_{L^s_\nu, c_\nu}(\varphi, \psi) = h_{L^{\mathrm{st}}_0, \bar c}(\varphi^{\mathrm{st}}, \psi^{\mathrm{st}}).$$
4. *Let $(\rho^{\mathrm{st}}_\nu, \rho^{\mathrm{wk}}_\nu)$ be the rotation vector of any c_ν-minimal measure of L^s_ν. Then we have*
$$\lim_{\nu\to\infty} (\rho^{\mathrm{wk}}_\nu - B^T_\nu A^{-1}_\nu \rho^{\mathrm{st}}_\nu - \tilde C_\nu c^{\mathrm{wk}}_\nu) = 0,$$
and any accumulation point ρ of ρ^{st}_ν is contained in the set $\partial\alpha_{H^{\mathrm{st}}}(\bar c)$.

The proof of item 2 requires additional discussion and is presented in Section 6.2. In Section 6.1 we prove item 1, 3 and 4.

6.1. The Mañe set and the barrier function. — We first state an alternate definition of the Aubry and Mañe sets due to Fathi (see also [6]). Let u be a weak KAM solution for the Lagrangian L. We define $\overline{\mathcal{G}}(L,u)$ to be the set of points $(\varphi,v) \in \mathbb{T}^d \times \mathbb{R}^d$ such that there exists a (u, L, α_L)-calibrated curve $\gamma : (-\infty, 0] \to \mathbb{T}^d$, with $(\varphi, v) = (\gamma(0), \dot\gamma(0))$. Let ϕ_t denote the Euler-Lagrange flow of L, then

$$\tilde{\mathcal{I}}(L,u) = \bigcap_{t\leqslant 0} \phi_t(\overline{\mathcal{G}}(L,u)), \quad \tilde{\mathcal{A}}_L = \bigcap_u \tilde{\mathcal{I}}(L,u), \quad \tilde{\mathcal{N}}_L = \bigcup_u \tilde{\mathcal{I}}(L,u), \tag{6.1}$$

where the union and intersection are over all weak KAM solutions of L. The Aubry set and Mañe set of $c \in H^1(\mathbb{T}^d,\mathbb{R})$ is defined as

$$\tilde{\mathcal{A}}_L(c) = \tilde{\mathcal{A}}_{L-c\cdot v}, \quad \tilde{\mathcal{N}}_L(c) = \tilde{\mathcal{N}}_{L-c\cdot v}.$$

The projected Aubry and Mañe sets are the projection of these sets to $\mathbb{T}^d$.

We now turn to the setting of Proposition 6.1. Let L^s_ν, L^{st}_0, c_ν, $\bar c$ be as in the assumption. The strategy of the proof is similar to the one in [7].

Lemma 6.2. — *Let u_ν be a weak KAM solution of $L^s_\nu - c_\nu \cdot v$. Assume that $(\varphi_\nu, v_\nu) \in \tilde{\mathcal{I}}(L^s_\nu - c_\nu\cdot v, u_\nu)$ satisfies $(\varphi_\nu, v_\nu) \to (\varphi, v) = (\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}, v^{\mathrm{st}}, v^{\mathrm{wk}})$, and $u_\nu(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}) \to u^{\mathrm{st}}(\varphi^{\mathrm{st}})$. Then*

$$(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) \in \tilde{\mathcal{I}}(L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}}, u^{\mathrm{st}}).$$

Proof. — We first show that $(\varphi_\nu, v_\nu) \in \overline{\mathcal{G}}(L^s_\nu - c_\nu\cdot v, u_\nu)$ implies $(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) \in \overline{\mathcal{G}}(L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}}, u^{\mathrm{st}})$. Indeed, there exists $\gamma_\nu : (-\infty, 0] \to \mathbb{T}^d$, each $(u_\nu, L^s_\nu - c_\nu \cdot v, \alpha_{L^s_\nu}(c_\nu))$-calibrated, with $(\gamma_\nu, \dot\gamma_\nu)(0) = (\varphi, v)$. We follow the same line as proof of item 3 in Theorem 2.5 (Section 5), then by restricting to a subsequence, γ^{st}_ν converges in $C^1_{loc}((-\infty,0],\mathbb{T}^d)$ to a $(u^{\mathrm{st}}, L^{\mathrm{st}}_0 - \bar c\cdot v^{\mathrm{st}}, \alpha_{H^{\mathrm{st}}_0}(\bar c))$-calibrated curve γ^{st}. In particular, $(\gamma^{\mathrm{st}}_\nu, \dot\gamma^{\mathrm{st}}_\nu) \to (\gamma^{\mathrm{st}}, \dot\gamma^{\mathrm{st}})$, which implies $(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) \in \overline{\mathcal{G}}(L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}}, u^{\mathrm{st}})$.

Let ϕ^ν_t denote the Euler-Lagrange flow of L^s_ν, and ϕ^{st}_t the flow for L^{st}. Let π denote the projection to the strong components $(\varphi^{\mathrm{st}}, v^{\mathrm{st}})$, then from Lemma 4.3 $\pi\phi^\nu_t \to \phi^{\mathrm{st}}_t$ uniformly. As a result for a fixed $T > 0$ and $(\varphi_\nu, v_\nu) \in \tilde{\mathcal{I}}(L^{\mathrm{st}}_\nu - c_\nu \cdot v, u_\nu)$, we have

$$(\varphi_\nu, v_\nu) = (\varphi^{\mathrm{st}}_\nu, \varphi^{\mathrm{wk}}_\nu, v^{\mathrm{st}}_\nu, v^{\mathrm{wk}}_\nu) \in \phi^\nu_{-T}\left(\overline{\mathcal{G}}(L^s_\nu - c_\nu\cdot v, u_\nu)\right),$$

hence $(\varphi^{\mathrm{st}}_\nu, v^{\mathrm{st}}_\nu) \to (\varphi^{\mathrm{st}}, v^{\mathrm{st}}) \in \varphi^{\mathrm{st}}_{-T}\left(\overline{\mathcal{G}}(L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}}, u^{\mathrm{st}})\right)$. Since $T > 0$ is arbitrary, we obtain $(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) \in \tilde{\mathcal{I}}(L^{\mathrm{st}}_0 - \bar c\cdot v^{\mathrm{st}}, u^{\mathrm{st}})$. □

Proof of Proposition 6.1, part I. — We first prove item 1.

Suppose $\tilde\varphi_\nu \in \tilde{\mathcal{N}}_{H^s_\nu}(c_\nu)$, then there exists weak KAM solutions u_ν of $L^s_\nu - c_\nu\cdot v$, such that $(\varphi_\nu, v_\nu) \in \tilde{\mathcal{I}}(L^s_\nu - c_\nu\cdot v, u_\nu)$. By Theorem 2.5, after restricting to a subsequence, we have $u_\nu(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}) \to u^{\mathrm{st}}(\varphi^{\mathrm{st}})$. By Lemma 6.2, $(\varphi^{\mathrm{st}}_\nu, v^{\mathrm{st}}_\nu) \to (\varphi^{\mathrm{st}}, v^{\mathrm{st}})$ implies $(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) \in \tilde{\mathcal{I}}(L^{\mathrm{st}}_0 - \bar c\cdot v^{\mathrm{st}}, u^{\mathrm{st}}) \subset \tilde{\mathcal{N}}_{H^{\mathrm{st}}_0}(\bar c)$.

For item 3, suppose $\varphi_\nu = (\varphi_\nu^{\mathrm{st}}, \varphi_\nu^{\mathrm{wk}}) \in \mathcal{A}_{H_\nu^s}(c_\nu)$ satisfies $\varphi_\nu^{\mathrm{st}} \to \varphi^{\mathrm{st}} \in \mathcal{A}_{H_0^{\mathrm{st}}}(\bar{c})$. Then $h_{L_\nu^s, c_\nu}(\varphi_\nu, \cdot)$ is a weak KAM solution of $L_\nu^s - c \cdot v$ (see [12], Theorem 5.3.6). By Theorem 2.5, by restricting to a subsequence, there exists a weak KAM solution u^{st} of $L_0^{\mathrm{st}} - \bar{c} \cdot v^{\mathrm{st}}$ such that

$$\lim_{\nu \to \infty} h_{L_\nu^s, c_\nu}(\varphi_\nu, \psi^{\mathrm{st}}, \psi^{\mathrm{wk}}) - h_{L_\nu^s, c_\nu}(\varphi_\nu, 0, 0) = u^{\mathrm{st}}(\psi^{\mathrm{st}}).$$

We may further assume that $h_{L_\nu^s, c_\nu}(\varphi_\nu, 0, 0) \to C \in \mathbb{R}$. Since $\mathcal{A}_{H_0^{\mathrm{st}}}(\bar{c})$ has only one static class, there exists a constant $C_1 > 0$ such that

$$u^{\mathrm{st}}(\psi^{\mathrm{st}}) + C_1 = h_{L^{\mathrm{st}}, \bar{c}}(\varphi^{\mathrm{st}}, \psi^{\mathrm{st}}).$$

Using the fact that $\varphi_\nu \in \mathcal{A}_{H_\nu^s}(c_\nu)$, we get $h_{L_\nu^s, c_\nu}(\varphi_\nu, \varphi_\nu) = 0$. Taking the limit,

$$u^{\mathrm{st}}(\varphi^{\mathrm{st}}) = -C_1 = h_{L^{\mathrm{st}}, \bar{c}}(\varphi^{\mathrm{st}}, \varphi^{\mathrm{st}}) - C = -C.$$

Therefore,

$$\lim_{\nu \to \infty} h_{L_\nu^s, c_\nu}(\varphi_\nu^{\mathrm{st}}, \varphi_\nu^{\mathrm{wk}}, \psi^{\mathrm{st}}, \psi^{\mathrm{wk}}) = h_{L^{\mathrm{st}}, \bar{c}}(\varphi^{\mathrm{st}}, \psi^{\mathrm{st}}).$$

Item 4: Let $\rho_\nu = (\rho_\nu^{\mathrm{st}}, \rho_\nu^{\mathrm{wk}})$ be the rotation vector of minimal measures of $L_\nu^s - c_\nu \cdot v$, then from Proposition 5.6,

$$\lim_{\nu \to \infty} \rho_\nu^{\mathrm{wk}} - B_\nu^T A_\nu^{-1} \rho_\nu^{\mathrm{st}} - \tilde{C}_\nu c_\nu^{\mathrm{wk}} = 0.$$

Moreover, assume that $\rho_\nu^{\mathrm{st}} \to \rho^{\mathrm{st}} \in \mathbb{R}^m$, then by taking the limit in the second conclusion of Proposition 5.6, we get

$$\alpha_{H_0^{\mathrm{st}}}(\bar{c}) + \beta_{H_0^{\mathrm{st}}}(\rho^{\mathrm{st}}) - \bar{c} \cdot \rho^{\mathrm{st}} = 0,$$

using the Fenchel duality, ρ^{st} is a subdifferential of the convex function $\alpha_{H_0^{\mathrm{st}}}$ at $\bar{c}$. □

6.2. Semi-continuity of the Aubry set. — Our strategy of the proof mostly follow [7].

Given a compact metric space $\mathcal{X}$, a semi-flow ϕ_t on $\mathcal{X}$, and $\varepsilon, T > 0$, an (ε, T)-chain consists of $x_0, \dots, x_N \in \mathcal{X}$ and $T_0, \dots, T_{N-1} \geqslant T$, such that $d(\phi_{T_i} x_i, x_{i+1}) < \varepsilon$. We say that $x \mathfrak{C}_{\mathcal{X}} y$ if for any $\varepsilon, T > 0$, there exists an (ε, T)-chain with $x_0 = x$ and $x_N = y$. The relation $\mathfrak{C}_{\mathcal{X}}$ is called *the chain transitive relation* (see [11]).

The family of maps $\bar{\phi}_t = \phi_t$ defines a semi-flow on the set $\overline{\mathcal{G}(L - c \cdot v, u)}$, and therefore defines a chain transitive relation. Given $\varphi, \psi \in \mathbb{T}^d$ and a weak KAM solution u of $L - c \cdot v$, we say that $\varphi \mathfrak{C}_u \psi$ if there exists $\tilde{\varphi} = (\varphi, v), \tilde{\psi} = (\psi, w) \in \mathbb{T}^d \times \mathbb{R}^d$ such that

$$\tilde{\varphi} \mathfrak{C}_{\mathcal{X}} \tilde{\psi}, \text{ where } \mathcal{X} = \overline{\mathcal{G}(L - c \cdot v, u)}.$$

Item 1 in the following proposition is due to Mañe, and item 2 is due to Mather. The version presented here is due to Bernard ([7]).

Proposition 6.3. — *Let L be a Tonelli Lagrangian, then:*

1. *Let* $\varphi \in \mathcal{A}_L(c)$ *and* u *be a weak KAM solution of* $L - c\cdot v$, *we have* $\varphi \mathfrak{C}_u \varphi$.
2. *Suppose* $\mathcal{A}_L(c)$ *has only finitely many static classes, and there exists a weak KAM solution* u *such that* $\varphi \mathfrak{C}_u \varphi$. *Then* $\varphi \in \mathcal{A}_L(c)$.

Proposition 6.3 implies that, when $\mathcal{A}_L(c)$ has finitely many static classes, the Aubry set coincides with the set $\{\varphi : \varphi \mathfrak{C}_u \varphi\}$. We will prove semi-continuity for this set.

Definition. — *Let* $\mathcal{X}$ *be a compact metric space with a semi-flow* ϕ_t. *A family of piecewise continuous curves* $x_\nu : [0, T_\nu] \to \mathcal{X}$ *is said to accumulate locally uniformly to* $(\mathcal{X}, \phi_t)$ *if for any sequence* $S_\nu \in [0, T_\nu]$, *the curves* $x_\nu(t + S_\nu)$ *has a subsequence which converges uniformly on compact sets to a trajectory of* ϕ_t.

Lemma 6.4 ([7]). — *Suppose* $x_\nu : [0, T_\nu] \to \mathcal{X}$ *accumulates locally uniformly to* (X, ϕ_t), $x_\nu(0) \to x$ *and* $x_\nu(T_\nu) \to y$, *then* $x \mathfrak{C}_X y$.

Proof of Proposition 6.1, part II. — We prove item 2. Let $\varphi_\nu = (\varphi_\nu^{\mathrm{st}}, \varphi_\nu^{\mathrm{wk}}) \in \mathcal{A}_{H_\nu^s}(c_\nu)$ and $\varphi_\nu^{\mathrm{st}} \to \varphi^{\mathrm{st}}$, we show that $\varphi^{\mathrm{st}} \in \mathcal{A}_{H_0^{\mathrm{st}}}(\bar c)$. According to Proposition 6.3, $\varphi_\nu \mathfrak{C}_u \varphi_\nu$. Let $\tilde\varphi_\nu$ be the unique point in $\tilde{\mathcal{A}}_{H_\nu^s}(c_\nu)$ projecting to φ_ν, then there exists weak KAM solutions u_ν of $L_\nu^s - c_\nu \cdot v$, such that $\tilde\varphi_\nu \mathfrak{C} \tilde\varphi_\nu$ in $\overline{\mathcal{G}}(L_\nu^s - c_\nu \cdot v, u_\nu)$. Fix $\varepsilon_\nu \to 0$ and $M_\nu \to \infty$, then for each ν, there exists

$$T_{\nu,1} < \cdots < T_{\nu,N_\nu}, \quad T_{\nu,j+1} - T_{\nu,j} > M_\nu,$$

and a piecewise C^1 curve $\gamma_\nu = (\gamma_\nu^{\mathrm{st}}, \gamma_\nu^{\mathrm{wk}}) : [0, T_\nu] \to \mathbb{T}^d$, satisfying

1. $\gamma_\nu | (T_{\nu,j}, T_{\nu,j+1})$ satisfies the Euler-Lagrange equation of L_ν^s;
2. $d\big(((\gamma_\nu(T_{\nu,j}-), \dot\gamma_\nu(T_{\nu,j}-)),\ ((\gamma_\nu(T_{\nu,j}+), \dot\gamma_\nu(T_{\nu,j}+))\big) < \varepsilon_\nu$.

Using Lemma 4.3, the projection of the Euler-Lagrange flow of L_ν^s to $(\varphi^{\mathrm{st}}, v^{\mathrm{st}})$ converges uniformly over compact interval to the Euler-Lagrange flow of L_0^{st}. This, combined with item 2 and Lemma 6.2, implies that $(\gamma_\nu^{\mathrm{st}}, \dot\gamma_\nu^{\mathrm{st}})$ accumulates locally uniformly to

$$(\overline{\mathcal{G}}(L_0^{\mathrm{st}} - \bar c \cdot v^{\mathrm{st}}, u^{\mathrm{st}}), \phi_{-t}^{\mathrm{st}})$$

where ϕ_t^{st} is the Euler-Lagrange flow of L_0^{st}. Therefore, $\varphi_\nu^{\mathrm{st}} \to \varphi^{\mathrm{st}}$ impies $\varphi^{\mathrm{st}} \mathfrak{C}_u \varphi^{\mathrm{st}}$. Using Proposition 6.3 again, we get $\varphi^{\mathrm{st}} \in \mathcal{A}_{H_0^{\mathrm{st}}}(\bar c)$. □

7. Technical estimates on weak KAM solutions

In this section we prove Proposition 5.3 and 5.4. For $(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega_{\kappa,q}^{m,d}(\mathcal{B}^{\mathrm{st}}) \cap \{\|U^{\mathrm{st}}\| \leqslant R\}$, recall the notations $H^s = \mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})$, $H^{\mathrm{st}} = \mathcal{H}^{\mathrm{st}}(p, U^{\mathrm{st}})$, $L^s = L_{H^s}$, $L^{\mathrm{st}} = L_{H^{\mathrm{st}}}$.

7.1. Approximate Lipschitz property in the strong component. — In this section we prove Proposition 5.4 using Proposition 5.3. Proposition 5.3 is proved in the next two sections.

We first state a lemma of action comparison between an extremal curve and its "linear drift".

Lemma 7.1. — *Let $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Tonelli Hamiltonian, $T \geqslant 1$, and $\gamma : [0,T] \to \mathbb{T}^d$ be an extremal curve. Then for any $1 \leqslant i \leqslant d$, $h > 0$, and a unit vector $f \in \mathbb{R}^d$,*

$$\int_0^T L(\gamma + \frac{th}{T}f, \dot\gamma + \frac{h}{T}f)dt - \int_0^T L(\gamma,\dot\gamma)dt$$

$$\leqslant (\partial_v L(\gamma(T),\dot\gamma(T))\cdot f)h + \left(\|f\cdot(\partial^2_{vv}L)f\|\frac{1}{T} + \|f\cdot(\partial^2_{\varphi v}L)f\| + T\|f\cdot(\partial^2_{\varphi\varphi}L)f\|\right)h^2.$$

Proof. — We compute

$$\begin{aligned} L(\gamma + \frac{th}{T}f, \dot\gamma + \frac{h}{T}f) - L(\gamma,\dot\gamma) &\leqslant \partial_\varphi L(\gamma,\dot\gamma)\cdot\frac{th}{T}f + \partial_v L(\gamma,\dot\gamma)\frac{h}{T}f \\ &\quad \|f\cdot(\partial^2_{vv}L)f\|\frac{h^2}{T^2} + \|f\cdot(\partial^2_{\varphi v}L)f\|\frac{th^2}{T^2} + \|f\cdot(\partial^2_{\varphi\varphi}L)f\|\frac{t^2h^2}{T^2}. \end{aligned}$$

It follows from the Euler-Lagrange equation that

$$\partial_\varphi L(\gamma,\dot\gamma)\cdot\frac{th}{T} + \partial_v L(\gamma,\dot\gamma)\frac{h}{T} = \frac{d}{dt}\left(\partial_v L\frac{th}{T}\right),$$

and our estimate follows from direct integration. ☐

The following lemma establishes a relation between "approximate semi concavity" with approximate Lipschitz property.

Lemma 7.2. — *For $C,\delta > 0$, assume that all $\varphi \in \mathbb{T}^d$ the function $u : \mathbb{T}^d \to \mathbb{R}$ satisfies the following condition: there exists $l \in \mathbb{R}^d$ such that*

$$u(\varphi + y) - u(\varphi) \leqslant l\cdot y + C\|y\|^2 + \delta, \quad y \in \mathbb{R}^d,$$

Then $\|l\| \leqslant \sqrt{d}(C+\delta)$, and u is $(2\sqrt{d}(C+\delta),\delta)$ approximately Lipschitz.

Proof. — Assume that $l = (l_1,\dots,l_d)$. For each $1 \leqslant i \leqslant d$, we pick $y = -e_i\frac{l_i}{|l_i|}$, where e_i is the coordinate vector in φ_i. Then

$$0 = u(\varphi + e_i) - u(\varphi) \leqslant -|l_i| + C + \delta,$$

so $|l_i| \leqslant C + \delta$. As a result $\|l\| \leqslant \sqrt{d}(C+\delta)$. For any $y \in [0,1]^d$, we have $\|y\| \leqslant \sqrt{d}$ and

$$u(\varphi + y) - u(\varphi) \leqslant (\sqrt{d}(C+\delta) + C\|y\|)\|y\| + \delta < 2\sqrt{d}(C+\delta)\|y\| + \delta. \qquad ☐$$

Proof of Proposition 5.4. — Since u is a weak KAM solution, for any $\varphi \in \mathbb{T}^d$, let $\gamma = (\gamma^{\text{st}}, \gamma^{\text{wk}}) : (-\infty, 0] \to \mathbb{T}^d$ be a $(u, L^s - c \cdot v, \alpha_H(c))$-calibrated curve with $\gamma(0) = \varphi = (\varphi^{\text{st}}, \varphi^{\text{wk}})$. Then for any $T > 0$

$$u(\varphi) = u(\gamma(-T)) + \int_{-T}^{0} (L^s - c \cdot v + \alpha_{H^s}(c))(\gamma, \dot\gamma) dt.$$

Using (4.6), we get

(7.1)
$$u(\varphi) = u(\gamma(-T)) + \int_{-T}^{0} (L^{\text{st}} - \bar c \cdot v^{\text{st}})(\gamma^{\text{st}}, \dot\gamma^{\text{st}}) dt + (\alpha_{H^s}(c) - \frac{1}{2} c^{\text{wk}} \cdot \tilde C^{-1} c^{\text{wk}}) T$$
$$+ \int_{-T}^{0} \frac{1}{2} (\dot\gamma^{\text{wk}} - B^T A^{-1} \dot\gamma^{\text{st}} - \tilde C c^{\text{wk}}) \cdot \tilde C^{-1} (\dot\gamma^{\text{wk}} - B^T A^{-1} \dot\gamma^{\text{st}} - \tilde C c^{\text{wk}}) + U^{\text{wk}}(\gamma(t)) dt.$$

We now produce an upper bound using a special test curve. Let $\gamma_0^{\text{st}} : [-T, 0] \to \mathbb{T}^m$ be such that

$$\int_{-T}^{0} (L^{\text{st}} - \bar c \cdot v^{\text{st}})(\gamma_0^{\text{st}}, \dot\gamma_0^{\text{st}}) dt = \min_{\zeta} \int_{-T}^{0} (L^{\text{st}} - \bar c \cdot v^{\text{st}})(\zeta, \dot\zeta) dt \tag{7.2}$$

where the minimum is over all $\zeta(-T) = \gamma^{\text{st}}(-T)$ and $\zeta(0) = \gamma^{\text{st}}(0)$.

We define $\xi = (\xi^{\text{st}}, \xi^{\text{wk}}) : [-T, 0] \to \mathbb{T}^d$ as follows.

1. For $y \in \mathbb{R}^d$,

$$\xi^{\text{st}}(t) = \gamma_0^{\text{st}}(t) + \frac{T + t}{T} y.$$

   The curve ξ^{st} is a linear drift over γ_0^{st} with $h = \|y\|$ and $f = \frac{y}{\|y\|}$ (see Lemma 7.1).

2. Define

$$\xi^{\text{wk}}(t) = \gamma^{\text{wk}}(-T) + B^T A^{-1} (\xi^{\text{st}}(t) - \gamma_0^{\text{st}}(-T)) + \tilde C c^{\text{wk}} (T + t).$$

   We note that $\xi^{\text{wk}}(-T) = \gamma^{\text{wk}}(-T)$ and

$$\dot\xi_0^{\text{wk}} - B^T A^{-1} \xi^{\text{st}} - \tilde C c^{\text{wk}} = 0.$$

Using the fact that u is dominated by $L^s - c\cdot v + \alpha_{H^s}(c)$, we have

$$\begin{aligned} u(\varphi^{\mathrm{st}}+y,\xi^{\mathrm{wk}}(0)) &\leqslant u(\gamma(-T)) + \int_{-T}^0 (L^s - c\cdot v+\alpha_{H^s}(c))(\xi,\dot\xi)dt \\ &= u(\gamma(-T)) + \int_{-T}^0 (L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}})(\xi^{\mathrm{st}},\dot\xi^{\mathrm{st}})dt \\ &\quad + \int_{-T}^0 \frac12 (\dot\xi^{\mathrm{wk}} - B^TA^{-1}\dot\xi^{\mathrm{st}} - \tilde C c^{\mathrm{wk}}) \\ &\qquad \cdot \tilde C^{-1}(\dot\xi^{\mathrm{wk}} - B^TA^{-1}\dot\xi^{\mathrm{st}} - \tilde C c^{\mathrm{wk}})dt \\ &\quad + (\alpha_{H^s}(c) - \frac12 c^{\mathrm{wk}}\cdot\tilde C^{-1}c^{\mathrm{wk}})T + \int_{-T}^0 U^{\mathrm{wk}}(\xi)dt \end{aligned}$$

and note that the third line in the above formula vanishes, using the definition of ξ^{wk}. Combine with (7.1), we get

$$\begin{aligned} &u(\varphi^{\mathrm{st}}+y,\xi^{\mathrm{wk}}(0)) - u(\varphi^{\mathrm{st}},\varphi^{\mathrm{wk}}) \\ &\leqslant \int_{-T}^0 (L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}})(\xi^{\mathrm{st}},\dot\xi^{\mathrm{st}})dt - \int_{-T}^0 (L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}})(\gamma^{\mathrm{st}},\dot\gamma^{\mathrm{st}})dt + 2\|U^{\mathrm{wk}}\|_{C^0}. \end{aligned}$$

From (7.2) we get

$$\begin{aligned} &u(\varphi^{\mathrm{st}}+y,\xi^{\mathrm{wk}}(0)) - u(\varphi^{\mathrm{st}},\varphi^{\mathrm{wk}}) \\ &\leqslant \int_{-T}^0 (L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}})(\xi^{\mathrm{st}},\dot\xi^{\mathrm{st}})dt - \int_{-T}^0 (L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}})(\gamma_0^{\mathrm{st}},\dot\gamma_0^{\mathrm{st}})dt + 2\|U^{\mathrm{wk}}\|_{C^0}. \end{aligned}$$

Since γ_0^{st} is an extremal of $L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}}$, the linear drift lemma (Lemma 7.1) applies. Noting that $\|\partial^2_{v^{\mathrm{st}}v^{\mathrm{st}}}L^{\mathrm{st}}\| \leqslant \|A^{-1}\|$, $\|\partial^2_{\varphi^{\mathrm{st}}\varphi^{\mathrm{st}}}L\| \leqslant \|U^{\mathrm{st}}\|_{C^2} \leqslant R$, and $\partial^2_{\varphi^{\mathrm{st}}v^{\mathrm{st}}}L = 0$. We obtain from Lemma 7.1 that

$$\begin{aligned} \int_{-T}^0 (L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}})(\xi^{\mathrm{st}},\dot\xi^{\mathrm{st}})dt - \int_{-T}^0 (L^{\mathrm{st}} - \bar c\cdot v^{\mathrm{st}})(\gamma_0^{\mathrm{st}},\dot\gamma_0^{\mathrm{st}})dt& \\ \leqslant l\cdot y + (\|A^{-1}\| + \|U^{\mathrm{st}}\|_{C^2})\|y\|^2&, \end{aligned}$$

where $l = \partial_v L^{\mathrm{st}}(\gamma_0^{\mathrm{st}}(0),\dot\gamma_0^{\mathrm{st}}(0))$. Note that $\|A^{-1}\| + \|U^{\mathrm{st}}\|_{C^2}$ is a constant depending only on $\mathcal{B}^{\mathrm{st}}, Q, R$.

