ISSN 0012-9593

quatrième série - tome 42 fascicule 1 janvier-février 2009

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Thierry GALLAY & Romain JOLY

Global stability of travelling fronts for a damped wave equation with bistable nonlinearity

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4$^{\text{e}}$ série, t. 42, 2009, p. 103 à 140

GLOBAL STABILITY OF TRAVELLING FRONTS FOR A DAMPED WAVE EQUATION WITH BISTABLE NONLINEARITY

BY Thierry GALLAY AND Romain JOLY

ABSTRACT. – We consider the damped wave equation $\alpha u_{tt} + u_t = u_{xx} - V'(u)$ on the whole real line, where V is a bistable potential. This equation has travelling front solutions of the form $u(x,t) = h(x - st)$ which describe a moving interface between two different steady states of the system, one of which being the global minimum of V. We show that, if the initial data are sufficiently close to the profile of a front for large $|x|$, the solution of the damped wave equation converges uniformly on $\mathbb{R}$ to a travelling front as $t \to +\infty$. The proof of this global stability result is inspired by a recent work of E. Risler [38] and relies on the fact that our system has a Lyapunov function in any Galilean frame.

RÉSUMÉ. – Nous étudions l'équation hyperbolique amortie $\alpha u_{tt} + u_t = u_{xx} - V'(u)$ sur la droite réelle, où V est un potentiel bistable. Cette équation possède des ondes progressives de la forme $u(x,t) = h(x - st)$ qui décrivent le mouvement d'une interface séparant deux états d'équilibre du système, dont l'un est le minimum global de V. Nous montrons que, si les données initiales sont suffisamment proches du profil du front pour $|x|$ grand, alors la solution de l'équation hyperbolique amortie converge uniformément sur $\mathbb{R}$ vers une onde progressive lorsque $t \to +\infty$. La démonstration de ce résultat de stabilité globale s'inspire d'un travail récent de E. Risler [38] et repose sur l'existence pour notre système d'une fonction de Lyapunov dans tout référentiel en translation uniforme.

1. Introduction

The aim of this paper is to describe the long-time behavior of a large class of solutions of the semilinear damped wave equation

$$\alpha u_{tt} + u_t = u_{xx} - V'(u), \tag{1.1}$$

where $\alpha > 0$ is a parameter, $V : \mathbb{R} \to \mathbb{R}$ is a smooth bistable potential, and the unknown $u = u(x,t)$ is a real-valued function of $x \in \mathbb{R}$ and $t \geq 0$. Equations of this form appear in many different contexts, especially in physics and in biology. For instance, Equation (1.1) describes the continuum limit of an infinite chain of coupled oscillators, the propagation of voltage along a nonlinear transmission line [4], and the evolution of an interacting population if the spatial spread of the individuals is modelled by a velocity jump process instead of the usual Brownian motion [18, 21, 24].

As was already observed by several authors, the long-time asymptotics of the solutions of the damped wave equation (1.1) are quite similar to those of the corresponding reaction-diffusion equation $u_t = u_{xx} - V'(u)$. In particular, if $V'(u)$ vanishes rapidly enough as $u \to 0$, the solutions of (1.1) originating from small and localized initial data converge as $t \to +\infty$ to the same self-similar profiles as in the parabolic case [11, 23, 27, 34, 35]. The analogy persists for solutions with nontrivial limits as $x \to \pm\infty$, in which case the long-time asymptotics are often described by uniformly translating solutions of the form $u(x,t) = h(x - st)$, which are usually called *travelling fronts*. Existence of such solutions for hyperbolic equations of the form (1.1) was first proved by Hadeler [19, 20], and a few stability results were subsequently obtained by Gallay & Raugel [9, 10, 12, 13].

While local stability is an important theoretical issue, in the applications one is often interested in *global convergence results* which ensure that, for a large class of initial data with a prescribed behavior at infinity, the solutions approach travelling fronts as $t \to +\infty$. For the scalar parabolic equation $u_t = u_{xx} - V'(u)$, such results were obtained by Kolmogorov, Petrovski & Piskunov [29], by Kanel [25, 26], and by Fife & McLeod [7, 8] under various assumptions on the potential. All the proofs use in an essential way comparison theorems based on the maximum principle. These techniques are very powerful to obtain global information on the solutions, and were also successfully applied to monotone parabolic systems [41, 44] and to parabolic equations on infinite cylinders [39, 40].

