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Francis C. S. BROWN

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MULTIPLE ZETA VALUES AND PERIODS OF MODULI SPACES $\overline{\mathfrak{M}}_{0,n}$

BY FRANCIS C. S. BROWN

ABSTRACT. – We prove a conjecture due to Goncharov and Manin which states that the periods of the moduli spaces $\mathfrak{M}_{0,n}$ of Riemann spheres with n marked points are multiple zeta values. We do this by introducing a differential algebra of multiple polylogarithms on $\mathfrak{M}_{0,n}$ and proving that it is closed under the operation of taking primitives. The main idea is to apply a version of Stokes' formula iteratively to reduce each period integral to multiple zeta values.

We also give a geometric interpretation of the double shuffle relations, by showing that they are two extreme cases of general product formulae for periods which arise by considering natural maps between moduli spaces.

RÉSUMÉ. – Nous démontrons une conjecture de Goncharov et Manin qui prédit que les périodes des espaces de modules $\mathfrak{M}_{0,n}$ des courbes de genre 0 avec n points marqués sont des valeurs zêta multiples. Nous introduisons une algèbre différentielle de fonctions polylogarithmes multiples sur $\mathfrak{M}_{0,n}$ dans laquelle il existe des primitives. L'idée principale est d'appliquer une version de la formule de Stokes récursivement pour réduire chaque intégrale de périodes à une combinaison linéaire de valeurs zêta multiples.

Nous donnons également une interprétation géométrique des double relations de mélange pour les valeurs zêta multiples. En considérant des applications naturelles entre les espaces des modules, on déduit des formules de produit générales entre leurs périodes. Les doubles relations de mélange s'obtiennent comme deux cas particuliers de cette construction.

1. Introduction

Let $n = \ell + 3 \geq 4$, and let $\mathfrak{M}_{0,n}$ denote the moduli space of curves of genus 0 with n marked points. There is a smooth compactification $\overline{\mathfrak{M}}_{0,n}$, defined by Deligne, Knudsen and Mumford, such that the complement

$$\overline{\mathfrak{M}}_{0,n} \backslash \mathfrak{M}_{0,n}$$

is a normal crossing divisor. Let $A, B \subset \overline{\mathfrak{M}}_{0,n}\backslash\mathfrak{M}_{0,n}$ denote two sets of boundary divisors which share no irreducible components. In [27], Goncharov and Manin show that the relative cohomology group

$$H^\ell(\overline{\mathfrak{M}}_{0,n}\backslash A, B\backslash B\cap A) \tag{1.1}$$

defines a mixed Tate motive which is unramified over $\mathbb{Z}$.

On the other hand, let $n_1, \ldots, n_r \in \mathbb{N}$, and suppose that $n_r \geq 2$. The multiple zeta value $\zeta(n_1, \ldots, n_r)$ is the real number defined by the convergent sum

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1}\ldots k_r^{n_r}}. \tag{1.2}$$

Its weight is the quantity $n_1 + \cdots + n_r$, and its depth is the number of indices r. We will say that the period $2i\pi$ has weight 1. A very general conjecture [25] claims that the periods of any mixed Tate motive unramified over $\mathbb{Z}$ are multiple zeta values. In the case of the motives (1.1) arising from moduli spaces, this says the following. Consider a real smooth compact submanifold $X_B \subset \overline{\mathfrak{M}}_{0,n}$ of dimension ℓ, whose boundary is contained in B and which does not meet A. It represents a class in $H_\ell(\overline{\mathfrak{M}}_{0,n}, B)$. Let $\omega_A \in \Omega^\ell(\overline{\mathfrak{M}}_{0,n}\backslash A)$ denote an algebraic form with singularities contained in A. In [27], Goncharov and Manin conjectured that the integral

$$I = \int_{X_B} \omega_A \tag{1.3}$$

is a linear combination of multiple zeta values, and proved that every multiple zeta value can occur as such a period integral. In this paper, we develop some general methods for computing periods and prove this conjecture as an application.

THEOREM 1.1. – *The integral I is a $\mathbb{Q}[2\pi i]$-linear combination of multiple zeta values of weight at most ℓ.*

This theorem thus lends weight to the conjecture on the periods of all mixed Tate motives which are unramified over $\mathbb{Z}$.