We now invoke Proposition 5.3 to get

$$|u(\varphi^{\mathrm{st}}+y,\xi^{\mathrm{wk}}(0)) - u(\varphi^{\mathrm{st}}+y,\varphi^{\mathrm{wk}})| \leqslant \delta|\xi^{\mathrm{wk}}(0)-\varphi^{\mathrm{wk}}| + \delta \leqslant 2\delta,$$

where $\delta = M_1^*\mu(\mathcal{B}^{\mathrm{wk}})^{-\frac{q}{2}-d+m}$ for some $M_1^* = M_1^*(\mathcal{B}^{\mathrm{st}},Q,\kappa,q,R)$. Combine all the estimates, we get

$$u(\varphi^{\mathrm{st}}+y,\varphi^{\mathrm{wk}}) - u(\varphi^{\mathrm{st}},\varphi^{\mathrm{wk}}) \leqslant l\cdot y + (\|A^{-1}\| + \|U^{\mathrm{st}}\|_{C^2})\|y\|^2 + 2\delta + 2\|U^{\mathrm{wk}}\|_C^0.$$

We note that in $\Omega^{m,d}_{\kappa,q}$ we have $\|U^{\mathrm{wk}}\|_{C^2} \leqslant \sum_{i=1}^{d-m} \|U^{\mathrm{wk}}_i\|_{C^2} \leqslant (d-m)\kappa(\mu(\mathcal{B}^{\mathrm{wk}}))^{-q}$. We may choose $M_2^* = M_2^*(\mathcal{B}^{\mathrm{st}}, Q, \kappa, q, R)$, such that

$$2\delta + 2\|U^{\mathrm{wk}}\| \leqslant M_2^* \mu(\mathcal{B}^{\mathrm{wk}}))^{-(\frac{q}{2}-d+m)} =: \delta'.$$

We now apply Lemma 7.2 to get $u(\cdot, \varphi^{\mathrm{wk}})$ is

$$(2\sqrt{d}(\|A^{-1}\| + \|U^{\mathrm{st}}\|_{C^2} + \delta'), \delta')$$

approximately Lipschitz. Define $M' = 2\sqrt{d}(\|A^{-1}\| + \|U^{\mathrm{st}}\|_{C^2} + M_2^*)$, and the proposition follows. □

7.2. Filtrated decomposition of the slow Lagrangian. — For the proof of Proposition 5.3, we need a filtrated decomposition of the Lagrangian L^s which treat all φ_i^{wk}, $1 \leqslant i \leqslant d-m$ separately. First, we have the following linear algebra identity.

LEMMA 7.3. — *Let* $S = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ *be a nonsingular symmetric matrix in block form. Then*

$$\begin{bmatrix} \mathrm{Id} & 0 \\ -B^T A^{-1} & \mathrm{Id} \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} \mathrm{Id} & -A^{-1}B \\ 0 & \mathrm{Id} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \tilde{C} \end{bmatrix},$$

where $\tilde{C} = C - B^T A^{-1} B$. *In particular,* $\tilde{C}$ *is positive definite if* S *is.*

Proof. — The proof is direct calculation. □

We write $H^s(\varphi, I) = K(I) - U(\varphi) = K(I) - U^{\mathrm{st}}(\varphi^{\mathrm{st}}) - U^{\mathrm{wk}}(\varphi)$ and $S = \partial^2_{II} K$. We describe a coordinate change block diagonalizing $\partial^2_{II} K$. Write S in the following block form

$$S = \begin{bmatrix} X_{d-m} & y_{d-m} \\ y_{d-m}^T & z_{d-m} \end{bmatrix}, \quad X_{d-m} \in M_{(d-1)\times(d-1)}, y_{d-m} \in \mathbb{R}^{d-1}, z_{d-m} \in \mathbb{R},$$

and for each $1 \leqslant i \leqslant d-m-1$, further decompose each X_{i+1} as

$$X_{i+1} = \begin{bmatrix} X_i & y_i \\ y_i^T & z_i \end{bmatrix}, \quad X_i \in M_{(m+i-1)\times(m+i-1)}, y_i \in \mathbb{R}^{m+i-1}, z_i \in \mathbb{R}.$$

Note that in this notation, $X_1 = \partial^2_{I^{\mathrm{st}} I^{\mathrm{st}}} K = A$ (see (2.4)).

Define, for $1 \leqslant i \leqslant d-m$,

$$E_i = \begin{bmatrix} \mathrm{Id}_{m+i-1} & -X_i^{-1} y_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Id}_{d-m-i} \end{bmatrix},$$

where Id_i denote the $i \times i$ identity matrix. Then by Lemma 7.3

$$\begin{aligned} E_{d-m}^T S E_{d-m} &= \begin{bmatrix} \mathrm{Id}_{d-1} & 0 \\ -y_{d-m}^T X_{d-m}^{-1} & 1 \end{bmatrix} \begin{bmatrix} X_{d-m} & y_{d-m} \\ y_{d-m}^T & z_{d-m} \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{d-1} & -X_{d-m}^{-1} y_{d-m} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} X_{d-m} & 0 \\ 0 & \tilde{z}_{d-m} \end{bmatrix}, \end{aligned}$$

where $\tilde{z}_{d-m} = z_{d-m} - y_{d-m}^T X_{d-m}^{-1} y_{d-m}$. Moreover, for each $1 \leqslant i \leqslant d-m-1$,

$$\begin{aligned} &\begin{bmatrix} \mathrm{Id}_{m+i-1} & 0 \\ -y_i^T X_i^{-1} & 1 \end{bmatrix} X_{i+1} \begin{bmatrix} \mathrm{Id}_{m+i-1} & -X_i^{-1} y_i \\ 0 & 1 \end{bmatrix} \\ &\qquad = \begin{bmatrix} \mathrm{Id}_{m+i-1} & 0 \\ -y_i^T X_i^{-1} & 1 \end{bmatrix} \begin{bmatrix} X_i & y_i \\ y_i^T & z_i \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{m+i-1} & -X_i^{-1} y_i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} X_i & 0 \\ 0 & \tilde{z}_i \end{bmatrix}. \end{aligned} \tag{7.3}$$

Let

$$E = E_{d-m} \cdots E_1 = \begin{bmatrix} \mathrm{Id}_m & -X_1^{-1} y_1 & -X_2^{-1} y_2 & \cdots & -X_{d-m}^{-1} y_{d-m} \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \tag{7.4}$$

then recursive computation yields

$$E^T S E = E_1^T \cdots E_{d-m}^T S E_{d-m} \cdots E_1 = \begin{bmatrix} X_1 & & & \\ & \tilde{z}_1 & & \\ & & \ddots & \\ & & & \tilde{z}_{d-m} \end{bmatrix} =: \tilde{S}. \tag{7.5}$$

We summarize the characterization of the Lagrangian in the following lemma. For $v = (v^{\mathrm{st}}, v_1^{\mathrm{wk}}, \ldots, v_{d-m}^{\mathrm{wk}}) \in \mathbb{R}^m \times \mathbb{R}^d$, we define

$$\lfloor v \rfloor_0 = v^{\mathrm{st}}, \quad \lfloor v \rfloor_i = (v^{\mathrm{st}}, v_1^{\mathrm{wk}}, \ldots, v_i^{\mathrm{wk}}),\ 1 \leqslant i \leqslant d-m. \tag{7.6}$$

Lemma 7.4. — *For $v, c \in \mathbb{R}^d$ we denote $w = E^T v$ and $\eta = E^{-1} c$, where E is defined in* (7.4)*. Explicitly, we have*

$$w = \begin{bmatrix} w^{\mathrm{st}} \\ w_1^{\mathrm{wk}} \\ \vdots \\ w_{d-m}^{\mathrm{wk}} \end{bmatrix} = \begin{bmatrix} v^{\mathrm{st}} \\ v_1^{\mathrm{wk}} - y_1^T X_1^{-1} \lfloor v \rfloor_0 \\ \vdots\ v_{d-m}^{\mathrm{wk}} - y_{d-m}^T X_{d-m}^{-1} \lfloor v \rfloor_{d-m-1} \end{bmatrix}, \tag{7.7}$$

and

$$\eta^{\mathrm{st}} = c^{\mathrm{st}} + A^{-1} B c^{\mathrm{wk}}, \quad \eta = (\eta^{\mathrm{st}}, \eta^{\mathrm{wk}}), c = (c^{\mathrm{st}}, c^{\mathrm{wk}}),$$

where A, B are defined in (2.4)*. Then we have*

$$(7.8)\quad L^s(\varphi, v) - c\cdot v$$
$$= L^{\mathrm{st}}(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) - \eta^{\mathrm{st}}\cdot v^{\mathrm{st}} + \sum_{i=1}^{d-m}\left(\frac{1}{2}\tilde z_i^{-1}(w_i^{\mathrm{wk}} - \tilde z_i\eta_i^{\mathrm{wk}})^2 - \frac{1}{2}z_i(\eta_i^{\mathrm{wk}})^2 + U_i^{\mathrm{wk}}(\varphi)\right).$$

Remark. — *This is a finer version of Lemma 4.2. In particular, the strong component $L^s - \eta^{\mathrm{st}}\cdot v^{\mathrm{st}}$ is identical to the $L^s - \bar c\cdot v^{\mathrm{st}}$, defined in Lemma 4.2.*

Proof. — Formula (7.7) can be read directly from the Definition (7.4) and $w = E^T v$. To show $\eta^{\mathrm{st}} = c^{\mathrm{st}} + A^{-1}Bc^{\mathrm{wk}}$, we compute

$$\begin{bmatrix} A & 0\\ 0 & * \end{bmatrix}\begin{bmatrix}\eta^{\mathrm{st}}\\ \eta^{\mathrm{wk}}\end{bmatrix} = \tilde S\eta = \tilde S E^{-1}c = E^T S c = \begin{bmatrix}\mathrm{Id}_m & 0\\ * & *\end{bmatrix}\begin{bmatrix} A & B\\ B^T & C\end{bmatrix}\begin{bmatrix}c^{\mathrm{st}}\\ c^{\mathrm{wk}}\end{bmatrix}.$$

The first block of the above equation yields $A\eta^{\mathrm{st}} = Ac^{\mathrm{st}} + Bc^{\mathrm{wk}}$, hence $\eta^{\mathrm{st}} = c^{\mathrm{st}} + A^{-1}Bc^{\mathrm{wk}}$.

We now prove (7.8). We have

$$L^s(\varphi, v) - c\cdot v = \frac{1}{2}v^T S^{-1} v - c^T v + U^{\mathrm{st}} + U^{\mathrm{wk}}$$
$$= \frac{1}{2}(E^T v)\tilde S^{-1}(E^T v) - (E^{-1}c)^T(E^T v) + U^{\mathrm{st}} + U^{\mathrm{wk}}$$
$$= \left(\frac{1}{2}w^{\mathrm{st}}\cdot A^{-1}w^{\mathrm{st}} - \eta^{\mathrm{st}}\cdot w^{\mathrm{st}} + U^{\mathrm{st}}\right) + \sum_{i=1}^{d-m}\left(\frac{1}{2}\tilde z_i^{-1}(w_i^{\mathrm{wk}})^2 - \eta_i^{\mathrm{wk}}w_i^{\mathrm{wk}} + U_i^{\mathrm{wk}}\right).$$

In the above formula, the first group is equal to $L^{\mathrm{st}} - \eta^{\mathrm{st}}\cdot v^{\mathrm{st}}$, noting $w^{\mathrm{st}} = v^{\mathrm{st}}$. Moreover

$$\frac{1}{2}\tilde z_i^{-1}(w_i^{\mathrm{wk}})^2 - \eta_i^{\mathrm{wk}}w_i^{\mathrm{wk}} = \frac{1}{2}\tilde z_i^{-1}(w_i^{\mathrm{wk}} - \tilde z_i\eta_i^{\mathrm{wk}})^2 - \frac{1}{2}\tilde z_i(\eta_i^{\mathrm{wk}})^2, \quad 1\leqslant i\leqslant d-m,$$

and (7.8) follows. ☐

We derive some useful estimates.

Lemma 7.5. — *There exists $M^* = M^*(\mathcal{B}^{\mathrm{st}}, Q, \kappa, q) > 1$ such that, for*

$$L^s = L_{\mathcal{H}^s}(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{st}}), \quad (\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{st}}) \in \Omega^{m,d}_{\kappa,q},$$

the following hold.

1. *For each $1\leqslant i\leqslant d-m$, we have $\sum_{j=i}^{d-m}\|U_j^{\mathrm{wk}}\|_{C^2} \leqslant M^*|k_i^{\mathrm{wk}}|^{-q}$.*
2. *For each $1\leqslant i\leqslant d-m$, $\tilde z_i^{-1} \leqslant M^*|k_i^{\mathrm{wk}}|^{2i}$.*

Proof. — For item 1, note that for each $j\geqslant i$, $|k_i^{\mathrm{wk}}|\leqslant \kappa|k_j^{\mathrm{wk}}|$, hence

$$\|U_j^{\mathrm{wk}}\|_{C^2} \leqslant \kappa|k_j^{\mathrm{wk}}|^{-q} \leqslant \kappa^{1+q}|k_i^{\mathrm{wk}}|^{-q}.$$

Item 1 holds for any $M^*\geqslant (d-m)\kappa^{1+q}$.

For item 2, inverting (7.3) we get

$$X_{i+1}^{-1} = \begin{bmatrix} \mathrm{Id}_{m+i-1} & -X_i^{-1}y_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_i^{-1} & 0 \\ 0 & \tilde{z}_i^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{m+i} & 0 \\ -y_{i+1}^T X_{i+1}^{-1} & 1 \end{bmatrix}.$$

Denote $f = (0, \ldots, 0, 1) \in \mathbb{T}^{m+i}$, then

$$f^T X_{i+1} f = f^T \begin{bmatrix} \mathrm{Id}_{m+i-1} & -X_i^{-1}y_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_i^{-1} & 0 \\ 0 & \tilde{z}_i^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{Id}_{m+i} & 0 \\ -y_{i+1}^T X_{i+1}^{-1} & 1 \end{bmatrix} f = \tilde{z}_i^{-1}.$$

Moreover, using the definition (see (1.3))

$$S = \partial^2_{II} K = \left[k_1^{\mathrm{st}} \cdots\ k_m^{\mathrm{st}}\ k_1^{\mathrm{wk}} \cdots\ k_{d-m}^{\mathrm{wk}}\right]^T Q \left[k_1^{\mathrm{st}} \cdots\ k_m^{\mathrm{st}}\ k_1^{\mathrm{wk}} \cdots\ k_{d-m}^{\mathrm{wk}}\right],$$

we have

$$\begin{aligned} X_{i+1} &= \left[k_1^{\mathrm{st}} \cdots\ k_m^{\mathrm{st}}\ k_1^{\mathrm{wk}} \cdots\ k_i^{\mathrm{wk}}\right]^T Q \left[k_1^{\mathrm{st}} \cdots\ k_m^{\mathrm{st}}\ k_1^{\mathrm{wk}} \cdots\ k_i^{\mathrm{wk}}\right] \\ &= \left[\bar{k}_1^{\mathrm{st}} \cdots\ \bar{k}_m^{\mathrm{st}}\ \bar{k}_1^{\mathrm{wk}} \cdots\ \bar{k}_i^{\mathrm{wk}}\right]^T Q_0 \left[\bar{k}_1^{\mathrm{st}} \cdots\ \bar{k}_m^{\mathrm{st}}\ \bar{k}_1^{\mathrm{wk}} \cdots\ \bar{k}_i^{\mathrm{wk}}\right] =: \bar{P}^T Q_0 \bar{P}, \end{aligned}$$

where $\bar{k}$ is the first n components of k. We have assumed $Q_0 \geqslant D^{-1}\mathrm{Id}$ for $D > 1$. By Lemma 3.3, there exists a constant $c_n > 1$ depending only on n such that

$$\begin{aligned} \|X_{i+1}^{-1}\| &= (\min_{\|v\|=1} v^T X_{i+1} v)^{-1} = (\min_{\|v\|=1} v^T \bar{P} Q_0 \bar{P})^{-1} \leqslant D\|\bar{P}^{-1}\|^2 \\ &\leqslant Dc_n |k_1^{\mathrm{st}}|^2 \cdots |k_m^{\mathrm{st}}|^2 |k_1^{\mathrm{wk}}|^2 \cdots |k_i^{\mathrm{wk}}|^2 \leqslant Dc_n \bar{M}^m \kappa^{i-1} |k_i^{\mathrm{wk}}|^{2i}, \end{aligned}$$

where $\bar{M} = |k_1^{\mathrm{st}}| + \cdots + |k_m^{\mathrm{st}}|$ depend only on $\mathcal{B}^{\mathrm{st}}$. □

7.3. Approximate Lipschitz property in the weak component. — In this section we prove Proposition 5.3. We fix $(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{st}}) \in \Omega^{m,d}_{\kappa,q} \cap \{\|U^{\mathrm{st}}\|_{C^2} \leqslant R\}$, and write $L^s = L_{\mathcal{H}^s}(\mathcal{B}^{\mathrm{wk}}, p, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{st}})$.

For $c \in \mathbb{R}^d$, we define

$$\begin{aligned} (7.9)\quad & L^s_{c,i}(\varphi^{\mathrm{st}}, \varphi_1^{\mathrm{wk}}, \ldots, \varphi_i^{\mathrm{wk}}, v^{\mathrm{st}}, v_1^{\mathrm{wk}}, \ldots, v_i^{\mathrm{wk}}) = L^s_{c,i}(\lfloor\varphi\rfloor_i, \lfloor v\rfloor_i) \\ &= L^{\mathrm{st}}(\varphi^{\mathrm{st}}, v^{\mathrm{st}}) - \eta^{\mathrm{st}}\cdot v^{\mathrm{st}} + \sum_{j=1}^{i} \left(\frac{1}{2}\tilde{z}_j^{-1}(w_j^{\mathrm{wk}} - \tilde{z}_j\eta_j^{\mathrm{wk}})^2 - \frac{1}{2}z_j(\eta_j^{\mathrm{wk}})^2 + U_j^{\mathrm{wk}}(\varphi)\right), \end{aligned}$$

then

$$\begin{aligned} (7.10)\quad & L^s(\varphi, v) - c\cdot v \\ &= L^s_{c,i}(\lfloor\varphi\rfloor_i, \lfloor v\rfloor_i) + \sum_{j=i+1}^{d-m} \left(\frac{1}{2}\tilde{z}_j^{-1}(w_j^{\mathrm{wk}} - \tilde{z}_j\eta_j^{\mathrm{wk}})^2 - \frac{1}{2}z_j(\eta_j^{\mathrm{wk}})^2 + U_j^{\mathrm{wk}}(\varphi)\right). \end{aligned}$$

Our proof of Proposition 5.3 follows an inductive scheme. Following our notational convention, denote $e_i^{\mathrm{wk}} = e_{i+m}$, which is the coordinate vector of φ_i^{wk}.

Lemma 7.6. — *Let $u:\mathbb{T}^d\to\mathbb{R}$ be a weak KAM solution of $L^s-c\cdot v$. Then for*

$$\delta_{d-m}:=2(\tilde z_{d-m}^{-1}\|U^{\mathrm{wk}}_{d-m}\|_{C^2})^{\frac12},$$

we have u is δ_{d-m}-semi-concave and δ_{d-m}-Lipschitz in $\varphi^{\mathrm{wk}}_{d-m}$.

Proof. — First we have

$$\partial^2_{\varphi^{\mathrm{wk}}_{d-m}\varphi^{\mathrm{wk}}_{d-m}}L^s=\partial^2_{\varphi^{\mathrm{wk}}_{d-m}\varphi^{\mathrm{wk}}_{d-m}}U^{\mathrm{wk}}_{d-m},\quad \partial^2_{\varphi^{\mathrm{wk}}_{d-m}v^{\mathrm{wk}}_{d-m}}L^s=0,\quad \partial^2_{v^{\mathrm{wk}}_{d-m}v^{\mathrm{wk}}_{d-m}}L^s=\tilde z_{d-m}^{-1}.$$

The first two equality follows directly from the definition, while the last one uses (7.7) and (7.8).

For any $\varphi\in\mathbb{T}^d$, let $\gamma:(-\infty,0]\to\mathbb{T}^d$ be a (u,L^s,c)-calibrated curve with $\gamma(0)=\varphi$. Then for any $T>0$

$$u(\varphi)=u(\gamma(-T))+\int_{-T}^0(L^s-c\cdot v+\alpha_{H^s}(c))(\gamma,\dot\gamma)dt.$$

Using the definition of the weak KAM solution,

$$u(\varphi+he_i^{\mathrm{wk}})\leqslant u(\gamma(-T))+\int_{-T}^0(L^s-c\cdot v+\alpha_{H^s}(c))(\gamma+\frac{th}{T}e^{\mathrm{wk}}_{d-m},\dot\gamma+\frac{h}{T}e^{\mathrm{wk}}_{d-m})dt.$$

Substract the two estimates, and apply Lemma 7.1 to $L^s-c\cdot v+\alpha_{H^s}(c)$ and γ, we get

$$\begin{aligned}u(\varphi+he_i^{\mathrm{wk}})-u(\varphi)&\leqslant(\partial_{v^{\mathrm{wk}}_{d-m}}L^s(\gamma(0),\dot\gamma(0))-c_{d-m})h\\&\quad+\left(\|\partial^2_{v_{d-m}v_{d-m}}L^s\|\frac1T+\|\partial^2_{\varphi^{\mathrm{wk}}_{d-m}v^{\mathrm{wk}}_{d-m}}L^s\|+T\|\partial^2_{\varphi^{\mathrm{wk}}_{d-m}\varphi^{\mathrm{wk}}_{d-m}}L^s\|\right)h^2\\&\leqslant(\partial_{v^{\mathrm{wk}}_{d-m}}L^s(\gamma(0)),\dot\gamma(0)-c^{\mathrm{wk}}_{d-m})h+\left(\tilde z_{d-m}^{-1}/T+\|U^{\mathrm{wk}}_{d-m}\|_{C^2}T\right)h^2,\end{aligned}$$

Take $T=(\tilde z_{d-m}\|U^{\mathrm{wk}}_{d-m}\|_{C^2})^{-\frac12}$, and write $l=\partial_{v^{\mathrm{wk}}_{d-m}}L^s(\gamma(0)),\dot\gamma(0)-c^{\mathrm{wk}}_{d-m}$, we get

$$u(\varphi+he_i^{\mathrm{wk}})-u(\varphi)\leqslant lh+\frac12\delta_{d-m}h^2.$$

The semi-concavity estimate follows. Using the fact that u is $\mathbb{Z}^d$ periodic, we take $h=l/|l|$ to get $|l|\leqslant\frac12\delta_{d-m}$. Therefore for $|h|\leqslant1$,

$$|u(\varphi+he_i^{\mathrm{wk}})-u(\varphi)|\leqslant(\frac12\delta_{d-m}+\frac12\delta_{d-m}h)h\leqslant\delta_{d-m}h.$$

This is the Lipschitz estimate. □

We now state the inductive step.

Proposition 7.7. — *Let $u : \mathbb{T}^d \to \mathbb{R}$ be a weak KAM solution of $L^s - c\cdot v$. Assume that for a given $1 \leqslant i \leqslant d-m-1$, u is (δ_j, δ_j) approximately Lipschitz in φ_j^{wk} for all $i+1 \leqslant j \leqslant d-m$. Then for*

$$\sigma_i = \left(\tilde{z}_i^{-1} \sum_{j=i}^{d-m} \|U_j^{\mathrm{wk}}\|_{C^2} \right)^{\frac{1}{2}}, \quad \delta_i = \sqrt{d}(6\sigma_i + 4\sum_{j=i+1}^{d-m} \delta_j),$$

we have u is (δ_i, δ_i) approximately Lipschitz in φ_i^{wk}.

Proof. — The proof is very similar to the proof of Proposition 5.4, but uses the finer decomposition in this section.