However, unlike its parabolic counterpart, the damped wave equation (1.1) has no maximum principle in general. More precisely, solutions of (1.1) taking their values in some interval $I \subset \mathbb{R}$ obey a comparison principle only if

$$4\alpha \sup_{u \in I} V''(u) \leq 1, \tag{1.2}$$

see [37] or [10, Appendix A]. In physical terms, this condition means that the relaxation time α is small compared to the period of the nonlinear oscillations. In particular, if I is a neighborhood of a local minimum $\bar{u}$ of V, inequality (1.2) implies that the linear oscillator $\alpha u_{tt} + u_t + V''(\bar{u})u = 0$ is strongly damped, so that no oscillations occur. It was shown in [10, 13] that the travelling fronts of (1.1) with a monostable nonlinearity are stable against large perturbations provided that the parameter α is sufficiently small so that the strong damping condition (1.2) holds for the solutions under consideration. In other words, the basin of attraction of the hyperbolic travelling fronts becomes arbitrarily large as $\alpha \to 0$, but if α is not assumed to be small there is no hope to use "parabolic" methods to obtain global stability results for the travelling fronts of the damped wave equation (1.1).

Recently, however, a different approach to the stability of travelling fronts has been developped by Risler [14, 38]. The new method is purely variational and is therefore restricted to systems that possess a gradient structure, but its main interest lies in the fact that it does not rely on the maximum principle. The power of this approach is demonstrated in the pioneering work [38] where global convergence results are obtained for the non-monotone reaction-diffusion system $u_t = u_{xx} - \nabla V(u)$, with $u \in \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$. The aim of the present article is to show that Risler's method can be adapted to the damped hyperbolic equation

(1.1) and allows in this context to prove global convergence results *without any smallness assumption* on the parameter α.

Before stating our theorem, we need to specify the assumptions we make on the nonlinearity in (1.1). We suppose that $V \in \mathcal{C}^3(\mathbb{R})$, and that there exist positive constants a and b such that

$$uV'(u) \geq au^2 - b, \quad \text{for all } u \in \mathbb{R}. \tag{1.3}$$

In particular, $V(u) \to +\infty$ as $|u| \to \infty$. We also assume

$$V(0) = 0, \quad V'(0) = 0, \quad V''(0) > 0, \tag{1.4}$$

$$V(1) < 0, \quad V'(1) = 0, \quad V''(1) > 0. \tag{1.5}$$

Finally we suppose that, except for $V(0)$ and $V(1)$, all critical values of V are positive:

$$\Big\{u \in \mathbb{R} \,\Big|\, V'(u) = 0\,,\ V(u) \leq 0\Big\} = \{0\,;\,1\}. \tag{1.6}$$

In other words V is a smooth, strictly coercive function which reaches its global minimum at $u = 1$ and has in addition a local minimum at $u = 0$. We call V a *bistable* potential because both $u = 0$ and $u = 1$ are stable equilibria of the one-dimensional dynamical system $\dot{u} = -V'(u)$. The simplest example of such a potential is represented in Fig. 1. Note however that V is allowed to have positive critical values, including local minima.

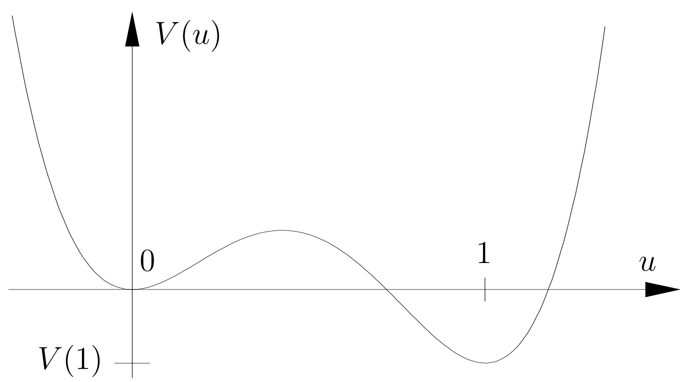

FIGURE 1. The simplest example of a potential V satisfying assumptions (1.3)–(1.6).

