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TESSELLATIONS OF RANDOM MAPS OF ARBITRARY GENUS

BY Grégory MIERMONT

Abstract. – We investigate Voronoi-like tessellations of bipartite quadrangulations on surfaces of arbitrary genus, by using a natural generalization of a bijection of Marcus and Schaeffer allowing one to encode such structures by labeled maps with a fixed number of faces. We investigate the scaling limits of the latter. Applications include asymptotic enumeration results for quadrangulations, and typical metric properties of randomly sampled quadrangulations. In particular, we show that scaling limits of these random quadrangulations are such that almost every pair of points is linked by a unique geodesic.

Résumé. – Nous examinons les propriétés de mosaïques de type Voronoï sur des quadrangulations bipartites de genre arbitraire. Ceci est rendu possible par une généralisation naturelle d'une bijection de Marcus et Schaeffer, permettant de décrire ces mosaïques par des cartes étiquetées avec un nombre fixé de faces, dont nous déterminons les limites d'échelle. Parmi les applications de ces résultats, figurent le comptage asymptotique des quadrangulations, ainsi que des propriétés métriques typiques de quadrangulations choisies au hasard. En particulier, nous montrons que les limites d'échelles de ces quadrangulations aléatoires sont telles que presque toute paire de points est liée par un unique chemin géodésique.

1. Introduction

1.1. Motivation

A map is a graph embedded into a compact orientable surface without boundary, yielding a cell decomposition of the surface, and considered up to orientation-preserving homeomorphisms. Random maps are considered in the physics literature on quantum gravity as discretized versions of random surfaces [2]. This approach allows one to perform computations of certain integrals with respect to an ill-defined measure on surfaces, by approximating them by finite sums over maps. From a mathematical perspective, this leads to the stimulating problem of existence of a measure on compact surfaces arising as the scaling limit of, say,

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uniform random triangulations of the sphere (or the compact orientable surface of genus g) with a large number of faces.

Important progress has been made in this direction in the recent years, thanks to bijective approaches initiated in Schaeffer's thesis [42]. They allowed Chassaing & Schaeffer [11] to obtain the scaling limits for the radius and profile of a uniform rooted planar quadrangulation with n faces, considered as a metric space by endowing its vertices with the usual graph distance. In particular, they showed that typical graph distances between vertices of such random quadrangulations are of order $n^{1/4}$. Generalizations of this result have been obtained for much more general families of random maps by Marckert, Weill and the author in [30, 34, 36, 45], relying on generalizations of Schaeffer's bijection by Bouttier, Di Francesco & Guitter [7].

An important step has been accomplished by Le Gall [26], who showed that scaling limits of these random quadrangulations, considered in the Gromov-Hausdorff sense [22], must be homeomorphic to a metric quotient of the so-called Brownian Continuum Random Tree of Aldous [1]. This random quotient was introduced under the name of *Brownian map* by Marckert & Mokkadem [32], who proved convergence of random quadrangulations towards this limit in a sense yet different from convergence in distribution in the Gromov-Hausdorff topology. Subsequently, Le Gall & Paulin [28] inferred that the Brownian map is homeomorphic to the two-dimensional sphere. At the present stage, it is however not known whether the scaling limit is uniquely determined as the Brownian map, which would lead to a satisfactory answer to the above mentioned problem.

A natural idea is to characterize the limit through its "finite-dimensional marginal distributions". From the point of view of metric measure spaces, this would mean to characterize the laws of mutual distances between an arbitrary number of randomly sampled points. This, however, seems hard to obtain. The approach of the present work, which deals with combinatorial, probabilistic and geometric aspects of maps, was initially motivated by another notion of finite-dimensional marginals, which takes into account the planar structure of the graphs that are considered, and roughly consists in studying the Voronoi tessellation on the map with sources taken at random.

Our central combinatorial result is a natural generalization of Schaeffer's bijective construction, to be introduced in Section 2. We do not restrict ourselves to the planar case, as the most natural framework is to consider maps of arbitrary (but fixed) genus g, so our bijection is really a generalized version of the Marcus-Schaeffer bijection in arbitrary genus [10, 33]. Let us explain briefly the idea of our construction. Recall that the Marcus-Schaeffer bijection encodes a genus-g bipartite quadrangulation with a distinguished vertex by a labeled map of genus g with one face, in such a way that the labels keep track of graph distances to the distinguished vertex in the quadrangulation. Our construction encodes a genus-g bipartite quadrangulation with k distinguished vertices by a labeled map of genus g with k faces (Theorem 4 and Corollary 1). In some sense, the faces of the encoding map correspond to the Voronoi tiles in the quadrangulation, with sources at the k distinguished vertices, and the labels allow to keep track of graph distance of a vertex to the source of the Voronoi tile it belongs to. An important fact is that there is some flexibility in the construction, allowing one to consider a generalized Voronoi tessellation in which the distances to the k sources are measured after addition of a *delay* depending on the source.

The resulting set of labeled maps with a fixed number of faces is combinatorially much simpler than that of bipartite quadrangulations, and we are able to study their scaling limits. This is done in Section 4 (Theorem 5 and Propositions 3, 4), after showing how to encode labeled maps by appropriate processes in Section 3. Sections 5 and 7 give three applications of these scaling limit results for labeled maps, to combinatorial and geometric properties of bipartite quadrangulations.