Since u is a weak KAM solution, then given any $\varphi \in \mathbb{T}^d$, there exists a calibrated curve $\gamma : (-\infty, 0] \to \mathbb{T}^d$ with $\gamma(0) = \varphi$. Then for any $T > 0$

$$u(\varphi) = u(\gamma(-T)) + \int_{-T}^0 (L^s - c\cdot v + \alpha_{H^s}(c))(\gamma, \dot\gamma)dt.$$

Let $h \in \mathbb{R}$, $\chi \in \mathbb{R}^d$, and a C^1 curve $\xi : [-T, 0] \to \mathbb{T}^d$ satisfies

$$\xi(-T) = \gamma(-T), \quad \xi(0) = \varphi + he_i^{\mathrm{wk}} + \chi,$$

then

(7.11)
$$\begin{aligned} u(\varphi + he_i^{\mathrm{wk}} + \chi) &\leqslant u(\gamma(-T)) + \int_{-T}^0 (L^s - c\cdot v + \alpha_{H^s}(c))(\xi, \dot\xi)dt \\ &\leqslant u(\gamma(-T)) + \int_{-T}^0 (L^s - c\cdot v + \alpha_{H^s}(c))(\gamma, \dot\gamma)dt \\ &\quad + \int_{-T}^0 (L^s - c\cdot v)(\xi, \dot\xi) - \int_{-T}^0 (L^s - c\cdot v)(\gamma, \dot\gamma)dt \\ &= u(\varphi) + \int_{-T}^0 (L^s - c\cdot v)(\xi, \dot\xi) - \int_{-T}^0 (L^s - c\cdot v)(\gamma, \dot\gamma)dt. \end{aligned}$$

We will first give the precise definition of ξ, then estimate (7.11), before finally obtain the desired estimate.

Definition of ξ. — Recall the Lagrangian $L_{c,i}^s : \mathbb{T}^{m+i} \times \mathbb{R}^{m+i} \to \mathbb{R}$ defined in (7.9). Let $\xi : [-T, 0] \to \mathbb{T}^{m+i}$ be an $L_{c,i}^s$ minimizing curve satisfying the constraint

$$\zeta(-T) = \lfloor\gamma\rfloor_i(-T), \quad \zeta(0) = \lfloor\gamma\rfloor_i(0),$$

where $\lfloor\cdot\rfloor_i$ is defined in (7.6). For $h \in \mathbb{R}$, we define ξ in the following way.

1. The first $m+i$ components of ξ is ζ with an added linear drift in e_i^{wk}, more precisely,

$$\lfloor\xi\rfloor_i(t) = \zeta(t) + \frac{th}{T} e_i^{\mathrm{wk}}. \tag{7.12}$$

2. We define the other components inductively. For $i < j \leqslant d-m$, suppose $\lfloor \xi \rfloor_{j-1}(t) = (\xi^{\mathrm{st}}, \xi_1^{\mathrm{wk}}, \dots, \xi_{j-1}^{\mathrm{wk}})(t)$ has been defined. We define
$$\xi_j^{\mathrm{wk}}(t) = \gamma_j^{\mathrm{wk}}(t) + y_j^T X_j^{-1} \lfloor \xi \rfloor_{j-1}(t) - y_j^T X_j^{-1} \lfloor \gamma \rfloor_{j-1}(t).$$

For each $i < j \leqslant d-m$, we have

$$\begin{cases} \xi_j^{\mathrm{wk}}(-T) = \gamma_j^{\mathrm{wk}}(-T), \\ \dot\xi_j^{\mathrm{wk}} - y_j^T X_j^{-1} \lfloor \dot\xi \rfloor_{j-1} = \dot\gamma_j^{\mathrm{wk}} - y_j^T X_j^{-1} \lfloor \dot\gamma \rfloor_{j-1}. \end{cases} \tag{7.13}$$

We define $\chi = \xi(0) - \varphi - h e_i^{\mathrm{wk}}$, and note that from (7.12),
$$\lfloor \chi \rfloor_i = \lfloor \xi \rfloor_i(0) - \lfloor \gamma \rfloor_i(0) - h e_i^{\mathrm{wk}} = 0.$$

Action comparison. — We now compute

$$\begin{aligned} &\int_{-T}^0 (L^s - c\cdot v)(\xi, \dot\xi)dt - \int_{-T}^0 (L^s - c\cdot v)(\gamma, \dot\gamma)dt \\ &\quad = \int_{-T}^0 L^s_{c,i}(\lfloor \xi \rfloor_i, \lfloor \dot\xi \rfloor_i)dt - \int_{-T}^0 L^s_{c,i}(\lfloor \gamma \rfloor_i, \lfloor \dot\gamma \rfloor_i)dt \\ &\qquad + \sum_{j=i+1}^{d-m} \int_{-T}^0 \big(U_j^{\mathrm{wk}}(\xi(t)) - U_j^{\mathrm{wk}}(\gamma(t))\big)\, dt \\ &\qquad + \frac12 \sum_{j=i+1}^{d-m} \tilde z_j^{-1} \int_{-T}^0 \big((\xi_j^{\mathrm{wk}} - y_j^T X_j^{-1} \lfloor \dot\xi \rfloor_{j-1} - \tilde z_j \eta_j^{\mathrm{wk}})^2 \\ &\qquad\quad - (\gamma_j^{\mathrm{wk}} - y_j^T X_j^{-1} \lfloor \dot\gamma \rfloor_{j-1} - \tilde z_j \eta_j^{\mathrm{wk}})^2\big) \\ &\quad \leqslant \int_{-T}^0 L^s_{c,i}(\lfloor \xi \rfloor_i, \lfloor \dot\xi \rfloor_i)dt - \int_{-T}^0 L^s_{c,i}(\lfloor \gamma \rfloor_i, \lfloor \dot\gamma \rfloor_i)dt + 2T \sum_{j=i+1}^{d-m} \|U_j^{\mathrm{wk}}\|_{C^0}. \end{aligned} \tag{7.14}$$

In the above formula, the equality is due to (7.10). Moreover, observe that from (7.13), the third line of the above formula vanishes. The inequality follows by replacing U_j^{wk} with its upper bound $\|U_j^{\mathrm{wk}}\|_{C^0}$.

We now have

$$\begin{aligned} &\int_{-T}^0 L^s_{c,i}(\lfloor \xi \rfloor_i, \lfloor \dot\xi \rfloor_i)dt - \int_{-T}^0 L^s_{c,i}(\lfloor \gamma \rfloor_i, \lfloor \dot\gamma \rfloor_i)dt \\ &\qquad = \int_{-T}^0 L^s_{c,i}(\lfloor \xi \rfloor_i, \lfloor \dot\xi \rfloor_i)dt - \int_{-T}^0 L^s_{c,i}(\zeta, \dot\zeta)dt \\ &\qquad\quad + \int_{-T}^0 L^s_{c,i}(\zeta, \dot\zeta)dt - \int_{-T}^0 L^s_{c,i}(\lfloor \gamma \rfloor_i, \lfloor \dot\gamma \rfloor_i)dt \\ &\qquad \leqslant \int_{-T}^0 L^s_{c,i}(\lfloor \xi \rfloor_i, \lfloor \dot\xi \rfloor_i)dt - \int_{-T}^0 L^s_{c,i}(\zeta, \dot\zeta)dt, \end{aligned}$$

noting that ζ is minimizing for $L^s_{c,i}$.

Since ζ is minimizing and hence extremal for $L^s_{c,i}$, from the definition of ξ in (7.12), Lemma 7.1 applies. Hence

$$\int_{-T}^{0} L^s_{c,i}(\lfloor\xi\rfloor_i, \lfloor\dot{\xi}\rfloor_i)dt - \int_{-T}^{0} L^s_{c,i}(\zeta,\dot{\zeta})dt \leqslant l\cdot h + \left(\frac{1}{T}\tilde{z}_i^{-1} + T\|\sum_{j=i}^{d-m} U_j^{\mathrm{wk}}\|_{C^2}\right)h^2,$$

where $l = \partial_{v_i}(L^s_{c,i})(\zeta(0),\dot{\zeta}(0))$. As in the proof of Lemma 7.6, we choose $T = \left(\tilde{z}_i \sum_{j=i}^{d-m}\|U_j^{\mathrm{wk}}\|_{C^2}\right)^{-\frac{1}{2}}$, we get
(7.15)

$$\int_{-T}^{0} L^s_{c,i}(\lfloor\xi\rfloor_i, \lfloor\dot{\xi}\rfloor_i)dt - \int_{-T}^{0} L^s_{c,i}(\zeta,\dot{\zeta})dt \leqslant l\cdot h + \sigma_i h^2, \sigma_i = \left(\tilde{z}_i^{-1}\sum_{j=i}^{d-m}\|U_j^{\mathrm{wk}}\|_{C^2}\right)^{\frac{1}{2}}.$$

Combine with (7.14), and use the upper bound $\sum_{j=i+1}^{d-m}\|U_j^{\mathrm{wk}}\|_{C^0} \leqslant \sum_{j=i}^{d-m}\|U_j^{\mathrm{wk}}\|_{C^2}$, we get

$$\int_{-T}^{0}(L^s - c\cdot v)(\xi,\dot{\xi})dt - \int_{-T}^{0}(L^s - c\cdot v)(\gamma,\dot{\gamma})dt \leqslant l\cdot h + \sigma_i h^2 + 2\sigma_i.$$

Estimating the weak KAM solution. — Combine the last formula with (7.11), we get

$$u(\varphi + he_i^{\mathrm{wk}} + \chi) - u(\varphi) \leqslant l\cdot h + \sigma_i h^2 + \sigma_i.$$

Since $\lfloor\chi\rfloor_i = 0$, using the inductive assumption,

$$|u(\varphi + he_i^{\mathrm{wk}} + \chi) - u(\varphi + he_i^{\mathrm{wk}})| \leqslant 2\sum_{j=i+1}^{d-m}\delta_j.$$

Therefore

$$u(\varphi + he_i^{\mathrm{wk}}) - u(\varphi) \leqslant l\cdot h + \sigma_i h^2 + 2\sigma_i + 2\sum_{j=i+1}^{d-m}\delta_j.$$

We now use Lemma 7.2 to get for

$$\delta_i = 2\sqrt{d}(3\sigma_i + 2\sum_{j=i+1}^{d-m}\delta_j),$$

u is (δ_i,δ_i) approximately Lipschitz in φ_i^{wk}. □

Proof of Proposition 5.3. — We have shown by induction that for all $1 \leqslant i \leqslant d-m$, u is (δ_i,δ_i) approximately Lipschitz in φ_i^{wk}, where δ_i are defined inductively in Lemma 7.6 and Proposition 7.7.

By Lemma 7.5, for each $1\leqslant i\leqslant d-m$

$$\sigma_i = (\tilde{z}_i^{-1}\|U_i^{\mathrm{wk}}\|_{C^2})^{\frac{1}{2}} \leqslant M^*|k_i^{\mathrm{wk}}|^{-\frac{q}{2}+i-m}.$$

Then $\delta_{d-m} = 2\sigma_{d-m} \leqslant M^*|k^{\mathrm{wk}}_{d-m}|^{-\frac{q}{2}+d-m}$. For each $1 \leqslant i \leqslant d-m$, we have (7.16)

$$\delta_i = \sqrt{d}(6\sigma_i + 4\sum_{j=i+1}^{d-m}\delta_i) \leqslant (6\sqrt{d})^{i-m}\sum_{j=i}^{d-m}\sigma_i \leqslant M^*(6\sqrt{d})^{i-m}|k^{\mathrm{wk}}_i|^{-\frac{q}{2}+d-m}.$$

For any $\varphi^{\mathrm{wk}}, \psi^{\mathrm{wk}} \in \mathbb{T}^{d-m}$ and $\varphi^{\mathrm{st}} \in \mathbb{T}^m$,

$$|u(\varphi^{\mathrm{st}}, \varphi^{\mathrm{wk}}) - u(\varphi^{\mathrm{st}}, \psi^{\mathrm{wk}})| \leqslant \sum_{i=1}^{d-m}\delta_i|\varphi^{\mathrm{wk}}_i - \psi^{\mathrm{wk}}_i| + \sum_{i=1}^{d-m}\delta_i$$

Since $\sum_{i=1}^{d-m}\delta_i \leqslant (d-m)M^*(6\sqrt{d})^{i-m}(\mu(\mathcal{B}^{\mathrm{wk}}))^{-\frac{q}{2}+d-m}$, the proposition follows by replacing M^* by $(d-m)M^*(6\sqrt{d})^{i-m}$. □

Appendix A. Normally hyperbolic invariant manifolds

In this section we state a version of the center manifold theorem and prove Theorem 2.4. While the central manifold theorem is classical, we need an version whose center direction is a non-compact set equipped with a Riemannian metric. This is done in the first two subsections. In the last subsection, we perform a reduction on our system H^s from (2.3) to apply the central manifold theorem.

A.1. Normally hyperbolic invariant manifolds via isolation block. — We state an abstract theorem on existence of normally hyperbolic invariant manifolds for a smooth map F, based on construction of Conley's isolating block (see McGehee, [26]).

We introduce a set of notations. We have three components $x \in \mathbb{R}^s, y \in \mathbb{R}^u, z \in \Omega^c \subset \mathbb{R}^c$, where Ω^c is a (possibly unbounded) convex set. We assume that Ω^c admits a C^1 complete Riemannian metric g. We also consider a Riemannian metric on the product space $W = \mathbb{R}^s \times \mathbb{R}^u \times \Omega^c$ by taking the tensor product of g and Ω^c, and the standard Euclidean metric on $\mathbb{R}^s, \mathbb{R}^u$.

Fix some $r > 0$ and let $D^s \subset \mathbb{R}^s$ and $D^u \subset \mathbb{R}^u$ be *closed* balls of radius r at the origin in $\mathbb{R}^s$ and $\mathbb{R}^u$ (r is considered fixed and we omit the dependence). Denote $D^{\mathrm{sc}} = D^s \times \Omega^c$, $D^{uc} = D^u \times \Omega^c$, and $D = D^{\mathrm{sc}} \times D^u$.

Consider a C^1 smooth map

$$F : D = D^s \times D^u \times \Omega^c \to \mathbb{R}^s \times \mathbb{R}^u \times \Omega^c,$$

we state a set of conditions guaranteeing the set

$$W^{\mathrm{sc}}(F) = \{Z \in D : F^k(Z) \in D \text{ for all } k > 0\},$$

called the center-stable manifold, is a graph i.e.,

$$W^{\mathrm{sc}}(F) = \{(X, Y) \in D^{\mathrm{sc}} \times D^u : w^{\mathrm{sc}}(X) = Y\}$$

for a C^1 function w^{sc}.

[C1] $\pi_{sc}F(D^{\mathrm{sc}} \times D^u) \subset D^{\mathrm{sc}}$.
[C2] F maps $D^{\mathrm{sc}} \times \partial D^u$ into $D^{\mathrm{sc}} \times \mathbb{R}^u \backslash D^u$ and is a homotopy equivalence.

The first two conditions guarantee a *topological isolating block*: F stretches $D^{\mathrm{sc}} \times B^u$ along the unstable component D^u and is a weak contraction along the center-stable component D^{sc}.

Now we state the *cone conditions*. For some $\mu > 0$

$$C_\mu^u(Z) = \{v = (v^c, v^s, v^u) \in T_Z D : \mu\|v^u\|^2 \geqslant \|v^c\|^2 + \|v^s\|^2\}.$$

Note that

$$(C_\mu^u(Z))^c = \{v = (v^c, v^s, v^u) \in T_Z D : \mu^{-1}(\|v^c\|^2 + \|v^s\|^2) \geqslant \|v^u\|^2\} =: C^{\mathrm{sc}}_{\mu^{-1}}(Z).$$

Let us also define

$$K_u^\mu(x_1, y_1, z_1) = \{(x_2, y_2, z_2) : \mu\|y_2 - y_1\|^2 \geqslant \|x_2 - x_1\|^2 + \operatorname{dist}(z_1, z_2)^2\},$$

where the distance is induced by the Riemannian metric g.

We assume there exist $\mu > 1$ and $\chi > 1$ with the property that for any $Z_1, Z_2 \in D$ such that $Z_2 \in K_u^\mu(Z_1)$ we have

[C3] $F(Z_2) \in K_\mu^u(F(Z_1))$.
[C4] $\|\pi_u(F(Z_2) - F(Z_1))\| \geqslant \chi\|\pi_u(Z_2 - Z_1)\|$.

Proposition A.1. — *(Lipschitz center-stable manifold theorem) Suppose F satisfies conditions [C1-C4], then $W^{\mathrm{sc}}(F)$ is given by the graph of a Lipschitz function*

$$W^{\mathrm{sc}}(F) = \{(x, y, z) \in D : w^{\mathrm{sc}}(x, z) = y\}.$$

Moreover, for if $W^{\mathrm{sc}}(F)$ is C^1, we have

$$T_Z W^{\mathrm{sc}}(F) \in C^{\mathrm{sc}}_{\mu^{-1}}(Z).$$

In order to obtain the center-unstable manifold, consider the involution $I : (x, y, z) \mapsto (y, x, z)$ and assume $\operatorname{inv}(F) = I \circ F^{-1} \circ I^{-1}$ satisfies the same conditions.

Theorem A.2. — *Assume that F and $\operatorname{inv}(F)$ satisfy the conditions [C1-C4], there exists a C^1 function $w^c : M \to D$ such that*

$$W^c(F) := W^{\mathrm{sc}}(F) \cap W^{uc}(F) = \{(x, y, z) \in D : w^c(z) = (x, y)\}.$$

Proof. — Proposition A.1 implies the existence of Lipschitz functions $w^{uc} : D^{uc} \to D$ and $w^{\mathrm{sc}} : D^{\mathrm{sc}} \to D$, with

$$W^{\mathrm{sc}}(F) = \{x = w^{\mathrm{sc}}(y, z)\}, \quad W^{uc}(F) = W^{\mathrm{sc}}(\operatorname{inv}(F)) = \{y = w^{uc}(x, z)\}.$$

Then standard arguments (see Theorem 5.18 in [27]) implies these functions are C^1. The fact that $\mu > 1$ and

$$T_Z W^{\mathrm{sc}}(F) \in C^{\mathrm{sc}}_{\mu^{-1}}(Z), \quad T_Z W^u(F) \in C^{\mathrm{sc}}_{\mu^{-1}}(Z)$$

implies $W^{\mathrm{sc}}(F)$ and $W^{uc}(F)$ intersect transversally, and $W^{\mathrm{sc}}(F) \cap W^{uc}(F)$ is a graph over the center component M. □

A.2. Existence of Lipschitz invariant manifolds. — We prove Proposition A.1. Let $\mathcal{V}$ be a collection of sets $\Gamma \subset D$ satisfying the following conditions:
(a) $\pi_u \Gamma = D^u$,
(b) $Z_2 \in K^u_\mu(Z_1)$ for all $Z_1, Z_2 \in \Gamma$, where π_u is the projection to the unstable component.

These conditions ensures $\pi_u : \Gamma \to D^u$ is one-to-one and onto, therefore, Γ is a graph over D^u. Moreover, condition (b) further implies that the graph is Lipschitz. In particular, each $\Gamma \in \mathcal{V}$ is a topological disk.

Lemma A.3. — *Let $\Gamma \in \mathcal{V}$, then $F(\Gamma) \cap D \in \mathcal{V}$.*

Proof. — By [C4] for any Z_1 and Z_2 we have that $F(Z_2)$ belongs to the cone $K^u_{F(Z_1)}$ of $F(Z_1)$. Thus, it suffices to show that $D^u \subset \pi_u(F(\Gamma) \cap D)$. The proof is by contradiction. Suppose there is $Z_* \in B^u$ such that $Z_* \notin \pi_u(F(\Gamma))$.

We have the following commutative diagram

$$
\begin{array}{ccc}
\partial\Gamma & \overset{i_1}{\hookrightarrow} & \Gamma \\
\downarrow \pi_u \circ F & & \downarrow \pi_u \circ F. \\
\mathbb{R}^u \backslash D^u & \overset{i_2}{\hookrightarrow} & \mathbb{R}^u \backslash \{Z_*\}
\end{array}
\tag{A.1}
$$

From [C2] and using the fact that B^s and Ω^c are contractible, $\pi_u \circ F|\Gamma$ is a homotopy equivalence. Note that i_2 is a homotopy equivalences, and $\pi_u \circ F|\Gamma$ is a homeomorphism onto its image. Let h and g be the homotopy inverses of $\pi_u \circ F|\partial\Gamma$ and i_2, then $h \circ g \circ (\pi_u \circ F)$ defines a homotopy inverse of i_1. As a result Γ is homotopic to $\partial\Gamma$, this is a contradiction. $\square$

Proposition A.1 follows from the following statement.

Proposition A.4. — *The mapping $\pi_{sc} : W^{\mathrm{sc}}(F) \to D^{\mathrm{sc}}$ is one-to-one and onto, therefore, it is the graph of a function w^{sc}. Moreover, w^{sc} is Lipschitz and*

$$T_Z W^{\mathrm{sc}}(F) \in (C^u_\mu(Z))^c = C^{\mathrm{sc}}_{\mu^{-1}}(Z), \quad Z \in W^{\mathrm{sc}}(F).$$

Proof. — For each $X \in D^{\mathrm{sc}}$, we define $\Gamma_X = (\pi_{sc})^{-1}X$, clearly $\Gamma_X \in \mathcal{V}$. We first show $\Gamma_X \cap W^{\mathrm{sc}}(F)$ is nonempty and consists of a single point. Assume first that $\Gamma_X \cap W^{\mathrm{sc}}(F)$ is empty. Then by definition of $W^{\mathrm{sc}}(F)$, there is $n \in \mathbb{N}$ such that $F^n(\Gamma_X) \cap D = \varnothing$. However, by Lemma A.3, $\bigcap_{i=1}^n F^i(\Gamma_X) \cap D \in \mathcal{V}$ is always nonempty, a contradiction. We now consider two points $Z_1, Z_2 \in W^{\mathrm{sc}}(F)$ with $\pi_u Z_1 = \pi_u Z_2$. Note that $F^k(Z_1), F^k(Z_2) \in D$ for all $k \geqslant 0$, and $Z_2 \in K^u_\mu(Z_1)$, by [C4] we have

$$2 \geqslant \|\pi_u(F^k(Z_1) - F^k(Z_2))\| \geqslant \chi^k \|\pi_u(Z_1 - Z_2)\|$$

for all k, which implies $Z_1 = Z_2$.

The last argument actually shows $Z_2 \notin K^u_\mu(Z_1)$ for all $Z_1, Z_2 \in W^{\mathrm{sc}}(F)$. For any $\epsilon > 0$, for $Z_1 = (X_1, Y_1), Z_2 = (X_2, Y_2) \in W^{\mathrm{sc}}(F)$ with $\mathrm{dist}(X_1, X_2)$ small, we have $\|Y_1 - Y_2\| \leqslant \mu^{-\frac{1}{2}}\mathrm{dist}(X_1, X_2)$. This implies both the Lipschitz and the cone properties in our proposition. □

A.3. NHIC for the dominant system. — We prove Theorem 2.4 in this section. First, an overview of notations.

1. The strong Hamiltonian is $H^{\mathrm{st}} = \mathcal{H}^{\mathrm{st}}(p_0, \mathcal{B}^{\mathrm{st}}, U^{\mathrm{st}})$ defined on $\mathbb{T}^m \times \mathbb{R}^m$, and its associated Lagrangian vector field is X^{st} (see (2.7)). We denote the time-1-map of X^{st} by G_0^{st} and lift it to the universal cover $\mathbb{R}^m \times \mathbb{R}^m$ without changing its name.
2. The vector field X^{st} is extended trivially to $(\mathbb{T}^m \times \mathbb{R}^m) \times (\mathbb{T}^{d-m} \times \mathbb{R}^{d-m})$ (see (2.10)). The time-1-map is denoted G_0, and we have $G_0 = G_0^{\mathrm{st}} \times \mathrm{Id}$. We will also lift it to the universal cover with the same name.
3. The slow Hamiltonian is $H^s = \mathcal{H}^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})$, and consider its Lagrangian vector field X^s_{Lag}.

   We apply a coordinate change $(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, v^{\mathrm{wk}}) = \Phi(\varphi^{\mathrm{st}}, v^{\mathrm{st}}, \varphi^{\mathrm{wk}}, I^{\mathrm{wk}})$ as in (2.8), and a rescaling Φ_Σ as defined in (2.12). The new vector field is denoted $\tilde{X}^s = (\Phi_\Sigma^{-1})_*(\Phi^{-1})_* X^s_{\mathrm{Lag}}$ (see (2.9), (2.12)). We denote its time-1-map G, which is considered a map on the Euclidean space $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R}^{d-m}$.
4. Below we also use notations from Section 2.4.

By Theorem 2.3, we have:

Corollary A.5. — *Assume that $(\mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}}) \in \Omega^{m,d}_{\kappa,q}(\mathcal{B}^{\mathrm{st}})$, then for any $\delta_1 > 0$, there exists $M > 0$ such that for all $(\mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})$ with $\mu(\mathcal{B}^{\mathrm{wk}}) > M$, uniformly on $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R}^{d-m}$, we have*

$$\|\Pi_{(\varphi^{\mathrm{st}}, v^{\mathrm{st}})}(G - G_0)\| < \delta_1, \quad \|DG - DG_0\| < \delta_1.$$

By assumption, the Hamiltonian flow H^{st}, (and hence G_0^{st}) admits an NHIC $\chi^{\mathrm{st}}(\mathbb{T}^l \times B^l_{1+a})$ with exponent α, β, where χ^{st} is an embedding. We use local coordinates in a tubular neighborhood to simplify the setting. (See also the left block of (A.3) below)

Lemma A.6. — *There exist a tubular neighborhood $N(\Lambda^{\mathrm{st}}_a) \subset \mathbb{T}^m \times \mathbb{R}^m$ of Λ^{st}_a and a C^1 diffeomorphism*

$$h^{\mathrm{st}} : B^{m-l}_1 \times B^l_1 \times (\mathbb{T}^l \times B^l_{1+a}) \to N(\Lambda^{\mathrm{st}}_a)$$

such that:

1. *$h^{\mathrm{st}}(0, 0, z) = \chi^{\mathrm{st}}(z)$, in particular, $h^{\mathrm{st}}(\mathcal{C}^{\mathrm{st}}_a) := h^{\mathrm{st}}(\{0\} \times \{0\} \times (\mathbb{T}^l \times B^l_{1+a})) = \Lambda^{\mathrm{st}}_a$.*
2. *For the map $F^{\mathrm{st}}_0 := (h^{\mathrm{st}}) \circ G^{\mathrm{st}}_0 \circ (h^{\mathrm{st}})^{-1}$:*
   (a) *$\mathcal{C}^{\mathrm{st}}_a$ is an NHIC for F^{st}_0 with the same exponents α, β.*

(b) *The associated stable/unstable bundles take the form*

$$E^s = \mathbb{R}^l \times \{0\} \times \{0\}, \quad E^u = \{0\} \times \mathbb{R}^l \times \{0\}.$$

In particular, DF_0^{st} is a block diagonal matrix in the blocks corresponding to the three components.