Under assumptions (1.4)–(1.6), it is well-known that the parabolic equation $u_t = u_{xx} - V'(u)$ has a family of travelling fronts of the form $u(x,t) = h(x - c_*t - x_0)$ connecting the stable equilibria $u = 1$ and $u = 0$, see e.g. [1]. More precisely, there exists a unique speed $c_* > 0$ such that the boundary value problem

$$\begin{cases} h''(y) + c_*h'(y) - V'(h(y)) = 0, & y \in \mathbb{R}, \\ h(-\infty) = 1, \quad h(+\infty) = 0, \end{cases} \tag{1.7}$$

has a solution $h : \mathbb{R} \to (0,1)$, in which case the profile h itself is unique up to a translation. Moreover $h \in \mathcal{C}^4(\mathbb{R})$, $h'(y) < 0$ for all $y \in \mathbb{R}$, and $h(y)$ converges exponentially toward its limits as $y \to \pm\infty$. As was observed in [10, 19], for any $\alpha > 0$ the damped hyperbolic equation (1.1) has a corresponding family of travelling fronts given by

$$u(x,t) \;=\; h(\sqrt{1+\alpha c_*^2}\, x - c_* t - x_0), \quad x_0 \in \mathbb{R}. \tag{1.8}$$

Remark that the actual speed of these waves is not c_*, but $s_* = c_*/\sqrt{1+\alpha c_*^2}$. In particular s_* is smaller than $1/\sqrt{\alpha}$ (the slope of the characteristics of Equation (1.1)), which means that the travelling fronts (1.8) are always "subsonic". In what follows we shall refer to c_* as the "parabolic speed" to distinguish it from the physical speed s_*.

We are now in position to state our main result:

THEOREM 1.1. – *Let $\alpha > 0$ and let $V \in \mathcal{C}^3(\mathbb{R})$ satisfy (1.3)–(1.6) above. Then there exist positive constants δ and ν such that, for all initial data $(u_0, u_1) \in H^1_{\mathrm{ul}}(\mathbb{R}) \times L^2_{\mathrm{ul}}(\mathbb{R})$ satisfying*

$$\limsup_{\xi \to -\infty} \int_\xi^{\xi+1} \Big((u_0(x)-1)^2 + u_0'(x)^2 + u_1(x)^2\Big)\,\mathrm{d}x \le \delta, \tag{1.9}$$

$$\limsup_{\xi \to +\infty} \int_\xi^{\xi+1} \Big(u_0(x)^2 + u_0'(x)^2 + u_1(x)^2\Big)\,\mathrm{d}x \le \delta, \tag{1.10}$$

Equation (1.1) has a unique global solution (for positive times) such that $u(\cdot,0) = u_0$, $u_t(\cdot,0) = u_1$. Moreover, there exists $x_0 \in \mathbb{R}$ such that

$$\sup_{x\in\mathbb{R}} \Big| u(x,t) - h(\sqrt{1+\alpha c_*^2}\, x - c_* t - x_0)\Big| \;=\; \mathcal{O}(e^{-\nu t}), \quad \textit{as } t \to +\infty. \tag{1.11}$$

REMARKS. – 1. Loosely speaking Theorem 1.1 says that, if the initial data (u_0, u_1) are close enough to the global equilibrium $(1,0)$ as $x \to -\infty$ and to the local equilibrium $(0,0)$ as $x \to +\infty$, the solution $u(x,t)$ of (1.1) converges uniformly in space and exponentially fast in time toward a member of the family of travelling fronts (1.8). In particular, any solution which looks roughly like a travelling front at initial time will eventually approach a suitable translate of that front. It should be noted, however, that our result does not give any constructive estimate of the time needed to reach the asymptotic regime described by (1.11). Depending on the shape of the potential and of the initial data, very long transients can occur before the solution actually converges to a travelling front.

2. The definition of the uniformly local Lebesgue space $L^2_{\mathrm{ul}}(\mathbb{R})$ and the uniformly local Sobolev space $H^1_{\mathrm{ul}}(\mathbb{R})$ will be recalled at the beginning of Section 2. These spaces provide a very convenient framework to study infinite-energy solutions of the hyperbolic equation (1.1), but their knowledge is not necessary to understand the meaning of Theorem 1.1. In a first reading one can assume, for instance, that u_0' and u_1 are bounded and uniformly continuous functions, in which case assumptions (1.9), (1.10) can be replaced by

$$\limsup_{x\to-\infty}(|u_0(x)-1| + |u_0'(x)| + |u_1(x)|) \;\le\; \delta, \quad \limsup_{x\to+\infty}(|u_0(x)| + |u_0'(x)| + |u_1(x)|) \;\le\; \delta.$$

Also, to simplify the presentation, we have expressed our convergence result (1.11) in the uniform norm, but the proof will show that the solution $u(x,t)$ of (1.1) converges to a travelling front in the uniformly local energy space $H^1_{\mathrm{ul}}(\mathbb{R}) \times L^2_{\mathrm{ul}}(\mathbb{R})$, see (9.5) below.