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Vincenzo DI GENNARO & Davide FRANCO

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MONODROMY OF A FAMILY OF HYPERSURFACES

BY VINCENZO DI GENNARO AND DAVIDE FRANCO

ABSTRACT. – Let Y be an $(m+1)$-dimensional irreducible smooth complex projective variety embedded in a projective space. Let Z be a closed subscheme of Y, and δ be a positive integer such that $\mathcal{I}_{Z,Y}(\delta)$ is generated by global sections. Fix an integer $d \geq \delta + 1$, and assume the general divisor $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ is smooth. Denote by $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ the quotient of $H^m(X;\mathbb{Q})$ by the cohomology of Y and also by the cycle classes of the irreducible components of dimension m of Z. In the present paper we prove that the monodromy representation on $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ for the family of smooth divisors $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ is irreducible.

RÉSUMÉ. – Soit Y une variété projective complexe lisse irréductible de dimension $m+1$, plongée dans un espace projectif. Soit Z un sous-schéma fermé de Y, et soit δ un entier positif tel que $\mathcal{I}_{Z,Y}(\delta)$ soit engendré par ses sections globales. Fixons un entier $d \geq \delta+1$, et supposons que le diviseur général $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ soit lisse. Désignons par $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ le quotient de $H^m(X;\mathbb{Q})$ par la cohomologie de Y et par les classes des composantes irréductibles de Z de dimension m. Dans cet article, nous prouvons que la représentation de monodromie sur $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ pour la famille des diviseurs lisses $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ est irréductible.

1. Introduction

In this paper we provide an affirmative answer to a question formulated in [9].

Let $Y \subseteq \mathbb{P}^N$ ($\dim\, Y = m+1$) be an irreducible smooth complex projective variety embedded in a projective space $\mathbb{P}^N$, Z be a closed subscheme of Y, and δ be a positive integer such that $\mathcal{I}_{Z,Y}(\delta)$ is generated by global sections. Assume that for $d \gg 0$ the general divisor $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ is smooth. In the paper [9] it is proved that this is equivalent to the fact that the strata $Z_{\{j\}} = \{x \in Z \,:\, \dim\, T_x Z = j\}$, where $T_x Z$ denotes the Zariski tangent space, satisfy the following inequality:

$$\dim\, Z_{\{j\}} + j \leq \dim\, Y - 1 \quad \text{for any} \quad j \leq \dim\, Y. \tag{1}$$

This property implies that, for any $d \geq \delta$, there exists a smooth hypersurface of degree d which contains Z ([9], 1.2. Theorem).

It is generally expected that, for $d \gg 0$, the Hodge cycles of the general hypersurface $X \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ depend only on Z and on the ambient variety Y. A very precise conjecture in this direction was made in [9]:

CONJECTURE 1 (Otwinowska - Saito). – Assume *deg* $X \geq \delta + 1$. Then the monodromy representation on $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ for the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing Z as above is irreducible.

We denote by $H^m(X;\mathbb{Q})^{\mathrm{van}}_{Z}$ the subspace of $H^m(X;\mathbb{Q})^{\mathrm{van}}$ generated by the cycle classes of the maximal dimensional irreducible components of Z modulo the image of $H^m(Y;\mathbb{Q})$ (using the orthogonal decomposition $H^m(X;\mathbb{Q}) = H^m(Y;\mathbb{Q}) \perp H^m(X;\mathbb{Q})^{\mathrm{van}}$) if $m = 2 \dim Z$, and $H^m(X;\mathbb{Q})^{\mathrm{van}}_{Z} = 0$ otherwise, and we denote by $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ the orthogonal complement of $H^m(X;\mathbb{Q})^{\mathrm{van}}_{Z}$ in $H^m(X;\mathbb{Q})^{\mathrm{van}}$. The conjecture above cannot be strengthened because, even in $Y = \mathbb{P}^3$, there exist examples for which $\dim H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z}$ is arbitrarily large and the monodromy representation associated to the linear system $|H^0(Y, \mathcal{I}_{Z,Y}(\delta))|$ is diagonalizable.

The authors of [9] observed that a proof for such a conjecture would confirm the expectation above and would reduce the Hodge conjecture for the general hypersurface $X_t \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ to the Hodge conjecture for Y. More precisely, by a standard argument, from Conjecture 1 it follows that when $m = 2 \dim Z$ and the vanishing cohomology of the general $X_t \in |H^0(Y, \mathcal{I}_{Z,Y}(d))|$ ($d \geq \delta + 1$) is not of pure Hodge type $(m/2, m/2)$, then the Hodge cycles in the middle cohomology of X_t are generated by the image of the Hodge cycles on Y together with the cycle classes of the irreducible components of Z. So, the Hodge conjecture for X_t is reduced to that for Y (compare with [9], Corollary 0.5). They also proved that the conjecture is satisfied in the range $d \geq \delta + 2$, or for $d = \delta + 1$ if hyperplane sections of Y have non trivial top degree holomorphic forms ([9], 0.4. Theorem). Their proof relies on Deligne's semisimplicity Theorem and on Steenbrink's Theory for semistable degenerations.

Arguing in a different way, we prove in this paper Conjecture 1 in full. More precisely, avoiding degeneration arguments, in Section 2 we will deduce Conjecture 1 from the following:

THEOREM 1.1. – *Fix integers* $1 \leq k < d$, *and let* $W = G \cap X \subset Y$ *be a complete intersection of smooth divisors* $G \in |H^0(Y, \mathcal{O}_Y(k))|$ *and* $X \in |H^0(Y, \mathcal{O}_Y(d))|$. *Then the monodromy representation on* $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp W}$ *for the family of smooth divisors* $X_t \in |H^0(Y, \mathcal{O}_Y(d))|$ *containing* W *is irreducible.*

Here we define $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp W}$ in a similar way as before, i.e. as the orthogonal complement in $H^m(X;\mathbb{Q})^{\mathrm{van}}$ of the image $H^m(X;\mathbb{Q})^{\mathrm{van}}_{W}$ of the map obtained by composing the natural maps $H_m(W;\mathbb{Q}) \to H_m(X;\mathbb{Q}) \cong H^m(X;\mathbb{Q}) \to H^m(X;\mathbb{Q})^{\mathrm{van}}$.

The proof of Theorem 1.1 will be given in Section 4 and consists in a Lefschetz type argument applied to the image of the rational map on Y associated to the linear system $|H^0(Y, \mathcal{I}_{W,Y}(d))|$, which turns out to have at worst isolated singularities. This approach was started in our paper [2] where we proved a particular case of Theorem 1.1, but the proof given here is independent and much simpler.

We begin by proving Conjecture 1 as a consequence of Theorem 1.1, and next we prove Theorem 1.1.

2. Proof of Conjecture 1 as a consequence of Theorem 1.1.

We keep the same notation we introduced before, and need further preliminaries.

NOTATIONS 2.1. – (i) Let $V_\delta \subseteq H^0(Y, \mathcal{I}_{Z,Y}(\delta))$ be a subspace generating $\mathcal{I}_{Z,Y}(\delta)$, and $V_d \subseteq H^0(Y, \mathcal{I}_{Z,Y}(d))$ $(d \geq \delta + 1)$ be a subspace containing the image of $V_\delta \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d-\delta))$ in $H^0(Y, \mathcal{I}_{Z,Y}(d))$. Let $G \in |V_\delta|$ and $X \in |V_d|$ be divisors. Put $W := G \cap X$. From condition (1), and [9], 1.2. Theorem, we know that if G and X are general then they are smooth. Moreover, by ([4], p. 133, Proposition 4.2.6. and proof), we know that if G and X are smooth then W has only isolated singularities.

(ii) In the case $m > 2$, fix a smooth $G \in |V_\delta|$. Let $H \in |H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l))|$ be a general hypersurface of degree $l \gg 0$, and put $Z' := Z \cap H$ and $G' := G \cap H$. Denote by $V'_d \subseteq H^0(G', \mathcal{I}_{Z',G'}(d))$ the restriction of V_d on G', and by $V''_d \subseteq H^0(G, \mathcal{I}_{Z,G}(d))$ the restriction of V_d on G. Since $H^0(G, \mathcal{I}_{Z,G}(d)) \subseteq H^0(G', \mathcal{I}_{Z',G'}(d))$, we may identify $V''_d = V'_d$. Put $W' := W \cap H \in |V'_d|$. Similarly as we did for the triple (Y, X, Z), using the orthogonal decomposition $H^{m-2}(W';\mathbb{Q}) = H^{m-2}(G';\mathbb{Q}) \perp H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}$, we define the subspaces $H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'}$ and $H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{\perp Z'}$ of $H^{m-2}(W';\mathbb{Q})$ with respect to the triple (G', W', Z'). Passing from (Y, X, Z) to (G', W', Z') will allow us to prove Conjecture 1 arguing by induction on m (see the proof of Proposition 2.4 below).

