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DUALITY OF SCHRAMM-LOEWNER EVOLUTIONS

BY JULIEN DUBÉDAT

ABSTRACT. – In this note, we prove a version of the conjectured duality for Schramm-Loewner Evolutions, by establishing exact identities in distribution between some boundary arcs of chordal SLE_κ, $\kappa > 4$, and appropriate versions of $\mathrm{SLE}_{\hat\kappa}$, $\hat\kappa = 16/\kappa$.

RÉSUMÉ. – On démontre dans cette note une version de la dualité conjecturée pour les évolutions de Schramm-Loewner, en établissant des identités en distribution exactes entre certains arcs de SLE_κ chordal, $\kappa > 4$, et des versions appropriées de $\mathrm{SLE}_{\hat\kappa}$, $\hat\kappa = 16/\kappa$.

1. Introduction

Schramm-Loewner Evolutions (or SLE), introduced by Schramm in 1999, are probability distributions, parameterized by $\kappa > 0$, on non-self traversing curves (the trace) connecting two boundary points in a planar, simply connected domain. They are characterized by a conformal invariance condition and a domain Markov property. See [7, 12, 16] for general SLE background.

The geometric properties of the trace vary with the parameter κ. In particular, when $\kappa \leq 4$, the trace is a.s. a simple curve; this is no longer the case if $\kappa > 4$ ([12]). The trace stopped at some finite time is then distinct from its boundary. The duality conjecture for SLE, roughly stated, is that a boundary arc of SLE_κ is locally absolutely continuous w.r.t. to (some version) of $\mathrm{SLE}_{\hat\kappa}$, $\hat\kappa = 16/\kappa$. This was suggested by Duplantier. In the case $(\kappa, \hat\kappa) = (8, 2)$, this follows from the exact combinatorial relation between Loop-Erased Random Walks and Uniform Spanning Trees and the identification of their scaling limits in terms of SLE ([8]). In the case $(\kappa, \hat\kappa) = (6, 8/3)$, it follows from the locality/restriction framework ([6]). An approach based on a relation with the free field has been proposed by Sheffield. A precise duality conjecture is stated in [1] and elaborated on in [3]; we prove slightly different versions here.

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In [3], it is shown that duality shares common features with reversibility and the question of defining multiple SLE strands in a common domain. This *local commutation* property states that two SLE strands can be grown in a domain to a positive size, in a way that does not depend on the order in which the SLE's are growing. Such systems of commuting SLE's are classified in [3]; in particular, two versions of SLE_κ, $\mathrm{SLE}_{\hat\kappa}$ can commute only if $\hat\kappa \in \{\kappa, 16/\kappa\}$.

While it is rather easy to check directly the local commutation identities implied by reversibility and some duality conjectures, the crucial difficulty consists in working backward and proving reversibility or duality from these local identities. One may think of this as a "local to global" problem.

Decisive progress was achieved by Zhan in [18], where he proves reversibility of chordal SLE_κ, $\kappa \leq 4$, i.e. that the range of the trace of an SLE_κ in D going from x to y has the same distribution as the range of the trace of SLE going from y to x in D. This was previously known for $\kappa \in \{2, 8/3, 4, 6, 8\}$. The argument involves a sequence of couplings of an $\mathrm{SLE}_\kappa(D, x, y)$ with an $\mathrm{SLE}_\kappa(D, y, x)$, such that each coupling in the sequence is absolutely continuous w.r.t. the trivial (independent) coupling, and the limiting coupling is exact (the ranges of the two traces are identical). The fact that similar techniques may be used to prove duality is also mentioned in [18]. The present article stems in part from an effort to clarify and extend this "local to global" argument.

After the present work appeared as a preprint, the manuscript [17] was brought to our attention. There, as here, ideas and techniques from [3, 18] are combined to obtain a certain number of duality identities, with some overlap with those stated here (in Proposition 10). Subsequently, a different construction of duality identities was given in [4], via the free field; this allows to establish "strong duality" identities, in which the conditional law of an SLE given a boundary arc is also specified. Such identities were first conjectured in [1].

Let $\gamma, \hat\gamma$ be traces of two SLE's satisfying the local commutation condition. Then, for U, V disjoint open subsets of the domain, one has a coupling of $(\gamma, \hat\gamma)$ which is "correct" on the time set $\{(s,t) : s \leq \tau, t \leq \hat\tau\}$, where τ, $\hat\tau$ are stopping times for the two SLE's, such that $\gamma^\tau \subset U$, $\hat\gamma^{\hat\tau} \subset V$. We construct a coupling of $(\gamma, \hat\gamma)$, which is "correct" on the time set $\{(s,t) : \gamma_{[0,s]} \cap \hat\gamma_{[0,t]} = \varnothing\}$. See Theorem 6 for a precise statement.

The duality identities follow from applying Theorem 6 to appropriate pairs of commuting SLE's, together with some *a priori* geometric information on the traces. Plainly, many identities may be generated in this fashion.

The identities considered here involve variants of SLE_κ: the $\mathrm{SLE}_\kappa(\underline{\rho})$ processes ($\underline{\rho} = \rho_1, \dots, \rho_n$). They satisfy a domain Markov property when keeping track of n marked points $z_1, \dots, z_n$ (in addition of the origin and the target of chordal SLE). The influence of z_i on the SLE trace is quantified by the real parameter ρ_i; this influence is attractive for $\rho_i < 0$ and repulsive for $\rho_i > 0$.

Let us consider a chordal SLE in the upper half-plane $\mathbb{H}$, going from 0 to infinity. In the phase $4 < \kappa < 8$, a boundary point, say 1, is "swallowed", *i.e.* gets disconnected from infinity by the trace at a random time τ_1 when the trace hits some point in $(1, \infty)$. The boundary arc straddling 1 is the boundary arc seen by 1 at time τ_1^-.

THEOREM 1. – *Consider a chordal* SLE_κ *in* $(\mathbb{H},0,\infty)$, $4<\kappa<8$; *let D be the leftmost visited point on* $(1,\infty)$. *Conditionally on D, the boundary arc straddling 1 is distributed as an* $\mathrm{SLE}_{\hat\kappa}(-\frac{\hat\kappa}{2},\hat\kappa-4,\hat\kappa-2)$ *in* $(\mathbb{H},D,\infty,0,1,D^+)$, *stopped when it hits* $(0,1)$.

In the phase $\kappa\geq 8$, a.s. every point in $\overline{\mathbb{H}}$ is visited by the trace. We isolate a boundary arc in a different way. Let G be the leftmost point on $(-\infty,0)$ visited by the trace before τ_1. We consider the boundary of K_{τ_G}, the hull of the SLE stopped when it first visits G; this boundary is an arc between G and a point in $(0,1)$.

THEOREM 2. – *Consider a chordal* SLE_κ *in* $(\mathbb{H},0,\infty)$, $\kappa\geq 8$. *Let G be the leftmost visited point on* $(-\infty,0)$ *before* τ_1. *Conditionally on G, the boundary of K_{τ_G} is distributed as an* $\mathrm{SLE}_{\hat\kappa}(\frac{\hat\kappa}{2},\frac{\hat\kappa}{2}-2,-\frac{\hat\kappa}{2},\hat\kappa-4)$ *in* $(\mathbb{H},G,\infty,G^-,G^+,0,1)$, *stopped when it hits* $(0,1)$.

The distributions of D and G are well known and easy to derive.

The article is organized as follows. Section 2 recalls some absolute continuity properties of chordal SLE. Local commutation is discussed in Section 3. Maximal couplings of commuting SLE's are constructed in Section 4. Geometric consequences (in particular duality) are drawn in Section 5. Some technical lemmas are postponed to Section 6.

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2. Absolute continuity for chordal SLE

In this section we consider some absolute continuity properties of chordal SLE, mostly based on [6]. Chordal SLE will also serve as a reference measure for variants we will study later; some familiarity with chordal SLE is assumed (see, e.g., [7, 12, 16]).

We adopt the following notation: $c=(D,x,y)$ is a configuration where D is a simply connected domain and x,y are distinct boundary points. Unless there is an ambiguity, the configuration is simply denoted by D. The chordal SLE_κ measure on $c=(D,x,y)$ is denoted by μ_c (κ is fixed). It is seen as a measure on Loewner chains up to increasing time change; or as a configuration-valued continuous process (up to time change); or as a measure on non self-traversing paths ([12]). This path (the SLE "trace") is denoted by γ, while the hull it generates is denoted by K ($D\setminus K_t$ is the connected component of $D\setminus\gamma_{[0,t]}$ having y on its boundary). Let U be a subdomain of D, agreeing with D in a neighborhood of x, and not containing y on its boundary. Then μ_c^U denotes the measure on paths induced by chordal SLE starting from x and stopped on exiting U; this happens at a random time τ, at which the hull is K_τ, the tip of the trace is γ_τ, and the configuration c_τ is $(D_\tau=D\setminus K_\tau,\gamma_\tau,y)$. More generally, for τ a stopping time, γ^τ denotes the trace stopped at τ (*i.e.* the process up to time τ), μ_c^τ the measure induced by stopping at τ. We will use γ to denote both the trace as a process and as a subset of $\overline{D}$ (the range of the process).

