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A DUALITY THEOREM FOR DIEUDONNÉ DISPLAYS

BY EIKE LAU

ABSTRACT. – We show that the Zink equivalence between p-divisible groups and Dieudonné displays over a complete local ring with perfect residue field of characteristic p is compatible with duality. The proof relies on a new explicit formula for the p-divisible group associated to a Dieudonné display.

RÉSUMÉ. – Nous montrons que l'équivalence de Zink entre les groupes p-divisibles et les displays de Dieudonné sur un anneau local complet à corps résiduel parfait de caractéristique p est compatible avec la dualité. La preuve repose sur une nouvelle formule explicite pour le groupe p-divisible associé à un display de Dieudonné.

Introduction

Let R be a complete local ring with maximal ideal $\mathfrak{m}$ and perfect residue field k of positive characteristic p. If $p = 2$ we assume that $pR = 0$.

As a generalisation of classical Dieudonné theory, Th. Zink defines in [11] a category of Dieudonné displays over R and shows that it is equivalent to the category of p-divisible groups over R. In the present article we give a unified formula for the group associated to a Dieudonné display and apply it to show that the equivalence is compatible with the natural duality operations on both sides. This is not clear from the original construction because that depends on decomposing a p-divisible group into its étale and infinitesimal part, which is not preserved under duality.

Let us recall the definition of a Dieudonné display. There is a unique subring $\mathbb{W}(R)$ of the Witt ring $W(R)$ that is stable under its Frobenius f and Verschiebung v, that surjects onto $W(k)$, and that contains an element $x \in W(\mathfrak{m})$ if and only if the components of x converge to zero $\mathfrak{m}$-adically. In [11] the ring $\mathbb{W}(R)$ is denoted by $\widehat{W}(R)$. Let $\mathbb{I}_R$ be the kernel of the natural homomorphism $\mathbb{W}(R) \to R$. A Dieudonné display over R is a quadruple

$$\mathscr{P} = (P, Q, F, F_1)$$

where P is a finite free $\mathbb{W}(R)$-module, Q a submodule containing $\mathbb{I}_R P$ such that P/Q is a free R-module, $F : P \to P$ and $F_1 : Q \to P$ are f-linear maps such that $F_1(v(w)x) = wF(x)$

for $x \in P$ and $w \in \mathbb{W}(R)$, and the image of F_1 generates P. Dieudonné displays over k are equivalent to Dieudonné modules (P, F, V) where $Q = V(P)$ and $F_1 = V^{-1}$.

Our formula is based on viewing both p-divisible groups and the modules P, Q as abelian sheaves for the flat topology on the opposite category of all R-algebras S with the following properties: the nilradical $\mathcal{N}(S)$ is a nilpotent ideal, it contains $\mathfrak{m}S$, and $S/\mathcal{N}(S)$ is a union of finite dimensional k-algebras; see section 1 for details. With that convention, the equivalence functor BT from Dieudonné displays to p-divisible groups is given by

$$(\star) \qquad \mathrm{BT}(\mathscr{P}) = [Q \xrightarrow{F_1\text{-incl}} P] \overset{L}{\otimes} \mathbb{Q}_p/\mathbb{Z}_p$$

where $[Q \to P]$ is a complex of sheaves in degrees $0, 1$. In other words, the cohomology of the right hand side of ($\star$) vanishes outside degree zero and the zero-th cohomology is the p-divisible group associated to $\mathscr{P}$. Instead of the flat topology one could also use the ind-étale topology, but for some arguments the former is more convenient.

Before stating the main result let us recall the duality of Dieudonné displays. We have the special Dieudonné display $\mathscr{G}_m = (\mathbb{W}(R), \mathbb{I}_R, f, v^{-1})$ that corresponds to the formal multiplicative group $\widehat{\mathbb{G}}_m$. A bilinear form $\mathscr{P}' \times \mathscr{P} \to \mathscr{G}_m$ is a bilinear map $\alpha : P' \times P \to \mathbb{W}(R)$ satisfying $\alpha(x', x) = v(\alpha(F_1'x', F_1x))$ for $x' \in Q'$ and $x \in Q$. For every $\mathscr{P}$ there is a dual $\mathscr{P}^t$ equipped with a perfect bilinear form $\mathscr{P}^t \times \mathscr{P} \to \mathscr{G}_m$, which determines $\mathscr{P}^t$ uniquely. The Serre dual of a p-divisible group G is denoted by $G^\vee$.

