ISSN 0012-9593

quatrième série - tome 42 fascicule 4 juillet-août 2009

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Fabrizio CATANESE & Frédéric MANGOLTE

Real singular Del Pezzo surfaces and 3-folds fibred by rational curves, II

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 42, 2009, p. 531 à 557

REAL SINGULAR DEL PEZZO SURFACES AND 3-FOLDS FIBRED BY RATIONAL CURVES, II

BY Fabrizio CATANESE AND Frédéric MANGOLTE

Abstract. – Let $W \to X$ be a real smooth projective 3-fold fibred by rational curves such that $W(\mathbb{R})$ is orientable. J. Kollár proved that a connected component N of $W(\mathbb{R})$ is essentially either Seifert fibred or a connected sum of lens spaces.

Answering three questions of Kollár, we give sharp estimates on the number and the multiplicities of the Seifert fibres (resp. the number and the torsions of the lens spaces) when X is a geometrically rational surface.

When N is Seifert fibred over a base orbifold F, our result generalizes Comessatti's theorem on smooth real rational surfaces: F cannot be simultaneously orientable and of hyperbolic type. We show as a surprise that, unlike in Comessatti's theorem, there are examples where F is non orientable, of hyperbolic type, and X is minimal.

Résumé. – Soit $W \to X$ une variété projective réelle non singulière munie d'une fibration en courbes rationnelles et telle que $W(\mathbb{R})$ soit orientable. J. Kollár a montré qu'une composante connexe N de $W(\mathbb{R})$ est essentiellement une variété de Seifert ou une somme connexe d'espaces lenticulaires.

Répondant à trois questions de Kollár, nous donnons une estimation optimale du nombre et des multiplicités des fibres de Seifert (resp. du nombre et des torsions des espaces lenticulaires) lorsque X est une surface géométriquement rationnelle.

Lorsque N admet une fibration de Seifert au-dessus d'un orbifold F, nos résultats généralisent le théorème de Comessatti sur les surfaces rationnelles réelles lisses : F ne peut pas être à la fois orientable et de type hyperbolique. Nous montrons, ce qui est une surprise, qu'à la différence du théorème de Comessatti, il existe des exemples où F est non orientable, de type hyperbolique, et X est minimale.

The research of the authors was supported by the D.F.G.-FORSCHERGRUPPE 790 "Classification of algebraic surfaces and compact complex manifolds" and the ANR grant "JCLAMA" of Agence Nationale de la Recherche. We profited of the hospitality of the Centro 'De Giorgi' of the Scuola Normale di Pisa in order to meet and prepare the final version of the article.

Introduction

Given a smooth real projective variety W of dimension n, we consider the topology of a connected component N of the set $W(\mathbb{R})$ of its real points.

John Nash proved in [12] that any compact connected differentiable manifold N is obtained in this way, and went over to ask whether the same would hold if one assumes W to be geometrically rational.

However, when W is a surface of negative Kodaira dimension, one is able, after the work of Comessatti [2] for geometrically rational surfaces, to deduce drastical restrictions for the topology of N. Namely, if N is orientable, then it is diffeomorphic to a sphere or to a torus: in other words, N cannot be simultaneously oriented and of hyperbolic type. In this note, we make a step towards a complete classification of the topological types for N when W is a rationally connected 3-fold fibred by rational curves (this is one of the higher dimensional analogues of Comessatti's theorem).

This study was initiated by János Kollár, in the third paper [8] of a ground-breaking series of articles applying the minimal model program to the study of the topology of real algebraic 3-folds.

Kollár's philosophy is that a very important condition in order to obtain restrictions upon the topological type of $W(\mathbb{R})$ is that W has terminal singularities and K_W is Cartier along $W(\mathbb{R})$.

Kollár proved in particular that if W is a smooth 3-fold fibred by rational curves (in particular, W has negative Kodaira dimension) and such that $W(\mathbb{R})$ is orientable, then a connected component N of $W(\mathbb{R})$ is essentially a Seifert fibred 3-manifold or the connected sum of a finite number of lens spaces. Note that in [5, 6] it was shown that conversely all the above manifolds N do occur for some smooth 3-fold W fibred by rational curves.

