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DELIGNE-LUSZTIG RESTRICTION OF A GELFAND-GRAEV MODULE

BY OLIVIER DUDAS

ABSTRACT. – Using Deodhar's decomposition of a double Schubert cell, we study the regular representations of finite groups of Lie type arising in the cohomology of Deligne-Lusztig varieties associated to tori. We deduce that the Deligne-Lusztig restriction of a Gelfand-Graev module is a shifted Gelfand-Graev module.

RÉSUMÉ. – À l'aide de la décomposition de Deodhar d'une double cellule de Schubert, nous étudions les représentations régulières des groupes finis de type de Lie apparaissant dans la cohomologie des variétés de Deligne-Lusztig associées à des tores. Nous en déduisons que la restriction de Deligne-Lusztig d'un module de Gelfand-Graev est un module de Gelfand-Graev décalé.

Introduction

Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraic closure $\mathbb{F}$ of a finite field of characteristic p. Let F be an isogeny of $\mathbf{G}$ such that some power is a Frobenius endomorphism. The finite group $G = \mathbf{G}^F$ of fixed points under F is called a finite group of Lie type. We fix a maximal torus $\mathbf{T}$ contained in a Borel subgroup $\mathbf{B}$ with unipotent radical $\mathbf{U}$, all of which assumed to be F-stable. The corresponding Weyl group will be denoted by W.

In an attempt to have a complete understanding of the character theory of G, Deligne and Lusztig have introduced in [7] a family of biadjoint morphisms R_w and ${}^*\mathrm{R}_w$ indexed by W, leading to an outstanding theory of induction and restriction between G and any of its maximal tori. Roughly speaking, they encode, into a virtual character, the different representations occurring in the cohomology of the corresponding Deligne-Lusztig variety. Unfortunately, the same construction does not give enough information in the modular setting, and one has to work at a higher level. More precisely, for a finite extension Λ of the ring $\mathbb{Z}_\ell$ of ℓ-adic integers, Bonnafé and Rouquier have defined in [4] the following functors (see §1.3 for the notation):

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$$\mathcal{R}_{\dot w} \,:\, \mathcal{D}^b(\Lambda\mathbf{T}^{wF}\text{-mod}) \longrightarrow \mathcal{D}^b(\Lambda G\text{-mod})$$

and

$$^*\mathcal{R}_{\dot w} \,:\, \mathcal{D}^b(\Lambda G\text{-mod}) \longrightarrow \mathcal{D}^b(\Lambda\mathbf{T}^{wF}\text{-mod})$$

between the derived categories of modules, which generalize the definition of Deligne-Lusztig induction and restriction.

In this article we study the action of the restriction functor on a special class of representations: the Gelfand-Graev modules, which are projective modules parametrized by the G-regular characters of U (see §1.4). More precisely, we prove in Section 3 the following result:

THEOREM. – *Let $\psi \,:\, U \longrightarrow \Lambda^\times$ be a G-regular linear character, and denote by Γ_ψ the associated Gelfand-Graev module of G. Then, for any w in W, one has*

$$^*\mathcal{R}_{\dot w}\Gamma_\psi \,\simeq\, \Lambda\mathbf{T}^{wF}[-\ell(w)]$$

in the derived category $\mathcal{D}^b(\Lambda\mathbf{T}^{wF}\text{-mod})$.

This result was already known for some specific elements of the Weyl group. In the case where w is the trivial element, the Deligne-Lusztig functor $^*\mathcal{R}_{\dot w}$ comes from a functor defined at the level of module categories, and the result can be proved in a completely algebraic setting (see [6, Proposition 8.1.6]). But more interesting is the case of a Coxeter element, studied by Bonnafé and Rouquier in [5]. Their proof relies on the following geometric properties for the corresponding Deligne-Lusztig variety $\mathrm{X}(w)$ (see [17]):

($\star$)
- $\mathrm{X}(w)$ is contained in the maximal Schubert cell $\mathbf{B}w_0\mathbf{B}/\mathbf{B}$;
- the quotient variety $U\backslash\mathrm{X}(w)$ is a product of $\mathbf{G}_m$'s.

Obviously, one cannot expect these properties to hold for any element w of the Weyl group (for instance, the variety $\mathrm{X}(1)$ is a finite set of points whose intersection with any F-stable Schubert cell is non-trivial). However, it turns out that for the specific class of representations we are looking at, we can restrict our study to a smaller variety which will be somehow a good substitute for $\mathrm{X}(w)$.