For some computations, it is convenient to fix a particular time parameterization, typically half-plane capacity of the hull (mapping conformally the domain to the upper half-plane). Otherwise we will reason up to bicontinuous (progressive, increasing) change of time. The class of stopping times is invariant under such time reparameterizations.

Later on, we will use tightness conditions, so we shall review some technical points now. Let (D, x, y) be a configuration, K a hull such that $(D \setminus K, x', y)$ is a configuration for some $x' \in \partial K$. By the Riemann mapping theorem, there is a conformal equivalence $\phi_K : D \setminus K \to D$; one can specify it uniquely by requiring its 2-jet at y to be trivial ($\phi_K(y) = y$, $\phi'_K(y) = 1$, $\phi''_K(y) = 0$ if ϕ_K extends smoothly at y; this condition is coordinate independent, so one can first "straighten" the boundary at y). One defines a topology on hulls as follows: (K_n) converges to K if $\phi_{K_n}^{-1}$ converges to ϕ_K^{-1} uniformly on compact sets of D. This is a version of Carathéodory convergence. A topology on chains $(K_t)_{t\geq 0}$ is given by the condition: $(K_t^n)_t$ converges to $(K_t)_t$ if $(t, w) \mapsto \phi_{K_t^n}^{-1}(w)$ converges uniformly on compact sets of $[0, T] \times D$. Then the Loewner equation maps continuously $C(\mathbb{R}^+, \mathbb{R})$ (with the usual topology of uniform convergence on compact sets) to the space of chains endowed with this topology. Thus the induced measure on chains is a Radon measure. From [12], we know that the chain is a.s. generated by a continuous non self-traversing path γ. For clarity, we will think of SLE as a measure on such paths, with the topology on chains described above.

To express densities, we need to define some conformal invariants. Let (D, x, y) be a configuration, z_x, z_y analytic local coordinates at the boundary (z_x mapping a neighborhood of x in D to the neighborhood of 0 in the upper semidisk). The Poisson excursion kernel is defined as

$$H_D(x, y) = \lim_{X \to x, Y \to y} \frac{G_D(X, Y)}{\Im(z_x(X))\Im(z_y(Y))}$$

where G_D is the Green function in D (with Dirichlet boundary conditions); this depends on the choice of z_x (or z_y) as a 1-form. (If z'_x is another local coordinate at x, dz'_x/dz_x is positive). If D and D' agree in a neighborhood of x, we choose the same local coordinate z_x, so that $H_{D'}(x, y')/H_D(x, y)$ does not depend on a choice of local coordinate at x. Similarly for $i, j = 1, 2$, consider configurations (D_{ij}, x_i, y_j) such that D_{ij} agrees with $D_{i,3-j}$ in a neighborhood of x_i and with $D_{3-i,j}$ in a neighborhood of y_j. Then the ratio:

$$\frac{H_{D_{11}}(x_1, y_1)H_{D_{22}}(x_2, y_2)}{H_{D_{12}}(x_1, y_2)H_{D_{21}}(x_2, y_1)}$$

is defined independently of any (coherent) choice of local coordinates at x_i, y_j. To simplify the notation, if $c = (D, x, y)$ is a configuration, we set $H(c) = H_D(x, y)$.

There is a σ-finite measure μ^{loop} on unrooted loops in $\mathbb{C}$, the Brownian loop measure ([6, 9]). As in [5], let us denote

$$m(D; K, K') = \mu^{\text{loop}}\{\delta : \delta \subset D, \delta \cap K \neq \varnothing, \delta \cap K' \neq \varnothing\}.$$

In accordance with [6], set $\alpha = \alpha_\kappa = \frac{6-\kappa}{2\kappa}$, $\lambda = \lambda_\kappa = \frac{(6-\kappa)(8-3\kappa)}{2\kappa}$.

Proposition 3. – *Assume that $c = (D, x, y)$ and $c' = (D', x, y')$ are configurations agreeing in a neighborhood U of x such that $\overline{U}$ is compact and at positive distance to the*