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NON-ORBIT EQUIVALENT ACTIONS OF $\mathbb{F}_n$

BY ADRIAN IOANA

ABSTRACT. – For any $2 \leq n \leq \infty$, we construct a concrete 1-parameter family of non-orbit equivalent actions of the free group $\mathbb{F}_n$. These actions arise as diagonal products between a generalized Bernoulli action and the action $\mathbb{F}_n \curvearrowright (\mathbb{T}^2, \lambda^2)$, where $\mathbb{F}_n$ is seen as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

RÉSUMÉ. – Pour tout $2 \leq n \leq \infty$, nous construisons une famille concrète à un paramètre, des actions non orbitalement équivalentes du groupe libre $\mathbb{F}_n$. Ces actions apparaissent comme produits diagonaux entre une action généralisée de Bernoulli et l'action $\mathbb{F}_n \curvearrowright (\mathbb{T}^2, \lambda^2)$, où $\mathbb{F}_n$ est vu comme un sous-groupe de $\mathrm{SL}_2(\mathbb{Z})$.

Introduction

Recall that two free ergodic measure preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ of two countable discrete groups Γ and Λ on two standard probability spaces X and Y are said to be *orbit equivalent* if there exists a probability space isomorphism $\theta : X \to Y$ such that $\theta(\Gamma x) = \Lambda\theta(x)$, for μ-almost every $x \in X$.

The orbit equivalence theory of measure preserving group actions has been an extremely active area in the past decade. New, spectacular rigidity results have been generated using tools ranging from ergodic theory and operator algebras to representation theory (see the surveys [8, 29, 33]). Recently, the problem of finding many non-orbit equivalent actions of a fixed non-amenable group Γ has attracted a lot of attention.

This question arose in the 1980's when it was shown that any infinite amenable group Γ has exactly one free ergodic measure preserving action, up to orbit equivalence—a result proved by Dye in the case Γ is abelian ([5]) and by Ornstein-Weiss in general ([22], see [3] for a generalization)—while some non-amenable groups (e.g. $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$) have uncountably many non-orbit equivalent actions ([1, 12, 34]). In recent years, several classes of non-amenable groups have been shown to share this property: property (T) groups ([13]), weakly rigid groups ([26]), non-amenable products of infinite groups ([30], see also [15, 20]) and mapping class groups ([19]).

In the case of the free groups, progress was slow for a while, only 4 non-orbit equivalent actions of $\mathbb{F}_n$–all concrete–being known in 2002 ([4, 13, 25]), before Gaboriau-Popa eventually proved the existence of uncountably many such actions ([11]). The key idea of their approach was to use the fact that the action of $\mathrm{SL}_2(\mathbb{Z})$ (as well as its restriction to any free subgroup $\mathbb{F}_n$) on the 2-torus $\mathbb{T}^2$ is rigid, in the sense of Popa ([25]).

However, since Gaboriau-Popa's proof also uses a separability argument, it only provides an existence result, leaving open the problem of finding specific actions of $\mathbb{F}_n$, which are not orbit equivalent. This problem has been emphasized in [27, Section 6], where two further examples were produced, raising the number of concrete non-orbit equivalent actions of $\mathbb{F}_n$ to 6.

The main result of this paper is the following:

THEOREM. – *Let $2 \leq n \leq \infty$. Fix an embedding $\mathbb{F}_n \subset SL_2(\mathbb{Z})$ and a surjective homomorphism $\pi : \mathbb{F}_n \to \mathbb{Z}$. Denote by σ the restriction of the natural action $SL_2(\mathbb{Z}) \curvearrowright (\mathbb{T}^2, \lambda^2)$ to $\mathbb{F}_n$, where λ^2 is the Haar measure on $\mathbb{T}^2$. For every $t \in (0,1)$, define the probability space $(X_t, \mu_t) = (\{0,1\}, r_t)^{\mathbb{Z}}$, where $r_t(\{0\}) = t, r_t(\{1\}) = 1-t$, and let β_t be the Bernoulli action of $\mathbb{Z}$ on (X_t, μ_t).*

Let α_t denote the diagonal product action of $\mathbb{F}_n$ on $(X_t \times \mathbb{T}^2, \mu_t \times \lambda^2)$ given by

$$\alpha_t(\gamma) = \beta_t(\pi(\gamma)) \times \sigma(\gamma), \forall \gamma \in \mathbb{F}_n.$$

Then $\{\alpha_t\}_{t \in (0,\frac{1}{2}]}$ is a 1-parameter family of free ergodic non-orbit equivalent actions of $\mathbb{F}_n$.

To put our main result in a better perspective, note that most non-amenable groups for which concrete uncountable families of non-orbit equivalent actions have been constructed admit in fact many actions which are *orbit equivalent superrigid*, i.e. such that their orbit equivalence class remembers the group and the action. Indeed, this is the case for weakly rigid groups ([26, 28]), non-amenable products of infinite groups ([30]) and mapping class groups ([19]). For the free groups, such an extreme rigidity phenomenon never occurs. On the contrary, any free ergodic action of $\mathbb{F}_n$ is orbit equivalent to actions of uncountably many non-isomorphic groups (see 2.27 in [20]).