(c) *Let g_0 denote the Euclidean metric. Then there exists a Riemannian metric g on $\mathbb{T}^l \times B^l_{1+a}$ such that the tensor metric $g_0 \otimes g_0 \otimes g$ on $B_1^l \times B_1^l \times (\mathbb{T}^l \times B^l_{1+a})$ is an adapted metric for the NHIC $\mathcal{C}_a^{\mathrm{st}}$.*

Proof. — We use the bundles E^u, E^s, and the parametrization χ^{st} of Λ_a^{st} to build a coordinate system for the normal bundle to Λ_a^{st}, which is diffeomorphic to the tubular neighborhood. We then pull back the adapted metric of Λ_a^{st} using this map to $\mathcal{C}_a^{\mathrm{st}}$. □

Denote $\Omega^{\mathrm{wk}} = \mathbb{R}^{d-m} \times \mathbb{R}^{d-m}$ and consider the trivial extension

$$h : B_1^l \times B_1^l \times ((\mathbb{T}^l \times B^l_{1+a}) \times \Omega^{\mathrm{wk}}) \to N(\Lambda_a^{\mathrm{st}}) \times \Omega^{\mathrm{wk}}$$

by $h(x, y, (z^{\mathrm{st}}, z^{\mathrm{wk}})) = (h^{\mathrm{st}}(x, y, z^{\mathrm{st}}), z^{\mathrm{wk}})$. Define the following maps

$$F_0 = h^{-1} \circ G_0 \circ h = (F_0^{\mathrm{st}}, \mathrm{Id}), \quad F = h^{-1} \circ G \circ h. \tag{A.2}$$

See diagram below, where "i" denote standard embeddings, $\Omega_a^{\mathrm{st}} = \mathbb{T}^l \times B^l_{1+a}$ and $\circlearrowleft \cdot$ denotes the (unperturbed) map of the given space.

(A.3)

$$\begin{array}{ccccc}
\Lambda_a^{\mathrm{st}} & \xrightarrow{i} & N(\Lambda_a^{\mathrm{st}}) \subset (\mathbb{T}^m \times \mathbb{R}^m) \circlearrowleft G_0^{\mathrm{st}} & \xrightarrow{i} & (\mathbb{T}^m \times \mathbb{R}^m) \times \Omega^{\mathrm{wk}} \circlearrowleft G_0 \\
\chi^{\mathrm{st}} \uparrow & & h^{\mathrm{st}} \uparrow & & h \uparrow \\
\Omega_a^{\mathrm{st}} & \xrightarrow{i} & B_1^{m-l} \times B_1^{m-l} \times \Omega_a^{\mathrm{st}} \circlearrowleft F_0^{\mathrm{st}} & \xrightarrow{i} & B_1^{m-l} \times B_1^{m-l} \times (\Omega_a^{\mathrm{st}} \times \Omega^{\mathrm{wk}}) \circlearrowleft F_0.
\end{array}$$

Finally, to apply Theorem A.2, we lift Ω_a^{st} to the universal cover $\mathbb{R}^l \times B^l_{1+a}$, and the maps F_0, F to the covering space without changing their names, namely

$$F, F_0 : B_1^l \times B_1^l \times (\Omega_a^{\mathrm{st}} \times \Omega^{\mathrm{wk}}) \circlearrowleft.$$

$\Omega^{\mathrm{st}}_{1+a} \times \Omega^{\mathrm{wk}}$ is our center component and is denoted by Ω. While the maps are defined on unbounded regions, we keep in mind that $F_0 = (F_0^{\mathrm{st}}, \mathrm{Id})$ where F_0^{st} is defined on a compact set $B_1^l \times B_1^l \times (\mathbb{T}^l \times B^l_{1+a})$.

We still need one reduction to apply Theorem A.2. Recall that $\Omega_a^{\mathrm{st}} = \mathbb{R}^l \times B^l_{1+a}$. Write $F_0 = (F_0^x, F_0^y, F_0^z)$, define

$$L(x, y, z) = (D_x F_0^x(0, 0, z) \cdot x, D_y F_0^y(0, 0, y) \cdot y, F_0^z(0, 0, z)) \tag{A.4}$$

this is the linearized map at $(0, 0, z)$ (we used $F_0(0, 0, z) = (0, 0, F_0^z)$, and DF_0 is block diagonal from Lemma A.6). since $F_0 = (F_0^{\mathrm{st}}, \mathrm{Id})$ and F_0^{st} is defined over a compact set, we obtain as $r \to 0$,

$$\|L - F_0\| = o(r), \quad \|DL - DF_0\| = o(1) \text{ on } B_r^l \times B_r^l \times (\Omega_a^{\mathrm{st}} \times \Omega^{\mathrm{wk}}). \tag{A.5}$$

Moreover, since F_0 preserves $\{0\} \times \{0\} \times \partial\Omega$, we get

$$L(B_1^l \times B_1^l \times \partial\Omega) \subset \mathbb{R}^l \times \mathbb{R}^l \times \partial\Omega. \tag{A.6}$$

Namely, the linearized map L preserves the boundary of the center component. Finally, we modify the map F so that it also fixes the center boundary. Let ρ be a standard mollifier satisfying

$$\begin{cases} \rho(x, y, (z^{\mathrm{st}}, z^{\mathrm{wk}})) = \rho(z^{\mathrm{st}}) = 0 & z^{\mathrm{st}} \in \Omega_0^{\mathrm{st}} \\ \rho(x, y, (z^{\mathrm{st}}, z^{\mathrm{wk}})) = \rho(z^{\mathrm{st}}) = 1 & z^{\mathrm{st}} \in \Omega_a^{\mathrm{st}} \setminus \Omega_{a/2}^{\mathrm{st}}. \end{cases}$$

Let

$$\tilde{F} = F(1-\rho) + L\rho, \tag{A.7}$$

we have:

Lemma A.7. — *For any $\mu > 1$, $\epsilon > 0$, and $r_0 > 0$, there exist $\delta_1 > 0$ and $0 < r < r_0$ such that if G and G_0 satisfy*

$$\|\Pi_{(\varphi^{\mathrm{st}}, v^{\mathrm{st}})}(G - G_0)\| < \delta_1, \quad \|DG - DG_0\| < \delta_1,$$

then the map $\tilde{F}$, defined by (A.2), (A.4), *and* (A.7), *satisfies conditions* [C1]–[C4] *with the parameters μ and $\chi = \alpha^{-1} - \epsilon$ on $B_r^l \times B_r^l \times \Omega$. The same hold for the map* $\mathrm{inv}(\tilde{F})$.

Proof. — First of all, from Lemma A.6, $DF_0(0,0,z) = \mathrm{diag}\{D_x F_0^x, D_y F_0^y, D_z F_0^z\}$ with

$$\|D_x F_0^x\|, \|(D_y F_0^y)^{-1}\|^{-1} \leqslant \alpha, \quad \|D_z F_0^z\|, \|(D_z F_0^z)^{-1}\| \leqslant \beta^{-1}. \tag{A.8}$$

Recall that $F_0 = (F_0^{\mathrm{st}}, \mathrm{Id})$ where F_0^{st} is defined over a compact set. Therefore, for sufficiently small $r > 0$, we have

$$\|\Pi_x F_0(x,y,z)\| \leqslant (\alpha + \epsilon)\|x\|, \quad \|\Pi_y DF_0(x,y,z)\| \geqslant (\alpha+\epsilon)^{-1}\|y\|,$$

hence

$$\Pi_x F_0(B_r^l \times B_r^l \times \Omega) \subset B_{(\alpha+\epsilon)r}^l, \quad \|\Pi_y F_0(B_r^l \times \partial B_r^l \times \Omega)\| \geqslant (\alpha+\epsilon)^{-1} r.$$

Since $\|\tilde{F} - F_0\| \leqslant \|(1-\rho)(F - F_0)\| + \|\rho(L - F_0)\|$,

$$\|\Pi_{(x,y)}(F - F_0)\| \leqslant C\|\Pi_{(\varphi^{\mathrm{st}}, v^{\mathrm{st}})}(G - G_0)\| \leqslant C\delta_1,$$

and $\|L - F_0\| = o(r)$, by choosing δ_1, r small enough, we get

$$\Pi_x \tilde{F}(B_r^l \times B_r^l \times \Omega) \subset B_r^l, \quad \|\Pi_y \tilde{F}(B_r^l \times \partial B_r^l \times \Omega)\| > r.$$

The first half of the above formula, combined with (A.6), gives [C1], and the second half gives [C2].

We now prove the cone conditions [C3] and [C4]. We first show the map $\tilde{F}$ is well approximated by the linearized map $DF_0(0,0,z)$. Given any $\epsilon > 0$, we use Corollary A.5 to choose δ_1 so small such that

$$\|D(F - F_0)\| + \|\Pi_{z^{\mathrm{st}}}(F - F_0)\|\|d\rho\| \leqslant C\|D(G - G_0)\| + \|\Pi_{\varphi^{\mathrm{st}}, v^{\mathrm{st}}}(G - G_0)\|\|d\rho\| < \epsilon/2$$

By (A.5), we can choose r_1 such that for $0 < r < r_1$,

$$\|D(F_0 - L)\| + \|F_0 - L\|\|d\rho\| < \epsilon/2$$

on $B^l_r \times B^l_r \times (\Omega^{\rm st}_a \times \Omega^{\rm wk})$. Then from $\tilde{F} = F + (L - F)\rho$, and the fact that ρ depends only on $z^{\rm st}$ gives

$$\begin{aligned}
&\|D\tilde{F}(x,y,z) - DL(x,y,z)\| \leqslant \|DF - DL\| + \|\Pi_{z^{\rm st}}(F - L)\|\|d\rho\| \\
&\leqslant \|D(F - F_0)\| + \|D(F_0 - L)\| + (\|\Pi_{z^{\rm st}}(F - F_0)\| + \|\Pi_{z^{\rm st}}(F_0 - L)\|)\|d\rho\| < \epsilon.
\end{aligned}$$

Consider $(x_1, y_1, z_1), (x_2, y_2, z_2) \in B^l_r \times B^l_r \times \Omega$, denote $(\Delta x, \Delta y, \Delta z) = (x_2, y_2, z_2) - (x_1, y_1, z_1)$ and $d = \|\Delta x\| + \|\Delta y\| + \mathrm{dist}(z_1, z_2)$. For d small enough

$$\begin{aligned}
&\|\tilde{F}(x_2, y_2, z_2) - \tilde{F}(x_1, y_1, z_1) - DF_0(0,0,z)(\Delta x, \Delta y, \Delta z)\| \\
&\quad = \|L(x_2, y_2, z_2) - L(x_1, y_1, z_1) + (\tilde{F} - L)(x_2, y_2, z_2) - (\tilde{F} - L)(x_1, y_1, z_1) \\
&\qquad - DF_0(0,0,z_1)(\Delta x, \Delta y, \Delta z)\| \\
&\quad \leqslant \|D(\tilde{F} - L)(x_1, y_1, z_1)\| d \leqslant \epsilon d.
\end{aligned}$$

To prove [C3], we first show the linear map preserves the unstable cone. For any $\mu > 1$ and $(v_x, v_y, v_z) \in T_{(x_1,y_1,z_1)} B^l_r \times B^l_r \times \Omega$ with $\mu\|v^y\|^2 \geqslant \|v^z\|^2 + \|v^x\|^2$, let $(v'_x, v'_y, v'_z) = DF_0(0,0,z)(v_x, v_y, v_z)$, we have

$$\begin{aligned}
\mu\|v'_y\|^2 &\geqslant \mu\alpha^{-1}\|v_y\|^2 \geqslant \alpha^{-1}(\|v_x\|^2 + \|v_z\|^2) \\
&\geqslant \alpha^{-1}(\|v'_x\|^2 + \beta\|v'_z\|^2) \geqslant \frac{\beta}{\alpha}(\|v'_x\|^2 + \|v'_z\|^2).
\end{aligned}$$

In other words, for any $\mu > 1$, we have $DF_0(0,0,z)C^u_\mu \subset C^u_{\alpha\mu/\beta}$.

Coming to the non-linear map $\tilde{F}$, for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in B^l_r \times B^l_r \times \Omega$, let $(x'_i, y'_i, z'_i) = \tilde{F}(x_i, y_i, z_i)$, and $(\Delta x, \Delta y, \Delta z)$, $(\Delta x', \Delta y', \Delta z')$ be the corresponding difference. If $(x_2, y_2, z_2) \in K^u_\mu(x_1, y_1, z_1)$, then $\mu\|\Delta y\|^2 \geqslant \|\Delta x\|^2 + \mathrm{dist}(z_1, z_2)^2$. In particular, $\mathrm{dist}(z_1, z_2)^2 \leqslant \mu\|\Delta y\|^2 \leqslant \mu r^2$. When r is small enough

$$\|(\Delta x', \Delta y', \Delta z') - DF_0(0,0,z_1)(\Delta x, \Delta y, \Delta z)\| \leqslant \epsilon d.$$

Furthermore, assume r is so small that $(1 - \epsilon)\mathrm{dist}(z, z') \leqslant \|\Delta z\|_{(0,0,z_1)} \leqslant (1 + \epsilon)\mathrm{dist}(z, z')$, where $\|\Delta z\|_{(0,0,z_1)}$ is measured using the local Riemannian metric. We drop the subscript from now on. Using the linear calculation, there exists a uniform constant $C > 1$ such that

$$\begin{aligned}
\mu\|\Delta y'\|^2 &\geqslant \frac{\beta}{\alpha}(\|\Delta x'\|^2 + \|\Delta z'\|^2) - C\epsilon^2 d^2 \\
&\geqslant \frac{\beta}{\alpha}(\|\Delta x'\|^2 + \|\Delta z'\|^2) - C(1+\mu)\epsilon^2\|\Delta y\|^2 \\
&\geqslant (1-\epsilon)\frac{\beta}{\alpha}(\|\Delta x'\|^2 + \mathrm{dist}(z'_1, z'_2)^2) - C^2(1+\mu)\epsilon^2\|\Delta y'\|^2,
\end{aligned}$$

noting that $\|D\tilde{F}^{-1}\|, \|D\tilde{F}\|$ are uniformly bounded. When ϵ is small enough, we get $\mu\|\Delta y\|^2 \geqslant \|\Delta x\|^2 + \operatorname{dist}(z_1', z_2')^2$. Thus, [C3] is proven.

[C4] follows directly from $\|(\Delta x', \Delta y', \Delta z') - DF_0(0,0,z_1)(\Delta x, \Delta y, \Delta z)\| \leqslant \epsilon d$ and (A.8). The proof for $\operatorname{inv}(\tilde{F})$ is identical and is omitted. □

Proof of Theorem 2.4. — For any $\delta > 0$, we choose $0 < r < \delta/C$ and $\mu > 1$, where C is a constant specified later. Apply Lemma A.7, there exists $M > 0$ such that whenever $\mu(\mathcal{B}^{\mathrm{wk}}) > M$, the map $\tilde{F}$ associated to $H^s(\mathcal{B}^{\mathrm{st}}, \mathcal{B}^{\mathrm{wk}}, p_0, U^{\mathrm{st}}, \mathcal{U}^{\mathrm{wk}})$ satisfies [C1]–[C4] on $B_r^l \times B_r^l \times (\Omega_a^{\mathrm{st}} \times \Omega^{\mathrm{wk}})$. As a result, we obtain a function $w^c : \Omega_a^{\mathrm{st}} \times \Omega^{\mathrm{wk}} \to B_r^{m-l} \times B_r^{m-l}$ such that

$$W^c = \operatorname{Graph}(w^c) = \{(x, y, (z^{\mathrm{st}}, z^{\mathrm{wk}})) : (x, y) = w^c(z^{\mathrm{st}}, z^{\mathrm{wk}})\}$$

is invariant under $\tilde{F}$, and is the maximally invariant set on $B_r^{m-l} \times B_r^{m-l} \times \Omega_a^{\mathrm{st}} \times \Omega^{\mathrm{wk}}$. Since $\tilde{F} = F$ on whenever $z^{\mathrm{st}} \in \Omega_{a/2}^{\mathrm{st}}$, any F invariant set in $U_1 := B_r^{m-l} \times B_r^{m-1} \times \Omega_{a/2}^{\mathrm{st}} \times \Omega^{\mathrm{wk}}$ is also $\tilde{F}$ invariant and hence is contained in W^c.

We now consider the map

$$\zeta : \Omega_0^{\mathrm{st}} \times \Omega^{\mathrm{wk}} \to (\mathbb{R}^m \times \mathbb{R}^m) \times (\mathbb{R}^{d-m} \times \mathbb{R}^{d-m}),$$
$$(z^{\mathrm{st}}, z^{\mathrm{wk}}) \mapsto h(w^c(z^{\mathrm{st}}, z^{\mathrm{wk}}), z^{\mathrm{st}}, z^{\mathrm{wk}}),$$

then $\zeta(\Omega_0^{\mathrm{st}} \times \Omega^{\mathrm{wk}})$ is weakly invariant for the vector field $\tilde{X}^s$, and maximally invariant on the set $U_2 := h(B_r^{m-l} \times B_r^{m-1} \times \Omega_0^{\mathrm{st}} \times \Omega^{\mathrm{wk}})$.

Finally, inverting the coordinate changes, we obtain for

$$\eta^s = \Phi \circ \Phi_\Sigma \circ \zeta, \quad \eta^s : \Omega_0^{\mathrm{st}} \times \Omega^{\mathrm{wk}} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{d-m} \times \mathbb{R}^{d-m},$$

$\eta^s(\Omega_0^{\mathrm{st}} \times \Omega^{\mathrm{wk}})$ is weakly invariant for $X^s_{\mathrm{Lag}} = (\Phi)_*(\Phi_\Sigma)_*\tilde{X}^s$, and maximally invariant on $U_3 = \Phi \circ \Phi_\Sigma(U_2)$.

Since h is identity in the weak component

$$U_2 = h^{\mathrm{st}}(B_r^{m-l} \times B_r^{m-1} \times \Omega_0^{\mathrm{st}}) \times \Omega^{\mathrm{wk}}.$$

The coordinate changes Φ and Φ_Σ does not change $(\varphi^{\mathrm{st}}, v^{\mathrm{st}})$, therefore,

$$U_3 = \Phi \circ \Phi_\Sigma(V_2) = h^{\mathrm{st}}(B_r^{m-l} \times B_r^{m-1} \times \Omega_0^{\mathrm{st}}) \times \Omega^{\mathrm{wk}} =: V \times \Omega^{\mathrm{wk}}.$$

Moreover, using the fact that $h^{\mathrm{st}}|_{\{0\}\times\{0\}\times\Omega_0^{\mathrm{st}}} = \chi^{\mathrm{st}}|_{\Omega_0^{\mathrm{st}}}$, and $\|w^c\|_{C^0} < r$ for small enough r we have for some $C > 0$, uniformly over all z^{wk}

$$\|h^{\mathrm{st}}(w^c(z^{\mathrm{st}}, z^{\mathrm{wk}}), z^{\mathrm{st}}) - \chi^{\mathrm{st}}(z^{\mathrm{st}})\| \leqslant Cr.$$

Since $h(x, y, z^{\mathrm{st}}, z^{\mathrm{wk}}) = (h^{\mathrm{st}}(x, y, z^{\mathrm{st}}), z^{\mathrm{wk}})$, we get $\|\Pi_{\varphi^{\mathrm{st}}, v^{\mathrm{st}}}\zeta - \chi^{\mathrm{st}}\|_{C^0} < Cr$, where we abuse notation by writing $\chi^{\mathrm{st}}(z^{\mathrm{st}}, z^{\mathrm{wk}}) = z^{\mathrm{st}}(z^{\mathrm{st}})$. Finally, since Φ, Φ_Σ are identity in $\varphi^{\mathrm{st}}, v^{\mathrm{st}}$, we have

$$\|\Pi_{\varphi^{\mathrm{st}}, v^{\mathrm{st}}}\Phi \circ \Phi_\Sigma \circ \zeta - \chi^{\mathrm{st}}\|_{C^0} = \|\Pi_{\varphi^{\mathrm{st}}, v^{\mathrm{st}}}\zeta - \chi^{\mathrm{st}}\|_{C^0} < Cr.$$

We obtain $\|\Pi_{\varphi^{\mathrm{st}}, v^{\mathrm{st}}}\eta^s - \chi^{\mathrm{st}}\| < \delta$. □

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TOPOLOGICAL SUBSTITUTION FOR THE APERIODIC RAUZY FRACTAL TILING

BY NICOLAS BÉDARIDE, ARNAUD HILION & TIMO JOLIVET

ABSTRACT. — We consider two families of planar self-similar tilings of different nature: the tilings consisting of translated copies of the fractal sets defined by an iterated function system, and the tilings obtained as a geometrical realization of a topological substitution (an object of purely combinatorial nature, defined in [6]). We establish a link between the two families in a specific case, by defining an explicit topological substitution and by proving that it generates the same tilings as those associated with the Tribonacci Rauzy fractal.

RÉSUMÉ (*Substitution topologique pour la pavage fractal apériodique de Rauzy*). — On considère deux familles de pavages auto-similaires de nature différente : ceux obtenus par translation de copies d'un ensemble fractal défini par un système de fonctions itérées, et ceux obtenus comme la réalisation géométrique d'une substitution topologique (un objet purement combinatoire, défini dans [6]). On établit un lien entre les deux familles dans un cas particulier, en définissant une substitution topologique explicitement puis en démontrant qu'elle engendre les mêmes pavages que ceux associés au fractal Tribonacci de Rauzy.

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1. Introduction

1.1. Main result and motivation. — Self-similar tilings of the plane are characterized by the existence of a common subdivision rule for each tile, such that the tiling obtained by subdivising each tile is the same as the original one, up to a contraction. These tilings have been introduced by Thurston [32] and they are studied in several fields including dynamical systems and theoretical physics, see [5]. A particular class of self-similar tilings arises from *substitutions*, which are "inflation rules" describing how to replace a geometrical shape by a union of other geometrical shapes (within a finite set of basic shapes). Among these, an important class consists of the planar tilings by the so-called Rauzy fractals associated with some one-dimensional substitutions. These fractals are used to provide geometrical interpretations of substitution dynamical systems. They also provide an interesting class of aperiodic self-similar tilings of the plane, see [16, 8].

The aim of this article is to establish a formal link between two self-similar tilings constructed from two different approaches:

- Using an *iterated function system (IFS)*, that is, specifying the shapes and the positions of the tiles with planar set equations (using contracting linear maps), which define the tiles as unions of smaller copies of other tiles. In particular, an IFS does make use of the Euclidean metric of the plane.
- Using a *topological substitution*, that is, specifying which tiles are allowed to be neighbors, and how the neighboring relations are transferred when we "inflate" the tiles by substitution to construct the tiling. With this kind of substitution, there is no use in anyway of a the Euclidean metric: the tiles do not have a metric shape (they are just topological disks).

In other words, we tackle the following question:

> *Given a tiling defined by an IFS, is there a topological substitution which generates an equivalent tiling? If yes, how can we construct it? In other words, when is it possible to describe the geometry of a self-similar tiling (geometrical constraints) by using a purely combinatorial rule (combinatorial constraints) ?*

In this article we answer this question for the tilings of the plane by translated copies of the Rauzy fractals associated with the Tribonacci substitution (which are defined by an IFS). We define a particular topological substitution σ (Figure 3.3, p. 588) and we prove that the Tribonacci fractal tiling $\mathcal{T}_{\mathsf{frac}}$ and the tiling $\mathcal{T}_{\mathsf{top}}$ generated by the topological substitution are equivalent in a strong way. More precisely:

- Associated with the Tribonacci substitution $s : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$, there is a dual substitution $\mathbf{E}$ (see Section 4.2) which acts on facets in $\mathbb{R}^3$. Iteration of this dual substitution gives rise to a stepped surface Σ_{step} (a surface which is a union of facets), that is included in the 1-neighborhood of some (linear) plane $\mathcal{P}$ in $\mathbb{R}^3$. Projecting the stepped surface Σ_{step} (and its facets) on $\mathcal{P}$ gives rise to a tiling $\mathcal{T}_{\mathsf{step}}$ of $\mathcal{P}$. It is known [2, 8] that the tiling $\mathcal{T}_{\mathsf{frac}}$ is strongly related to a tiling $\mathcal{T}_{\mathsf{step}}$.

- The topological substitution σ can be iterated on a tile C, giving rises to a 2-dimensional CW-complex $\sigma^\infty(C)$ homeomorphic to a plane, see Section 3.2. However, this complex is not embedded a priori in a plane, even if it turns out that $\sigma^\infty(C)$ can be effectively realized as a tiling $\mathcal{T}_{\mathsf{top}}$ of the plane, see Proposition 3.11. To locate a tile T in $\sigma^\infty(C)$ relatively to another one T', we build a vector (an *"position"*) $\omega_0(T, T') \in \mathbb{Z}^3$: by construction, this vector depends a priori on the choice of a combinatorial path from T to T' in $\sigma^\infty(C)$, and we have to prove that in fact it is independent of the path, see Section 5.1.
- Since it is already explained in the literature how to relate $\mathcal{T}_{\mathsf{frac}}$ and Σ_{step}, and since we explain how $\mathcal{T}_{\mathsf{top}}$ is build from $\sigma^\infty(C)$, the main result of the paper is Theorem 5.16 that states an explicit formula which define a bijection Ψ between tiles in $\sigma^\infty(C)$ and facets in Σ_{step}: we reproduce it just below.

THEOREM. — The map Ψ defined, for every tile T of $\sigma^\infty(C)$, by:

$$\Psi(T) = [\mathbf{M}_s^3(\omega_0(T, C) + \mathbf{u}_{\mathrm{type}(T)}), \theta(\mathrm{type}(T))]^* \tag{1.1}$$

is a bijection from the set of tiles of $\sigma^\infty(C)$ to the set of facets of Σ_{step}.

The notation used to state this theorem will be introduced along the paper. But we want to stress that the fact the formula (1.1) makes use of the position map ω_0 ensures that if two tiles T and T' are close in $\sigma^\infty(C)$, then their images $\Psi(T)$ and $\Psi(T')$ will be close in Σ_{step}. In fact, it is easy to convince oneself that something like that should be true by having a look at Figure 1.1, where three corresponding subsets of the tilings $\mathcal{T}_{\mathsf{top}}$, $\mathcal{T}_{\mathsf{frac}}$ and $\mathcal{T}_{\mathsf{step}}$ are given.

