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EQUILIBRIUM STATES FOR INTERVAL MAPS: THE POTENTIAL $-t\log|Df|$

BY HENK BRUIN AND MIKE TODD

ABSTRACT. – Let $f: I \to I$ be a C^2 multimodal interval map satisfying polynomial growth of the derivatives along critical orbits. We prove the existence and uniqueness of equilibrium states for the potential $\varphi_t : x \mapsto -t \log |Df(x)|$ for t close to 1, and also that the pressure function $t \mapsto P(\varphi_t)$ is analytic on an appropriate interval near $t = 1$.

RÉSUMÉ. – Soit $f: I \to I$ une application multimodale de classe C^2 dont les dérivées le long des orbites des points critiques sont à croissance polynomiale, où I est un intervalle. Nous démontrons l'existence et l'unicité d'un état d'équilibre pour le potentiel $\varphi_t : x \mapsto -t \log |Df(x)|$ lorsque t est proche de 1, et que la fonction de pression $t \mapsto P(\varphi_t)$ est analytique sur un intervalle approprié près de $t = 1$.

1. Introduction

Thermodynamic formalism ties potential functions φ to invariant measures of a dynamical system (X, f). The aim is to identify and prove uniqueness of a measure μ_φ that maximises the *free energy*, *i.e.,* the sum of the entropy and the integral over the potential. In other words

$$h_{\mu_\varphi}(f) + \int_X \varphi \, d\mu_\varphi = P(\varphi) := \sup_{\nu \in \mathcal{M}_{erg}} \left\{ h_\nu(f) + \int_X \varphi \, d\nu : -\int_X \varphi \, d\nu < \infty \right\}$$

where $\mathcal{M}_{erg}$ is the set of all ergodic f-invariant Borel probability measures. Such measures are called *equilibrium states*, and $P(\varphi)$ is the *pressure*. This theory was developed by Sinai, Ruelle and Bowen [43, 39, 3] in the context of Hölder potentials on hyperbolic dynamical systems, and has been applied to Axiom A systems, Anosov diffeomorphisms and other systems too, see e.g. [2, 21] for more recent expositions. Apart from uniqueness, it was shown in

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this context that the density $\frac{d\mu_\varphi}{dm_\varphi}$ of the invariant measure with respect to φ-conformal measure m_φ is a fixed point of the transfer operator $(\mathcal{L}_\varphi h)(x) = \sum_{f(y)=x} e^{\varphi(y)}\, h(y)$. Moreover, μ_φ is a Gibbs measure, *i.e.,* there are constants $K > 0$ and $P \in \mathbb{R}$ such that

$$\frac{1}{K} \leqslant \frac{\mu_\varphi(\mathbf{C}_n)}{e^{\varphi_n(x)-nP}} \leqslant K$$

for all $n \in \mathbb{N}$, all n-cylinder sets $\mathbf{C}_n$ and any $x \in \mathbf{C}_n$. Here $\varphi_n(x) := \varphi(f^{n-1}(x)) + \cdots + \varphi(x)$. We refer to P as the *Gibbs constant*.

In this paper we are interested in interval maps (I, f) with nonempty set Crit of critical points. These maps are, at best, only non-uniformly hyperbolic. We say that c is a non-flat critical point of f if there exists a diffeomorphism $g_c : \mathbb{R} \to \mathbb{R}$ with $g_c(0) = 0$ and $1 < \ell_c < \infty$ such that for x close to c, $f(x) = f(c) \pm |\varphi_c(x-c)|^{\ell_c}$. The value of ℓ_c is known as the *critical order* of c. Let $\ell_{max} = \max\{\ell_c : c \in \text{Crit}\}$. We define

$$\mathcal{H} := \left\{ f : I \to I \text{ is } C^2, \#\text{Crit} < \infty \text{ and all critical points are non-flat} \right\}.$$

For $f \in \mathcal{H}$, there is a finite partition $\mathcal{P}_1$ into maximal intervals on which f is monotone, called the *branch partition*. We will assume throughout that $\vee_n \mathcal{P}_n$ generates the Borel σ-algebra. Note that if $f \in \mathcal{H}$ has no attracting cycles then $\vee_n \mathcal{P}_n$ generates the Borel σ-algebra, see [30]. (The C^2 assumption precludes wandering sets, which are not very interesting from the measure theoretic point of view anyway.)

Fix $f \in \mathcal{H}$. The potential of our interest throughout is

$$\varphi_t : x \mapsto -t \log |Df(x)|.$$

The Lyapunov exponent of a measure μ is defined as $\lambda(\mu) := \int_I \log |Df|\, d\mu$. Let $\mathcal{M}_{erg} = \mathcal{M}_{erg}(f)$ be the set of all ergodic f-invariant probability measures, and

$$\mathcal{M}_+ = \{\mu \in \mathcal{M}_{erg} : \lambda(\mu) > 0,\ \text{supp}(\mu) \not\subset \text{orb}(\text{Crit})\}.$$

Measures μ with $\text{supp}(\mu) \subset \text{orb}(\text{Crit})$ are atomic. Atomic measures in $\mathcal{M}_{erg}$ must be supported on periodic cycles. So if $\text{supp}(\mu) \subset \text{orb}(\text{Crit})$ and $\lambda(\mu) > 0$, μ must be supported on a hyperbolic repelling periodic cycle, and thus the corresponding critical point must be preperiodic. (Note that for $t \leqslant 0$ such a situation can produce non-uniqueness of equilibrium states, see [25] and Section 7.)

1.1. Historical background

The principal examples of maps in $\mathcal{H}$ are unimodal maps with non-flat critical point. Equilibrium states (in particular of the potential φ_t) have been studied in this case by various authors [16, 22, 44, 7], using transfer operators. The transfer operator, in combination with Markov extensions (commonly known as Hofbauer towers), proved a powerful tool for so-called Collet-Eckmann unimodal maps (*i.e.,* the derivatives along the critical orbit grow exponentially, see (3) below) for Keller and Nowicki [22], who showed that an appropriately weighted version of the transfer operator is quasi-compact. To our knowledge, however, these methods cannot be applied to non-Collet-Eckmann maps.

A less direct approach was taken by Pesin and Senti, results which were announced in [34], with details given in preprint [33] and the final publication [35]. They used an inducing scheme (X, F, τ) (where τ is the inducing time and $F = f^\tau$), which is a hyperbolic expanding with full, albeit infinitely many, branches, to find a unique equilibrium state μ_{Φ_t} for the lifted potential Φ_t. This equilibrium state is then projected to the interval to give a measure μ_{φ_t}, a candidate equilibrium state for the system (I, f, φ_t). The down-side for the more general case is that μ_{φ_t} is only an 'equilibrium state' within the class of measures that are *compatible* to the inducing scheme, *i.e.,* the induced map $F = f^\tau$ is defined for all iterates μ-a.e. on X, and the inducing time τ is μ_F-integrable (here μ_F is the 'lift' of μ, see (4)). A priori, the 'equilibrium states' obtained in this way may not be true equilibrium states for the whole system, and different inducing schemes may lead to different measures μ_{φ_t}.

In this paper, with preprint versions since 2006, and in a companion paper [10], Hofbauer tower techniques are used to

- construct inducing schemes as first return maps on the Hofbauer tower;
- identify the class of compatible measures;
- compare various inducing schemes; and
- establish that candidate equilibrium states emerging from a single inducing scheme, indeed maximise free energy over all measures in $\mathcal{M}_+$.

In (versions leading up to) [35], identifying which measures are compatible to an inducing scheme is called the *liftability problem*. Most of the results in [35] apply only to measures compatible to a given inducing scheme. Only for specific unimodal maps, called *strongly regular* [35, Section 7.2], which are close to the Chebyshev polynomial and satisfy the Collet-Eckmann condition, is a genuine equilibrium state established. Strongly regular maps allow an inducing scheme (X, F) for which the number of branches X_i of inducing time $\tau_i = n$ increases at an arbitrarily slow exponential rate. This is used to show that measures with sufficiently large entropy are compatible to the inducing scheme, and hence that the obtained equilibrium state indeed maximises free energy over all of $\mathcal{M}_{erg}$.

Branch counting arguments for both Collet-Eckmann and non-Collet-Eckmann maps are given in Section 5 and especially Proposition 4 of this paper. Together with the Hofbauer tower ideas, this allows us to treat a much wider class of maps than [35]. On the other hand, for the strongly regular maps in [35], the control of the branch count for their specific inducing scheme enables Pesin and Senti to establish the existence and uniqueness of an equilibrium state μ_t for φ_t and t in a neighbourhood of $[0,1]$. Such a neighbourhood V is difficult to obtain for general interval maps where a priori there is no single inducing scheme to rely on for all $t \in V$. It should be noted that the results on general multimodal maps in [35, Section 8.1] apply only to Collet-Eckmann maps, as condition (23) of that paper shows, as well as only applying to measures compatible to the inducing scheme.

1.2. Main results

In our main theorems we will assume that f is *transitive*, *i.e.,* f has a dense orbit. If transitivity fails and instead the interval decomposes into finitely many transitive cycles of intervals, then our results remain valid for each transitive cycle, but uniqueness of equilibrium states may fail.