(iii) Let $\varphi : \mathcal{W} \to |V''_d|$ $(\mathcal{W} \subseteq G \times |V''_d|)$ be the universal family parametrizing the divisors $W = G \cap X \in |V''_d|$. Denote by $\sigma : \widetilde{\mathcal{W}} \to \mathcal{W}$ a desingularization of $\mathcal{W}$, and by $U_\varphi \subseteq |V''_d|$ a nonempty open set such that the restriction $(\varphi \circ \sigma)_{|U_\varphi} : (\varphi \circ \sigma)^{-1}(U_\varphi) \to U_\varphi$ is smooth. Next, let $\psi : \mathcal{W}' \to |V'_d|$ $(\mathcal{W}' \subseteq G \times |V'_d|)$ be the universal family parametrizing the divisors $W' = W \cap H \in |V'_d|$, and denote by $U_\psi \subseteq |V'_d|$ a nonempty open set such that the restriction $\psi_{|U_\psi} : \psi^{-1}(U_\psi) \to U_\psi$ is smooth. Shrinking U_φ and U_ψ if necessary, we may assume $U := U_\varphi = U_\psi \subseteq |V''_d| = |V'_d|$. For any $t \in U$ put $W_t := \varphi^{-1}(t)$, $\widetilde{W}_t := \sigma^{-1}(W_t)$, and $W'_t := \psi^{-1}(t)$. Observe that $W_t \cap \mathrm{Sing}(\mathcal{W}) \subseteq \mathrm{Sing}(W_t)$, so we may assume $W'_t = W_t \cap H \subseteq W_t \backslash \mathrm{Sing}(W_t) \subseteq \widetilde{W}_t$. Denote by ι_t and $\tilde{\iota}_t$ the inclusion maps $W'_t \to W_t$ and $W'_t \to \widetilde{W}_t$. The pull-back maps $\tilde{\iota}^*_t : H^{m-2}(\widetilde{W}_t;\mathbb{Q}) \to H^{m-2}(W'_t;\mathbb{Q})$ give rise to a natural map $\tilde{\iota}^* : R^{m-2}((\varphi \circ \sigma)_{|U})_*\mathbb{Q} \to R^{m-2}(\psi_{|U})_*\mathbb{Q}$ between local systems on U, showing that $\Im(\tilde{\iota}^*_t)$ is globally invariant under the monodromy action on the cohomology of the smooth fibers of ψ. Finally, we recall that the inclusion map ι_t defines a Gysin map $\iota^\star_t : H_m(W_t;\mathbb{Q}) \to H_{m-2}(W'_t;\mathbb{Q})$ (see [5], p. 382, Example 19.2.1).

REMARK 2.2. – Fix a smooth $G \in |V_\delta|$, and assume $m \geq 2$. The linear system $|V_d|$ induces an embedding of $G \backslash Z$ in some projective space: denote by Γ the image of $G \backslash Z$ through this embedding. Since $G \backslash Z$ is irreducible, then also Γ is, and so is its general hyperplane section, which is isomorphic to $(G \cap X) \backslash Z$ via $|V_d|$. So we see that, when $m \geq 2$, for any smooth $G \in |V_\delta|$ and any general $X \in |V_d|$, one has that $W \backslash Z$ is irreducible. In particular, when $m > 2$, then also W is irreducible.

LEMMA 2.3. – *Fix a smooth $G \in |V_\delta|$, and assume $m > 2$. Then, for a general $t \in U$, one has $\Im(\tilde{\iota}_t^*) = \Im(PD \circ \iota_t^\star)$, and the map $PD \circ \iota_t^\star$ is injective (PD means "Poincaré duality": $H_{m-2}(W_t';\mathbb{Q}) \cong H^{m-2}(W_t';\mathbb{Q})$).*