The first application is a new derivation of known asymptotic enumeration results (Theorem 1), to which the short Section 5 is devoted. These are initially due to Bender & Canfield [4], who obtained the asymptotic number of rooted maps of genus g with n edges [(1)] by recursive decomposition methods, of a very different nature from our bijective study. These results have also been obtained in the recent work of Chapuy, Marcus & Schaeffer [10], who completed the exact enumeration of maps of genus g that was initiated in [33]. The starting point of our study (using the Marcus-Schaeffer bijection, which corresponds to the case $k = 1$ of our bijection) is the same as in [10], but the rest of our approach is different, the arguments of Section 4 being of a more probabilistic nature.

Our other results concern the metric structure of randomly sampled bipartite quadrangulations. Let us indicate briefly what we mean by "random" in this paper. Most of the articles on the topic have focused on scaling limits as $n \to \infty$ of random families of maps conditioned to have n faces or n vertices. We prefer to randomize the number of faces, by using natural σ-finite measures on bipartite quadrangulations of fixed genus g, called the Boltzmann measures Q_g (the term being inspired from [3], see also [30, 34]), and which are obtained by assigning an appropriate weight to the faces. In the physics terminology, these measures correspond to the so-called *grand-canonical* measures, while the measures with fixed number of faces are the *microcanonical* measures. These measures are used to define natural "Boltzmann-Gibbs" distributions on quadrangulations, depending on an inverse temperature parameter β that allows one to tune the average size of the quadrangulation and take scaling limits.

The existence of these scaling limits, considered with respect to the Gromov-Hausdorff-Prokhorov topology on metric measure spaces, and stated in Theorem 2, is then obtained from the study of Section 4. This generalizes to arbitrary genera the fact that planar quadrangulations admit scaling limits for the Gromov-Hausdorff topology, as shown in Le Gall [26]. The theory of metrics on weighted metric spaces that is needed here is developed in Section 6.

The main result of the present paper, Theorem 3, gives qualitative information on the metric structure of the scaling limits of random bipartite quadrangulations. We show that these scaling limits are geodesic weighted metric spaces, in which two typical points are linked by a unique geodesic. The idea is to use our bijection for quadrangulations with $k = 2$ marked vertices, in order to show that, if x, y are typical points in the limiting random metric space (X, d) arising as a scaling limit of random bipartite quadrangulations, the intersection of a geodesic between x and y and a geometric locus of the form

$$\{z \in X : d(x,z) - d(y,z) = D\}\,, \qquad D \in [-d(x,y), d(x,y)]\,,$$

(1) This is equivalent to our result, since such maps are in one-to-one correspondence with bipartite quadrangulations of same genus with n faces [33, Proposition 1].

is a.s. reduced to a single point.

Theorem 3 is strongly reminiscent of the combinatorial results of the recent paper by Bouttier & Guitter [8]. They show that in the planar case $g = 0$, two typical vertices in a random quadrangulation with n faces cannot be linked by two geodesics which are "very far" in the scale $n^{1/4}$, as $n \to \infty$. Our result confirms that geodesics between typical points become unique in the scaling limit, answering one of the questions raised in Section 5 of [8], though in our slightly different setting of Boltzmann-distributed maps. We also mention the recent work by Le Gall [27] on similar topics as the present paper and [8], and providing results on the exceptional geodesics as well.

In this work, the cardinality of a set A is denoted by $|A|$.

1.2. Embedded graphs and maps

Let us introduce some formalism for embedded graphs and maps, which we partly borrow from [29].

Graphs on surfaces. Let S be a compact connected orientable two-dimensional surface without boundary. It is well-known that such surfaces are characterized, up to homeomorphism, by an integer $g \geq 0$, called the genus of S, and that the topology of S is that of the connected sum $\mathbb{S}_g$ of g tori. The surface of genus 0 is the two-dimensional sphere.

A *half-edge*, or oriented edge e in S is a continuous path $c : [0,1] \to S$, considered up to continuous increasing reparametrization, which is either injective on $[0,1]$, or is injective on $[0,1)$ and satisfies $c(0) = c(1)$. In the latter case, e is called a loop. The order on $[0,1]$ induces a natural orientation of the half-edge e, and its start and end points are defined as $e^- = c(0)$ and $e^+ = c(1)$, a definition that depends only on the half-edge e and not on the function c. Similarly, the image of e is defined as $\mathrm{Im}(e) = c([0,1])$, and the interior of e is $c((0,1))$. The reversal $\overline{e}$ of the half-edge e is defined as the reparametrization class of $c(1-\cdot)$, so that $\overline{e}^+ = e^-, \overline{e}^- = e^+$. An *edge* is a pair of the form $\mathbf{e} = \{e, \overline{e}\}$ where e is a half-edge.

An embedded graph in S, or simply a graph on S, is a pair $(V, \mathbf{E})$, where

- $\mathbf{E}$ is a non-empty, finite set of edges such that two distinct edges intersect, if at all, only at some of their endpoints
- V is the set of vertices, i.e. of end-points of elements of $\mathbf{E}$.

We let E be the set of half-edges corresponding to $\mathbf{E}$. The set E has even cardinality and comes with the involution $e \mapsto \overline{e}$. An orientation of the edges is a choice of one half-edge inside each edge to form a set $E_{1/2} \subset E$ with same cardinality as $\mathbf{E}$.

A graph on S determines faces, which are the connected components of the complement of $\bigcup_{e\in E} \mathrm{Im}(e)$ in S. Let F be the set of faces of the graph.

Maps. We say that the embedded graph $(V, \mathbf{E})$ is a *map* if all faces are simply connected (hence determining a cell complex structure on S). A map is necessarily a connected graph by [29, Lemma 1.5].

The degree of a face in a map is the number of edges it is incident to, where it is understood that edges that are incident twice to the same face should be counted twice. By Euler's formula,

$$|V| - |\mathbf{E}| + |F| = \chi(g)\,, \tag{1}$$