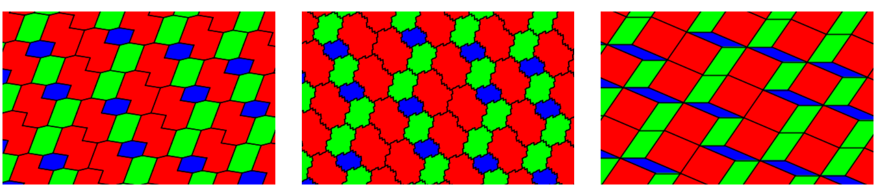

FIGURE 1.1. The three tilings $\mathcal{T}_{\mathsf{top}}$, $\mathcal{T}_{\mathsf{frac}}$ and $\mathcal{T}_{\mathsf{step}}$ (from left to right).

On Figure 1.1, it is also worth to notice that the underlying CW-complexes of $\mathcal{T}_{\mathsf{top}}$ and $\mathcal{T}_{\mathsf{step}}$ are not the same. Indeed, the valence of a vertex in $\mathcal{T}_{\mathsf{top}}$ is either 2 or 3, whereas the valence of a vertex in $\mathcal{T}_{\mathsf{step}}$ can be equal to 3, 4, 5 or 6. In that sense, the two tilings $\mathcal{T}_{\mathsf{step}}$ and $\mathcal{T}_{\mathsf{top}}$ are really different.

We have chosen to present our results on a specific substitution rather than in a general form because it makes presentation clearer and it avoids many "artificial" technicalities. Moreover, we do not know what a general answer to the above question may look like. However, we give some insight about this general question in Section 6.

1.2. Comparison of some different notions of substitutions. — The word "substitution" is used in many different ways in the literature. The list below reviews several such notions, going from the most geometrical one (IFS) to the most combinatorial one (topological substitutions). Indeed, as observed by Peyrière [26], having a combinatorial description of substitutive tiling turns out to be very useful in many situations. This list is not exhaustive, it only contains the notions of substitutions that we explicitly use in this article. See [18] for another survey about geometrical substitutions.

One-dimensional symbolic substitutions. — These substitutions are used to generated infinite one-dimensional words which are studied mostly for their word-theoretical and dynamical properties. An example is the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ defined in Section 4.3. See [16] for a classical reference. This is the only notion of the present list which is only symbolic (not geometrical).

Self-affine substitutions (iterated function systems). — Also known as *substitution Delone sets* [24], this notion is a particular class of iterated functions systems, where it is required that the geometrical objects defined by the IFS are compact sets which are the closure of their interior, in such a way that tilings can be defined. See Proposition 4.5 for an example of such a definition for the Tribonacci fractal.

Dual (or "generalized") substitutions. — These substitutions, introduced in [4] can be seen as a discrete version of self-affine substitutions. Instead of defining fractal tilings in a purely geometrical way (like with IFS), these substitutions act on unions of faces of unit cubes located at integer coordinates. We define the associated fractal sets and tilings by iterating the dual substitution and by taking a Hausdorff limit of the (renormalized) unions of unit cube faces. The fact that we deal with unit cube faces allows us to exploit some fine combinatorial and topological properties of the resulting patterns. This provides some powerful tools in the study of substitution dynamics and Rauzy fractal topology. Dual substitutions are usually denoted by $\mathbf{E}_1^*(\sigma)$, where σ is a one-dimensional symbolic substitution, See [8, 30] for many references and results, and Definition 4.2 for the particular example studied in this article.

Local substitution rules. — This notion has been used to tackle combinatorial questions about substitution dynamics [22, 2, 3, 7] and have also been studied in a more general context [14, 23]. Their aim is to get a "more combinatorial" version of dual substitutions. Instead of computing explicitely the coordinates of the image of each unit cube face (like we do for dual substitutions), we give some *local rules* (or *concatenation* rules) for "gluing together" the images of two adjacent faces. The map defined in Figure 5.2, p. 601 is an example of such a substitution (except that it is defined over topological tiles and not unit cube faces).

Topological substitutions. — Introduced in [6], topological substitutions do not make any use of geometry: the tiles are topological disks (with no Euclidean shape), the boundaries of which have a simplicial structure (made of vertices and edges). It is a notion less geometrically rigid than the previous ones. They act on CW-complexes, and the "gluing rules" are more abstract and combinatorial than local substitution rules. A topological substitution generates a CW-complex homeomorphic to the plane. If this complex can be geometrized as a tiling of the plane, we say that the tiling is a topological substitutive tiling. Topological substitutions allowed for instance to prove that there is no substitutive primitive tiling of the hyperbolic plane, even though an explicit example of a non-primitive topological substitution which generates a tiling of the hyperbolic plane is given in [6].

In order to distinguish this notion of substitution used in the present article from the other combinatorial notions discussed in this introduction, we use the term *topological substitution* instead of *combinatorial substitution*

The examples of topological substitutions given in the present article (Figure 3.3 and Figure 6.1) are interesting because they provide new examples of topological substitutive tilings, which can be realized as (substitutive) tilings of the plane.

Other related notions. — There is another notion, elaborated by Fernique and Ollinger [15] (and developped in details in the specific case of Tribonacci), which lies between local substitution rules and topological substitutions. For these so-called *combinatorial substitutions*, the Euclidean shape of the tiles is specified, and the matching rules are stated in terms of colors associated with some subintervals on the boundaries of the tiles and their images. We stress that, in that case, the Euclidean geometry is used both to give the shape of the tiles and to specify that two tiles with same shape differ with a translation of the plane.

Purely combinatorial notions of substitutions have already been defined. For instance, Priebe-Frank [17] introduced a very natural notion of (labeled) graph substitutions. In the case of a substitutive tiling, this graph substitution has to be understood as a substitution on the dual graph to the tiling. The main issue with this formalism is that there is no a priori control on the planarity of the graph obtained by iteration of the substitution, and thus in general the limit graph obtained by iteration can not be the dual graph to any tiling of the plane. Topological substitutions of [6] remedy this problem.

Topological substitutions have also some worth to be met cousins: the so-called *subdivision rules*, introduced by Cannon, Floyd and Parry in [10]. The natural context where these subdivision rules have hatched is the one of conformal geometry: on one hand, they can be seen as topological models for post-critically finite rational map of the Riemann sphere [11], on the other hand, they are likely useful to prove Cannon's conjecture for hyperbolic groups whose Gromov boundary is the 2-dimensional sphere as suggested by the results of [12]. Nevertheless, subdivision rules can be also used to produce conformal substitutive tilings of the plane, see [9, 27]. Even if, by iterating both a system of subdivision rules or a substitution, one get a 2-complex homeomorphic to the plane, these processes do differ in their nature: morally, in the case of subdivision rules the 2-complex is obtained as an inverse limit whereas in the case of a substitution it is obtained as a direct limit. It is not clear at all to the authors when, given a 2-complex obtained by one of two processes, one can also recover it using the other process.

1.3. Organization of the article. — In Section 2 we quickly review some usual facts about tilings. In Section 3 we recall the general definition of a topological substitution and we define the Tribonacci topological substitution we are interested in. In Section 4 we recall the definition of the Tribonacci dual substitution and its associated IFS Rauzy fractal. The link between the IFS and the topological substitution is finally studied in Section 5. In Section 6 we describe how our results can be extended to some other Rauzy fractal tilings, and we explain that finding a suitable topological substitution has some dynamical implications for the underlying one-dimensional substitution.

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2. Tilings

2.1. Basic definitions. — In this section we recall standard notions on tiling in $\mathbb{R}^2$. For the references about this material we refer the reader to [29, 31, 6, 19]. We denote by Γ a subgroup of transformations of $\mathbb{R}^2$: here, Γ will be the group of translations of $\mathbb{R}^2$. We keep Γ in the notation, just to have in mind that some classical tilings need rotations of tiles.

A *tile* is a compact subset of $\mathbb{R}^2$ which is the closure of its interior (in most of the basic examples, a tile is homeomorphic to a closed ball). We denote by ∂T

Here *conformal* means that the underlying geometry is the conformal geometry and not the Euclidean one. That is to say that the group Γ defining the tiling (see Definition 2.1) is not a subgroup of isometries of $\mathbb{R}^2$, but a subgroup of biholomorphisms of $\mathbb{C}$.

the boundary of a tile, i.e., $\partial T = T \setminus \mathring{T}$. Let $\mathcal{A}$ be a finite set of *labels*. A *labeled tile* is a pair (T, a) where T is a tile and a an element of $\mathcal{A}$. Two labeled tiles (T, a) and (T', a') are *equivalent* if $a = a'$ and there exists a translation $g \in \Gamma$ such that $T' = gT$. An equivalence class of labeled tiles is called a *prototile*: the class of (T, a) is denoted by $[T, a]$, or simply by $[T]$ when the context is sufficiently clear. We will say that (T, a) belongs to the prototile $[T, a]$. In some cases, one does not need the labeling to distinguish different prototiles: for example if we consider a family of prototiles such that the tiles in two different prototiles are not isometric.

Definition 2.1. — A tiling $\mathbf{X} = (\mathbb{R}^2, \Gamma, \mathcal{P}, \mathsf{T})$ of the plane modeled on a set of prototiles $\mathcal{P}$, is a set T of tiles, each belonging to a prototile in $\mathcal{P}$, such that:

- $\mathbb{R}^2 = \bigcup_{T \in \mathsf{T}} T$,
- two distinct tiles of T have disjoint interiors.

A connected finite union of (labeled) tiles is called a (labeled) *patch*. Two finite patches are *equivalent* if they have the same number k of tiles and these tiles can be indexed $T_1, \ldots, T_k$ and $T'_1, \ldots, T'_k$, such that there exists $g \in \Gamma$ with $T'_i = gT_i$ for every $i \in \{1, \ldots, k\}$. Two labeled patches are equivalent if moreover T_i, T'_i have same labeling for all $i \in \{1, \ldots, k\}$. An equivalence class of patches is called a *protopatch* and denoted $[P]$ if P is one of these patches.

The *support* of a patch P, denoted by $\operatorname{supp}(P)$, is the subset of $\mathbb{R}^2$ which consists of points belonging to a tile of P. A *subpatch* of a patch P is a patch which is a subset of the patch P.

Let $\mathbf{X} = (\mathbb{R}^2, \Gamma, \mathcal{P}, \mathsf{T})$ be a tiling, and let A be a subset of $\mathbb{R}^2$. A patch P *occurs in* A if there exists $g \in \Gamma$ such that for any tile $T \in P$, gT is a tile of T which is contained in A:

$$gT \in \mathsf{T} \quad \text{and} \quad \operatorname{supp}(gT) \subseteq A.$$

We note that any patch in the protopatch $[P]$ defined by P occurs in A. We say that the protopatch $[P]$ *occurs* in A. The *language* of $\mathbf{X}$, denoted $\Lambda_{\mathbf{X}}$, is the set of protopatches of $\mathbf{X}$.

When all tiles of $\mathcal{P}$ are euclidean polygons, the tiling is called a *polygonal tiling*.

2.2. Delone set defined by a tiling of the plane. — A Delone set in $\mathbb{R}^2$ is a set $\mathcal{D}$ of points such that there exists $r, R > 0$ such that every euclidean ball of radius r contains at most one point of $\mathcal{D}$ and every euclidean ball of radius R contains at least one point of $\mathcal{D}$.

When a tiling of the plane $\mathbf{X} = (\mathbb{R}^2, \Gamma, \mathcal{P}, \mathsf{T})$ is modeled on a finite set of prototiles $\mathcal{P}$, there is a standard way (among others) to derive a Delone set $\mathcal{D}$ from a tiling of the plane $\mathbf{X} = (\mathbb{R}^2, \Gamma, \mathcal{P}, \mathsf{T})$. We first choose a point in the interior of each prototile. This choice gives us a point x_T in each tile T of $\mathbf{X}$: $\mathcal{D} = \{x_t, T \in \mathsf{T}\}$ is a Delone set.

2.3. The 2-complex defined by a tiling of the plane. — Let X be a 2-dimensional CW-complex (see, for instance, [20] for basic facts about CW-complexes). The 0-cells will be called vertices, the 1-cells edges and the 2-cells faces. The subcomplex of X which consists of cells of dimension at most $k \in \{0, 1, 2\}$ is denoted by X^k (in particular $X^2 = X$). We denote by $|X^k|$ the number of k-cells in X^k.

Let $\mathbf{X} = (\mathbb{R}^2, \Gamma, \mathcal{P}, \mathsf{T})$ be a tiling of the plane. We suppose that the tiles are homeomorphic a a closed disk $\mathbb{D}^2$. This tiling defines naturally a 2-complex X in the following way. The set X^0 of vertices of X is the set of points in $\mathbb{R}^2$ which belong to (at least) three tiles of $\mathcal{T}$. Each connected component of the set $\bigcup_{T\in\mathsf{T}} \partial T \setminus X^0$ is an open arc. Any closed edge of X is the closure of one of these arcs.

Such an edge e is glued to the endpoints $x, y \in X^0$ of the arc. The set of faces of X is the set of tiles of $\mathbf{X}$. We remark that the boundary of a tile is a subcomplex of X^1 homeomorphic to the circle $\mathbb{S}^1$: this gives the gluing of the corresponding face on the 1-skeleton.

Let Y be a 2-dimensional CW-complex homeomorphic to the plane $\mathbb{R}^2$. A polygonal tiling $\mathbf{X}$ is a *geometric realization* of Y if the 2-complex X defined by $\mathbf{X}$ is isomorphic (as CW-complex) to Y. In that case, each face of the complex Y can be naturally labeled by the corresponding prototile of the tiling $\mathbf{X}$.

3. The Tribonacci topological substitution

Before giving the definition of the Tribonacci topological substitution in Section 3.2, we first recall some facts about (2-dimensional) topological substitutions in Section 3.1. These two sections can be read in parallel: along Section 3.1, we illustrate the notions with examples referring to Section 3.2.

3.1. Topological substitutions. — The mathematical content of this section is essentially contained in[6]: we include it here for completeness. The vocabulary we will use in the present setting is often common to the one of tilings: the context is in general sufficient to prevent any ambiguity.

3.1.1. *General definition.* — A *topological k-gon* ($k \geq 3$) is a 2-dimensional CW-complex made of one face, k edges and k vertices, which is homeomorphic to a closed disk $\mathbb{D}^2$, and such that the 1-skeleton is the boundary $\mathbb{S}^1$ of the closed disk. A *topological polygon* is a topological k-gon for some $k \geq 3$.

We consider a finite set $\mathcal{T} = \{T_1, \dots, T_d\}$ of topological polygons. The elements of $\mathcal{T}$ are called *prototiles*, and $\mathcal{T}$ is called the *set of prototiles*. If T_i is a n_i-gon, we denote by $E_i = \{e_{1,i}, \dots, e_{n_i,i}\}$ the set of edges of T_i. In practice, we will need later to consider these $e_{n_k,i}$ as oriented edges: we first fix an orientation on the boundary of T_i, and equip the $e_{n_k,i}$ with the induced orientation. We set $E_i^{-1} = \{e_{1,i}^{-1}, \dots, e_{n_i,i}^{-1}\}$ and $E_i^{\pm} = E_i \cup E_i^{-1}$, where e^{-1} denotes the edge e equipped with the reverse orientation.

A *patch P modeled* on $\mathcal{T}$ is a 2-dimensional CW-complex homeomorphic to the closed disk $\mathbb{D}^2$ such that for each closed face f of P, there exists a prototile $T_i \in \mathcal{T}$ and a homeomorphism $\tau_f : f \to T_i$ which respects the cellular structure. Then $T_i = \tau_f(f)$ is called the *type* of the face f, and denoted by type(f). The type of an edge e of T_i, denoted by type(e), is $\tau(e)$. An edge e of P is called a *boundary edge* if it is contained in the boundary $\mathbb{S}^1$ of the disk $\mathbb{D}^2 \cong P$. Such a boundary edge is contained in exactly one closed face of P. An edge e of P which is not a boundary edge is called an *interior edge*. An interior edge is contained in exactly two closed faces of P. In the following definition, and for the rest of this article, the symbol $\sqcup$ stands for the disjoint union.

Definition 3.1. — A *topological pre-substitution* is a triplet $(\mathcal{T}, \sigma(\mathcal{T}), \sigma)$ where:

1. $\mathcal{T} = \{T_1, \dots, T_d\}$ is a set of prototiles,
2. $\sigma(\mathcal{T}) = \{\sigma(T_1), \dots, \sigma(T_d)\}$ is a set of patches modeled on $\mathcal{T}$,
3. $\sigma : \displaystyle\bigsqcup_{i\in\{1,\dots,d\}} T_i \to \bigsqcup_{i\in\{1,\dots,d\}} \sigma(T_i)$ is a homeomorphism, which restricts to homeomorphisms $T_i \to \sigma(T_i)$, such that the image of a vertex of T_i is a vertex of the boundary of $\sigma(T_i)$.

Example 3.2. — In Figure 3.3, we show one example of a pre-topological substitution defined on 3 prototiles. This is the Tribonacci topological pre-substitution.

Compatible topological pre-substitution. — Let $\mathcal{T} = \{T_1, \dots, T_d\}$ be the set of prototiles of σ, and let $E_i^{\pm} = E_i \cup E_i^{-1}$ be the set of oriented edges of T_i ($i \in \{1, \dots, d\}$). We denote by $E^{\pm}$ the set of all oriented edges: $E^{\pm} = \bigsqcup_i E_i^{\pm}$.

A pair $(e, e') \in E^{\pm} \times E^{\pm}$ is *balanced* if $\sigma(e)$ and $\sigma(e')$ have the same length ($=$ the number of edges in the edge path). The *flip* is the involution of $E^{\pm} \times E^{\pm}$ defined by $(e, e') \mapsto (e', e)$, and the *reversion* is the involution of $E^{\pm} \times E^{\pm}$ defined by $(e, e') \mapsto (e^{-1}, e'^{-1})$. The quotient of $E^{\pm} \times E^{\pm}$ obtained by identifying a pair and its image by the flip and also a pair and its image by the reversion is denoted by E_2. We denote by $[e, e']$ the image of a pair $(e, e') \in E^{\pm} \times E^{\pm}$ in E_2. Since the flip and the reversion preserve balanced pairs, the notion of "being balance" is well defined for elements of E_2. The subset of E_2 which consists of balanced elements is called the *set of balanced pairs*, and denoted by $\mathcal{B}$. Let $[e, e'] \in \mathcal{B}$ a balanced pair. In other words, $\sigma(e)$ and $\sigma(e')$ are paths of edges which have same length say $p \geq 1$: $\sigma(e) = e_1 \dots e_p$, $\sigma(e') = e'_1 \dots e'_p$. Let $\varepsilon_i = \mathrm{type}(e_i)$ and let $\varepsilon'_i = \mathrm{type}(e'_i)$: $\varepsilon_i, \varepsilon'_i \in E^{\pm}$ for $i = 1 \dots p$. Then the $[\varepsilon_i, \varepsilon'_i]$ are called the *descendants* of $[e, e']$.

EXAMPLE 3.3. — For the Tribonacci topological pre-substitution, consider for example the edges $e = B_{45}, e' = C_{76}$. By Figure 3.3, we have $\sigma(e) = C_{34}C_{45}C_{56}, \sigma(e') = C_{10}C_{09}C_{98}$. Thus $[e, e']$ is a balanced pair. The descendants of this pair are $[C_{34}, C_{10}], [C_{45}, C_{09}], [C_{56}, C_{98}]$.

Now, we consider a patch P modeled on $\mathcal{T}$. An interior edge e of P defines an element $[\varepsilon, \varepsilon']$ of E_2. Indeed, let f and f' be the two faces adjacent to e in P. We denote by $\varepsilon = \tau_f(e)$ the edge of type(f) corresponding to e, and by $\varepsilon' = \tau_{f'}(e)$ the edge of type(f') corresponding to e. The edge e is said to be *balanced* if $[\varepsilon, \varepsilon']$ is balanced.

We define, by induction on $p \in \mathbb{N}$, the notion of a *p-compatible* topological pre-substitution σ. To any p-compatible topological pre-substitution σ we associate a new pre-substitution which will be denoted by σ^p.

DEFINITION 3.4. — a) Any pre-substitution $(\mathcal{T}, \sigma(\mathcal{T}), \sigma)$ is 1-*compatible.* We set $\sigma^1 = \sigma$.

b) A pre-substitution $(\mathcal{T}, \sigma(\mathcal{T}), \sigma)$ is said to be *p-compatible* ($p \geq 2$) if:
   1. $(\mathcal{T}, \sigma(\mathcal{T}), \sigma)$ is $(p-1)$-compatible
   2. for all $i \in \{1, \ldots, d\}$, every interior edge e of $\sigma^{p-1}(T_i)$ is balanced.

c) We suppose now that $(\mathcal{T}, \sigma(\mathcal{T}), \sigma)$ is a p-compatible pre-substitution ($p \geq 2$). Then we define $\sigma^p(T_i)$ ($i \in \{1, \ldots, d\}$) as the patch obtained in the following way:

   We consider the collection of patches $\sigma(\text{type}(f))$ for each face f of $\sigma^{p-1}(T_i)$. Then, if f and f' are two faces of $\sigma^{p-1}(T_i)$ adjacent along some edge e, we glue, edge to edge, $\sigma(\text{type}(f))$ and $\sigma(\text{type}(f'))$ along $\sigma(\tau_f(e))$ and $\sigma(\tau_{f'}(e))$. This is possible since the p-compatibility of σ ensures that the edge e is balanced. The resulting patch $\sigma^p(T_i)$ is defined by:

$$\sigma^p(T_i) = \left(\bigsqcup_{f \text{ face of } \sigma^{p-1}(T_i)} \sigma(\text{type}(f)) \right) \Big/ \sim$$

   where $\sim$ denotes the gluing.

   We define $\sigma^p(\mathcal{T})$ to be the set $\{\sigma^p(T_1), \ldots, \sigma^p(T_d)\}$.

REMARK 3.5. — Definition 3.4 is recursive. Indeed, conditions b) and c) should be denoted b_p) and c_p) since they do depend on p. Then the definition should be read in the following order: a), b_2), c_2), …, b_p), c_p), b_{p+1}), c_{p+1}), …

The map σ induces a natural map on the faces of each $\sigma^{p-1}(T_i)$ which factorizes to a map $\sigma_{i,p} : \sigma^{p-1}(T_i) \to \sigma^p(T_i)$ thanks to the p-compatibility hypothesis:

$$\begin{array}{ccc}
\displaystyle\bigsqcup_{f \text{ face of } \sigma^{p-1}(T_i)} \text{type}(f) & \xrightarrow{\sigma} & \displaystyle\bigsqcup_{f \text{ face of } \sigma^{p-1}(T_i)} \sigma(\text{type}(f)) \\
\sim\big\downarrow & & \big\downarrow\sim \\
\sigma^{p-1}(T_i) & \xrightarrow[\sigma_{i,p}]{} & \sigma^p(T_i)
\end{array}$$

We note that $\sigma_{i,p}$ is a homeomorphism which sends vertices to vertices. Then we define the map

$$\sigma_i^p : T_i \to \sigma^p(T_i)$$

as the composition: $\sigma_i^p = \sigma_{i,p} \circ \sigma_i^{p-1}$. This is an homemorphism which sends vertices to vertices. Then σ^p is naturally defined such that the restriction of σ^p on T_i is σ_i^p. We remark that $\sigma^1 = \sigma$. We have thus obtained a topological pre-substitution $(\mathcal{T}, \sigma^p(\mathcal{T}), \sigma^p)$. A topological pre-substitution is *compatible* if it is p-compatible for every integer p.

Checking compatibility. — In this subsection we give an algorithm which decides whether a pre-substitution is compatible.

Suppose that σ is p-compatible. We define W_p as the set of elements $[\varepsilon, \varepsilon'] \in E_2$ such that there exists $i \in \{1, \dots, d\}$, as well as two faces f and f' in $\sigma^p(T_i)$ glued along an edge e, such that $\tau_f(e) = \varepsilon$ and $\tau_{f'}(e) = \varepsilon'$. The topological pre-substitution σ is $(p+1)$-compatible if and only if W_p is contained in the set of balanced pairs $\mathcal{B}$: $W_p \subseteq \mathcal{B}$. Then:

- either $W_p \nsubseteq \mathcal{B}$: the algorithm stops, telling us that the substitution is not compatible,
- or $W_p \subseteq \mathcal{B}$: then we define $V_p = V_{p-1} \cup W_p$.

By convention we settle $V_0 = \emptyset$.

Suppose that σ is compatible. The sequence $(V_p)_{p \in \mathbb{N}}$ is an increasing sequence (for the inclusion) of subsets of the finite set E_2. Hence there exists some $p_0 \in \mathbb{N}$ such that $V_{p_0+1} = V_{p_0}$ (and thus $V_p = V_{p_0}$ for all $p \geq p_0$). The algorithm stops at step $p_0 + 1$ (where p_0 is the smallest integer such that $V_{p_0+1} = V_{p_0}$), telling that σ is compatible.

The *heredity graph of edges* of σ, denoted $\mathcal{E}(\sigma)$, is defined in the following way. The set of vertices of $\mathcal{E}(\sigma)$ is V_{p_0}. There is an oriented edge from vertex $[e, e']$ to vertex $[\epsilon, \epsilon']$ if $[\epsilon, \epsilon']$ is a descendant of $[e, e']$.

EXAMPLE 3.6. — For the Tribonacci topological pre-substitution, consider one again the edges $e = B_{45}, e' = C_{76}$. We have shown in a previous example that $\sigma(e) = C_{34}C_{45}C_{56}, \sigma(e') = C_{10}C_{09}C_{98}$. Thus $[B_{45}, C_{76}]$ is a vertex of the heredity graph of edges. There are three edges which start from this vertex and go to the vertices defined by the balanced pairs – see Figure 3.4 and the proof of Lemma 3.9 for more details.

Core of a topological pre-substitution. — Let P be a patch modeled on $\mathcal{T} = \{T_1, \dots, T_d\}$. The *thick boundary* $\mathsf{B}(P)$ of P is the closed sub-complex of P consisting of the closed faces which contain at least one vertex of the boundary ∂P of P. The *core* $\mathrm{Core}(P)$ of P is the closure in P of the complement of $\mathsf{B}(P)$: in particular, $\mathrm{Core}(P)$ is a closed subcomplex of P, see Figure 3.1.

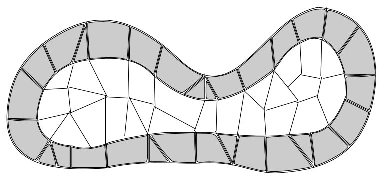


FIGURE 3.1. The thick boundary is the gray subcomplex and the core is the white subcomplex.

A topological pre-substitution $(\mathcal{T}, \sigma(\mathcal{T}), \sigma)$ has the *core property* if there exist $i \in \{1 \dots d\}$ and $k \in \mathbb{N}$ such that the core of $\sigma^k(T_i)$ is non-empty.

EXAMPLE 3.7. — For the Tribonacci topological pre-substitution we show in Figure 3.5 that the cores of $\sigma(C), \sigma^2(C)$ are empty. But the core of $\sigma^3(C)$ is not empty.

DEFINITION 3.8. — A *topological substitution* is a pre-substitution which is compatible and has the core property.

3.1.2. *Topological plane obtained by inflation.* — Consider a tile $T \in \mathcal{T}$ such that the core of $\sigma(T)$ contains a face of type T. Then, we can identify the tile T with a subcomplex of the core of $\sigma(T)$. By induction, $\sigma^k(T)$ is thus identified with a subcomplex of $\sigma^{k+1}(T)$ $(k \in \mathbb{N})$. We define $\sigma^\infty(T)$ as the increasing union:

$$\sigma^\infty(T) = \bigcup_{k=0}^{\infty} \sigma^k(T).$$

By construction, the complex $\sigma^\infty(T)$ is homeomorphic to $\mathbb{R}^2$. (Indeed, denoting $\sigma^k(T)$ by D_k, one observes that $\sigma^\infty(T)$ is an increasing union of closed disks D_k with D_k contained in the interior $\mathrm{Int}\ D_{k+1}$ of D_{k+1}, and with $D_{k+1} \smallsetminus \mathrm{Int}\ D_k$ homemorphic to the annulus $S^1 \times I$. This allows to build an homeomorphism between $\sigma^\infty(T)$ and $\mathbb{R}^2$ – see for instance [21, exercise 3 p. 207].) We say that such a complex is obtained *by inflation from* σ. Moreover this complex can be labeled by the types of the topological polygons. We notice that σ induces an homeomorphism of $\sigma^\infty(T)$.

We denote by $\mathcal{P}_\sigma$ the *set of patches in the complex* $\sigma^\infty(T)$. We notice that σ naturally defines a map $\mathcal{P}_\sigma \to \mathcal{P}_\sigma$, that is still denoted by σ. To be more precise, given a patch $P \in \mathcal{P}_\sigma$, there is some $k \in \mathbb{N}$ such that $P \subseteq \sigma^k(P) \subset \sigma^\infty(P)$, so that $\sigma(P) \subseteq \sigma^{k+1}(P) \subset \sigma^{\infty}(P)$: this patch $\sigma(P) \in \mathcal{P}_\sigma$ does not depend on the choice of k.

We denote by $\mathcal{T}_\sigma$ the *set of tiles in the complex* $\sigma^{\infty}(T)$: $\mathcal{T}_\sigma$ is a subset of $\mathcal{P}_\sigma$. See Figure 3.2 for examples of such topological complexes generated by topological substitutions.

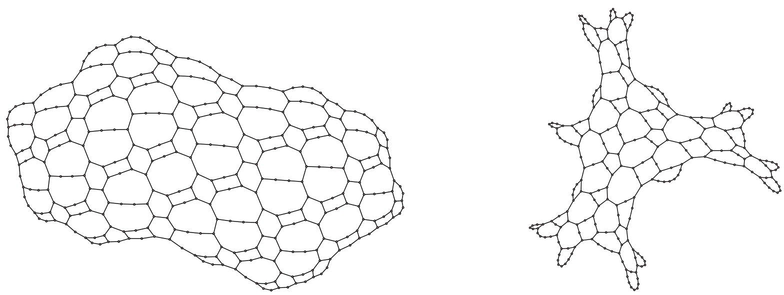


FIGURE 3.2. The topological complexes associated with $\sigma^6(C)$ (left) and $\tau^{10}(C)$ (right). The definitions of σ and τ are respectively given in Figure 3.3 (p. 588) and Figure 6.1 (p. 608).

3.2. The Tribonacci topological substitution. — We first define a topological substitution σ. Then we explain how to derive a tiling of $\mathbb{R}^2$ as a geometrical realization of the patches generated by σ. The topological substitution is defined on Figure 3.3 and the first iterations on the polygon C are given in Figure 3.5.

3.2.1. *Definition of the topological substitution.* — We consider the Tribonacci topological pre-substitution σ defined on Figure 3.3. There are three prototiles: two of them, A and B, are hexagons, while the third one, C, is a decagon. The images of these prototiles (together with the labeling of the vertices and the images of the vertices) are given on Figure 3.3. In practice, we will denote by A_i the vertex i of A, and by $A_{i(i+1)}$ the edge of A joining A_i and A_{i+1} (and so on for B and C).

LEMMA 3.9. — *The topological pre-substitution σ is a topological substitution.*

Proof. — Using the procedure described at the end of Subsection 3.1.1 (in Paragraph "Checking compatibility"), we first check that σ is compatible. We start with the pair of edges that are glued in the images of A, B and C. In fact, all these gluings occur in $\sigma(C)$: $[A_{45}, C_{54}]$, $[A_{23}, B_{05}]$, $[B_{45}, C_{76}]$, $[A_{43}, C_{56}]$.

We focus now on $[B_{45}, C_{76}]$. The image of the edge B_{45} is the path of edges $C_{34}C_{45}C_{56}$. The image of C_{76} is $C_{10}C_{09}C_{98}$. Both have length 3, and the gluing gives rise to the pairs of edges: $[C_{34}, C_{10}]$, $[C_{45}, C_{09}]$ and $[C_{56}, C_{98}]$.

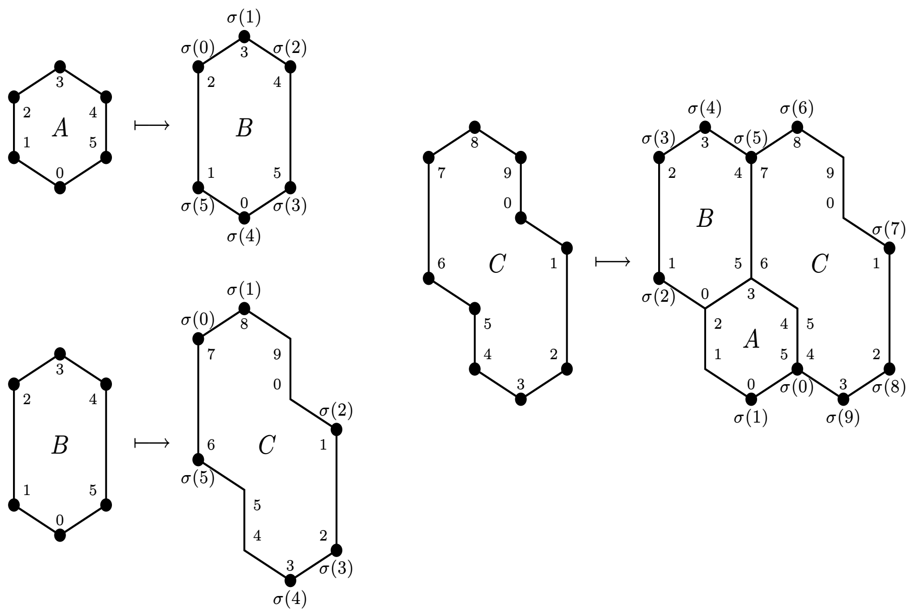


FIGURE 3.3. The Tribonacci topological substitution.

Carrying out the other pairs in the same way, and iterating the process, we check that σ is compatible. These computations are summed up in the heredity graph of edges, given in Figure 3.4.

The core property is checked for $\sigma^3(C)$: we see on Figure 3.5 that $\mathrm{Core}(\sigma^3(C)) \neq \emptyset$. Hence σ is a topological substitution. □

3.2.2. *Configurations at the vertices.* — We denote by V the set of vertices of the prototiles A, B, C. The *heredity graph of vertices* is an oriented graph denoted by $\mathcal{V}(\sigma)$. The set of vertices of $\mathcal{V}(\sigma)$ is the set V. Let $T, T' \in \{A, B, C\}$, and let v be a vertex of T and v' be a vertex of T': there is an oriented edge in $\mathcal{V}(\sigma)$ from v to v' if $\sigma(v)$ is a vertex of type v' of a tile of type T'.

The graph $\mathcal{V}(\sigma)$ is given in Figure 3.6. It can be used to control the valences of the vertices in $\sigma^\infty(C)$ thanks to the following property proved in [6]. A vertex $v \in V$ is a *divided vertex* if there are at least two oriented edges in $\mathcal{V}(\sigma)$ coming out of v. We denote by $V_{\mathcal{D}}$ the subset of V which consists of all divided vertices. The following properties are equivalent, see [6]:

- The complex $\sigma^\infty(C)$ has bounded valence.
- Every infinite oriented path in $\mathcal{V}(\sigma)$ crosses vertices of $V_{\mathcal{D}}$ only finitely many times.

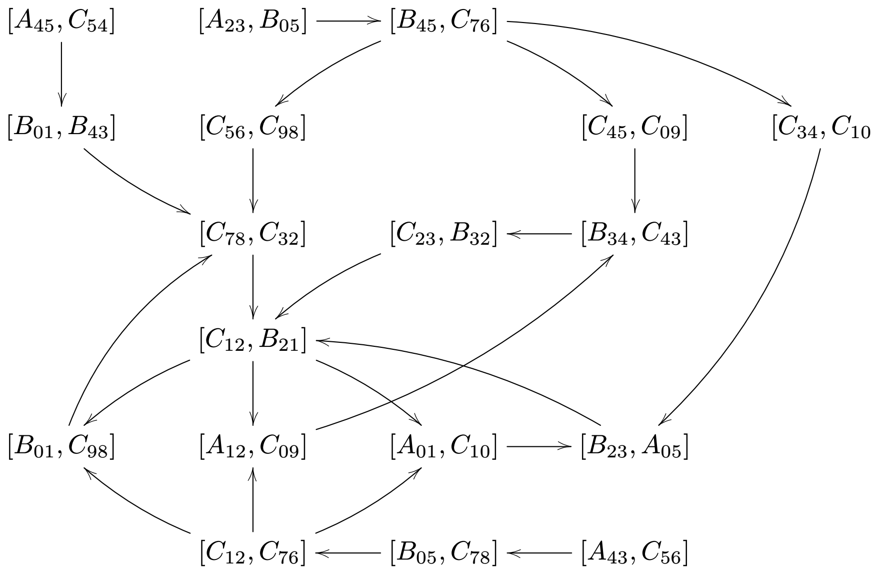


FIGURE 3.4. The heredity graph of edges $\mathcal{E}(\sigma)$ of the Tribonacci substitution σ.

- The oriented cycles of $\mathcal{V}(\sigma)$ do not cross any vertex of $V_{\mathcal{D}}$.

LEMMA 3.10. — *The valence of each vertex in $\sigma^\infty(C)$ is bounded.*

Proof. — We consider Figure 3.6. Remark that $V_{\mathcal{D}} = \{C_0, C_5\}$. Moreover, neither C_0 nor C_5 is crossed by an oriented cycle of $\mathcal{V}(\sigma)$. Hence the valence of the vertices of $\sigma^\infty(C)$ is bounded by the previous property. $\square$

Now we introduce another graph, which is called the *configuration graph of vertices* and is denoted by $\mathcal{C}(\sigma)$. We consider the equivalence relation on k-tuples of elements of V $(k \in \mathbb{N})$ generated by:

$$(x_1, \ldots, x_{k-1}, x_k) \sim (x_k, x_1, \ldots, x_{k-1}) \text{ and } (x_1, x_2, \ldots, x_k) \sim (x_k, \ldots, x_2, x_1).$$

Let $[x_1, \ldots, x_k]$ denote the equivalence class of $(x_1, \ldots, x_k)$. Let K be the maximal valence of a vertex in $\sigma^\infty(C)$. Let W be the set of equivalence classes of k-tuples with $2 \le k \le K$. A vertex v in the interior of a patch $\sigma^n(C)$ $(n \ge 1)$ defines an element $[x_1, \ldots, x_k] \in W$ (where k is the valence of v). Indeed, the faces adjacent to v are cyclically ordered, and x_i is the type of the vertex of the i-th face which is glued on v.

We define the oriented graph $\mathcal{C}(\sigma)$ as follows. Let W_0 be the subset of W defined by the vertices occuring in the interior of some $\sigma^n(C)$ for $n \ge 1$. An element of W_0 is called a *vertex configuration.* The set of vertices of $\mathcal{C}(\sigma)$ is W_0.

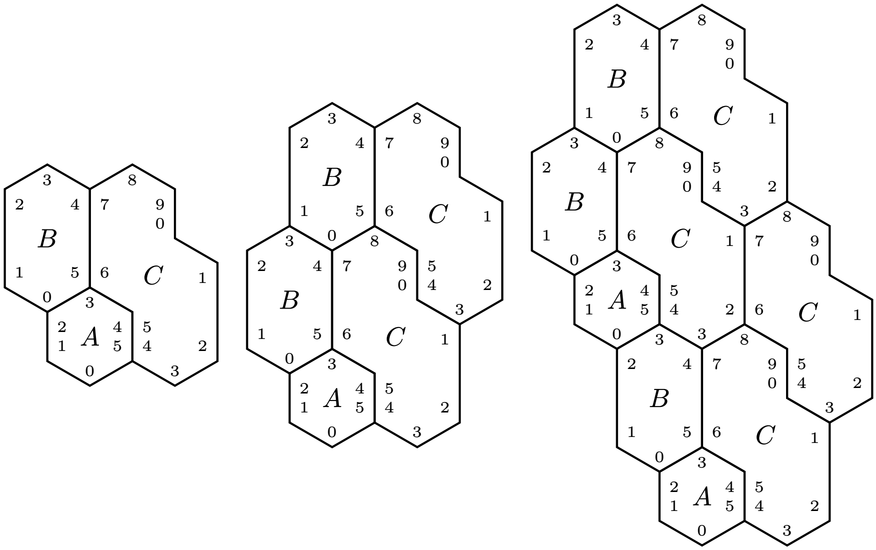


FIGURE 3.5. Iterating the Tribonacci topological substitution: $\sigma(C)$, $\sigma^2(C)$ and $\sigma^3(C)$.

$$
\begin{tikzcd}
C_0 \arrow[rr] \arrow[drr, start anchor=south, end anchor=north] & & C_4 \arrow[d] & \\
 & A_1 \arrow[r] & B_3 \arrow[r] & C_2 \arrow[dl] \\
 & A_5 \arrow[r] & B_1 \arrow[r] & C_8 \arrow[u] \\
 & A_3 \arrow[r] & B_5 \arrow[r] & C_6 \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
C_9 \arrow[d] & A_2 \arrow[d] & & \\
C_3 \arrow[d] & B_4 \arrow[l] & C_5 \arrow[l] \arrow[d] & \\
B_2 \arrow[r] & C_1 \arrow[dl] & C_7 \arrow[l] & \\
A_0 \arrow[u] & & B_0 \arrow[u] & A_4 \arrow[l]
\end{tikzcd}
$$

FIGURE 3.6. The heredity graph of the vertices $\mathcal{V}(\sigma)$.

For any $s \in W_0$, we choose some $T \in \mathcal{T}, n \geq 1$ and v a vertex in the interior of $\sigma^n(T)$ which defines s. Let s' the element of W_0 defined by $\sigma(v)$. There is an oriented edge in $\mathcal{C}(\sigma)$ from s to s'. We notice that this construction does not depend of the choice of T and n.

In practice, to build the graph $\mathcal{C}(\sigma)$, we first remark that a vertex v in the interior of some $\sigma^n(C)$ $(n \geq 1)$ is either the image of a vertex in the interior of $\sigma^{n-1}(C)$, or is in the interior of a path of edges which is the image of an interior edge of $\sigma^{n-1}(C)$. Thus we first make the list of vertex configurations for:

- vertices in the interior of the image of a tile: we get $[C_5, A_4]$ et $[C_6, A_3, B_5]$;
- vertices in the interior of the image of an interior edge. These ones can be derived from the vertices of $\mathcal{E}(\sigma)$ with a least 2 outing edges. There are 3 such vertices in of $\mathcal{E}(\sigma)$:
  - $[B_{45}, C_{76}]$ which gives rise to vertex configurations $[C_4, C_0]$ and $[C_5, C_9]$,
  - $[C_{12}, B_{21}]$ and $[C_{12}, C_{76}]$ which gives rise to vertex configurations $[C_4, C_0]$ and $[C_5, C_9]$.

Then we iteratively compute the vertex configurations obtained as the image under σ^n of the vertex configurations in $\{[C_5, A_4], [C_6, A_3, B_5], [C_4, C_0], [C_5, C_9], C_4, C_0], [C_5, C_9]\}$. The graph $\mathcal{C}(\sigma)$ is represented on Figure 3.7.

$$
\begin{tikzcd}
{[C_5, A_4]} \arrow[r] & {[C_7, B_4, B_0]} \arrow[r] & {[C_1, C_3, C_7]} \arrow[r] & {[A_0, B_2, C_1]} \arrow[loop above] \\
{[C_5, C_9]} \arrow[r] & {[C_7, B_4, C_3]} \arrow[r] & {[C_1, C_3, B_2]} \arrow[ur] & {[A_1, C_0]} \arrow[dl] \\
{[A_2, C_9, B_0]} \arrow[ur] & {[B_1, B_3, C_2]} \arrow[dl] & {[A_5, C_4, B_3]} \arrow[l] & {[C_0, C_4]} \arrow[l] \\
{[B_1, C_8, C_2]} \arrow[loop below] & {[C_2, C_6, C_8]} \arrow[l] & {[C_8, B_5, C_6]} \arrow[l] & {[C_6, A_3, B_5]} \arrow[l]
\end{tikzcd}
$$

FIGURE 3.7. The configuration graph of the vertices $\mathcal{C}(\sigma)$.

3.2.3. *A geometric realization of $\sigma^\infty(C)$*

PROPOSITION 3.11. — *The complex $\sigma^\infty(C)$ can be realized as a tiling of $\mathbb{R}^2$. This tiling is denoted by $\mathcal{T}_{\mathsf{top}}$.*

Proof. — We first recall that a 2-dimensional CW-complex with hexagonal faces such that each edge belongs to 2 faces and each vertex belongs to 3 faces is isomorphic to the 2-dimensional CW-complex X_{hex} defined by the tiling of $\mathbb{R}^2$ by regular euclidean hexagons.

Let C_{hex} the patch made of two hexagonal faces obtained by dividing C along an edge between vertices C_0 and C_5. Given a patch P made of tiles of types A, B and C, we build a patch P_{hex} made of hexagonal faces by replacing the faces of type C by C_{hex}.

We claim that $(\sigma^\infty(C))_{\text{hex}} = X_{\text{hex}}$. Indeed, the vertex configurations of $\sigma^\infty(C)$ are given by the vertices of the graph $\mathcal{C}$ (see Figure 3.7), and for every of them, we check that $P = P_{\text{hex}}$ for the corresponding patch P.

Since $(\sigma^\infty(C))_{\text{hex}} = X_{\text{hex}}$, it is straightforward to derive that $\sigma^\infty(C)$ can be geometrically realized as a tiling of $\mathbb{R}^2$, where A and B are realized by regular hexagons, and C by a decagon obtained by gluing two regular hexagons along an edge. □

Remark 3.12. — In particular, we can now precise Lemma 3.10: The valence of a vertex in $\sigma^\infty(C)$ is either 2 or 3.

3.3. The pointed topological substitution $\hat{\sigma}$. — Let $\mathcal{PP}$ be the set of pairs (P, T) where P is a patch in $\sigma^\infty(C)$ and T is a tile in $\sigma^\infty(C)$. We stress that T need not lie in P. Such a pair $(P, T) \in \mathcal{PP}$ is a *pointed patch*, and T is the *base tile* of the pointed patch (P, T).

For our purposes, we need to consider a kind of "pointed" version $\hat{\sigma}$ of σ: this will be a map

$$\begin{aligned} \hat{\sigma}: \quad & \mathcal{PP} \to \mathcal{PP} \\ & (P, T) \mapsto (\sigma(P), \mathsf{b}(T)). \end{aligned}$$

To completely define $\hat{\sigma}$, it remains now to define the map b. Let T be a tile of $\sigma^\infty(C)$. Then $\sigma(T)$ is a patch of $\sigma^\infty(C)$. If $\sigma(T)$ is a tile (this is the case when T is of type A or B), then we simply set $\mathsf{b}(T) = \sigma(T)$. If T is of type C, then we define $\mathsf{b}(T)$ as the tile of type C in $\sigma(T)$.

Remark 3.13. — In the example this last choice is a bit arbitrary: in particular, we could have considered other versions of $\hat{\sigma}$ where, for a tile T of type C, $\mathsf{b}(T)$ would be the tile of type A (or B) in $\sigma(T)$. In that case, we would have to modify accordingly the definition of the position map ω_0 of Section 5.1.3.

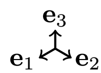
4. The Tribonacci dual substitution and its fractal tilings

We recall that a *substitution* is a morphism of the free monoid (of rank d). There is a general construction introduced in [22] and generalized by [4] that associates to a substitution a so-called *dual substitution*. To avoid to reproduce the general formalism of [4], we focus in Sections 4.2 and 4.3 on the *Tribonacci dual substitution* associated to the Tribonacci substitution $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. In particular, $d = 3$.

Dual substitutions act on facets of $\mathbb{R}^3$ (for the cellular decomposition of $\mathbb{R}^3$ given by $\mathbb{Z}^3$ translated copies of the unit cube) : the image of a facet is a set of facets. Hence, when iterating a dual substitution, one gets bigger sets of facets. A priori, there can be some overlaps: in that case, facets have to be count with some multiplicity, which leads to the notion of multiset of facets.

The data that encode a multiset of facet is given by: the type (an element of the set $\{1, 2, 3\}$), the position (an element of $\mathbb{Z}^3$) and the signed multiplicity (an element of $\mathbb{Z}$) of each facet. This setting is formalized in Section 4.1, leading to the equivalent notions of *weight functions* and *multisets of facets*. We detail the equivalence of the two points of view, so that later in Section 5 we will swap from one point of view to the other one according to the context.

4.1. Multisets of facets. — We denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the canonical basis of $\mathbb{R}^3$. In this article, this basis will be represented as follows in the different figures.



Let $\mathbf{x} \in \mathbb{Z}^3$ and let $i \in \{1,2,3\}$. The *facet* $[\mathbf{x}, i]^*$ of *vector* $\mathbf{x}$ and *type* i is a subset of $\mathbb{R}^3$ defined by:

$$[\mathbf{x}, 1]^* = \{\mathbf{x} + \lambda \mathbf{e}_2 + \mu \mathbf{e}_3 : \lambda, \mu \in [0,1]\} =$$ 

$$[\mathbf{x}, 2]^* = \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_3 : \lambda, \mu \in [0,1]\} =$$ 

$$[\mathbf{x}, 3]^* = \{\mathbf{x} + \lambda \mathbf{e}_1 + \mu \mathbf{e}_2 : \lambda, \mu \in [0,1]\} =$$ .

On each of the previous pictures, the symbol $\bullet$ represents the endpoint of the vector $\mathbf{x}$. We set $\mathcal{F} = \{[\mathbf{x}, i]^*, \mathbf{x} \in \mathbb{Z}^3, i \in \{1,2,3\}\}$.

Let $\mathcal{W}$ be the set of maps from $\mathcal{F}$ to $\mathbb{Z}_{\geq 0}$: such a map is called a *weight function*. A weight function $w \in \mathcal{W}$ gives a weight $w([\mathbf{x}, i]^*) \in \mathbb{Z}_{\geq 0}$ to any facet. Equipped with the addition of maps, $\mathcal{W}$ is a monoid.

A *multiset of facets* is a map $m : \mathbb{Z}^3 \to \mathbb{Z}^3_{\geq 0}$. We denote by $\mathcal{M}$ the set of multisets of facets. The set $\mathcal{M}$, equipped with the addition of maps, is a monoid.

Multisets of facets and weight functions are equivalent objects. Indeed, a multiset $m \in \mathcal{M}$ defines a weight function $w_m \in \mathcal{W}$ by declaring that $w_m([\mathbf{x}, i]^*)$ is the ith coordinate of $m(\mathbf{x})$. The map

$$\begin{aligned} \mathcal{M} &\to \mathcal{W} \\ m &\mapsto w_m \end{aligned}$$

is an isomorphism of monoids. The inverse of the map is given by

$$\begin{aligned} \mathcal{W} &\to \mathcal{M} \\ w &\mapsto \Big(\mathbf{x} \mapsto \big(w([\mathbf{x}, 1]^*), w([\mathbf{x}, 2]^*), w([\mathbf{x}, 3]^*)\big)\Big) \end{aligned}$$

The group $\mathbb{Z}^3$ acts naturally on $\mathcal{M}$: if $\mathbf{v} \in \mathbb{Z}^3$, $m \in \mathcal{M}$ then

$$m + \mathbf{v} : \mathbf{x} \mapsto m(\mathbf{x} - \mathbf{v}).$$

The *support* $\operatorname{supp}(w)$ of a weight function w is the union of facets which have positive weight:

$$\operatorname{supp}(w) = \bigcup_{w([\mathbf{x}, i]^*) > 0} [\mathbf{x}, i]^*.$$

It is a subset of $\mathbb{R}^3$. The *support* $\operatorname{supp}(m)$ of a multiset of facets m is the support of the corresponding weight function:

$$\operatorname{supp}(m) = \operatorname{supp}(w_m).$$

Let $\mathcal{W}^\circ \subset \mathcal{W}$ be the subset of weight functions which take values in $\{0,1\}$. We denote by $\mathcal{M}^\circ$ the corresponding subset of $\mathcal{M}$: a multiset of facets m is in $\mathcal{M}^\circ$ if and only if for all $\mathbf{x} \in \mathbb{Z}^3$, the coordinates of $m(\mathbf{x})$ are in $\{0,1\}$.

Remark 4.1. — We notice that a multiset of facets in $\mathcal{M}^\circ$ (or a weight function in $\mathcal{W}^\circ$) is totally determined by its support.

4.2. Dual substitutions. — In this Section we quickly review a construction due to Arnoux-Ito [4] that associates to a unimodular substitution s what is called a dual substitution $\mathbf{E}_1^*(s)$. For details we refer to [4, 8]. In particular, this construction can be applied to the Tribonacci substitution to lead the dual substitution $\mathbf{E}$ defined below. The definition of a substitution will be given in Section 4.3.