Let us give some consequences of this theorem, which are already known but can be deduced in an elementary way from our result. From the quasi-isomorphism one can first obtain a canonical algebra homomorphism from the endomorphism algebra of a Gelfand-Graev module to the algebra $\Lambda\mathbf{T}^{wF}$. Tensoring by the fraction field K of Λ, it can be shown that we obtain the Curtis homomorphism ${}_K\mathrm{Cur}_w : \mathrm{End}_{KG}(K\Gamma_\psi) \longrightarrow K\mathbf{T}^{wF}$, thus giving a modular and conceptual version of this morphism (see [3, Theorem 2.7]).

The character-theoretic version of the theorem is obtained in a drastic way, by tensoring the quasi-isomorphism by K and by looking at the induced equality in the Grothendieck group of the category of $\Lambda\mathbf{T}^{wF}$-modules. Applying the Alvis-Curtis duality gives then a new method for computing the values of the Green functions at a regular unipotent element (see [7, Theorem 9.16]). This is the key step for showing that a Gelfand-Graev character has a unique irreducible component in each rational series $\mathcal{E}(G,(s)_{\mathbf{G}^{*F^*}})$.

Beyond these applications, our approach aims at understanding each of the cohomology groups of the Deligne-Lusztig varieties, leading to concentration and disjointness properties, in the spirit of Broué's conjectures. For example, by truncating by unipotent characters, one can deduce that the Steinberg character is concentrated in the cohomology group in middle degree. This result was already proved in [10, Proposition 3.3.15], by a completely different method, since their proof relies on the computation of eigenvalues of Frobenius. By refining our method, we should be able to deal with some other unipotent characters and enlarge the scope of our result.

This paper is divided into three parts: in the first section, we introduce the basic notations about the modular representation theory of finite groups of Lie type. Then, we focus on an extremely rich decomposition for double Schubert cells, introduced by Deodhar in [9]. To this end, we shall use the point of view of [18] and the Bialynicki-Birula decomposition, since it is particularly adapted to our case. This is the crucial ingredient for proving the main theorem. Indeed, we show in the last section that the maximal piece of the induced decomposition on $\mathrm{X}(w)$ satisfies the properties $(\star)$, and that it is the only one carrying regular characters in its cohomology.

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1. Preliminaries

1.1. Cohomology of a quasi-projective variety

Let Λ be a commutative ring and H a finite group. We denote by ΛH-mod the abelian category of finitely generated ΛH-modules, and by $\mathcal{D}^b(\Lambda H\text{-mod})$ the derived category of the corresponding bounded complexes. From now on, we assume that Λ is a finite extension of the ring $\mathbb{Z}_\ell$ of ℓ-adic integers, for a prime ℓ different from p. To any quasi-projective variety X defined over $\mathbb{F}$ and acted on by H, one can associate a classical object in this category, namely the cohomology with compact support of X, denoted by $\mathrm{R}\Gamma_c(\mathrm{X},\Lambda)$. It is quasi-isomorphic to a bounded complex of modules which have finite rank over Λ.

We give here some quasi-isomorphisms we shall use in Section 3. The reader will find references or proofs of these properties in [4, Section 3] and [7, Proposition 6.4] for the third assertion. The last one can be deduced from [13, Exposé XVIII, 2.9].

Proposition 1.1. – *Let X and Y be two quasi-projective varieties acted on by H. Then one has the following isomorphisms in the derived category $\mathcal{D}^b(\Lambda H\text{-mod})$:*

(i) *The Künneth formula:*

$$\mathrm{R}\Gamma_c(\mathrm{X}\times\mathrm{Y},\Lambda) \simeq \mathrm{R}\Gamma_c(\mathrm{X},\Lambda) \overset{\mathrm{L}}{\otimes} \mathrm{R}\Gamma_c(\mathrm{Y},\Lambda)$$

where $\overset{\mathrm{L}}{\otimes}$ denotes the left-derived functor of the tensor product over Λ.