Proof. – By ([13], p. 385, Proposition 16.23) we know that $\Im(\tilde{\iota}_t^*)$ is equal to the image of the pull-back $H^{m-2}(W_t \backslash \mathrm{Sing}(W_t);\mathbb{Q}) \to H^{m-2}(W_t';\mathbb{Q})$. On the other hand, by ([3], p. 157 Proposition 5.4.4., and p. 158 (PD)) we have natural isomorphisms involving intersection cohomology groups:

$$H^{m-2}(W_t \backslash \mathrm{Sing}(W_t);\mathbb{Q}) \cong IH^{m-2}(W_t) \cong IH^m(W_t)^\vee \cong H^m(W_t;\mathbb{Q})^\vee \cong H_m(W_t;\mathbb{Q}). \tag{2}$$

So we may identify the pull-back $H^{m-2}(W_t \backslash \mathrm{Sing}(W_t);\mathbb{Q}) \to H^{m-2}(W_t';\mathbb{Q})$ with $PD \circ \iota_t^\star$. This proves that $\Im(\tilde{\iota}_t^*) = \Im(PD \circ \iota_t^\star)$. Moreover, since W_t' is smooth, then $IH^{m-2}(W_t') \cong H^{m-2}(W_t';\mathbb{Q})$ ([3], p. 157). So, from (2), we may identify $PD \circ \iota_t^\star$ with the natural map $IH^{m-2}(W_t) \to IH^{m-2}(W_t \cap H)$, which is injective in view of Lefschetz Hyperplane Theorem for intersection cohomology ([3], p. 158 (I), and p. 159, Theorem 5.4.6) (recall that $W_t' = W_t \cap H$). □

We are in position to prove Conjecture 1.

Fix a smooth $G \in |V_\delta|$, and a general $X \in |V_d|$. Put $W = G \cap X$. Since the monodromy group of the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing W is a subgroup of the monodromy group of the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing Z, in order to deduce Conjecture 1 from Theorem 1.1, it suffices to prove that $H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp Z} = H^m(X;\mathbb{Q})^{\mathrm{van}}_{\perp W}$. Equivalently, it suffices to prove that $H^m(X;\mathbb{Q})^{\mathrm{van}}_{Z} = H^m(X;\mathbb{Q})^{\mathrm{van}}_{W}$. This is the content of the following:

PROPOSITION 2.4. – *For any smooth $G \in |V_\delta|$ and any general $X \in |V_d|$, one has $H^m(X;\mathbb{Q})^{\mathrm{van}}_{Z} = H^m(X;\mathbb{Q})^{\mathrm{van}}_{W}$.*

Proof. – First we analyze the cases $m = 1$ and $m = 2$, and next we argue by induction on $m > 2$ (recall that $\dim Y = m + 1$).

The case $m = 1$ is trivial because in this case $\dim Z \leq \dim W = 0$.

Next assume $m = 2$. In this case $\dim Y = 3$ and $\dim Z \leq 1$. Denote by $Z_1, \dots, Z_h$ ($h \geq 0$) the irreducible components of Z of dimension 1 (if there are). Fix a smooth $G \in |V_\delta|$ and a general $X \in |V_d|$, and put $W = G \cap X = Z_1 \cup \dots \cup Z_h \cup C$, where C is the residual curve, with respect to $Z_1 \cup \dots \cup Z_h$, in the complete intersection W. By Remark 2.2 we know that C is irreducible. Then, as (co)cycle classes, $Z_1, \dots, Z_h, C$ generate $H^2(X;\mathbb{Q})^{\mathrm{van}}_W$, and $Z_1, \dots, Z_h$ generate $H^2(X;\mathbb{Q})^{\mathrm{van}}_Z$. Since $Z_1 + \dots + Z_h + C = \delta H_X$ in $H^2(X;\mathbb{Q})$ (H_X= general hyperplane section of X in $\mathbb{P}^N$), and this cycle comes from $H^2(Y;\mathbb{Q})$, then $Z_1 + \dots + Z_h + C = 0$ in $H^2(X;\mathbb{Q})^{\mathrm{van}}$, and so $H^2(X;\mathbb{Q})^{\mathrm{van}}_Z = H^2(X;\mathbb{Q})^{\mathrm{van}}_W$. This concludes the proof of Proposition 2.4 in the case $m = 2$.

Now assume $m > 2$ and argue by induction on m. First we observe that the intersection pairing on $H^{m-2}(W';\mathbb{Q})^{\mathrm{van}}_{Z'}$ is non-degenerate: this follows from Hodge Index Theorem,