The proof of the theorem has two main parts which we now briefly outline. Assume therefore that $\theta = (\theta_1, \theta_2) : X_s \times \mathbb{T}^2 \to X_t \times \mathbb{T}^2$ is an orbit equivalence between α_s and α_t, for some $s < t \in (0, \frac{1}{2}]$. First we prove that θ_i "locally" (i.e. on a set $A_i \subset X_s \times \mathbb{T}^2$ of positive measure) depends only on the i-th coordinate, for $i \in \{1,2\}$. This is achieved by playing against each other contrasting properties of the actions β_s and σ. Thus, for $i = 1$ we use that β_t is an action of an amenable group, while σ is strongly ergodic (see Lemma 2.2 and Proposition 2.3) and for $i = 2$, we use that β_s is a Bernoulli action, whereas σ is rigid (see Proposition 3.1).

For the second part, assume for simplicity that θ_i depends only on the i-th coordinate (i.e. A_i has full measure), for $i \in \{1,2\}$. Letting $w : \mathbb{F}_n \times (X_s \times \mathbb{T}^2) \to \mathbb{F}_n$ be the cocycle associated with θ, it follows that $\chi = \pi \circ w$ depends only on the $\mathbb{F}_n$-coordinate. Thus, χ is a homomorphism $\mathbb{F}_n \to \mathbb{Z}$ which satisfies $\theta_1(\gamma x) = \chi(\gamma)\theta_1(x)$, for all $\gamma \in \mathbb{F}_n$ and almost every $x \in X_s$. This is further used to prove that β_s is isomorphic to the restriction $\beta_{t|m\mathbb{Z}}$, for some $m \geq 1$. In the general case, we first show that after multiplying θ with a $\mathbb{F}_n$-valued function one can assume that θ_i depends only on the i-th coordinate and then proceed as above. This argument, applied to a more general situation, is the subject of Section 4. Finally, a simple application of entropy gives that $s \geq t$, a contradiction.

Note that our main result holds for any non-amenable group Γ which admits both an infinite amenable quotient Δ that has no non-trivial finite normal subgroup and a free, weakly mixing, strongly ergodic, rigid action $\Gamma \curvearrowright (Y, \nu)$ (Theorem 5.1).

Recently, a combination of results and ideas from [14], [10] and [6] has led to a complete quantitative answer to the problem motivating this paper: any non-amenable group Γ admits uncountably many free ergodic non-orbit equivalent actions ([6]). Note, however, that the question of finding explicit such actions for an arbitrary Γ is still open.

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1. Preliminaries

In this section we review some of the notions and results that we will later use. All groups Γ that we consider hereafter are countable discrete, all probability spaces (X, μ) are standard (unless specified otherwise) and all actions $\Gamma \curvearrowright (X, \mu)$ are measure preserving.

1.1. Orbit equivalence and cocycles

Assume that $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are two free orbit equivalent actions. Let $\theta : X \to Y$ be an orbit equivalence, i.e. a probability space isomorphism such that $\theta(\Gamma x) = \Lambda\theta(x)$, for μ-almost every (a.e.) $x \in X$. For every $\gamma \in \Gamma$ and $x \in X$, denote by $w(\gamma, x)$ the unique (by freeness) element of Λ such that $\theta(\gamma x) = w(\gamma, x)\theta(x)$. The map $w : \Gamma \times X \to \Lambda$ is measurable, satisfies

$$w(\gamma_1\gamma_2, x) = w(\gamma_1, \gamma_2 x)w(\gamma_2, x),$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $x \in X$, and is called the *Zimmer cocycle* associated with θ. In general, a measurable map $w : \Gamma \times X \to \Lambda$ verifying the above relation is called a *cocycle*. Two cocycles $w_1, w_2 : \Gamma \times X \to \Lambda$ are said to be *cohomologous* (in symbols, $w_1 \sim w_2$) if there exists a measurable map $\phi : X \to \Lambda$ such that $w_1(\gamma, x) = \phi(\gamma x)w_2(\gamma, x)\phi(x)^{-1}$, for all $\gamma \in \Gamma$ and a.e. $x \in X$.

The simplest instance when two actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are orbit equivalent is when they are *conjugate*, i.e. there exist a probability space isomorphism $\theta : X \to Y$ and a group isomorphism $\delta : \Gamma \to \Lambda$ such that $\theta(\gamma x) = \delta(\gamma)\theta(x)$, for all $\gamma \in \Gamma$ and a.e. $x \in X$. Moreover, if $\Gamma = \Lambda$ and δ is the trivial isomorphism, then we say that the Γ-actions on X and Y are *isomorphic.* Much of orbit equivalence rigidity theory aims at proving that, for certain classes of actions, orbit equivalence implies conjugacy. In doing so, the analysis of the associated Zimmer cocycle plays an important role. For example, a general principle proved in [28, Proposition 5.11] asserts that if the Zimmer cocycle associated with an orbit equivalence between two weakly mixing actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ is cohomologous to a group homomorphism $\delta : \Gamma \to \Lambda$, then the actions must be (virtually) conjugate.