The rough idea of our method is as follows. The set of real points $\mathfrak{M}_{0,n}(\mathbb{R})$ is tesselated by a number of open cells X_n which can naturally be identified with a Stasheff polytope, or associahedron. First consider the case where the domain of integration in (1.3) is a single cell X_n (this actually suffices for the version of the conjecture considered in [27]). The key is then to apply a version of Stokes' theorem to the closed polytope $\overline{X}_n \subset \overline{\mathfrak{M}}_{0,n}(\mathbb{R})$. Since each face of $\overline{X}_n$ is itself a product of associahedra $\overline{X}_a \times \overline{X}_b$, we repeatedly take primitives to obtain a cascade of integrals over associahedra of smaller and smaller dimension. In order to do this, we need to construct a graded algebra $L(\mathfrak{M}_{0,n})$ of multiple polylogarithm functions on $\mathfrak{M}_{0,n}$ in which primitives exist. At each stage of the induction, the dimension of the domain of integration decreases by one, and the weight of the integrand increases by one. At the final stage, we evaluate a multiple polylogarithm at the point 1, and this gives a linear combination of multiple zeta values. This gives an effective algorithm for computing such integrals. Our approach also works in greater generality, and our results should extend without difficulty, for example, to the case of configuration spaces related to other Coxeter groups.

1.1. General overview

This paper is essentially a study of the de Rham theory of the motivic fundamental group of $\mathfrak{M}_{0,n}$. Previously, the focus has mainly been on the projective line minus roots of unity, and in particular $\mathfrak{M}_{0,4} \cong \mathbb{P}^1\backslash\{0,1,\infty\}$ ([13], [14], [25, 26], [45]). The advantage of considering the moduli spaces $\mathfrak{M}_{0,n}$ is that we can bring to bear the full richness of their geometry. We show, for example, that the double shuffle relations for multiple zeta values are just two special cases of generalised product relations arising naturally from functorial maps between moduli spaces.

An essential part of this work is devoted to multiple polylogarithms, which are functions first defined by Goncharov for all $n_1, \ldots, n_\ell \in \mathbb{N}$ by the power series:

$$\mathrm{Li}_{n_1,\ldots,n_\ell}(x_1,\ldots,x_\ell) = \sum_{0<k_1<\cdots<k_\ell} \frac{x_1^{k_1}\ldots x_\ell^{k_\ell}}{k_1^{n_1}\ldots k_\ell^{n_\ell}}, \qquad \text{where} \quad |x_i|<1. \tag{1.4}$$

By analytic continuation, they define multi-valued functions on $\mathfrak{M}_{0,n}$, where $n=\ell+3$. One of our main objects of study in this paper is the larger set $L(\mathfrak{M}_{0,n})$ of all homotopy-invariant iterated integrals on $\mathfrak{M}_{0,n}$. It forms a differential algebra of multi-valued functions on $\mathfrak{M}_{0,n}$, in which the set of functions (1.4) is strictly contained. From the point of view of differential Galois theory, $L(\mathfrak{M}_{0,n})$ defines a maximal unipotent Picard-Vessiot theory on $\mathfrak{M}_{0,n}$. We then define the universal algebra of multiple polylogarithms $B(\mathfrak{M}_{0,n})$ to be a modified version of Chen's reduced bar construction. It is a differential graded Hopf algebra which is an abstract algebraic version of $L(\mathfrak{M}_{0,n})$. One of our key results states that the de Rham cohomology of $B(\mathfrak{M}_{0,n})$ is trivial. From this we deduce the existence of primitives in $L(\mathfrak{M}_{0,n})$. We also need to understand the regularised restriction of polylogarithms to the faces of $\overline{X}_n$. This requires a canonical regularisation theorem, and amounts to studying what happens when singularities of an iterated integral collide. We are thus led to work on certain blow-ups of $\mathfrak{M}_{0,n}$, described below. It follows that the structure of $L(\mathfrak{M}_{0,n})$, and hence the function theory of multiple polylogarithms, is intimately related to the combinatorics of the associahedron.

1.2. Detailed summary of results

In Section 2, we review some aspects of the geometry of the moduli spaces $\mathfrak{M}_{0,n}$, and study certain blow-ups obtained from them. Let S denote a set with n elements, each labelling a marked point on the projective line $\mathbb{P}^1$, and write $\mathfrak{M}_{0,S}=\mathfrak{M}_{0,n}$. A *dihedral structure* on S is an identification of S with the set of edges (or vertices) of an unoriented n-gon. For each such dihedral structure δ, we embed $\mathfrak{M}_{0,S}$ in the affine space $\mathbb{A}^\ell$, where $\ell=n-3$, and blow up parts of the boundary in $\mathbb{A}^\ell\backslash\mathfrak{M}_{0,S}$ to obtain an intermediary space

$$\mathfrak{M}_{0,S} \subset \mathfrak{M}^\delta_{0,S} \subset \overline{\mathfrak{M}}_{0,S},$$

where $\mathfrak{M}^\delta_{0,S}$ is an affine scheme defined over $\mathbb{Z}$. We prove that the set of $\mathfrak{M}^\delta_{0,S}$, for varying δ, forms a set of smooth affine charts on $\overline{\mathfrak{M}}_{0,S}$. In order to define them, we introduce *dihedral coordinates*, which are one of the key tools used throughout this paper. These are functions

$$u_{ij} : \mathfrak{M}_{0,S} \to \mathbb{P}^1\backslash\{0,1,\infty\}, \qquad \text{where} \quad \{i,j\}\in\chi_{S,\delta},$$

