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## *On the group of real analytic diffeomorphisms*

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# ON THE GROUP OF REAL ANALYTIC DIFFEOMORPHISMS

BY TAKASHI TSUBOI

ABSTRACT. – The group of real analytic diffeomorphisms of a real analytic manifold is a rich group. It is dense in the group of smooth diffeomorphisms. Herman showed that for the $n$-dimensional torus, its identity component is a simple group. For $U(1)$ fibered manifolds, for manifolds admitting special semi-free $U(1)$ actions and for 2- or 3-dimensional manifolds with nontrivial $U(1)$ actions, we show that the identity component of the group of real analytic diffeomorphisms is a perfect group.

RÉSUMÉ. – Le groupe des difféomorphismes analytiques réels d'une variété analytique réelle est un groupe riche. Il est dense dans le groupe des difféomorphismes lisses. Herman a montré que, pour le tore de dimension $n$, sa composante connexe de l'identité est un groupe simple. Pour les variétés $U(1)$ fibrées, pour les variétés admettant une action semi-libre spéciale de $U(1)$, et pour les variétés de dimension 2 ou 3 admettant une action non-triviale de $U(1)$, on montre que la composante de l'identité du groupe des difféomorphismes analytiques réels est un groupe parfait.

## 1. Introduction and statement of the result

Let $\mathrm{Diff}^\omega(M)$ denote the group of real analytic diffeomorphisms of a real analytic manifold $M$. The group $\mathrm{Diff}^\omega(M)$ is an open subset of the space of real analytic maps $\mathrm{Map}^\omega(M,M)$ with the $C^1$ topology. The group $\mathrm{Diff}^\omega(M)$ with the $C^1$ topology has a manifold structure modelled on the space $\mathcal{X}^\omega(M)$ of real analytic vector fields on $M$. Hence $\mathrm{Diff}^\omega(M)$ is locally contractible (see Proposition 11.9). It is well-known that $\mathrm{Diff}^\omega(M)$ is dense in the group $\mathrm{Diff}^\infty(M)$ of smooth diffeomorphisms in the $C^1$ topology (See Corollary 11.8). Hence $\mathrm{Diff}^\omega(M)$ is a huge complicated group.

Let $\mathrm{Diff}^\omega(M)_0$ denote the identity component of $\mathrm{Diff}^\omega(M)$. For the $n$-dimensional torus $T^n$, Herman [10] in 1974 showed that $\mathrm{Diff}^\omega(T^n)_0$ is a simple group. For 30 years since then,

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there are no new results on the simplicity of the groups of real analytic diffeomorphisms. However, Herman conjectured and we may still conjecture that for any compact connected manifold $M$, the identity component $\mathrm{Diff}^\omega(M)_0$ of the group of real analytic diffeomorphisms is simple.

Now, in this paper, we change the question. If an infinite group is simple, then it is perfect. Hence we may ask a weaker question and may try to show the perfectness of the group of real analytic diffeomorphisms. Note that, in the case of the group of smooth diffeomorphisms, the perfectness implies the simplicity ([5], see also [2]), however, we cannot apply this argument to the group of real analytic diffeomorphisms.

For this question, our present results are as follows.

THEOREM 1.1. – *Let $M$ be a real analytically $U(1)$ fibered real analytic closed manifold. Then the identity component $\mathrm{Diff}^\omega(M)_0$ of the group of real analytic diffeomorphisms of $M$ is a perfect group.*

We consider other manifolds with well-understood $U(1)$ actions. Let $N$ be a compact $(n-1)$-dimensional manifold with boundary $\partial N$. Let $M$ be the $n$-dimensional manifold obtained from $N \times U(1)$ by identifying $\{x\} \times U(1)$ to a point for $x \in \partial N$. This $M$ has a real analytic structure with the obvious real analytic $U(1)$ action. We call this $U(1)$ action a special semi-free $U(1)$ action. Spheres and direct products with spheres admit special semi-free $U(1)$ actions.

THEOREM 1.2. – *Let $M$ be a real analytic manifold which admits a special semi-free $U(1)$ action. Then the identity component $\mathrm{Diff}^\omega(M)_0$ of the group of real analytic diffeomorphisms of $M$ is a perfect group.*

If the dimension of $M$ is 2 or 3, we can show the perfectness of $\mathrm{Diff}^\omega(M)_0$ if $M$ admits a nontrivial $U(1)$ action.

THEOREM 1.3. – *Let $M$ be a real analytic manifold of dimension 2 or 3 which admits a nontrivial $U(1)$ action. Then the identity component $\mathrm{Diff}^\omega(M)_0$ of the group of real analytic diffeomorphisms of $M$ is a perfect group.*

These theorems are shown in the following way.

First, we show the perfectness of the group of orbit preserving diffeomorphisms for the $U(1)$ bundles (Theorem 2.2) and a similar result for the orbit preserving diffeomorphisms for the manifolds admitting special semi-free $U(1)$ actions (Theorem 5.1). These theorems for orbit preserving diffeomorphisms are proved by using the famous Arnold theorem [1] for the Diophantine rotations and a similar Theorem 5.3 for the rotations of concentric circles, which we prove in Section 10. We also need certain explicit orbitwise actions of elements of $SL(2;\boldsymbol{R})$, and the existence of such nice actions gives the restriction to the $U(1)$ actions for which we can show our results by now.

To show our main theorems, we perturb the given $U(1)$ action by real analytic diffeomorphisms and obtain finitely many $U(1)$ actions such that the tangent space $T_xM^n$ of any point $x$ of the manifold $M^n$ is spanned by the generating vector fields of the resultant $U(1)$ actions.

For $n = \dim(M^n)$ and $U(1)$ actions generated by the vector fields $\xi_1, \ldots, \xi_n$, we have the determinant $\Delta = \det(\xi_{ij})$ with respect to an orthonormal frame $\dfrac{\partial}{\partial x_j}$ for a real analytic Riemannian metric on $M^n$, where $\xi_i = \displaystyle\sum_{j=1}^{n} \xi_{ij} \frac{\partial}{\partial x_j}$ $(i = 1, \ldots, n)$.

On the open set where $\Delta \neq 0$, a diffeomorphism sufficiently close to the identity can be written as a composition of orbit preserving diffeomorphisms. In fact, we show that for real analytic diffeomorphisms $f$ such that $f - \mathrm{id}$ are divisible by a certain power of $\Delta$, $f$ can be written as a composition of orbit preserving real analytic diffeomorphisms. This is done by an inverse mapping theorem for real analytic maps with singular Jacobians (Theorems 6.7 or 6.12).

Now we need to decompose a real analytic diffeomorphism sufficiently close to the identity as a composition of real analytic diffeomorphisms which satisfy the assumption of Theorems 6.7 or 6.12. This is done by using the regimentation Lemma 7.1, which replaces the fragmentation lemma ([2], [13]) for the smooth diffeomorphisms.

Then we use the perfectness of the group of orbit preserving diffeomorphisms of $U(1)$ bundles (Theorem 2.2) or a similar theorem (Theorem 5.1) for manifolds admitting special semi-free actions, to show our main theorems (Section 8).

Our method can treat the real analytic manifolds with a little more general $U(1)$ actions. We say that two elements of a group are homologous if they represent the same element in the abelianization of the group. In Section 9, we show that, if the manifold admits a nontrivial $U(1)$ action, any real analytic diffeomorphism isotopic to the identity is homologous to a diffeomorphism which is an orbitwise rotation (Proposition 9.1). Then we show Theorem 1.3 by showing Propositions 9.2, 9.3 and Theorem 9.4.

We think that $\mathrm{Diff}^{\omega}(M)_0$ is perfect if $M$ admits a nontrivial $U(1)$ action. But for the moment we need a structure theorem for the orbifold $M/U(1)$ in the construction of a nice multi-section outside of a codimension 2 suborbifold to show that orbitwise rotations are homologous to zero.

## 2. Orbit preserving diffeomorphisms of $U(1)$ bundles

As we mentioned, for the $n$-dimensional torus $T^n$, Herman [10] in 1974 noticed that the result of Arnold [1] implies $\mathrm{Diff}^{\omega}(T^n)_0$ is a simple group. Hence it is perfect.

We note that Herman's proof ([10]) uses the fact that the commutator subgroup $[\mathrm{Diff}^{\omega}(T^n)_0, \mathrm{Diff}^{\omega}(T^n)_0]$ of $\mathrm{Diff}^{\omega}(T^n)_0$ is its dense subgroup. In fact, for the group $\mathrm{Diff}^{\infty}(M)$ of $C^{\infty}$ diffeomorphisms of a smooth manifold $M$, its identity component $\mathrm{Diff}^{\infty}(M)_0$ is perfect by the result of Thurston ([20], [2]). Since $\mathrm{Diff}^{\omega}(M)$ is dense in $\mathrm{Diff}^{\infty}(M)$, the commutator subgroup $[\mathrm{Diff}^{\omega}(M)_0, \mathrm{Diff}^{\omega}(M)_0]$ is dense in $\mathrm{Diff}^{\omega}(M)_0$.

For the real analytic diffeomorphisms of $T^n$, Arnold [1] already noticed the followings.

THEOREM 2.1 (Arnold[1]). – *Let $\alpha \in \boldsymbol{R}^n$ satisfy the Diophantine condition. For a real analytic family $\Phi(w)$ $(w \in W)$ of analytic diffeomorphisms of $T^n$ close to the identity, there is an analytic family $(\psi(w), \lambda(w)) \in \mathrm{Diff}^{\omega}(T^n)_0 \times T^n$ such that*

$$\Phi(w) = R_{\lambda(w)-\alpha} \circ \psi(w) \circ R_{\alpha} \circ \psi(w)^{-1},$$

*where $R_*$ denotes the rotation by $*$ on $T^n = \boldsymbol{R}^n/\boldsymbol{Z}^n$.*

Here a real vector $\alpha \in \boldsymbol{R}^n$ is said to satisfy the Diophantine condition if there exist positive real numbers $C$ and $\beta$ such that $|\alpha \bullet \ell - m| \geq C\|\ell\|^{-\beta}$ for any $\ell \in \boldsymbol{Z}^n \setminus \{0\}$ and $m \in \boldsymbol{Z}$.

Since $R_\lambda$ can be written as a commutator in

$$PSL(2;\boldsymbol{R})^n = PSL(2;\boldsymbol{R}) \times \cdots \times PSL(2;\boldsymbol{R})$$

depending real analytically on $\lambda$, $\Phi(w)$ can be written as a product of 2 commutators depending real analytically on $w \in W$.

This means that for a compact manifold $N$ and the product $N \times T^n$ with the product foliation $\mathcal{F} = (\{*\} \times T^n)$ $(* \in N)$, the group $\mathrm{Diff}^\omega(\mathcal{F})_0$ is a perfect group, where $\mathrm{Diff}^\omega(\mathcal{F})$ denotes the group of real analytic diffeomorphisms mapping each fiber of the projection $N \times T^n \longrightarrow N$ to itself and the subscript $_0$ denotes the identity component.

We first generalize the perfectness result for the group of orbit preserving diffeomorphisms of a $U(1)$ bundle.

THEOREM 2.2. – *Let $p : M \longrightarrow B$ be a real analytic principal $U(1)$ bundle over a closed manifold $B$. Let $\mathrm{Diff}^\omega(\mathcal{F})$ denote the group of real analytic diffeomorphisms mapping each fiber of the projection $p : M \longrightarrow B$ to itself. The identity component $\mathrm{Diff}^\omega(\mathcal{F})_0$ of $\mathrm{Diff}^\omega(\mathcal{F})$ is a perfect group.*

## 3. Proof of Theorem 2.2

*Proof of Theorem 2.2 for trivial $U(1)$ bundles.* – Theorem 2.2 for the trivial $U(1)$ bundle is just a reformation of Arnold's Theorem 2.1. In this case, $M = B \times U(1)$ and $\mathcal{F} = (\{*\} \times U(1))_{*\in B}$. An element of $\mathrm{Diff}^\omega(\mathcal{F})_0$ is written as the real analytic family $\Phi(w)$ $(w \in B)$ of real analytic diffeomorphism of $U(1)$. It is enough to show that $\Phi(w)$ near the identity can be written as a product of commutators.

Take a Diophantine rotation $R_\alpha$ in the direction of the fibers of the $U(1)$ bundle. The element $\Phi(w)$ near the identity is written as

$$\Phi(w) = R_{\lambda(w)-\alpha} \circ \psi(w) \circ R_\alpha \circ \psi(w)^{-1}.$$

Here, $\lambda(w)$ is determined uniquely by the condition that the rotation number of $R_{-\lambda(w)} \circ (R_\alpha \circ \Phi(w))$ coincides with that of $R_\alpha$, $\alpha$ mod 1. In the proof in [1] of Arnold's Theorem 2.1, the conjugating diffeomorphism $\psi(w)$ is obtained uniquely so that the base point $1 \in U(1)$ is fixed ($\psi(w)(1) = 1$). Thus $\psi(w)$ $(w \in B)$ is a real analytic family of real analytic diffeomorphisms. In the expression $\Phi(w) = R_{\lambda(w)-\alpha} \circ \psi(w) \circ R_\alpha \circ \psi(w)^{-1}$, $R_{\lambda(w)}$ can be written as a product of two commutators in $\mathrm{Map}^\omega(B, SL(2,\boldsymbol{R}))$ by the following Lemma 3.1. Thus Theorem 2.2 for trivial $U(1)$ bundles is shown. □

LEMMA 3.1. – *A rotation $\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$ $(X^2 + Y^2 = 1)$ close to the identity can be written as a product of 2 commutators using products of rotations and diagonal matrices.*