ISSN 0012-9593

quatrième série - tome 42 *fascicule 2* *mars-avril 2009*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Gérard FREIXAS i MONTPLET

An arithmetic Riemann-Roch theorem for pointed stable curves

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4e série, t. 42, 2009, p. 335 à 369

AN ARITHMETIC RIEMANN-ROCH THEOREM FOR POINTED STABLE CURVES

BY GÉRARD FREIXAS I MONTPLET

ABSTRACT. – Let $(\mathcal{O},\Sigma,F_\infty)$ be an arithmetic ring of Krull dimension at most 1, $\mathcal{S}=\operatorname{Spec}\mathcal{O}$ and $(\pi:\mathcal{X}\to\mathcal{S};\sigma_1,\dots,\sigma_n)$ an n-pointed stable curve of genus g. Write $\mathcal{U}=\mathcal{X}\setminus\cup_j\sigma_j(\mathcal{S})$. The invertible sheaf $\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1+\cdots+\sigma_n)$ inherits a hermitian structure $\|\cdot\|_{\mathrm{hyp}}$ from the dual of the hyperbolic metric on the Riemann surface $\mathcal{U}_\infty$. In this article we prove an arithmetic Riemann-Roch type theorem that computes the arithmetic self-intersection of $\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1+\cdots+\sigma_n)_{\mathrm{hyp}}$. The theorem is applied to modular curves $X(\Gamma)$, $\Gamma=\Gamma_0(p)$ or $\Gamma_1(p)$, $p\geq 11$ prime, with sections given by the cusps. We show $Z'(Y(\Gamma),1)\sim e^a\pi^b\Gamma_2(1/2)^cL(0,\mathcal{M}_\Gamma)$, with $p\equiv 11 \mod 12$ when $\Gamma=\Gamma_0(p)$. Here $Z(Y(\Gamma),s)$ is the Selberg zeta function of the open modular curve $Y(\Gamma)$, a,b,c are rational numbers, $\mathcal{M}_\Gamma$ is a suitable Chow motive and $\sim$ means equality up to algebraic unit.

RÉSUMÉ. – Soient $(\mathcal{O},\Sigma,F_\infty)$ un anneau arithmétique de dimension de Krull au plus 1, $\mathcal{S}=\operatorname{Spec}\mathcal{O}$ et $(\pi:\mathcal{X}\to\mathcal{S};\sigma_1,\dots,\sigma_n)$ une courbe stable n-pointée de genre g. Posons $\mathcal{U}=\mathcal{X}\setminus\cup_j\sigma_j(\mathcal{S})$. Le faisceau inversible $\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1+\cdots+\sigma_n)$ hérite une structure hermitienne $\|\cdot\|_{\mathrm{hyp}}$ du dual de la métrique hyperbolique sur la surface de Riemann $\mathcal{U}_\infty$. Dans cet article nous prouvons un théorème de Riemann-Roch arithmétique qui calcule l'auto-intersection arithmétique de $\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1+\cdots+\sigma_n)_{\mathrm{hyp}}$. Le théorème est appliqué aux courbes modulaires $X(\Gamma)$, $\Gamma=\Gamma_0(p)$ ou $\Gamma_1(p)$, $p\geq 11$ premier, prenant les cusps comme sections. Nous montrons $Z'(Y(\Gamma),1)\sim e^a\pi^b\Gamma_2(1/2)^cL(0,\mathcal{M}_\Gamma)$, avec $p\equiv 11 \mod 12$ lorsque $\Gamma=\Gamma_0(p)$. Ici $Z(Y(\Gamma),s)$ est la fonction zêta de Selberg de la courbe modulaire ouverte $Y(\Gamma)$, a,b,c sont des nombres rationnels, $\mathcal{M}_\Gamma$ est un motif de Chow approprié et $\sim$ signifie égalité à unité près.

1. Introduction

Let $(\mathcal{O},\Sigma,F_\infty)$ be an arithmetic ring of Krull dimension at most 1 [22, Def. 3.1.1]. This means that $\mathcal{O}$ is an excellent, regular, Noetherian integral domain, Σ is a finite non-empty set of monomorphisms $\sigma:\mathcal{O}\hookrightarrow\mathbb{C}$ and $F_\infty:\mathbb{C}^\Sigma\to\mathbb{C}^\Sigma$ is a conjugate-linear involution of

$\mathbb{C}$-algebras such that the diagram

$$\begin{tikzcd} & \mathbb{C}^\Sigma \arrow[d, "F^\infty"] \\ \mathcal{O} \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & \\ & \mathbb{C}^\Sigma \end{tikzcd}$$

commutes. Here δ is induced by the set Σ. Define $\mathcal{S} = \operatorname{Spec}\mathcal{O}$ and let $(\pi : \mathcal{X} \to \mathcal{S}; \sigma_1, \ldots, \sigma_n)$ be an n-pointed stable curve of genus g, in the sense of Knudsen and Mumford [35, Def. 1.1]. Assume that $\mathcal{X}$ is regular. Write $\mathcal{U} = \mathcal{X} \setminus \cup_j \sigma_j(\mathcal{S})$. To $\mathcal{X}$ and $\mathcal{U}$ we associate the complex analytic spaces

$$\mathcal{X}_\infty = \bigsqcup_{\sigma\in\Sigma} \mathcal{X}_\sigma(\mathbb{C}),\ \ \mathcal{U}_\infty = \bigsqcup_{\sigma\in\Sigma} \mathcal{U}_\sigma(\mathbb{C}).$$

Notice that F_∞ acts on $\mathcal{X}_\infty$ and $\mathcal{U}_\infty$. The stability hypothesis guarantees that every connected component of $\mathcal{U}_\infty$ has a hyperbolic metric of constant curvature -1. The whole family is invariant under the action of F_∞. Dualizing we obtain an arakelovian –i.e. invariant under F_∞– hermitian structure $\|\cdot\|_{\mathrm{hyp}}$ on $\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1+\cdots+\sigma_n)$. Contrary to the requirements of classical Arakelov theory [23], the metric $\|\cdot\|_{\mathrm{hyp}}$ is *not smooth*, but has some mild singularities of logarithmic type. Actually $\|\cdot\|_{\mathrm{hyp}}$ is a *pre-log-log* hermitian metric in the sense of Burgos-Kramer-Kühn [8, Sec. 7]. Following loc. cit., there is a first arithmetic Chern class $\widehat{\mathrm{c}}_1(\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1+\cdots+\sigma_n)_{\mathrm{hyp}})$ that lives in a pre-log-log arithmetic Chow group $\widehat{\mathrm{CH}}^1_{\mathrm{pre}}(\mathcal{X})$. The authors define an intersection product

$$\widehat{\mathrm{CH}}^1_{\mathrm{pre}}(\mathcal{X}) \otimes_{\mathbb{Z}} \widehat{\mathrm{CH}}^1_{\mathrm{pre}}(\mathcal{X}) \overset{\cdot}{\longrightarrow} \widehat{\mathrm{CH}}^2_{\mathrm{pre}}(\mathcal{X})$$

and a pushforward map

$$\pi_* : \widehat{\mathrm{CH}}^2_{\mathrm{pre}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{S}).$$

This paper is concerned with the class $\pi_*(\widehat{\mathrm{c}}_1(\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1+\cdots+\sigma_n)_{\mathrm{hyp}})^2)$.

In their celebrated work [24], Gillet and Soulé –with deep contributions of Bismut– proved an arithmetic analogue of the Grothendieck-Riemann-Roch theorem. Their theorem deals with the push-forward of a smooth hermitian vector bundle by a proper and generically smooth morphism of arithmetic varieties. The associated relative complex tangent bundle is equipped with a *smooth* Kähler structure. With the notations above, if $n = 0$ and $g \geq 2$, then the metric $\|\cdot\|_{\mathrm{hyp}}$ is smooth and the arithmetic Grothendieck-Riemann-Roch theorem may be applied to $\omega_{\mathcal{X}/\mathcal{S},\mathrm{hyp}}$ and the "hyperbolic" Kähler structure on $\mathcal{X}_\infty$. The result is a relation between $\pi_*(\widehat{\mathrm{c}}_1(\omega_{\mathcal{X}/\mathcal{S},\mathrm{hyp}})^2) \in \widehat{\mathrm{CH}}^1(\mathcal{S})$ and the class $\widehat{\mathrm{c}}_1(\lambda(\omega_{\mathcal{X}/\mathcal{S}}), \|\cdot\|_Q)$, where $\|\cdot\|_Q$ is the Quillen metric corresponding to our data. However, for $n > 0$ the singularities of $\|\cdot\|_{\mathrm{hyp}}$ prevent from applying the theorem of Gillet and Soulé.

The present article focuses on the so far untreated case $n > 0$. We prove an arithmetic analogue of the Riemann-Roch theorem that relates $\pi_*(\widehat{\mathrm{c}}_1(\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1+\cdots+\sigma_n)_{\mathrm{hyp}})^2)$ and $\widehat{\mathrm{c}}_1(\lambda(\omega_{\mathcal{X}/\mathcal{S}}), \|\cdot\|_Q)$. The Quillen type metric $\|\cdot\|_Q$ is defined by means of the Selberg zeta function of the connected components of $\mathcal{U}_\infty$ (see Definition 2.2). In contrast with the result of Gillet and Soulé, our formula involves the first arithmetic Chern class of a new hermitian line bundle ψ_W. The corresponding invertible sheaf is the pull-back of the so called *tautological psi* line bundle on the moduli stack $\overline{\mathcal{M}}_{g,n}$, by the classifying morphism $\mathcal{S} \to \overline{\mathcal{M}}_{g,n}$. The underlying hermitian structure is dual to Wolpert's renormalization of the hyperbolic metric [63, Def. 1] (see also Definition 2.1 below). The class $\widehat{\mathrm{c}}_1(\psi_W)$ reflects the appearance of the continuous spectrum in the spectral resolution of the hyperbolic laplacian. After the necessary normalizations and definitions given in Section 2, the main theorem is stated as follows:

Theorem A. – *Let $g, n \geq 0$ be integers with $2g-2+n > 0$, $(\mathcal{O}, \Sigma, F_\infty)$ an arithmetic ring of Krull dimension at most 1 and $\mathcal{S} = \operatorname{Spec} \mathcal{O}$. Let $(\pi : \mathcal{X} \to \mathcal{S}; \sigma_1, \ldots, \sigma_n)$ be an n-pointed stable curve of genus g, with $\mathcal{X}$ regular. For every closed point $\wp \in \mathcal{S}$ denote by $n_\wp$ the number of singular points in the geometric fiber $\mathcal{X}_\wp$ and put $\Delta_{\mathcal{X}/\mathcal{S}} = [\sum_\wp n_\wp \wp] \in \mathrm{CH}^1(\mathcal{S})$. Then the identity*

$$12\widehat{\mathrm{c}}_1(\lambda(\omega_{\mathcal{X}/\mathcal{S}})_Q) - \Delta_{\mathcal{X}/\mathcal{S}} + \widehat{\mathrm{c}}_1(\psi_W) = \pi_*\left(\widehat{\mathrm{c}}_1(\omega_{\mathcal{X}/\mathcal{S}}(\sigma_1 + \cdots + \sigma_n)_{\mathrm{hyp}})^2\right) + \widehat{\mathrm{c}}_1\left(\mathcal{O}(C(g,n))\right)$$

holds in the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{S})$.

The theorem is deduced from the Mumford isomorphism on $\overline{\mathcal{M}}_{g,n}$ (Theorem 3.10) and a metrized version that incorporates the appropriate hermitian structures (Theorem 6.1) [(1)]. The techniques employed combine the geometry of the boundary of $\overline{\mathcal{M}}_{g+n,0}$ –through the so called clutching morphisms– and the behavior of the small eigenvalues of the hyperbolic laplacian on degenerating families of compact surfaces. By a theorem of Burger [6, Th. 1.1] we can replace the small eigenvalues by the lengths of the pinching geodesics. Then Wolpert's pinching expansion of the family hyperbolic metric [62, Exp. 4.2] provides an expression of these lengths in terms of a local equation of the boundary divisor $\partial\mathcal{M}_{g+n,0}$. This gives a geometric manner to treat the small eigenvalues. Another consequence of theorems 3.10 and 6.1 is a significant case of the local index theorem of Takhtajan-Zograf [55]–[56] (Theorem 6.8 below).

Natural candidates to which Theorem A applies are provided by arithmetic models of modular curves, taking their cusps as sections. We focus on the curves $X(\Gamma)/\mathbb{C}$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ is a congruence subgroup of the type $\Gamma_0(p)$ or $\Gamma_1(p)$. We assume that $p \geq 11$ is a prime number. If $\Gamma = \Gamma_0(p)$, we further suppose $p \equiv 11 \mod 12$. These conditions guarantee in particular that $\Gamma_0(p)$ acts without elliptic fixed points and $X(\Gamma)$ has genus $g \geq 1$. To $X(\Gamma)$ we attach two kinds of zeta functions:

- let $Y(\Gamma) := X(\Gamma) \setminus \{\text{cusps}\}$ be the open modular curve. Then $Y(\Gamma)$ is a hyperbolic Riemann surface of finite type. We denote by $Z(Y(\Gamma), s)$ the Selberg zeta function of $Y(\Gamma)$ (see Section 2). It is a meromorphic function defined over $\mathbb{C}$, with a simple zero at $s = 1$;
- let $\mathrm{Prim}_2(\Gamma)$ be a basis of normalized Hecke eigenforms for Γ. To $f \in \mathrm{Prim}_2(\Gamma)$ we can attach a Chow motive $\mathcal{M}(f)$ over $\mathbb{Q}$, with coefficients in a suitable finite extension F of $\mathbb{Q}(\mu_p)$, independent of f [(2)]. If χ is a Dirichlet character with values in $F^\times$, we

(1) In particular, with the formalism of [7, Sec. 4.3], the assumption of regularity of $\mathcal{X}$ can be weakened to $\pi : \mathcal{X} \to \mathcal{S}$ generically smooth.

(2) The construction of $\mathcal{M}(f)$ amounts to the decomposition of the jacobian $\mathrm{Jac}(X(\Gamma))$ under the action of the Hecke algebra. More generally, Deligne [11, Sec. 7] and Scholl [49, Th. 1.2.4] associate a Grothendieck –i.e. homological– motive to any normalized new Hecke eigenform of weight $k \geq 2$, level n and character χ.

denote by $\mathbb{Q}(\chi)$ its Artin motive. For instance we may take $\chi = \overline{\chi_f}$, for the Dirichlet character χ_f associated to $f \in \mathrm{Prim}_2(\Gamma)$. If Sym^2 denotes the square symmetrization projector and (2) the Tate twist by 2, we put

$$\mathcal{M}_\Gamma := \bigoplus_{f \in \mathrm{Prim}_2(\Gamma)} \mathrm{Sym}^2 \mathcal{M}(f) \otimes \mathbb{Q}(\overline{\chi_f})(2) \in \mathrm{Ob}(M_{\mathrm{rat}}(\mathbb{Q})_F).$$

The motivic L-function of $\mathcal{M}_\Gamma$, $L(s, \mathcal{M}_\Gamma)$, can be defined –with the appropriate definition of the local factor at p– so that we have the relation

$$L(s, \mathcal{M}_\Gamma) = \prod_{f \in \mathrm{Prim}_2(\Gamma)} L(s+2, \mathrm{Sym}^2 f, \overline{\chi_f}). \tag{1.1}$$

The reader is referred to [9], [11, Sec. 7], [28, Sec. 5], [49] and [52] for details. (3) Denote by Γ_2 the Barnes double Gamma function [3] (see also [47] and [59]).

Theorem B. – *Let $p \geq 11$ be a prime number and $\Gamma = \Gamma_0(p)$ or $\Gamma_1(p)$. Assume $p \equiv 11 \mod 12$ whenever $\Gamma = \Gamma_0(p)$. Then there exist rational numbers a, b, c such that*

$$Z'(Y(\Gamma), 1) \sim_{\overline{\mathbb{Q}}^\times} e^a \pi^b \Gamma_2(1/2)^c L(0, \mathcal{M}_\Gamma),$$

where $\alpha \sim_{\overline{\mathbb{Q}}^\times} \beta$ means $\alpha = q\beta$ for some $q \in \overline{\mathbb{Q}}^\times$ (4).

The proof relies on Theorem A and the computation of Bost [5] and Kühn [37] for the arithmetic self-intersection number of $\omega_{X_1(p)/\mathbb{Q}(\mu_p)}(\mathrm{cusps})_{\mathrm{hyp}}$. Undertaking the proof of Bost and Kühn –under the form of Rohrlich's modular version of Jensen's formula [45]– and applying Theorem A to $(\mathbb{P}^1_{\mathbb{Z}}; 0, 1, \infty)$, one can also show the equality

$$Z'(\Gamma(2), 1) = 4\pi^{5/3} \Gamma_2(1/2)^{-8/3}, \tag{1.2}$$

where $Z(\Gamma(2), s)$ is the Selberg zeta function of the congruence group $\Gamma(2)$. The details are given in our thesis [19, Ch. 8]. However, our method fails to provide the exact value of $Z'(\mathrm{PSL}_2(\mathbb{Z}), 1)$.

To the knowledge of the author, the special values $Z'(Y(\Gamma), 1)$ remained unknown. Even though it was expected that they encode interesting arithmetic information (see [26] and [46]), it is quite remarkable that they can be expressed in terms of the special values $L(0, \mathcal{M}_\Gamma)$. The introduction of $\mathcal{M}_\Gamma$ in the formulation of the theorem was suggested by Beilinson's conjectures (see [53] for an account) and two questions of Fried [21, Sec. 4, p. 537 and App., Par. 4]. Fried asks about the number theoretic content of the special values of Ruelle's zeta function and an interpretation in terms of regulators (5). Also Theorem B may be seen as an analogue of the product formula for number fields $\prod_\nu |x|_\nu = 1$. This analogy alone deserves further study.

So far there have been other attempts of proof of Theorem A. This is the case of [60, Part II]. The method of loc. cit. seems to lead to an analogous statement up to an unknown universal constant. The advantage of our approach is that explicit computations –such as Theorem B and (1.2)– are allowed. Moreover, in contrast with [60, Fund. rel. IV′, p. 280],

(3) The factors $L(s+2, \mathrm{Sym}^2 f, \overline{\chi_f})$ have already been studied by Hida [28], Shimura [52] and Sturm [54].
(4) The exponents a, b, c can actually be computed in terms of p.
(5) The Ruelle zeta function $R(s)$ of a hyperbolic Riemann surface is related to the Selberg zeta function $Z(s)$ by $R(s) = Z(s)/Z(s+1)$. For instance, $R'(1) = Z'(1)/Z(2)$.