(ii) *The quotient variety* $H\backslash \mathrm{X}$ *exists. Moreover, if the order of the stabilizer of any point of* X *is prime to* ℓ*, then*

$$\mathrm{R}\Gamma_c(H\backslash \mathrm{X},\Lambda) \simeq \Lambda \overset{\mathrm{L}}{\otimes}_{\Lambda H} \mathrm{R}\Gamma_c(\mathrm{X},\Lambda).$$

(iii) *If the action of* H *on* X *is the restriction of an action of a connected group and if the order of the stabilizer of any point of* X *in* H *is prime to* ℓ*, then*

$$\mathrm{R}\Gamma_c(\mathrm{X},\Lambda) \simeq \Lambda \overset{\mathrm{L}}{\otimes}_{\Lambda H} \mathrm{R}\Gamma_c(\mathrm{X},\Lambda).$$

(iv) *Let* $\pi : Y \longrightarrow X$ *be an* H*-equivariant smooth morphism of finite type. If the fibers of* π *are isomorphic to affine spaces of constant dimension* n*, then*

$$\mathrm{R}\Gamma_c(\mathrm{Y},\Lambda) \simeq \mathrm{R}\Gamma_c(\mathrm{X},\Lambda)[-2n].$$

If N is a finite group acting on X on the right and on Y on the left, we can form the amalgamated product $\mathrm{X}\times_N \mathrm{Y}$, as the quotient of $\mathrm{X}\times\mathrm{Y}$ by the diagonal action of N. Assume that the actions of H and N commute and that the order of the stabilizer of any point for the diagonal action of N is prime to ℓ. Then $\mathrm{X}\times_N \mathrm{Y}$ is an H-variety and we deduce from the above properties that

$$\mathrm{R}\Gamma_c(\mathrm{X}\times_N \mathrm{Y},\Lambda) \simeq \mathrm{R}\Gamma_c(\mathrm{X},\Lambda) \overset{\mathrm{L}}{\otimes}_{\Lambda N} \mathrm{R}\Gamma_c(\mathrm{Y},\Lambda) \tag{1}$$

in the derived category $\mathcal{D}^b(\Lambda H\text{-mod})$.

1.2. Algebraic groups

We keep the basic assumptions of the introduction: $\mathbf{G}$ is a connected reductive algebraic group, together with an isogeny F such that some power is a Frobenius endomorphism. In other words, there exists a positive integer δ such that F^δ defines a split $\mathbb{F}_q$-structure on $\mathbf{G}$ for a certain power q of the characteristic p. For any F-stable algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$, we will denote by H the finite group of fixed points $\mathbf{H}^F$.

We fix a Borel subgroup $\mathbf{B}$ containing a maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that both $\mathbf{B}$ and $\mathbf{T}$ are F-stable. They define a root sytem Φ with basis Δ, and a set of positive (resp. negative) roots Φ^+ (resp. Φ^-). Note that the corresponding Weyl group W is endowed with an action of F, compatible with the isomorphism $W \simeq N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Therefore, the image by F of a root is a positive multiple of some other root, which will be denoted by $\phi^{-1}(\alpha)$, defining thus a bijection $\phi : \Phi \longrightarrow \Phi$. Since $\mathbf{B}$ is also F-stable, this map preserves Δ and Φ^+. We will also use the notation $[\Delta/\phi]$ for a set of representatives of the orbits of ϕ on Δ.

Let $\mathbf{U}$ (resp. $\mathbf{U}^-$) be the unipotent radical of $\mathbf{B}$ (resp. the opposite Borel subgroup $\mathbf{B}^-$). For any root α, we denote by $\mathbf{U}_\alpha$ the corresponding one-parameter subgroup and $u_\alpha : \mathbb{F} \longrightarrow \mathbf{U}_\alpha$ an isomorphism of algebraic groups. Note that the groups $\mathbf{U}$ and $\mathbf{U}^-$ are F-stable whereas $\mathbf{U}_\alpha$ might not be. However, we may, and we will, choose the family $(u_\alpha)_{\alpha\in\Phi}$ such that the restriction to $\mathbf{U}_\alpha$ of the action of F satisfies $F(u_\alpha(\zeta)) = u_{\phi(\alpha)}(\zeta^{q_\alpha^\circ})$ where q_α° is some power of p defined by the relation $F(\phi(\alpha)) = q_\alpha^\circ\,\alpha$. We define d_α to be the length of the orbit of α under the action of ϕ and we set $q_\alpha = q_\alpha^\circ q_{\phi(\alpha)}^\circ \cdots q_{\phi^{d_\alpha-1}(\alpha)}^\circ$. Then $\mathbf{U}_\alpha$ is stable by F^{d_α} and $\mathbf{U}_\alpha^{F^{d_\alpha}} \simeq \mathbb{F}_{q_\alpha}$.

Let us consider the derived group $D(\mathbf{U})$ of $\mathbf{U}$. For any total order on Φ^+, the product map induces the following isomorphism of varieties: