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SIZE MINIMIZING SURFACES

BY Thierry De PAUW

Abstract. – We prove a new existence theorem pertaining to the Plateau problem in 3-dimensional Euclidean space. We compare the approach of E.R. Reifenberg with that of H. Federer and W.H. Fleming. A relevant technical step consists in showing that compact rectifiable surfaces are approximatable in Hausdorff measure and in Hausdorff distance by locally acyclic surfaces having the same boundary.

Résumé. – Nous obtenons un nouveau théorème d'existence relatif au problème de Plateau dans l'espace euclidien de dimension 3. Ce faisant, nous comparons les approches d'E.R. Reifenberg d'une part, et de H. Federer et W.H. Fleming d'autre part. Un pas technique important consiste à démontrer qu'on peut approcher tout ensemble compact et rectifiable, en mesure de Hausdorff et en distance de Hausdorff, par une surface localement acyclique ayant le même bord.

PART I
INTRODUCTION

1. Foreword

The Plateau problem can be stated informally like this: Given a *boundary* $B \subseteq \mathbb{R}^3$, we seek a surface $S \subseteq \mathbb{R}^3$ *spanning* B and having least *area* among all such surfaces. Solving the problem partly consists in making sense of the italicized words. One expects that the minimizing surfaces model soap films, which are the objects J. Plateau was interested in, [22]. In his classical book [3], R. Courant reports on the work of J. Douglas where surfaces are understood as continuous mappings. This setting is shown to be an actual restriction for instance in W.H. Fleming's paper [14].

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We start by recalling why an application of the direct method of the calculus of variations may be a troublesome task. Indeed some minimizing sequence may have "thin tentacles", or "filigree", that will not contribute for a lot of area but yet might persist for some substantial part of the limit (e.g. in the sense of Hausdorff distance). Think of B being a (planar) circle and let S denote the 2-dimensional flat disk bounded by B. Referring to the observation that the nearest point projection on the plane containing B and S has Lipschitz constant 1 (and therefore does not increase area), we infer that S is the unique area minimizer in any reasonable sense (1). Choose, for S_j, $j = 1, 2, \dots$, the flat disk S from which j non-overlapping small disks have been removed and replaced with "curvy conical" surfaces (the tentacles) whose vertices are points $a_1, \dots, a_j$ chosen in advance. This can be done in order for the total contribution in area of the tentacles to be bounded by j^{-1}, so that $S_1, S_2, \dots$ is indeed a minimizing sequence. The reader may enjoy tickling their imagination by staring at Figures 1.3.1–1.3.4, in [20]. Letting $a_1, a_2, \dots$ be a dense sequence in space we see that we can arrange for the Hausdorff limit of that particular minimizing sequence to be the whole space $\mathbb{R}^3$. Therefore the required semicontinuity of area does not hold. One way to circumvent the problem is to modify the minimizing sequence (cutting off the tentacles and patching the holes with controlled disks); another way consists in considering a weaker concept of convergence of surfaces so that the filigree disappear in the limit. We explain below the two points of view.

Two nearly simultaneous theories were published in 1960. One by E.R. Reifenberg [23], and the other by H. Federer and W.H. Fleming [12]. Both dealt with general dimensions and codimensions — and as a matter of fact this was one of their main striking features —, yet in this paper we will purposely restrict ourselves to 2-dimensional surfaces in $\mathbb{R}^3$. We now turn to giving a brief account of these contributions.

2. The approach of E.R. Reifenberg

Assume for the sake of illustration that $B \subseteq \mathbb{R}^3$ is a simple closed Jordan curve, and $S \supseteq B$ is a compact set. We say that B *bounds* S if the homomorphism $H_1(B;G) \to H_1(S;G)$ induced in homology by the inclusion map is trivial. Upon a moment of reflection it should be clear that this indeed says that S fills the hole in B (see the intriguing example [23, Appendix, Example 9] though). The definition also readily depends on G, a fixed "coefficients group". Furthermore, as we shall see soon enough the choice of a particular homology theory is not indifferent. In this setting *area* is understood as the 2-dimensional spherical measure $\mathscr{S}^2$ (see [11, 2.10.2(2)] for a definition (2)). Letting $S_1, S_2, \dots$ be any sequence of competitors (i.e. compact sets bounded by B in the above sense) converging to some S in Hausdorff distance we first want to make sure that the boundary condition is preserved in the limit. This will be the case if we consider Čech homology groups in the definition of "B bounds S" (see Proposition 17.1). After possibly projecting the sets S_j, $j = 1, 2, \dots$, on the convex hull of B we infer from the Blaschke selection