THEOREM 1. – *Let $f \in \mathcal{H}$ be transitive with negative Schwarzian derivative and let $\varphi_t := -t\log|Df|$ for $t \in \mathbb{R}$. Suppose that for some $C > 0$ and $\beta > 2\ell_{max} - 1$,*

$$|Df^n(f(c))| \geqslant Cn^\beta \quad \textit{for all } c \in \mathrm{Crit} \textit{ and } n \geqslant 1. \tag{1}$$

Then there exists $t_1 < 1$ such that the following hold for all $t \in (t_1, 1]$:

(a) (I, f, φ_t) *has an equilibrium state* $\mu_{\varphi_t} \in \mathcal{M}_+$*;*

(b) *if* $t_1 < t < 1$*, then* μ_{φ_t} *is the unique equilibrium state in* $\mathcal{M}_{erg}$ *and a compatible inducing scheme with respect to which* μ_{φ_t} *has exponential tails;*

(c) *if* $t = 1$*, then there may be other equilibrium states in* $\mathcal{M}_{erg} \setminus \mathcal{M}_+$*. However, for* $\mu_{\varphi_1} \in \mathcal{M}_+$ *there is a compatible inducing scheme with respect to which* μ_{φ_1} *has polynomial tails;*

(d) *the map* $t \mapsto P(\varphi_t)$ *is analytic on* $(t_1, 1)$.

We refer to this situation as the *summable case*. Note that for $t = 1$ the measure $\mu_{\varphi_1} \in \mathcal{M}_+$ is an absolutely continuous invariant measure (acip). Therefore this result improves on the polynomial case of [8, Proposition 4.1], since in that theorem the polynomial decay of the tails was given under the above conditions, but also assuming that the critical points must all have the same order. Results of [9] enable us to drop this assumption. As was shown in [8], this tail decay rate implies that the decay of correlations is at least polynomial.

As in the theorem, for $t = 1$ equilibrium states with zero Lyapunov exponent are possible, see Section 7 for details. Conversely the following easy lemma shows that for $t < 1$ this is not the case.

Lemma 1. – *For $f \in \mathcal{H}$ satisfying* (1) *and for $t < 1$, any equilibrium state μ for φ_t must have $\lambda(\mu) > 0$.*

Proof. – The pressure function $t \mapsto P(\varphi_t)$ is convex, continuous and non-increasing. As in [9], condition (1) implies the existence of an acip μ_1 with $\lambda(\mu_1) > 0$, which is also an equilibrium state for the potential $\varphi_1 = -\log|Df|$. It follows that

$$P(\varphi_t) \geqslant (1-t)\lambda(\mu_1) \qquad \text{for all } t \in \mathbb{R}, \tag{2}$$

so if $t < 1$ we have $P(\varphi_t) > 0$. By [36], we have $\lambda(\mu) \geqslant 0$ for any invariant measure, so Ruelle's inequality [38] implies that $h_\mu(f) \leqslant \lambda(\mu)$. Thus (for $t < 1$) equilibrium states have positive Lyapunov exponent because $\lambda(\mu) = 0$ implies $P(\varphi_t) = 0$. □

Notice that for $t \leqslant 0$, the potential $-t\log|Df|$ is upper semicontinuous, and the entropy function $\mu \mapsto h_\mu(f)$ is upper semicontinuous, as explained in [21]. This guarantees the existence of equilibrium states for (I, f) when $t \leqslant 0$, regardless of whether (1) holds or not.

A stronger condition than (1) is the *Collet-Eckmann condition* which states that there exist $C, \alpha > 0$ such that

$$|Df^n(f(c))| \geqslant Ce^{\alpha n} \text{ for all } c \in \mathrm{Crit} \text{ and } n \in \mathbb{N}. \tag{3}$$

This condition implies that $\lambda(\mu) > 0$ for every $\mu \in \mathcal{M}_{erg}$, see e.g. [32] (and [12] for the proof in the multimodal case). In the unimodal case, the difference between Collet-Eckmann and non-Collet-Eckmann maps can be seen from the behaviour of the pressure function at $t = 1$, as follows from [32]. Indeed, if (1) holds but not (3), then there are periodic orbits with Lyapunov exponents arbitrarily close to 0, and hence $P(\varphi_t) = 0$ for $t \geqslant 1$. This is regardless of the existence of equilibrium states, which, for $t > 1$, can only be measures for which $\lambda(\mu) = h_\mu(f) = 0$. This means that the function $t \mapsto P(\varphi_t)$ is not differentiable at $t = 1$: we