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Bassam FAYAD & Raphaël KRIKORIAN

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HERMAN'S LAST GEOMETRIC THEOREM

BY BASSAM FAYAD AND RAPHAËL KRIKORIAN

ABSTRACT. – We present a proof of Herman's Last Geometric Theorem asserting that if F is a smooth diffeomorphism of the annulus having the intersection property, then any given F-invariant smooth curve on which the rotation number of F is Diophantine is accumulated by a positive measure set of smooth invariant curves on which F is smoothly conjugated to rotation maps. This implies in particular that a Diophantine elliptic fixed point of an area preserving diffeomorphism of the plane is stable. The remarkable feature of this theorem is that it does not require any twist assumption.

RÉSUMÉ. – Nous présentons une preuve du dernier théorème géométrique d'Herman qui affirme que, si un difféomorphisme F de l'anneau possède la propriété d'intersection, alors toute courbe C^∞ F-invariante, sur laquelle le nombre de rotation de F est diophantien, est accumulée par un ensemble de mesure positive de courbes invariantes C^∞ sur lesquelles F est C^∞-conjuguée à une rotation. Ceci implique en particulier la stabilité des points fixes elliptiques diophantiens des difféomorphismes du plan qui préservent l'aire. Le caractère remarquable de ce théorème est qu'il ne requiert aucune condition de torsion.

1. Introduction

In his 1998 ICM address [8], M. Herman asked the following question: "*Let f be a C^∞- diffeomorphism preserving the Lebesgue measure of $\mathbb{T}^1 \times [-1,1]$, homotopic to the identity, that has a finite number of periodic points (...) and is such that the rotation number $\rho(f_{|\mathbb{T}^1\times[-1,1]}) = \alpha$ satisfies a diophantine condition. Is f C^∞-conjugated to $R_\alpha(\theta, r) = (\theta+\alpha, r)$?*

I would expect a counter-example, but there is some evidence in the opposite direction.

We will show elsewhere this is the case if f is C^∞-close to R_α and [in this case] *f is always C^∞-conjugated to R_α near $\mathbb{T}^1 \times \{\pm 1\}$.*"

By "Herman's Last Geometric Theorem" (1), we shall refer to the latter local rigidity result (see Corollary 1 for an exact statement), together with Herman's discovery that an invariant

(1) This denomination was suggested to us by A. Katok

diophantine circle of an area preserving planar diffeomorphism is always accumulated by a positive measure set of invariant circles (see exact statement in Theorem 1).

It is possible to trace back Herman's first statement of the theorem no later than 1995 in his "Séminaire de Systèmes Dynamiques" at the Université Paris VII, and later on in the same seminar at various occasions. To our knowledge, Herman never wrote a complete proof of the theorem and the only available material was a set of notes (given to the participants of the aforementioned seminar) where he explained the strategy of the proof. It is based on this strategy that we give here a complete proof of "Herman's last geometric Theorem". Of course, the content of the paper is under the responsibility of the authors.

Aknowledgement. – We are greatful to Anatole Katok and Jean-Paul Thouvenot for their continuous interest in the progress of this paper.

1.1. Stability and Ergodicity

Probably the best way to introduce Herman's last geometric theorem is in its relation to the stability question of elliptic fixed points. Indeed, the study of the (Lyapunov) stability of fixed points is a fundamental problem in the theory of dynamical systems and its applications.

In the case of an area-preserving plane diffeomorphism f, the fixed points are classified accroding to the eigenvalues of the Jacobian *df* at these fixed points in the following way. If the eigenvalues of *df* are distinct, then the fixed point is said to be *hyperbolic* if they are real, and the point is said to be *elliptic* if they lie on the unit circle. In the exceptional case of two equal eigenvalues ± 1, the point is called *parabolic*.

While it has been known since very long that hyperbolic fixed points are unstable, the question of stability of elliptic fixed points remained essentially unsolved until the discovery of KAM theory (named after Kolmogorov, Arnol′d and Moser).