indexed by the set of chords $\chi_{S,\delta}$ in the n-gon defined by δ. Together, they define an embedding $(u_{ij})_{\chi_{S,\delta}} : \mathfrak{M}_{0,S} \rightarrow \mathbb{A}^{n(n-3)/2}$, and the scheme $\mathfrak{M}^{\delta}_{0,S}$ is the Zariski closure of the image of this map. For example, in the case $n = 5$, we can identify $\mathfrak{M}_{0,S} = \{(t_1, t_2) \in \mathbb{P}^1 \times \mathbb{P}^1 : t_1 t_2 (1 - t_1)(1 - t_2)(t_1 - t_2) \neq 0,\ t_1, t_2 \neq \infty\}$. The pentagon (S, δ) has five chords, labelled $\{13, 24, 35, 41, 52\}$ (fig. 1), and we have

$$u_{13} = 1 - t_1, \quad u_{24} = \frac{t_1}{t_2}, \quad u_{35} = \frac{t_2 - t_1}{t_2(1 - t_1)}, \quad u_{41} = \frac{1 - t_2}{1 - t_1}, \quad u_{52} = t_2.$$

The scheme $\mathfrak{M}^{\delta}_{0,5}$ is then defined by the five cyclically symmetric equations in $\mathbb{A}^5$:

$$u_{13} + u_{24}u_{52} = 1,\ u_{24} + u_{35}u_{13} = 1, \ldots, u_{52} + u_{13}u_{14} = 1.$$

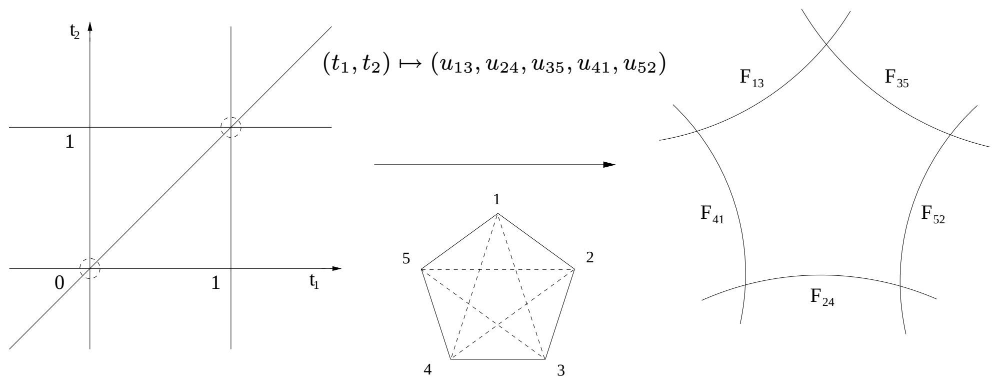

FIGURE 1. Dihedral coordinates on $\mathfrak{M}_{0,5}$. The scheme $\mathfrak{M}^{\delta}_{0,5}$ (right) is defined to be the Zariski closure of the image of the embedding $\{u_{ij}\} : \mathfrak{M}_{0,5} \hookrightarrow \mathbb{A}^5$ defined by the set of dihedral coordinates, which are indexed by chords in a pentagon (middle). This map has the effect of blowing up the points $(0, 0)$ and $(1, 1)$. A cell $X_{S,\delta}$ is given by the region $0 < t_1 < t_2 < 1$ (left). After blowing-up it becomes a pentagon with sides $F_{ij} = \{u_{ij} = 0\}$.

Now consider the set of real points $\mathfrak{M}_{0,S}(\mathbb{R})$. There is a bounded cell $X_{S,\delta} \subset \mathfrak{M}_{0,S}(\mathbb{R})$ defined by the region $\{0 < u_{ij} < 1\}$. One shows that $\mathfrak{M}_{0,S}(\mathbb{R})$ is the disjoint union of the open cells $X_{S,\delta}$ of dimension $\ell = n - 3$, as δ runs over the set of dihedral structures on S, so a dihedral structure corresponds to choosing a connected component of $\mathfrak{M}_{0,S}(\mathbb{R})$. The closure of the cell $\overline{X}_{S,\delta}$ satisfies

$$\overline{X}_{S,\delta} = \{0 \leq u_{ij} \leq 1\} \subset \mathfrak{M}^{\delta}_{0,S}(\mathbb{R}), \tag{1.5}$$

and $\mathfrak{M}^{\delta}_{0,S} \backslash \mathfrak{M}_{0,S}$ is the union of all divisors meeting the boundary of $X_{S,\delta}$. Therefore $\overline{X}_{S,\delta}$ is a convex polytope, and its boundary divisors give an explicit algebraic model of the associahedron. It is well-known that the combinatorics of the associahedron is given by triangulations of polygons. But because dihedral coordinates are already defined in terms of polygons, the main combinatorial properties of the associahedron, and its dihedral symmetry, follow