Consider

$$\mathbf{M}_s = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This matrix has characteristic polynomial $X^3 - X^2 - X - 1$. Its dominant eigenvalue β is a Pisot number: $\beta > 1$ and its conjugates $\alpha, \overline{\alpha} \in \mathbb{C}$ are such that $|\alpha| < 1$. The euclidean space $\mathbb{R}^3$ is hence decomposed as the direct sum of the *expanding line* (spanned by the left β-eigenvector $\mathbf{v}_\beta$ of $\mathbf{M}_s$) and the *contracting plane* $\mathcal{P}$ associated with the complex eigenvalues $\alpha, \overline{\alpha}$. Let $\pi_\beta : \mathbb{R}^3 \to \mathcal{P}$ be the projection on $\mathcal{P}$ along the line $\mathbb{R}\mathbf{v}_\beta$. We denote by $\mathbf{h} : \mathcal{P} \to \mathcal{P}$ the restriction the action of $\mathbf{M}_s$ to $\mathcal{P}$, which is contracting because $|\alpha| < 1$. Remark that $M_s, \mathbf{h}, \pi_\beta$ commute.

Definition 4.2. — We define

$$\mathbf{E} : \begin{cases} [\mathbf{x}, 1]^* \mapsto \mathbf{M}_s^{-1}\mathbf{x} + \big([\mathbf{0}, 1]^* \cup [\mathbf{0}, 2]^* \cup [\mathbf{0}, 3]^*\big) \\ [\mathbf{x}, 2]^* \mapsto \mathbf{M}_s^{-1}\mathbf{x} + [\mathbf{e}_3, 1]^* \\ [\mathbf{x}, 3]^* \mapsto \mathbf{M}_s^{-1}\mathbf{x} + [\mathbf{e}_3, 2]^*. \end{cases}$$

Alternatively $\mathbf{E}$ can be defined using multisets as following:

- The image of $[\mathbf{x}, 1]^*$ by $\mathbf{E}$ is the multiset $(\mathbb{Z}^3, m)$ where

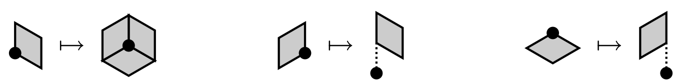
$$m(\mathbf{y}) = \begin{cases} \mathbf{0} & \text{if } \mathbf{y} \neq \mathbf{M}_s^{-1}\mathbf{x} \\ (1,1,1) & \text{if } \mathbf{y} = \mathbf{M}_s^{-1}\mathbf{x}. \end{cases}$$

- The image of $[\mathbf{x},2]^*$ by $\mathbf{E}$ is the multiset $(\mathbb{Z}^3, m)$ where
$$m(\mathbf{y}) = \begin{cases} \mathbf{0} & \text{if} \quad \mathbf{y} \neq \mathbf{M}_s^{-1}\mathbf{x} \\ (0,0,1) & \text{if} \quad \mathbf{y} = \mathbf{M}_s^{-1}\mathbf{x}. \end{cases}$$
- The image of $[\mathbf{x},3]^*$ by $\mathbf{E}$ is the multiset $(\mathbb{Z}^3, m)$ where
$$m(\mathbf{y}) = \begin{cases} \mathbf{0} & \text{if} \quad \mathbf{y} \neq \mathbf{M}_s^{-1}\mathbf{x} \\ (0,1,0) & \text{if} \quad \mathbf{y} = \mathbf{M}_s^{-1}\mathbf{x}. \end{cases}$$

We extend $\mathbf{E}$ to $\mathcal{M}$ by declaring that the image of a union of faces is the union of the images of these faces (the multiplicities of faces add up). We also note for future application that for all $\mathbf{x}, \mathbf{u} \in \mathbb{Z}^3$, and for all $i \in \{1,2,3\}$, we have

$$\mathbf{E}\big([\mathbf{x},i]^* + \mathbf{u}\big) = \mathbf{E}[\mathbf{x},i]^* + \mathbf{M}_s^{-1}\mathbf{u} \tag{4.1}$$

In practice, in order to simplify the notation, we represent $\mathbf{E}$ by the following pictures



where the black dots in the preimages stand for $\mathbf{x}$, and the black dots in the images stand for $\mathbf{M}_s^{-1}\mathbf{x}$. We denote the Euclidean scalar product of to vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ by $\langle \mathbf{u}, \mathbf{v} \rangle$, and we define $\mathcal{U}$ as the multiset of facets in $\mathcal{M}^\circ$ whose support is $[\mathbf{0},1]^* \cup [\mathbf{0},2]^* \cup [\mathbf{0},3]^*$ (see Remark 4.1).

Proposition 4.3 ([4, 2, 3]). — *We have*

- *For every integer n, $\mathbf{E}^n(\mathcal{U})$ belongs to $\mathcal{M}^\circ$, so it can be considered as a subset of $\mathbb{R}^3$ (see Remark 4.1).*
- *For every integer n, $\mathbf{E}^n(\mathcal{U})$ is a subset of $\mathbf{E}^{n+1}(\mathcal{U})$. The increasing sequence of $\mathbf{E}^n(\mathcal{U})$ converges and we denote*
$$\Sigma_{\mathsf{step}} = \lim_n \mathbf{E}^n(\mathcal{U}) = \bigcup_{n \in \mathbb{N}} \mathbf{E}^n(\mathcal{U}).$$
This set Σ_{step} is called the stepped surface.
- *Moreover:*
$$\Sigma_{\mathsf{step}} = \bigcup_{i \in \{1,2,3\}} \bigcup_{\substack{\mathbf{x} \in \mathbb{Z}^3 \\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle}} [\mathbf{x}, i]^*.$$
- *The restriction $\pi_\beta : \Sigma_{\mathsf{step}} \to \mathcal{P}$ of π_β to Σ_{step} is an homeomorphism.*
- *This map induces a tiling of the plane $\mathcal{P}$: we denote this tiling by $\mathcal{T}_{\mathsf{step}}$. The set of tiles of $\mathcal{T}_{\mathsf{step}}$ is:*
$$\bigcup_{i \in \{1,2,3\}} \{\pi_\beta([\mathbf{0}, i]^*) + \pi_\beta(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^3, 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

We say that a vector $\mathbf{x} \in \mathbb{Z}^3$ lies in Σ_{step} if there exists $i \in \{1,2,3\}$ such that $[\mathbf{x}, i]^*$ is a subset of Σ_{step}. Proposition 4.3 implies that the set of vectors lying in Σ_{step} is precisely $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$, where

$$\mathcal{V}_i = \{\mathbf{x} \in \mathbb{Z}^3 \ : \ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

REMARK 4.4. — We set $\mathcal{D}_i = \pi_\beta(\mathcal{V}_i)$. Then $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ is a Delone set in $\mathcal{P}$. The tiling $\mathcal{T}_{\mathsf{step}}$ is obtained by putting in $\mathcal{P}$ a tile of type i (i.e., a translated image of $\pi_\beta([\mathbf{0}, i]^*)$) at each vector in $\mathcal{D}_i$.

4.3. Link between the tiling $\mathcal{T}_{\mathsf{step}}$ and the Rauzy fractal. — In this section we recall basic facts about Rauzy fractals and substitutions [16]. We consider the free monoid $\{1,2,3\}^*$ and the *Tribonacci substitution* $s : \{1,2,3\}^* \to \{1,2,3\}^*$, which is a morphism defined by

$$s : 1 \mapsto 12 \quad 2 \mapsto 13 \quad 3 \mapsto 1.$$

Denote by $u = 12131\ldots$ the infinite word on the alphabet $\{1,2,3\}$ such that $s(u) = u$. In fact, for all $n \in \mathbb{N}$, $s^n(1)$ is a prefix of $s^{n+1}(1)$, so that $u = \lim_{+\infty} s^n(1)$. We denote by $u_i \in \{1,2,3\}$ the i-th letter in u: $u = u_1 u_2 u_3 \ldots$ with $u_1 = 1$, $u_2 = 2$, $u_3 = 3$.

Let us define $\mathbf{M}_s$ the *incidence matrix* of s: its ith column vector is equal to $\mathbf{P}(s(i))$, where $\mathbf{P}$ be the abelianization map from $\{1,2,3\}^*$ to $\mathbb{Z}^3$ defined by $\mathbf{P}(w) = (|w|_1, |w|_2, |w|_3)$ and $|w|_i$ stands for the number of occurrences of i in w. Consistently with the notation of Section 4.2, we have:

$$\mathbf{M}_s = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In the following proposition, we go on using the notation introduced in Section 4.2. In particular, the map $\mathbf{h} : \mathcal{P} \to \mathcal{P}$ is the restriction the action of $\mathbf{M}_s$ to contracting eigenplane $\mathcal{P}$.

PROPOSITION 4.5 ([28, 13, 4]). — *We have:*

- *The sets*
  - $\mathcal{R}_i = \overline{\left\{\pi_\beta\big(\mathbf{P}(u_1 \ldots u_{j-1})\big) : j \in \mathbb{N},\ u_j = i\right\}}$ *for* $i \in \{1,2,3\}$,
  - $\mathcal{R} = \overline{\left\{\pi_\beta\big(\mathbf{P}(u_1 \ldots u_{j-1})\big), j \in \mathbb{N}\right\}} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$

  are compact subsets of $\mathcal{P}$ (where $\overline{A}$ denotes the closure of a subset A in $\mathcal{P}$).
- *The $\mathcal{R}_i$'s are the solution of the following IFS:*

$$\begin{cases} \mathcal{R}_1 = \mathbf{h}\mathcal{R}_1 \cup \mathbf{h}\mathcal{R}_2 \cup \mathbf{h}\mathcal{R}_3 \\ \mathcal{R}_2 = \mathbf{h}\mathcal{R}_1 + \pi_\beta(e_1) \\ \mathcal{R}_3 = \mathbf{h}\mathcal{R}_2 + \pi_\beta(e_1). \end{cases}$$

- *There exists a tiling of the plane $\mathcal{P}$, that will be denoted by $\mathcal{T}_{\mathsf{frac}}$, whose set of tiles is:*

$$\bigcup_{i\in\{1,2,3\}} \{\mathcal{R}_i + \pi_\beta(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^3, 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

The set $\mathcal{R}$ is called the *Rauzy fractal* of s and $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ are the *subtiles* of $\mathcal{R}$.

Remark 4.6. — Comparing Proposition 4.5 and Proposition 4.3, we see that the positions of the tiles in $\mathcal{T}_{\mathsf{frac}}$ and $\mathcal{T}_{\mathsf{step}}$ are given by the same formula. Indeed, the tiling $\mathcal{T}_{\mathsf{frac}}$ is obtained by putting in $\mathcal{P}$ a tile of type i (i.e., a translated image of $\mathcal{R}_i$) at each vector in $\mathcal{D}_i$, where the sets $\mathcal{D}_i$ are precisely the ones of Remark 4.4. This explicits the strong relation between the two tilings $\mathcal{T}_{\mathsf{frac}}$ and $\mathcal{T}_{\mathsf{step}}$.

5. Link between topological and dual substitutions

5.1. The position map. — In this section we define the *position map* ω_0 from the set $\mathfrak{P}$ of paths of tiles in $\sigma^\infty(C)$ to $\mathbb{Z}^3$. (See Sections 5.1.2 and 5.1.3 below for precise definitions of $\mathfrak{P}$ and ω_0.)

We use the term *position map* because ω_0 will be used to give a geometric interpretation of the relative positions of two tiles in a common patch. This geometric interpretation is given by a vector in $\mathbb{Z}^3$ (the *position* of a tile with respect to another one). This will simplify the work done in Section 5.2 where we associate geometric patches of stepped surfaces to abstract topological patches.

5.1.1. *Notation.* — Let σ be the Tribonacci topological substitution defined in Section 3.2. We denote by P a patch of $\sigma^\infty(C)$, or possibly $P = \sigma^\infty(C)$.

Definition 5.1. — Consider a positive integer n. A *path of tiles* γ in P is a sequence $T_0, \ldots, T_n$ of tiles of P such that two consecutive tiles T_i, T_{i+1} are different and share (at least) one common edge for all $i \in \{0, \ldots, n-1\}$. The integer $n+1$ is the *length* of the path $\gamma = T_0, \ldots, T_n$. The set $\{T_0, \ldots, T_n\} \subset P$ is called the *support of* γ in P. When $T_0 = T_n$, γ is a *loop of tiles.* The integer n is the *length* of the loop $\gamma = T_0, \ldots, T_n$. The path $\gamma = T_0, \ldots, T_n$ and the path $\gamma' = T'_0, \ldots, T'_m$ can be *concatenated* if $T'_0 = T_n$. The *concatenation* of these paths is the path of tiles $\gamma\gamma' = T_0, \ldots, T_n, T'_1, \ldots, T'_m$.

Let γ be a loop of tiles in $\sigma^\infty(C)$. Among the connected components of the complementary of the support of γ, there is exactly one, denoted by $\mathsf{C}_\infty(\gamma)$, which contains an infinite number of tiles. We denote by $\mathsf{C}_0(\gamma)$ the complement of $\mathsf{C}_\infty(\gamma)$: it is a patch, in particular it is homeomorphic to a disk. Alternatively $\mathsf{C}_0(\gamma)$ is the smallest subpatch of $\sigma^\infty(C)$ containing the support of γ. We define the *area of* γ to be the number of tiles in $\mathsf{C}_0(\gamma)$:

$$\mathrm{Area}(\gamma) = |\mathsf{C}_0(\gamma)|.$$

Now we define an equivalence relation on paths of tiles which will define a *protopath of tiles*: The path $\gamma = T_0, \ldots, T_n$ and the path $\gamma' = T'_0, \ldots, T'_m$ are equivalent if

- $m = n$;
- for every $i \in \{0, \ldots, n\}$, T_i and T'_i have the same prototile type;
- the gluing edges of T_i and T_{i+1} have the same type as the gluing edges of T'_i and T'_{i+1}.

In the same way we define the notion of *protoloop*. The notions of concatenation, area and length naturally extend to protopaths.

5.1.2. *Additivity.* — Let P be patch in $\sigma^\infty(C)$. By definition, P is homeomorphic to a disk and its boundary is homeomorphic to the circle S^1. Let T be a tile in P, the *wreath* of T in P is the subset of $P \setminus \{T\}$ made of tiles that have at least one vertex in common with T. We denote it by $\mathrm{Wreath}_P(T)$. A *cut tile* of P is a tile whose wreath in P is not connected.

Let T be a cut tile of P. Then $P \setminus T$ has at least 2 connected components, and each of these components is a patch.

Lemma 5.2. — *Let P be a finite patch in $\sigma^\infty(C)$. There exists one tile of P which is not a cut tile and has one edge in the boundary of P.*

Proof. — Pick a tile T_0 in P which has a vertex in the boundary of P. If T_0 is not a cut tile, we are done. Otherwise, because P is homeomorphic to a disk, $P \setminus T_0$ has at least two connected components: we choose one of them that we denote by P_1. Pick a tile T_1 in P_1 which has a vertex in the boundary of P. If T_1 is not a cut tile, we are done. Otherwise, $P \setminus T_1$ has at least two connected components, and at least one of them is included in P_1: we choose one of these ones, that we denote by P_2. If every tile with one edge in the boundary is a cut-tile we obtain an infinite number of nested connected components which is a contradiction with the fact that P contains a finite number of tiles. □

Let $\mathfrak{P}$ be the set of protopaths of tiles in $\sigma^\infty(C)$. A map $\omega : \mathfrak{P} \to \mathbb{Z}^3$ is *additive* if for every $\gamma, \gamma' \in \mathfrak{P}$ that can be concatenated, we have $\omega(\gamma\gamma') = \omega(\gamma) + \omega(\gamma')$. Hence, an additive map $\omega : \mathfrak{P} \to \mathbb{Z}^3$ is uniquely defined by the image of the protopaths of length 2.

Definition 5.3. — Let $\mathfrak{P}_0 \subseteq \mathfrak{P}$ be the subset consisting of protoloops $\gamma = T_0, \ldots, T_n$ (with $T_0 = T_n$) of tiles in $\sigma^\infty(C)$ such that

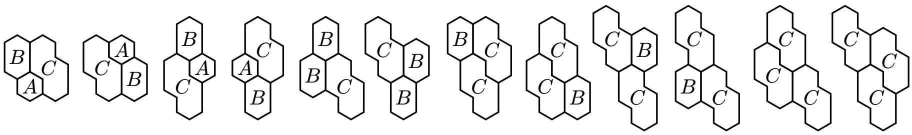
- for every $i \in \{1, \ldots, n-1\}$, $T_i \in \mathrm{Wreath}_{\mathsf{C}_0(\gamma)}(T_0)$,
- for every $i \neq j \in \{1, \ldots, n-1\}$, $T_i \neq T_j$.

We notice that $\mathfrak{P}_0$ contains all protoloops of lentgh 2.

LEMMA 5.4. — *The set $\mathfrak{P}_0$ is finite.*

Proof. — The valence of every vertex in $\sigma^\infty(C)$ is bounded (by 3, see Lemma 3.10), we deduce that the cardinlity is finite. □

It is possible to produce an explicit list of the elements in $\mathfrak{P}_0$. We detail below all the elements of $\mathfrak{P}_0$ of length 3.



LEMMA 5.5. — *Let $\omega : \mathfrak{P} \to \mathbb{Z}^3$ an additive map such that ω vanishes on the elements of $\mathfrak{P}_0$. Then ω vanishes on every protoloop in $\sigma^\infty(C)$.*

Proof. — The proof is by induction on the area of the protoloop of tiles γ in $\sigma^\infty(C)$. According to Definition 5.1, a loop of tiles has length a least 2, and thus also area at least 2. Moreover, a loop of tiles γ with area 2 is the concatenation of a certain number of copies of a same loop of tiles of length 2 $\gamma' = T_0, T_1, T_0$. Since any loop of length 2 is in $\mathfrak{P}_0$, we get that $\omega(\gamma') = 0$. By additivity of ω, we derive that $\omega(\gamma) = 0$.

Suppose that ω vanishes on every loop in $\sigma^\infty(C)$ of area at most k. Let $\gamma = T_0, \dots, T_n$ be a loop of area $k+1$. By Lemma 5.2 there exists a tile T in $\mathsf{C}_0(\gamma)$ which is not a cut-tile and has one edge in the boundary of $\mathsf{C}_0(\gamma)$. The tile T may occur several times in γ, and for each occurrence we will successively act as follows.

Let $T_i = T$ be an occurrence of T in γ.

- If $T_{i-1} = T_{i+1}$, we set $\gamma' = T_0, \dots, T_{i-1} = T_{i+1}, \dots, T_n$. Then by additivity of ω,
$$\omega(\gamma) = \omega(\gamma') + \omega(T_{i-1}, T_i) + \omega(T_i, T_{i+1}) = \omega(\gamma').$$
- If $T_{i-1} \neq T_{i+1} \in \mathrm{Wreath}_{\mathcal{C}_0(\gamma)}(T)$, see Figure 5.1. Since T is not a cut tile of $\mathcal{C}_0(\gamma)$, there exists a path of tiles $T_{i-1}, T'_1, \dots, T'_d, T_{i+1}$ in $\mathrm{Wreath}_{\mathcal{C}_0(\gamma)}(T)$ joining T_{i-1} and T_{i+1}. Then
$$\gamma'' = T, T_{i-1}, T'_1, \dots, T'_d, T_{i+1}, T$$
is a loop of tiles, and since T has at least an edge in the boundary of $\mathcal{C}_0(\gamma)$, we see that $\gamma'' \in \mathfrak{P}_0$. In particular, $\omega(\gamma'') = 0$. We set
$$\gamma' = T_0, \dots, T_{i-1}, T'_1, \dots, T'_d, T_{i+1}, \dots, T_n.$$
Then by additivity of ω,
$$\omega(\gamma) = \omega(\gamma') + \omega(\gamma'') = \omega(\gamma').$$

After proceeding as above for each occurrence of T in γ, we end up with a loop of tiles γ_0 such that $\omega(\gamma) = \omega(\gamma_0)$ and $\mathrm{Area}(\gamma_0) \leq k$ (since the support of γ_0 is included in $\mathcal{C}_0(\gamma) \smallsetminus \{T\}$). We conclude using the induction hypothesis on γ_0. □

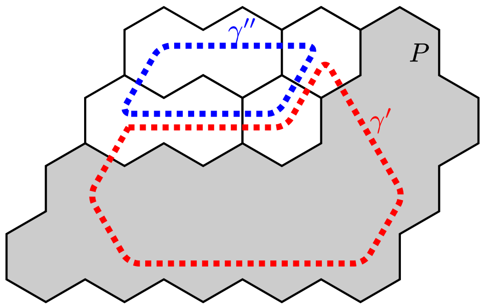


FIGURE 5.1. Scheme of proof of Lemma 5.5

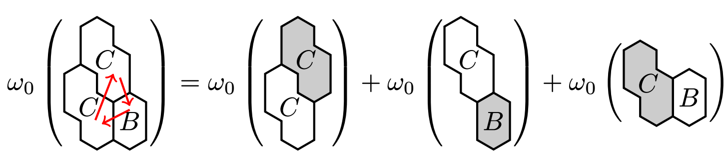
5.1.3. *The position map ω_0*

DEFINITION 5.6. — First we define ω_0 on the set of protopaths of length 2 in $\sigma^\infty(C)$. They form a finite set due to the heredity graph of edges. In Figure 5.2 we explicitly give this set and define this list and define ω_0 on it. For each protopath $\gamma = (T_0, T_1)$ of Figure 5.2, T_0 is the white tile. Moreover, we set $\omega_0(T_1, T_0) = -\omega_0(T_0, T_1)$.

We are now ready to define the map $\omega_0 : \mathfrak{P} \to \mathbb{Z}^3$. For every protopath $\gamma = T_1, T_2, \ldots, T_n$ of length $n \geq 2$, we set

$$\omega_0(\gamma) = \sum_{i=1}^{n-1} \omega_0(T_i, T_{i+1}).$$

Finally, to make sure that ω_0 vanishes on protoloops of $\sigma^\infty(C)$, it remains to check that the map ω_0 vanishes on the elements of $\mathfrak{P}_0$ (thanks to Lemma 5.5). This a finite process, since Lemma 5.4 ensures that $\mathfrak{P}_0$ is finite. We detail below an instance of the kind of easy computations that have to be carried on:



$$= (-1, 0, 2) + (0, 2, -3) + (1, -2, 1) = (0, 0, 0).$$

According to Lemma 5.5, we thus obtain the following proposition.

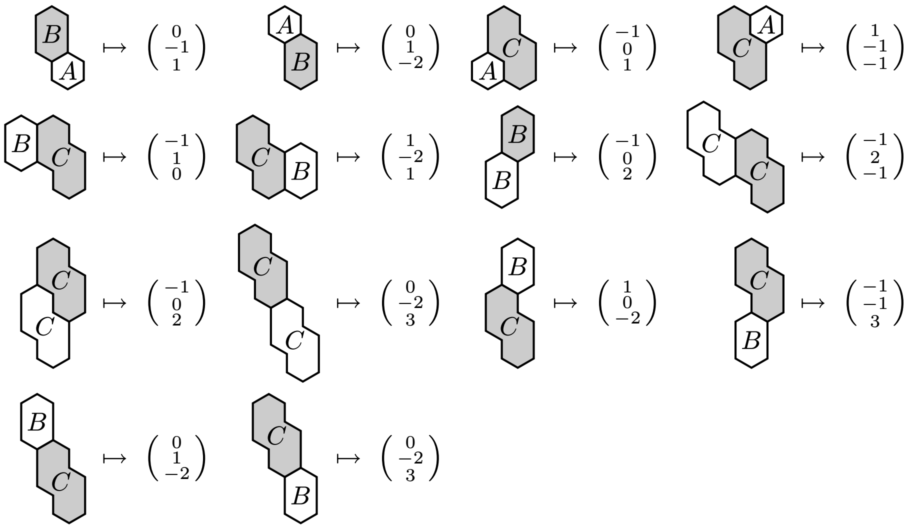


FIGURE 5.2. Definition of the map ω_0 over protopaths of length 2. The orientation of the path is indicated using colors: the first tile is white (see Definition 5.6).

PROPOSITION 5.7. — *The map $\omega_0 : \mathfrak{P} \to \mathbb{Z}^3$ defined previously is additive, and vanishes on each protoloop of tiles.* □

REMARK 5.8. — Given T, T' two tiles in $\sigma^\infty(C)$, we set

$$\omega_0(T, T') = \omega_0(\gamma)$$

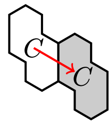
where γ is any path of tiles joining T to T'. Indeed, if γ, γ' are two such paths, Proposition 5.7 ensures that $\omega_0(\gamma) = \omega_0(\gamma')$ since ω_0 vanishes on the loop of tiles $\gamma'\gamma^{-1}$.

The following proposition will be used afterwards.

PROPOSITION 5.9. — *Let T, T' be tiles in $\sigma^\infty(C)$. Then*

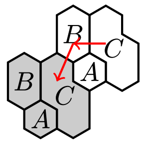
$$\omega_0\big(\mathsf{b}(T), \mathsf{b}(T')\big) = \mathbf{M}_s^{-1}\omega_0(T, T'). \tag{5.1}$$

Proof. — Proposition 5.7 ensures that ω_0 is additive. It is thus sufficient to prove (5.1) for adjacent tiles T, T'. Moreover, $\omega_0(T, T')$ only depends on the protopath (T, T') defined by the adjacent tiles T, T'. There is only a finite number of protopaths of length 2 to consider: those which are listed on Figure 5.2. We detail below an instance of the kind of easy computations that have to be carried on. Suppose that (T, T') is the following protopath:



.

By definition of ω_0 (Figure 5.2) we have $\omega_0(T, T') = (-1, 2, -1)$. We compute $\omega_0(\mathsf{b}(T), \mathsf{b}(T')$ by inspecting the image of (T, T') by σ:



.