Prior to that, Birkhoff had introduced an important tool for the study of stability, the so called *normal forms*. They give a simple description, up to canonical change of coordinates, of the map near an elliptic fixed point, in the spirit of Taylor series for real functions. For a smooth map F fixing the origin, a normal form expression of order N is given in polar coordinates (θ, r) by

$$(\theta, r) \mapsto \left(\theta + \sum_{i=0}^{N-1} a_i r^i + \varphi_1(\theta, r), r + \varphi_2(\theta, r)\right)$$

where φ_1 and φ_2 vanish with their derivatives up to order $N-1$ at $r=0$.

Birkhoff proved that if a C^∞ map F has an *irrational* elliptic fixed point, i.e. with eigenvalues that are not roots of unity, then it admits, after canonical coordinate changes, normal forms at any order. He further showed that there exists a formal power series that conjugates F to a *complete* normal form $(\theta + \sum_{i=0}^{\infty} a_i r^i, r)$. Clearly, a map that is *exactly* a normal form $(\theta + \sum_{i=0}^{\infty} a_i r^i, r)$ is completely integrable and thus stable at the origin. Not surprisingly, complete integrability turns out to be too much to ask (it was known to Poincaré that resonant tori usually break up under small perturbations of a completely integrable system) [(2)]

(2) Note however that, in the holomorphic case, complete integrability is equivalent to stability; see Siegel's theorem in the next section.

and it was shown by Siegel that the formal power series that conjugate F to a complete normal form are in general divergent.

Nevertheless, Birkhoff normal forms proved to be very useful in the result of stability discovered by Moser [10] in line with Kolmogorov's seminal approach asserting the persistence of a positive measure set of invariant circles when a completely integrable system is perturbed, provided a non-degeneracy condition is imposed on the initial system (here the Birkhoff normal form). One invariant circle being sufficient for Lyapunov stability, it indeed follows from usual KAM theory that if the series a_i contains nonzero terms (torsion) then an irrational elliptic fixed point is stable. Actually, Moser proved the stability of an elliptic fixed points in finite regularity (C^4), provided that the eigenvalues merely avoid the six roots of unity of order $1, 2, 3, 4$, and that $a_1 \neq 0$ in the Birkhoff normal form of order 2. The latter is of course a generic transversality condition.

On the other hand, Anosov and Katok constructed in [1] smooth area preserving diffeomorphisms of the unit disc in $\mathbb{R}^2$, with an irrational elliptic fixed point at the origin, that are ergodic. These examples showed that the existence of torsion was necessary in establishing stability in the KAM setting, at least when no arithmetical conditions, besides avoiding the first six roots of unity or even having irrational arguments, are imposed on the eigenvalues.

In fact, besides being infinitely tangent to the rotation at the origin, the Anosov-Katok examples were obtained only for a family of rotation numbers (arguments of the eigenvalues) at the origin that contained a dense G_δ-set of the circle but that avoided all Diophantine numbers.

While the strength of Moser's result lies in the fact that stability is insured by the finite number of conditions stated above, its non-zero torsion condition involves the behavior of the map in the neighborhood of the fixed point. A tantalizing question naturally arose, to decide whether as it is the case with instability for hyperbolic fixed points, a sole information on the Jacobian at a fixed elliptic point could be enough to insure stability.

This is precisely what was established in the real analytic category by H. Rüssmann who proved in [11] the following dichotomy, that implies stability, if the rotation number of the fixed elliptic point satisfies a Brjuno condition: *either the Birkhoff normal form has some non zero term, in which case Moser's Theorem applies, or the Birkhoff form completely vanishes and the map is analytically linearizable in the neighborhood of the fixed point*.

This dichotomy clearly fails in the smooth category, as is shown by the following example (in cylindrical coordinates): $(r, \theta) \mapsto (r, \theta + \alpha + e^{-1/r})$.

Thus, the question of whether an elliptic fixed point with a Diophantine rotation number (satisfying no *a priori* twist condition) is always stable remained unsolved for smooth maps until Herman gave it an affirmative answer as a corollary of his last geometric theorem

THEOREM 1. – *Let F be a smooth diffeomorphism of the annulus having the intersection property. Then given a smooth curve Γ invariant by F on which the rotation number α of F is Diophantine, it holds that Γ is accumulated by a positive measure set of smooth invariant curves on which F is smoothly conjugated to rotation maps.*

The result stems actually from the following alternative: either there is an open neighborhood of Γ on which F is conjugated to a rigid rotation of the annulus of rotation number equal to that of F on Γ or Γ is accumulated by smooth invariant curves on which F is smoothly conjugated to rotation maps with frequencies covering a positive measure set inside a Diophantine class obtained by slightly relaxing the Diophantine condition on α.