When W belongs to the subclass of rationally connected 3-folds fibred by rational curves, Kollár proved some additional restrictions upon N and made three further conjectures. In our first note [1] we proved two of the optimal estimates that Kollár conjectured to hold. In the present note we prove the third estimate, which is the most important one since it allows us to conclude in particular that, if N is a Seifert fibred 3-manifold, then the base orbifold cannot be simultaneously oriented and of hyperbolic type.

Let us now introduce our results in more detail.

Let N be an oriented three dimensional compact connected topological manifold without boundary. Take a decomposition $N = N' \#^a \mathbb{P}^3(\mathbb{R}) \#^b (S^1 \times S^2)$ with $a+b$ maximal and observe that this decomposition is unique by a theorem of Milnor [10].

We shall focus our attention on the case where N' is Seifert fibred or a connected sum of lens spaces. We consider the integers $k := k(N)$ and $n_l := n_l(N)$, $l = 1 \dots k$ defined as follows:

1. if $g\colon N' \to F$ is a Seifert fibration, k denotes the number of multiple fibres of g and $n_1 \le n_2 \le \cdots \le n_k$ denote the respective multiplicities;
2. if N' is a connected sum of lens spaces, k denotes the number of lens spaces and $n_1 \le n_2 \le \cdots \le n_k$, $n_l \ge 3$, $\forall l$, the orders of the respective fundamental groups (thus we have a decomposition $N' = \#_{l=1}^k (L(n_l, q_l))$ for some $1 < q_l < n_l$ relatively prime to n_l).

Observe that when N' is a connected sum of lens spaces, the number k and the numbers n_l, $l = 1, \dots k$, are well defined (again by Milnor's theorem). In the case of a Seifert fibred manifold N', these integers may a priori depend upon the choice of a Seifert fibration.

Three results of our two notes are summarized by the following.

THEOREM 0.1. – *Let $W \to X$ be a real smooth projective 3-fold fibred by rational curves over a geometrically rational*[(1)] *surface X. Suppose that $W(\mathbb{R})$ is orientable. Then, for each connected component $N \subset W(\mathbb{R})$, $k(N) \leq 4$ and $\sum_l (1 - \frac{1}{n_l(N)}) \leq 2$. Furthermore, if N' is Seifert fibred over $S^1 \times S^1$, then $k(N) = 0$.*

This theorem answers, as we already said, some questions posed by Kollár, see [8, Remark 1.2 (1,2,3)]. In the first note, we proved the estimate $k(N) \leq 4$ and we showed that $k(N) = 0$ if N' is Seifert fibred over the torus. Thus Theorem 0.1 follows from [1, Corollary 0.2, and Theorem 0.3] and from Theorem 0.2 of the present paper using results of [8] as in [1]. The present note is mainly devoted to the proof of the inequality $\sum_l (1 - \frac{1}{n_l(N)}) \leq 2$, see Lemma 6.1.

The proof of this inequality goes as follows: let $W \to X$ be a real smooth projective 3-fold fibred by rational curves over a geometrically rational surface X. Using the same arguments as in [1, Sec. 3], we reduce the proof of the estimate for the integers $n_l(N)$ to an inequality depending on the indices of certain singular points of a real component M of the topological normalization of $X(\mathbb{R})$ (see Definition 1.1). In this process, the number $k(N)$ can be made to correspond to the number of real singular points on M which are of type A_μ^+, and globally separating when μ is odd; each number $n_l(N) - 1$ corresponds to the index μ_l of the singularity $A_{\mu_l}^+$ of M. The main part of the paper is devoted to the proof of the following.

THEOREM 0.2. – *Let X be a projective surface defined over $\mathbb{R}$. Suppose that X is geometrically rational with Du Val singularities. Then a connected component M of the topological normalization $\overline{X(\mathbb{R})}$ contains at most 4 singular points x_l of type $A_{\mu_l}^+$ which are globally separating for μ_l odd. Furthermore, their indices satisfy*

$$\sum \left(1 - \frac{1}{\mu_l + 1}\right) \leq 2\ .$$

Let us now give an interpretation of the above results in terms of geometric topology (see e.g. [13] for the basic definitions and classical results). Suppose that N' admits a Seifert fibration with base orbifold F. From our main Theorem 0.1 we infer that, if the underlying manifold $|F|$ is orientable, then the Euler characteristic of the compact 2-dimensional orbifold F is nonnegative (see Proposition 7.1). Thus, by the uniformization theorem for compact 2-dimensional orbifolds, F admits a spherical structure or an euclidean structure.

In general, a 3-manifold N does not possess a geometric structure, but, if it does, then the geometry involved is unique. Moreover, it turns out that every Seifert fibred manifold admits a geometric structure. The geometry of N is modeled on one of the six following models (see [13] for a detailed description of each geometry):

$$S^3, S^2 \times \mathbb{R}, E^3, \mathrm{Nil}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{\mathrm{SL}_2\,\mathbb{R}},$$

(1) By [4] these assumptions are equivalent to: W rationally connected and fibred by rational curves.

where E^3 is the 3-dimensional euclidean space and $\mathbb{H}^2$ is the hyperbolic plane. The six above geometries are called the Seifert geometries. The appropriate geometry for a Seifert fibration is determined by the Euler characteristic of the base orbifold and by the Euler number of the Seifert bundle [13, Table 4.1].

Let W be a real projective 3-fold fibred by rational curves and such that $W(\mathbb{R})$ is orientable, let $N \subset W(\mathbb{R})$ be a connected component and let N' be the manifold defined as above. Suppose moreover that N' possesses a geometric structure. By Theorem [8, Th. 1.1], the geometry of N' is one of the six Seifert geometries. Conversely, by [6], any orientable three dimensional manifold endowed with any Seifert geometry is diffeomorphic to a real component of a real projective 3-fold fibred by rational curves. But, when W is rationally connected, the following corollary of our main theorem gives further restrictions.

Corollary 0.3. – *Let W be a real smooth projective rationally connected 3-fold fibred by rational curves. Suppose that $W(\mathbb{R})$ is orientable and let N be a connected component of $W(\mathbb{R})$. Then neither N nor N' can be endowed with a $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$ structure or with a $\mathbb{H}^2\times\mathbb{R}$ structure whose base orbifold F is orientable.*

Observe moreover that in [8] all compact 3-manifolds with S^3 or E^3 geometry, and some manifolds with Nil geometry, are realized as a real component of a real smooth projective rationally connected 3-fold fibred by rational curves.

There remains of course the question about what happens when N is Seifert fibred over a non-orientable orbifold F: is the orbifold still not of hyperbolic type? In the last section we show that the answer to this question is negative. We produce indeed an example of a smooth 3-fold W, fibred by rational curves over a Du Val Del Pezzo surface X, where $W(\mathbb{R})$ is orientable, and contains a connected component which is Seifert fibred over a non-orientable base orbifold of hyperbolic type.

The striking fact is here that X is a real minimal surface: this contrasts with Comessatti's theorem: since indeed a real minimal nonsingular geometrically rational surface cannot have a component which is of hyperbolic type.

Theorem 0.4. – *There exists a minimal real Du Val Del Pezzo surface X of degree 1 having exactly two singular points, of type A_2^+, and such that the real part $X(\mathbb{R})$ has a connected component containing the two singular points and which is homeomorphic to a real projective plane.*

Let W' be the projectivized tangent bundle of X: then W' has terminal singularities, $W'(\mathbb{R})$ is contained in the smooth locus of W', in particular if W is obtained resolving the singular points of W', then $W(\mathbb{R}) = W'(\mathbb{R})$.

Moreover $W(\mathbb{R})$ is orientable and contains a connected component N which is Seifert fibred over a non orientable orbifold of hyperbolic type (the real projective plane with two points of multiplicity 3).

Briefly, now, the contents of the paper.

Sections 1 and 2 are devoted to the reduction of the proof of the main theorem to the assertion of non existence of seven configurations of singular points on a real component of a Du Val Del Pezzo surface of degree 1.