It is thus very useful to have a criterion for a cocycle to be cohomologous to a group homomorphism. The following theorem, due to S. Popa (see [28, Theorem 3.1]), provides such a criterion. Before stating it, recall that an action $\Gamma \curvearrowright (X,\mu)$ is called *weakly mixing* if for every finite collection of measurable sets $A_1, A_2, \dots, A_n \subset X$ and every $\varepsilon > 0$, we can find $\gamma \in \Gamma$ such that $|\mu(A_i \cap \gamma A_j) - \mu(A_i)\mu(A_j)| \leq \varepsilon$, for all $i, j \in \{1,\dots,n\}$. Also, the action $\Gamma \curvearrowright (X,\mu)$ is called *mixing* if for every measurable sets $A_1, A_2 \subset X$ we have that $\lim_{\gamma\to\infty} |\mu(A_1 \cap \gamma A_2) - \mu(A_1)\mu(A_2)| = 0$.

THEOREM 1.1 ([28]). – *Let $\Gamma \curvearrowright (X,\mu)$ be a weakly mixing action and let $\Gamma \curvearrowright (Y,\nu)$ be another action. Let Λ be a countable group and let $w : \Gamma \times (X \times Y) \to \Lambda$ be a cocycle for the diagonal product action of Γ on $X \times Y$. Denote by $w^l, w^r : \Gamma \times (X \times X \times Y) \to \Lambda$ the cocycles for the diagonal product action $\Gamma \curvearrowright X \times X \times Y$ given by $w^l(\gamma, x_1, x_2, y) = w(\gamma, x_1, y)$ and $w^r(\gamma, x_1, x_2, y) = w(\gamma, x_2, y)$, for all $\gamma \in \Gamma, x_1, x_2 \in X$ and $y \in Y$.*

If $w^l \sim w^r$, then w is cohomologous to a cocycle which is independent on the X-variable.

1.2. The group measure space construction

Let $\Gamma \curvearrowright (X,\mu)$ be a measure preserving action of a countable group Γ on a standard probability space (X,μ). Let $\mathcal{H} = L^2(X,\mu)\overline{\otimes}\ell^2\Gamma$. For every $\gamma \in \Gamma$ and $f \in L^\infty(X,\mu)$, define the operators $u_\gamma, L_f \in \mathbb{B}(\mathcal{H})$ by

$$u_\gamma(g \otimes \delta_{\gamma'}) = \gamma(g) \otimes \delta_{\gamma\gamma'},$$
$$L_f(g \otimes \delta_{\gamma'}) = fg \otimes \delta_{\gamma'}, \forall \gamma' \in \Gamma, \forall g \in L^2(X,\mu),$$

where, as usual, $\gamma(g) = g \circ \gamma^{-1}$. Since $u_\gamma u_{\gamma'} = u_{\gamma\gamma'}, u_\gamma L_f u_\gamma^* = L_{\gamma(f)}$, for all $\gamma, \gamma' \in \Gamma$ and $f \in L^\infty(X,\mu)$, the linear span of $\{L_f u_\gamma | f \in L^\infty(X,\mu), \gamma \in \Gamma\}$ is a $*$-subalgebra of $\mathbb{B}(\mathcal{H})$. The strong operator closure of this algebra, denoted $L^\infty(X,\mu) \rtimes \Gamma$, is called the *group measure space* von Neumann algebra associated with the action $\Gamma \curvearrowright (X,\mu)$ ([21]). The vector state $\tau(y) = \langle y(1 \otimes \delta_e), 1 \otimes \delta_e \rangle$ gives a normal faithful trace on $L^\infty(X,\mu) \rtimes \Gamma$, which is therefore a finite von Neumann algebra. Furthermore, if the action $\Gamma \curvearrowright (X,\mu)$ is free and ergodic, then $L^\infty(X,\mu) \rtimes \Gamma$ is a II_1 factor and $L^\infty(X,\mu)$ is a Cartan subalgebra, i.e. maximal abelian and regular.

Following [7], two free ergodic measure preserving actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are orbit equivalent if and only if the corresponding Cartan subalgebra inclusions are isomorphic, i.e.

$$(L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma) \simeq (L^\infty(Y,\nu) \subset L^\infty(Y,\nu) \rtimes \Lambda).$$

Moreover, if $\theta : X \to Y$ is an orbit equivalence between the actions, then the induced isomorphism of abelian von Neumann algebras $\theta^* : L^\infty(Y,\nu) \ni f \to f \circ \theta \in L^\infty(X,\mu)$ extends to an isomorphism $\theta^* : L^\infty(Y,\nu) \rtimes \Lambda \to L^\infty(X,\mu) \rtimes \Gamma$. We next note that a more general statement of this type is true. Recall first that a measurable map $q : X \to Y$ between two probability spaces (X,μ) and (Y,ν) is called a *quotient map* if it is measure preserving and onto. In this case, the map $q^* : L^\infty(Y,\nu) \ni f \to f \circ q \in L^\infty(X,\mu)$ is an embedding of abelian von Neumann algebras.