(1) In the present paper we consider only area induced by the Euclidean metric of $\mathbb{R}^3$.

(2) In case S is $(\mathscr{H}^2, 2)$ rectifiable then $\mathscr{S}^2(S) = \mathscr{H}^2(S)$ where the latter is the 2-dimensional Hausdorff measure of S, [11, 3.2.26].

principle that some subsequence $S_{j(1)}, S_{j(2)}, \ldots$ converges in Hausdorff distance. Before referring to this principle E.R. Reifenberg performs a careful cutting and pasting surgery on the sets S_j, $j = 1, 2, \ldots$, in order that semicontinuity of area holds along the modified minimizing sequence $\tilde{S}_j$, $j = 1, 2, \ldots$. Checking that the sets $\tilde{S}_j$ verify the same boundary condition as S_j turns out to rely on the Exactness Axiom of Eilenberg-Steenrod (among many other things of course). This axiom is verified when G is a compact abelian group (see [8, Chap. IX, Theorem 7.6]) but not necessarily otherwise (in particular exactness does not hold when $G = \mathbb{Z}$, see [8, Chap. X, §4]). Thus existence theory in this setting is restricted to the case when G is compact and abelian, and in fact E.R. Reifenberg concentrates on $G = \mathbb{Z}_2$ and G the group of reals modulo 1.

We are now ready to state a corollary of E.R. Reifenberg's work. Letting $B \subseteq \mathbb{R}^3$ be a closed simple Jordan curve and G be a compact abelian group, the following variational problem admits a minimizer:

$$(\mathscr{P}_{R,G,B}) \begin{cases} \text{minimize } \mathscr{S}^2(S) \text{ among compact sets } S \supseteq B \\ \text{such that } \check{H}_1(i_{B,S}) : \check{H}_1(B;G) \to \check{H}_1(S;G) \text{ is trivial} \end{cases}$$

where $i_{B,S}$ denotes the inclusion $B \to S$. Moreover Reifenberg proves that if S^* is a (proper) minimizer of the problem then in a neighborhood of $\mathscr{S}^2$-almost every $x \in S^* \setminus B$, the set S^* is a topological disk. In a subsequent analysis [24] he was able to improve this regularity result to showing that at such point S^* is in fact a real analytic graph.

3. The approach of H. Federer and W.H. Fleming

Here boundaries and surfaces are meant as currents. An m-dimensional current in $\mathbb{R}^3$ is a continuous linear form on the space $\mathscr{D}^m(\mathbb{R}^3)$ of smooth differential forms of degree m with compact support. A current $T \in \mathscr{D}_m(\mathbb{R}^3)$ is called rectifiable whenever the following holds. There exist

1. A bounded $\mathscr{H}^m$ measurable $(\mathscr{H}^m, m)$ *rectifiable* set $M \subseteq \mathbb{R}^3$;
2. An $\mathscr{H}^m$ measurable *orientation* $\xi : M \to \wedge_m \mathbb{R}^3$;
3. An $\mathscr{H}^m$ measurable *multiplicity* $\theta : M \to \mathbb{Z} \setminus \{0\}$;

such that

$$\mathbf{M}(T) := \int_M |\theta| d\mathscr{H}^m < \infty \tag{1}$$

and

$$\langle T, \omega \rangle = \int_M \langle \omega, \xi \rangle \theta d\mathscr{H}^m$$