Besides the elegance and conciseness of the result, its importance lies in the fact that in many of the physical situations where quasi-periodic stability is involved, the non degeneracy of torsion is either hard to prove or at least untrue at the first orders.

The technique used to prove the theorem is based on a general approach to KAM theory where useful dynamical informations are obtained from Whitney dependent normal forms (which are derived from a systematic use of Hamilton's Implicit function theorem in judicious Fréchet spaces). This approach proved to be very helpful in dealing with delicate KAM problems such as, for example, Herman's rigorous approach to a proof of Arnol'd's results on the stability of the solar system (a proof of which was nicely written by Jacques Féjoz [5]).

Before stating more precisely the main results of this paper, let us mention that the ergodic examples of Anosov and Katok on the unit disc were extended in [4] to cover all Liouville rotation numbers at the origin (and the boundary) which gives, together with Herman's last geometric theorem, an additional example of the complete dichotomy between Diophantine stable and Liouville unstable paradigms.

THEOREM 2 ([4]). – *For any given Liouville number α, there exists a smooth area-preserving diffeomorphism of the unit disk, preserving the boundary and having rotation number α on the boundary, which is weakly mixing with respect to Lebesgue measure.*

In fact, the method of the proof of Theorem 1 shows that given a Diophantine class to which α belongs, there exists a class of differentiability of the map F that insures the validity of Theorem 1, with however invariant curves that will have less regularity than the map F itself. On the other hand, given a Diophantine class, it is also possible to construct by quantitative Anosov-Katok methods, as the one used in [4], weakly mixing examples as in Theorem 2 but with finite regularity.

1.2. Results

We now pass to a more detailed description and precise statement of Herman's results.

We denote the circle by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote by $\mathrm{Diff}^r_+(\mathbb{T})$, $r \in \mathbb{N} \cup \{\infty\}$ the group of orientation preserving diffeomorphisms of the circle of class C^r. We represent the lifts of these diffeomorphisms as elements of $D^r(\mathbb{T})$, the group of C^r-diffeomorphisms $\tilde{f}$ of the real line such that $\tilde{f} - \mathrm{Id}_{\mathbb{R}}$ is $\mathbb{Z}$-periodic.

Following Poincaré, one can define the rotation number of a circle homeomorphism f as the uniform limit $\rho(f) = \lim_{|j|\to\infty}(\tilde{f}^j(x) - x)/j \mathrm{mod}[1]$, where $\tilde{f}^j$ ($j \in \mathbb{Z}$) denotes the j-th iterate of a lift $\tilde{f}$ to $\mathbb{R}$ of f. A rotation map of the circle with angle α, that we denote by $R_\alpha : x \mapsto x + \alpha$, has clearly a rotation number equal to α.

Denote the infinite annulus by $\mathbb{A} = \mathbb{T}\times\mathbb{R}$. We shall use coordinates (θ, r) on $\mathbb{A}$. We denote by $\mathrm{Diff}^\infty_0(\mathbb{A})$ the set of diffeomorphisms of the annulus that are homotopic to the identity (see Section 2). Denote by $C^\infty(\mathbb{R})$ the set of smooth real maps $f : \mathbb{R} \to \mathbb{R}$ and by $C^\infty(\mathbb{T}, \mathbb{R})$ the set of smooth real maps $f \in C^\infty(\mathbb{R})$ that are 1-periodic.

We denote by Γ_0 the circle $\mathbb{T}\times\{0\}$ in $\mathbb{A}$. More generally, we shall call *circle* in $\mathbb{A}$ any closed curve $\Gamma = \{(\theta, \gamma(\theta))\}_{\theta\in\mathbb{T}}$, where γ belongs to $C^\infty(\mathbb{T}, \mathbb{R})$. For $c \in \mathbb{R}$, we denote by $\mathcal{G}^c$ the set of circles $\Gamma = \{(\theta, \gamma(\theta)),\ \theta \in \mathbb{T}\}$ such that $\int_{\mathbb{T}} \gamma(\theta)d\theta = c$.