THEOREM. – *For every Dieudonné display $\mathscr{P}$ over R there is a natural isomorphism*

$$\Psi : \mathrm{BT}(\mathscr{P}^t) \cong \mathrm{BT}(\mathscr{P})^\vee.$$

The proof is independent of the fact that the functor BT from Dieudonné displays to p-divisible groups defined by ($\star$) is actually an equivalence. Let us indicate how to define the homomorphism Ψ. Denote by $Z(\mathscr{P})$ the complex $[Q \to P]$ in ($\star$). To the tautological bilinear form $\mathscr{P}^t \times \mathscr{P} \to \mathscr{G}_m$ one can directly assign a homomorphism of complexes $Z(\mathscr{P}^t) \otimes Z(\mathscr{P}) \to Z(\mathscr{G}_m)$, which gives after tensoring twice with $\mathbb{Q}_p/\mathbb{Z}_p$ a homomorphism

$$\mathrm{BT}(\mathscr{P}^t) \overset{L}{\otimes} \mathrm{BT}(\mathscr{P}) \to \mathrm{BT}(\mathscr{G}_m) \overset{L}{\otimes} \mathbb{Q}_p/\mathbb{Z}_p \cong \widehat{\mathbb{G}}_m[1].$$

By the cohomological theory of biextensions, such a homomorphism is equivalent to a homomorphism Ψ as above. That Ψ is an isomorphism must be shown only if the group $\mathrm{BT}(\mathscr{P})$ is étale or of multiplicative type or bi-infinitesimal. The first two cases are straightforward; the bi-infinitesimal case relies on the theorem of Cartier [3] on the Cartier dual of the Witt ring functor.

Over arbitrary rings in which p is nilpotent, infinitesimal p-divisible groups are equivalent to *displays* according to [13] and [5]. The bi-infinitesimal case of the above theorem is closely related to the duality theorem in [13] for the display associated to a bi-infinitesimal p-divisible group. This in turn has been anticipated by Norman [10] who shows a similar duality theorem for the Cartier module of a bi-infinitesimal p-divisible group, provided the module is displayed (which is always the case by the said equivalence). These duality results all depend on the theory of biextensions developed in [9], that appears here in the cohomological form it was given in SGA 7.

The present proof that the functor BT defined by ($\star$) is an equivalence of categories consists in verifying that it reproduces the equivalence constructed in [11]. However, it should be possible to relate the crystals associated to a Dieudonné display $\mathscr{P}$ and to the p-divisible group $\mathrm{BT}(\mathscr{P})$. Then the fact that BT is an equivalence will follow directly from the Grothendieck-Messing deformation theory of p-divisible groups [8], and the duality theorem for Dieudonné displays will be related to the crystalline duality theorem [1]. We hope to return to this point soon. Let us also note that Caruso [4] proved a duality theorem for Breuil modules of finite flat p-group schemes [2] by using the crystalline duality theorem. Breuil modules of p-divisible groups are related to Dieudonné displays by [12].

This article is organised as follows. In Section 1 the formula for BT is explained, in Section 2 it is shown to give an equivalence of categories, in Section 3 the duality theorem is proved, and Section 4 is concerned with functoriality in the base. In an appendix we discuss briefly the deformational duality theorem [7] since variants of it are used in the text.

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1. Exposition of the main formula

We begin with a number of general definitions and notations. Let p be a prime. For any ring A let $W(A)$ be the ring of p-Witt vectors and I_A the kernel of the first Witt polynomial $w_0 : W(A) \to A$. If A is perfect of characteristic p, I_A is generated by p. Let f be the Frobenius of $W(A)$ and v the Verschiebung. If $\mathfrak{a} \subset A$ is a nilpotent ideal, let $\widehat{W}(\mathfrak{a}) \subseteq W(\mathfrak{a})$ be the subgroup of Witt vectors with only finitely many non-zero components. More generally, if A is $\mathfrak{a}$-adically complete and separated, let $\widehat{W}(\mathfrak{a}) \subseteq W(\mathfrak{a})$ be the subgroup of Witt vectors whose components converge to zero $\mathfrak{a}$-adically; in other words, $\widehat{W}(\mathfrak{a}) = \varprojlim \widehat{W}(\mathfrak{a}/\mathfrak{a}^n)$. In any case $\widehat{W}(\mathfrak{a})$ is an ideal in $W(A)$.

Definition 1.1. – Let A be a ring and $\mathfrak{a} \subset A$ an ideal. The pair $(A, \mathfrak{a})$ is called admissible if A is $\mathfrak{a}$-adically complete and separated and $A/\mathfrak{a}$ is perfect of characteristic p. If $p = 2$ we also require that $pA = 0$.

Lemma 1.2. – *If $(A, \mathfrak{a})$ is admissible then there is a unique f-stable subring $\mathbb{W}(A)$ of $W(A)$ such that $\mathbb{W}(A) \cap W(\mathfrak{a}) = \widehat{W}(\mathfrak{a})$ and $\mathbb{W}(A)$ maps surjectively onto $W(A/\mathfrak{a})$. The subring $\mathbb{W}(A)$ is also stable under v.*

This is proved in [11] if A is noetherian and $A/\mathfrak{a}$ is a field, but neither of these assumptions is used in the proof. $\mathbb{W}(A)$ is constructed as follows: Since $A/\mathfrak{a}$ is perfect, the projection $W(A) \to W(A/\mathfrak{a})$ has a unique splitting, necessarily f-equivariant, thus an f-equivariant decomposition of abelian groups $W(A) \cong W(A/\mathfrak{a}) \oplus W(\mathfrak{a})$, under which $\mathbb{W}(A)$ is mapped to $W(A/\mathfrak{a}) \oplus \widehat{W}(\mathfrak{a})$. The condition that 2 is invertible or zero in A is only needed to guarantee that $\mathbb{W}(A)$ is v-stable. We have

$$\mathbb{W}(A) = \varprojlim \mathbb{W}(A/\mathfrak{a}^n)$$

by uniqueness or by the construction. Let $\mathbb{I}_A$ be the kernel of $w_0 : \mathbb{W}(A) \to A$.