whenever $\omega \in \mathscr{D}^m(\mathbb{R}^3)$. By M being $(\mathscr{H}^m, m)$ *rectifiable* we mean that $\mathscr{H}^m(M) < \infty$ and there are finitely many or countably many m-dimensional submanifolds of class C^1, $M_1, M_2, \ldots$, such that $\mathscr{H}^m(M \setminus \cup_j M_j) = 0$. This implies M has an m-dimensional approximate tangent space $\mathrm{Tan}^m(M, x)$ at $\mathscr{H}^m$-almost every $x \in M$ (see e.g. [11, 3.2.16,3.2.19] or [26, 11.4,11.6]). At such points $x \in M$ an *orientation* $\xi(x)$ consists in a unit m vector generating $\mathrm{Tan}^m(M, x)$. The integer *multiplicity* θ can be thought of as the number of sheets passing through a point. The combinatorics of "sheets" and "multiplicities" accounts for the way the boundary of T is computed: The boundary operator ∂ of currents is defined by duality

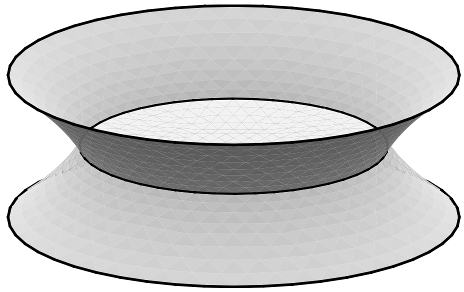

FIGURE 1. Size minimizing but not mass minimizing

of exterior differentiation, thereby generalizing Stokes' theorem for smooth orientable surfaces M. In this context the area of a 2-dimensional rectifiable current T is understood as the *mass* $\mathbf{M}(T)$ defined in (1) — the $\mathscr{H}^2$ measure of the underlying set M counting multiplicities.

The group of m-dimensional rectifiable currents in $\mathbb{R}^3$ is denoted by $\mathscr{R}_m(\mathbb{R}^3)$. We say $T \in \mathscr{R}_m(\mathbb{R}^3)$ is an *integral current* if also $\partial T \in \mathscr{R}_{m-1}(\mathbb{R}^3)$ [(3)]. The group of m-dimensional integral currents in $\mathbb{R}^3$ is denoted by $\mathbf{I}_m(\mathbb{R}^3)$. The compactness theorem relevant to the Plateau problem is the following.

3.1. THEOREM (Federer-Fleming). – *Let $T_1, T_2, \dots$ be a sequence of 2-dimensional integral currents in $\mathbb{R}^3$ whose supports are all contained in some fixed compact set, and such that* $\sup_j \mathbf{M}(T_j) + \mathbf{M}(\partial T_j) < \infty$. *There then exists a subsequence $T_{j(1)}, T_{j(2)}, \dots$ converging weakly* to a 2-dimensional integral current T in $\mathbb{R}^3$.*

The weak* convergence to a *current* of some subsequence of $T_1, T_2, \dots$ follows from the uniform mass bound together with the Banach-Alaoglu theorem and the separability of $\mathscr{D}^2(\mathbb{R}^3)$. Thus the deep content of the theorem is that the limit T is rectifiable as well. We notice that the boundary operator ∂ is continuous with respect to weak* convergence and that mass is lower semicontinuous. The latter follows from the following formula:

$$\mathbf{M}(T) = \sup\{\langle T, \omega\rangle : \omega \in \mathscr{D}^m(\mathbb{R}^3) \text{ and } \|\omega(x)\| \leqslant 1 \text{ for all } x \in \mathbb{R}^3\}$$

where $\|\cdot\|$ is a suitable norm on $\wedge_m \mathbb{R}^3$. Thus the following variational problem admits a minimizer:

$$(\mathscr{P}_{FF,\partial T_0}) \begin{cases} \text{minimize } \mathbf{M}(T) \\ \text{among } T \in \mathscr{R}_2(\mathbb{R}^3) \text{ with } \partial T = \partial T_0. \end{cases}$$

Here $T_0 \in \mathscr{R}_2(\mathbb{R}^3)$ is fixed. The filigree disappear automatically in the weak* limit due to cancelations of orientations of nearby points in "horizontal sections of the tentacles".

(3) The condition is void when $m = 0$.