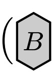
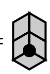
By choosing the path of length three above and by reading Figure 5.2, we compute

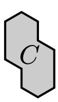
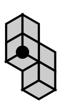
$$\omega_0(\mathsf{b}(T), \mathsf{b}(T')) = (1, -1, 0) + (1, 0, -2) = (2, -1, -2),$$

so the proposition holds in this case because $\mathbf{M}_s^{-1}(-1, 2, -1) = (2, -1, -2)$. $\square$

5.2. From the topological patches to the stepped surface. — Let (P, T) a pointed patch formed by a patch P of $\sigma^\infty(C)$ and a tile T of $\sigma^\infty(C)$. We are going to associate to (P, T) a multiset of facets $\varphi_0(P, T) \in \mathcal{M}$, see Section 4.1. When P is equal to the tile T, we simply denote the pointed patch (T, T) by T.

Let $\mathcal{T}_\sigma$ denote the set of tiles of $\sigma^\infty(C)$, as in Section 3.1.2. First, we define a map $\Phi : \mathcal{T}_\sigma \to \mathcal{M}$ so that two tiles of the same type have the same image. This map Φ is defined by setting:

$\Phi\Big($  $\Big) =$  $= \mathbf{E}([\mathbf{0}, 1]^*)$, $\Phi\Big($  $\Big) =$  $= \mathbf{E}^2([\mathbf{0}, 1]^*)$,

$\Phi\Big($  $\Big) =$  $= \mathbf{E}^3([\mathbf{0}, 1]^*)$.

Alternatively:

$$\Phi(A) = \mathbf{E}^3([\mathbf{0}, 3]^*) + \mathbf{e}_3 - \mathbf{e}_1, \quad \Phi(B) = \mathbf{E}^3([\mathbf{0}, 2]^*) + \mathbf{e}_2 - \mathbf{e}_1, \quad \Phi(C) = \mathbf{E}^3([\mathbf{0}, 1]^*).$$

In the pictures representing multisets, the symbol $\bullet$ indicates the origin of $\mathbb{R}^3$. For instance the image of A is the multiset $m : \mathbb{Z}^3 \to \mathbb{Z}^3_{\geq 0}$ defined by

$$m(\mathbf{y}) = \begin{cases} \mathbf{0} & \text{if } \mathbf{y} \neq (1, 0, -1) \\ (1, 1, 1) & \text{if } \mathbf{y} = (1, 0, -1). \end{cases}$$

For a patch P and a tile T, we consider T' another tile and γ a path of tiles from T to T'. By Proposition 5.7, the vector $\omega_0(\gamma)$ only depends on T and T', and not on the choice of the path γ. Thus we denote it by $\omega_0(T, T')$.

DEFINITION 5.10. — Let P be a patch of $\sigma^\infty(C)$, and let T be a tile of $\sigma^\infty(C)$. The multiset of facets $\varphi_0(P,T) \in \mathcal{M}$ is defined by:

$$\varphi_0(P,T) = \sum_{T' \in P} \Big(\Phi(T') + \omega_0(T,T') \Big).$$

REMARK 5.11. — By definition, we notice that $\varphi_0(T,T) = \Phi(T)$ for every tile T in $\sigma^\infty(C)$.

The next two lemmas state useful properties of the map φ_0. By definition of φ_0 and by additivity of ω_0, we derive immediately the following lemma.

LEMMA 5.12. — *Let P be a patch of $\sigma^\infty(C)$, and let T, T' be tiles of $\sigma^\infty(C)$. Then we have $\varphi_0(P,T) = \varphi_0(P,T') + \omega_0(T,T')$.*

Let P_1, P_2 be patches in $\sigma^\infty(C)$. We denote by $P_1 \cap P_2$ the (possibly empty) patch in $\sigma^\infty(C)$ made of tiles belonging to both P_1 and P_2: this is the standard definition of "intersection of patches".

LEMMA 5.13. — *Let P_1, P_2 be patches of $\sigma^\infty(C)$, and let T be a tile of $\sigma^\infty(C)$.*

- *If P_1 and P_2 have no tile in common, then $\varphi_0(P_1 \cup P_2, T) = \varphi_0(P_1,T) + \varphi_0(P_2,T)$.*
- *In the general situation we have $\varphi_0(P_1 \cup P_2, T) = \varphi_0(P_1,T) + \varphi_0(P_2,T) - \varphi_0(P_1 \cap P_2, T)$.*

Proof. — The second point is a direct consequence of the first one. By definition, and because P_1, P_2 have no tile in common, we have

$$\begin{aligned}\varphi_0(P_1 \cup P_2, T) &= \sum_{T' \in P_1 \cup P_2} \Big(\Phi(T') + \omega_0(T,T') \Big) \\ &= \sum_{T' \in P_1} \Big(\Phi(T') + \omega_0(T,T') \Big) + \sum_{T' \in P_2} \Big(\Phi(T') + \omega_0(T,T') \Big) \\ &= \varphi_0(P_1 \cup P_2, T) = \varphi_0(P_1,T) + \varphi_0(P_2,T). \end{aligned}$$ □

5.3. Commutation between σ, E, φ_0

PROPOSITION 5.14. — *Let P be a simply connected patch of $\sigma^\infty(C)$ and T a tile. We have*

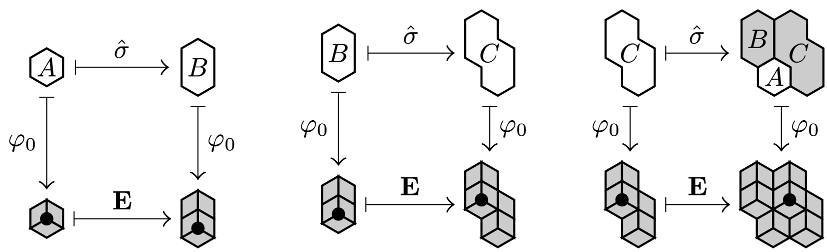
$$\varphi_0 \circ \hat{\sigma}(P,T) = \mathbf{E} \circ \varphi_0(P,T). \tag{5.2}$$

In the previous formula, we formally consider that the map $\mathbf{E}$ acts on multisets.

Proof. — A direct verification shows that for every tile T in $\sigma^\infty(C)$, we have

$$\varphi_0 \circ \hat{\sigma}(T,T) = \mathbf{E} \circ \varphi_0(T,T), \tag{5.3}$$

as detailed on the following diagrams (where the base tile of a patch is the white tile).



We now establish the relation (5.2) when P is a tile T'. Let T, T' be tiles of $\sigma^\infty(C)$. Recall that, by definition of the pointed substitution $\hat{\sigma}$, we have $\hat{\sigma}(T',T) = (\sigma(T'), \mathsf{b}(T))$. By Lemma 5.12 and Proposition 5.9, we have

$$\begin{aligned}\varphi_0(\sigma(T'), \mathsf{b}(T)) &= \varphi_0(\sigma(T'), \mathsf{b}(T')) + \omega_0(\mathsf{b}(T), \mathsf{b}(T')) \\ &= \varphi_0(\hat{\sigma}(T',T')) + \mathbf{M}_s^{-1}\omega_0(T,T').\end{aligned}$$

Using relation (5.3) and Equation 4.1, we get that:

$$\begin{aligned}\varphi_0(\hat{\sigma}(T',T)) &= \mathbf{E}(\varphi_0(T',T')) + \mathbf{M}_s^{-1}\omega_0(T,T') \\ &= \mathbf{E}(\varphi_0(T',T') + \omega_0(T,T')).\end{aligned}$$

Using again Lemma 5.12, we get:

$$\varphi_0(\sigma(T'), \mathsf{b}(T)) = \mathbf{E}(\varphi_0(T',T)). \tag{5.4}$$

We now prove the relation (5.2) in full generality. Let P be a patch in $\sigma^\infty(C)$ and let T be a tile of $\sigma^\infty(C)$. Since

$$\sigma(P) = \bigcup_{T'' \in P} \sigma(T''),$$

Lemma 5.13 ensures that

$$\begin{aligned}\varphi_0(\hat{\sigma}(P,T)) &= \varphi_0(\sigma(P), \mathsf{b}(T)) \\ &= \sum_{T'' \in P} \varphi_0(\sigma(T''), \mathsf{b}(T)) \\ &= \sum_{T'' \in P} \varphi_0(\hat{\sigma}(T'',T)).\end{aligned}$$

We conclude by using relation (5.4) and the additivity of the map $\mathbf{E}$:

$$\begin{aligned}\varphi_0(\hat{\sigma}(P,T)) &= \sum_{T''\in P} \mathbf{E}\Big(\varphi_0(T'',T)\Big)\\ &= \mathbf{E}\left(\sum_{T''\in P} \varphi_0(T'',T)\right)\\ &= \mathbf{E}(\varphi_0(P,T)).\end{aligned}$$

□

5.4. From $\sigma^\infty(C)$ to Σ_{step}

5.4.1. *The map induced by φ_0.* — Proposition 5.14 implies that for every integer n,

$$\varphi_0 \circ \hat{\sigma}^n(C,C) = \mathbf{E}^n \circ \varphi_0(C,C) = \mathbf{E}^n(\mathcal{U}).$$

In what follows, it is convenient to identify a multiset and its support as explained in Remark 4.1. Since $\mathbf{E}^n(\mathcal{U})$ converges to the stepped surface $\Sigma_{\mathsf{step}} = \bigcup_{n\in\mathbb{N}} \mathbf{E}^n(\mathcal{U})$ (see Proposition 4.3) and since $\sigma^n(C)$ converges to $\sigma^\infty(C)$, the map φ_0 induces a map, still denoted by φ_0 such that:

- for every tile T of $\sigma^\infty(C)$, $\varphi_0(T,C)$ is a subset of Σ_{step},
- $\Sigma_{\mathsf{step}} = \displaystyle\bigcup_{T \text{ tile of } \sigma^\infty(C)} \varphi_0(T,C)$.

Lemma 5.15. — *For every tile T of $\sigma^\infty(C)$, there exists a unique facet $[\mathbf{x},i]^*$ in Σ_{step} such that*

$$\varphi_0(T,C) = \mathbf{E}^3([\mathbf{x},i]^*).$$

Proof. — First, we recall that Lemma 3 of [4] states that the map $\mathbf{E}$ is "injective": precisely, if $\mathbf{E}([\mathbf{x},i]^*)$ and $\mathbf{E}([\mathbf{x}',i']^*)$ have a facet in common, then $\mathbf{x} = \mathbf{x}'$ and $i = i'$. This provides the unicity of the facet $[\mathbf{x},i]^*$ in the lemma (if it exists).

We know that $\varphi_0(T,C) = \Phi(T) + \omega_0(T,C)$. By definition of Φ, there exists an integer i such that $\varphi_0(T,C) = \mathbf{E}^3([0,i]^*) + \mathbf{u}_i + \omega_0(T,C)$ with

$$\mathbf{u}_1 = \mathbf{e}_3 - \mathbf{e}_1 = \mathbf{M}_s^{-3}(-2\mathbf{e}_1 - \mathbf{e}_2), \mathbf{u}_2 = \mathbf{e}_2 - \mathbf{e}_1 = \mathbf{M}_s^{-3}(-\mathbf{e}_1), \mathbf{u}_3 = \mathbf{0}.$$

That is to say:

$$\varphi_0(T,C) = \mathbf{E}^3([\mathbf{x},i]^*) \quad \text{with} \quad \mathbf{x} = \mathbf{M}_s^3(\omega_0(T,c) + \mathbf{u}_i).$$

We claim that $[\mathbf{x},i]^*$ lies in Σ_{step}. Indeed since $\Sigma_{\mathsf{step}} = \mathbf{E}^3(\Sigma_{\mathsf{step}})$, we know that there exists facets of Σ_{step} which images by $\mathbf{E}^3$ cover $\varphi_0(T,C)$. By Lemma 3 of [4] again, we conclude that the facet $[\mathbf{x},i]^*$ is lying in Σ_{step}. □

5.4.2. *The map Ψ.* — We are now in position to define a bijection Ψ from the set of tiles of $\sigma^\infty(C)$ to the set of facets of Σ_{step}:

$$\Psi(T) = [\mathbf{x}, i]^*, \quad \text{where} \quad \varphi_0(T) = \mathbf{E}^3([\mathbf{x}, i]^*).$$

Moreover, since $\Phi(C) = \mathbf{E}^3([\mathbf{0}, 1]^*)$, $\Phi(B) = \mathbf{E}^3([\mathbf{0}, 2]^*)$ and $\Phi(A) = \mathbf{E}^3([\mathbf{0}, 3]^*)$, we see that

$$\begin{aligned} \mathrm{type}(T) = A &\Leftrightarrow \mathrm{type}(\Psi(T)) = 3, \\ \mathrm{type}(T) = B &\Leftrightarrow \mathrm{type}(\Psi(T)) = 2, \\ \mathrm{type}(T) = C &\Leftrightarrow \mathrm{type}(\Psi(T)) = 1. \end{aligned}$$

We set $\theta(A) = 3$, $\theta(B) = 2$, $\theta(C) = 1$. We summarize the previous discussion in the following proposition:

THEOREM 5.16. — *The map Ψ defined, for every tile T of $\sigma^\infty(C)$, by:*

$$\Psi(T) = [\mathbf{M}_s^3(\omega_0(T, C) + \mathbf{u}_{type(T)}), \theta(type(T))]^*$$

is a bijection from the set of tiles of $\sigma^\infty(C)$ to the set of facets of Σ_{step}.

5.5. Link between two tilings $\mathcal{T}_{\mathsf{top}}$ and $\mathcal{T}_{\mathsf{step}}$

5.5.1. *Theorem 5.16 revisited.* — We recall that, according to Proposition 3.11, $\sigma^\infty(C)$ can be realized as the tiling $\mathcal{T}_{\mathsf{top}}$, and according to Proposition 4.3 the tiling $\mathcal{T}_{\mathsf{step}}$ is the "image" of the stepped surface Σ_{step} by the projection $\pi_\beta : \Sigma_{\mathsf{step}} \to \mathcal{P}$. Hence, Theorem 5.16 explains exactly how the two tilings $\mathcal{T}_{\mathsf{top}}$ and $\mathcal{T}_{\mathsf{step}}$ are related. In particular:

- The map Ψ sends tiles of the same type to tiles of the same type.
- In $\mathcal{T}_{\mathsf{top}}$, the way to locate a tile T with respect to another tile T', via the identification with $\sigma^\infty(C)$, is by using the position $\omega_0(T, T')$. In $\mathcal{T}_{\mathsf{step}}$, the corresponding tiles $\Psi(T)$ and $\Psi(T')$ then will differ from the vector $\pi_\beta(\omega_0(T, T'))$.

5.5.2. *Via the Delone set $\mathcal{D} = \bigcup_{i \in \{1,2,3\}} \mathcal{D}_i$.* — We would like to explicit the link between $\mathcal{T}_{\mathsf{top}}$ and $\mathcal{T}_{\mathsf{step}}$ in the same spirit of what we explain in Remark 4.6 and Remark 4.4. For that, we first notice that, alternatively, $\sigma^\infty(C)$ can be geometrized as follow.

According to Lemma 5.15, the set $\{\pi_\beta(\varphi_0(T, C)) \mid T \text{ tile of } \sigma^\infty(C)\}$ tiles the plane $\mathcal{P}$, and the resulting tiling is a geometric realization of $\sigma^\infty(C)$. The definition of Ψ in Section 5.2 gives us a base point in $\Phi(A)$, $\Phi(B)$ and $\Phi(C)$, and consequently, gives rise to a base point $\mathbf{x}_T$ in each $\varphi_0(T, C)$. The set $\{\pi_\beta(\mathbf{x}_T) \mid T \text{ tile of } \sigma^\infty(C)\}$ is a Delone set in $\mathcal{P}$. However, it is not equal to the set $\mathcal{D}$ of Remark 4.4: indeed, it is equal to $\mathbf{M}_s^{-3}\mathcal{D}$, see the proof of Lemma 5.15.

This leads us to do what follows. For each tile T of $\sigma^\infty(C)$, we consider

$$T_{\mathsf{geo}} = \pi_\beta(\mathbf{M}_s^3(\varphi_0(T, C))) \subset \mathcal{P}.$$

The set $\{T_{\mathsf{geo}} \mid T \text{ tile of } \sigma^\infty(C)\}$ tiles the plane $\mathcal{P}$. This tilling is again a geometric realization of $\sigma^\infty(C)$, and we denote it by $\mathcal{T}'_{\mathsf{top}}$. By construction, $\pi_\beta(\mathbf{M}_s^3\mathbf{x}_T)$

lies in T_{geo}, and the set $\{\pi_\beta(\mathbf{M}_s^3\mathbf{x}_T) \mid T \text{ tile of } \sigma^\infty(C)\}$ is precisely the Delone set $\mathcal{D}$. Moreover, Theorem 5.16 ensures that the subset consisting of vectors $\pi_\beta(\mathbf{M}_s^3\mathbf{x}_T)$ with $\theta(\text{type}(T)) = i$ ($i \in \{1,2,3\}$) is precisely the set $\mathcal{D}_i$ of Remark 4.4.

To sum up: the tiling $\mathcal{T}'_{\mathsf{top}}$ is obtained by putting in $\mathcal{P}$ a tile of type i (i.e., a translated image of T_{geo} with $\theta(\text{type}(T)) = i$) at each vector in $\mathcal{D}_i$. This explicits the strong relation between the two tilings $\mathcal{T}'_{\mathsf{top}}$ and $\mathcal{T}_{\mathsf{step}}$, and thus also with $\mathcal{T}_{\mathsf{frac}}$ via Remark 4.6.

5.5.3. *The map Ψ as a "two-dimensional sliding block code".* — It is worth to remark that, by definition of Ψ, and because the position map ω_0 is additive, there is an elementary way to rebuild $\mathcal{T}_{\mathsf{step}}$ from $\mathcal{T}_{\mathsf{top}}$ just by looking at local configurations of tiles. We give in Figure 5.3 all the information needed for doing that.

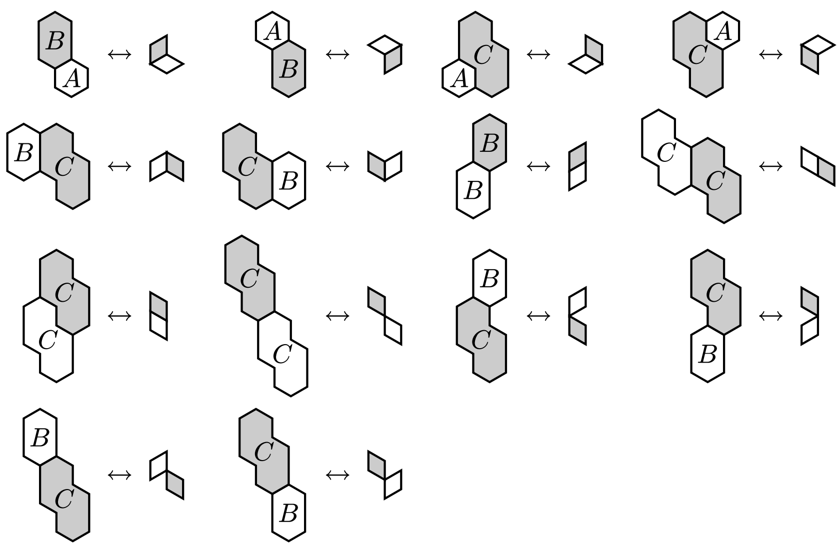


FIGURE 5.3. Ψ and Ψ^{-1} as a two-dimensional sliding block code.

In practice, to construct $\mathcal{T}_{\mathsf{step}}$ from $\mathcal{T}_{\mathsf{top}}$, we can ignore the formula of Theorem 5.16 and simply use the recipe given in Figure 5.3. We choose a tile T in $\mathcal{T}_{\mathsf{top}}$, and put in the plane a tile of $\mathcal{T}_{\mathsf{step}}$ of same type as T. Then we choose a tile T' adjacent to T and use Figure 5.3 to place correctly the corresponding tile $\Psi(T')$ relatively to $\Psi(T)$. And we go on, inductively rebuilding $\mathcal{T}_{\mathsf{step}}$.

This can also be done to define Ψ^{-1}, see Figure 5.3.

This point of view on Ψ remind us of the so-called *sliding block code* in classical symbolic dynamics, see for instance [25]. In that spirit, ψ could be called a *two-dimensional sliding block code.*

6. Concluding remarks

Towards more general results. — The results presented in this article are specifically about the tilings associated with the Tribonacci substitution and its associated Rauzy fractal tiling. We have been able to get such results for some other examples of Pisot substitutions, such as the one shown in Figure 6.1.

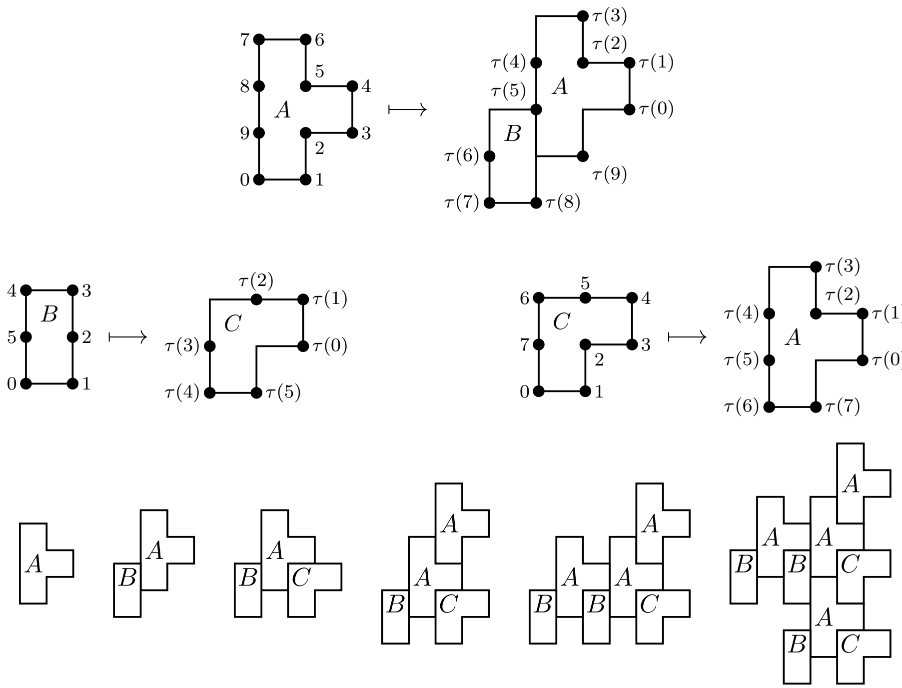
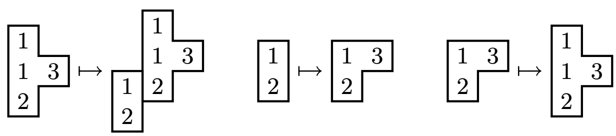


FIGURE 6.1. Definition of the topological substitution τ obtained from the dual substitution associated with $1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 2$ (top). Six iterations from the tile A are shown (bottom).

We describe how we derived the topological substitution τ from the dual substitution $\mathbf{E}_1^*(t)$ associated with the symbolic substitution $t : 1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 2$.

1. Start with a single facet $[\mathbf{0}, 1]^*$ and compute $\mathbf{E}_1^*(t)^k([\mathbf{0}, 1]^*)$ with k large enough, in such a way that the patch $\mathbf{E}_1^*(t)^k([\mathbf{0}, 1]^*)$ contains every possible neighboring couples of facets. This is shown in Figure 6.2 (left).
2. Compute some more iterates by $\mathbf{E}_1^*(t)$ to "inflate" the tiles from single facets to patches of facets (metatiles). This is shown in Figure 6.2 (center), where each metatile has the same color as its single-facet preimage. We must iterate $\mathbf{E}_1^*(t)$ sufficiently many times (3 times in this case), so that every intersection between two tiles is either empty or consists of edges (single points are not allowed).
3. Iterate $\mathbf{E}_1^*(t)$ one more time to "read" how the metatiles should be substituted. This is where we extract the information to define the topological substitution τ in two steps:
   (a) We define the image of each tile by noticing that the image tiles are either one of the other metatiles, or a union of two metatiles:



   (b) We define the boundaries' images by comparing the common edges between two adjacent metatiles and the common edges between their images in Figure 6.2 (center and right).

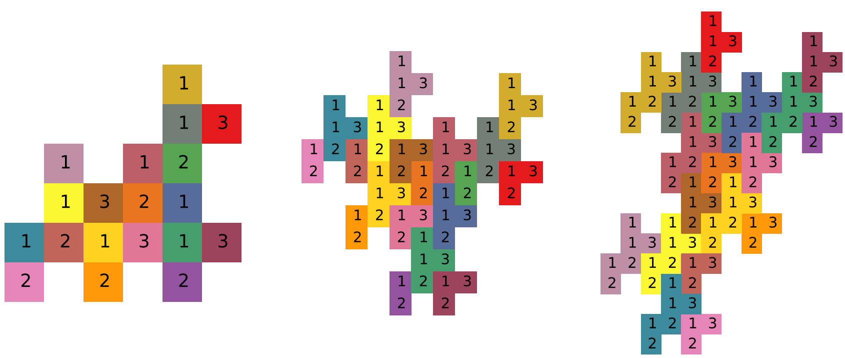


FIGURE 6.2. Deriving the topological substitution τ from the dual substitution $\mathbf{E}_1^*(t)$. From left to right: $\mathbf{E}_1^*(t)^7([\mathbf{0}, 1]^*)$, $\mathbf{E}_1^*(t)^{10}([\mathbf{0}, 1]^*)$ and $\mathbf{E}_1^*(t)^{11}([\mathbf{0}, 1]^*)$.

Limits of this approach. — Despite the fact that Rauzy fractals tilings have finite local complexity [1], the method described above is not guaranteed to work in general for an arbitrary dual substitution. The main problem is that in many cases, the topology of the patterns produced by the dual substitution can be complicated (disconnected or not simply connected, for example). This can cause Step 2 above to fail.

These difficulties are linked with some questions about the dynamics of the underlying Pisot substitution. Indeed, it can be proved that the underlying Pisot substitution has pure discrete spectrum if and only if the patterns generated by its associated dual substitution contain arbitrarily large balls [8]. This property is difficult to check for dual substitutions, but easy to check for topological substitutions (see the core property in Section 3). See [1] for more information about the Pisot conjecture and its different formulations.

Another possible approach to the original question raised in the introduction would be, given an IFS with a topologically complicated attractor, to construct another IFS with a topologically simpler attractor which gives a similar tiling, and then apply the method described above. For example, the tilings associated with the Tribonacci substitution and the "flipped Tribonacci" substitution $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 1$ are closely related: the tile positions are equal (but the neighbor relations change), even though the topology of the flipped Tribonacci fractal is complicated. However we do not know if this feasible in general, even in the case of Rauzy fractals.

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