Definition 1.3. – Assume that $(A, \mathfrak{a})$ is admissible. A Dieudonné display over A is a quadruple $\mathscr{P} = (P, Q, F, F_1)$ such that

P is a finitely generated projective $\mathbb{W}(A)$-module,
Q is a submodule of P containing $\mathbb{I}_A P$,
P/Q is projective as an A-module,
$F : P \to P$ and $F_1 : Q \to P$ are f-linear maps,
$F_1(v(w)x) = wF(x)$ for $w \in \mathbb{W}(A)$ and $x \in P$,
$F_1(Q)$ generates P as a $\mathbb{W}(A)$-module.

These axioms also imply $F(x) = pF_1(x)$ for $x \in Q$.

Remarks. – (1) Every pair of $\mathbb{W}(A)$-modules (P, Q) satisfying the first three of the above conditions admits a decomposition $P = L \oplus T$ such that $Q = L \oplus \mathbb{I}_A T$, called normal decomposition. Its existence is straightforward if $\mathfrak{a} = 0$, thus A perfect; if $\mathfrak{a}$ is nilpotent one can use that $\widehat{W}(\mathfrak{a})$ is nilpotent as well; the general case follows by passing to the limit.

(2) For a $\mathbb{W}(A)$-module M let $M^{(1)} = \mathbb{W}(A) \otimes_{f, \mathbb{W}(A)} M$, and for an f-linear homomorphism of $\mathbb{W}(A)$-modules $F : M \to N$ let $F^\sharp : M^{(1)} \to N$ be its linearisation. In analogy with [13] Lemma 9, the structure of a Dieudonné display on a pair (P, Q) as above with given normal decomposition $P = L \oplus T$ is equivalent to the isomorphism

$$(F_1^\sharp, F^\sharp) : L^{(1)} \oplus T^{(1)} \xrightarrow{\sim} P.$$

(3) We have the following notion of base change. If $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$ are admissible pairs, every ring homomorphism $g : A \to B$ with $g(\mathfrak{a}) \subseteq \mathfrak{b}$ induces a ring homomorphism $\mathbb{W}(g) : \mathbb{W}(A) \to \mathbb{W}(B)$. The base change of a Dieudonné display $\mathscr{P}$ over A by g is then $\mathscr{P}_B = (P_B, Q_B, F_B, F_{1,B})$ where

$$P_B = \mathbb{W}(B) \otimes_{\mathbb{W}(A)} P, \qquad Q_B = \operatorname{Ker}(P_B \to B \otimes_A P/Q),$$

and F_B, $F_{1,B}$ are the unique f-linear extensions of F, F_1, whose existence follows from a normal decomposition as explained in [13] Definition 20.

(4) For a Dieudonné display $\mathscr{P}$ over A there is a unique $\mathbb{W}(A)$-linear map $V^\sharp : P \to P^{(1)}$ such that $V^\sharp(F_1 x) = 1 \otimes x$ for $x \in Q$, cf. [13] Lemma 10. Uniqueness is clear; if $P = L \oplus T$ is a normal decomposition, $V^\sharp$ can be defined to be

$$P \xrightarrow{(F_1^\sharp, F^\sharp)^{-1}} L^{(1)} \oplus T^{(1)} \xrightarrow{(1,p)} L^{(1)} \oplus T^{(1)} = P^{(1)}.$$

We have $F^\sharp V^\sharp = p$ and $V^\sharp F^\sharp = p$. If A is perfect, F_1 is bijective, and its inverse defines an f^{-1}-linear map $V : P \to P$ whose linearisation is $V^\sharp$.

Assume now that R is a local ring with maximal ideal $\mathfrak{m}$ and residue field k such that $(R, \mathfrak{m})$ is admissible, i.e. R is $\mathfrak{m}$-adically complete, k is perfect of characteristic p, and $p = 2$ implies $pR = 0$.

Definition 1.4. – Let $\mathcal{C}_R$ be the category of all R-algebras S such that the nilradical $\mathcal{N}(S)$ is nilpotent, $\mathcal{N}(S)$ contains $\mathfrak{m}S$, and $S_{\mathrm{red}} = S/\mathcal{N}(S)$ is a union of finite dimensional, necessarily étale, k-algebras.