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Robert LAZARSFELD & Mircea MUSTAŢĂ

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CONVEX BODIES ASSOCIATED TO LINEAR SERIES

BY ROBERT LAZARSFELD AND MIRCEA MUSTAŢĂ

ABSTRACT. – In his work on log-concavity of multiplicities, Okounkov showed in passing that one could associate a convex body to a linear series on a projective variety, and then use convex geometry to study such linear systems. Although Okounkov was essentially working in the classical setting of ample line bundles, it turns out that the construction goes through for an arbitrary big divisor. Moreover, this viewpoint renders transparent many basic facts about asymptotic invariants of linear series, and opens the door to a number of extensions. The purpose of this paper is to initiate a systematic development of the theory, and to give some applications and examples.

RÉSUMÉ. – Dans son travail sur la log-concavité des multiplicités, Okounkov montre au passage que l'on peut associer un corps convexe à un système linéaire sur une variété projective, puis utiliser la géométrie convexe pour étudier ces systèmes linéaires. Bien qu'Okounkov travaille essentiellement dans le cadre classique des fibrés en droites amples, il se trouve que sa construction s'étend au cas d'un grand diviseur arbitraire. De plus, ce point de vue permet de rendre transparentes de nombreuses propriétés de base des invariants asymptotiques des systèmes linéaires, et ouvre la porte à de nombreuses extensions. Le but de cet article est d'initier un développement systématique de la théorie et de donner quelques applications et exemples.

Introduction

In his interesting papers [34] and [36], Okounkov showed in passing that one could associate a convex body to a linear series on a projective variety, and then use convex geometry to study such linear systems. Although Okounkov was essentially working in the classical setting of ample line bundles, it turns out that the construction goes through for an arbitrary big divisor. Moreover, one can recover and extend from this viewpoint most of the fundamental results from the asymptotic theory of linear series. The purpose of this paper is to initiate a systematic development of this theory, and to give a number of applications and examples.

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We start by describing Okounkov's construction. Let X be a smooth irreducible projective variety of dimension d defined over an uncountable algebraically closed field $\mathbf{K}$ of arbitrary characteristic. (1) The construction depends upon the choice of a fixed flag

$$Y_\bullet \;:\; X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \ldots \supseteq Y_{d-1} \supseteq Y_d = \{\mathrm{pt}\},$$

where Y_i is a smooth irreducible subvariety of codimension i in X. Given a big divisor (2) D on X, one defines a valuation-like function

$$(*)\;\; \nu = \nu_{Y_\bullet} = \nu_{Y_\bullet, D} \;:\; \big(H^0\big(X, \mathcal{O}_X(D)\big) - \{0\}\big) \longrightarrow \mathbf{Z}^d \;,\;\; s \mapsto \nu(s) = \big(\nu_1(s), \ldots, \nu_d(s)\big)$$

as follows. First, set $\nu_1 = \nu_1(s) = \mathrm{ord}_{Y_1}(s)$. Then s determines in the natural way a section

$$\tilde{s}_1 \in H^0\big(X, \mathcal{O}_X(D - \nu_1 Y_1)\big)$$

that does not vanish identically along Y_1, and so we get by restricting a non-zero section

$$s_1 \;\in\; H^0\big(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1)\big).$$

Then take

$$\nu_2(s) \;=\; \mathrm{ord}_{Y_2}(s_1),$$

and continue in this manner to define the remaining $\nu_i(s)$. For example, when $X = \mathbf{P}^d$ and $Y_\bullet$ is a flag of linear spaces, $\nu_{Y_\bullet}$ is essentially the lexicographic valuation on polynomials.

Next, define

$$\mathrm{vect}(|D|) \;=\; \mathrm{Im}\big(\big(H^0\big(X, \mathcal{O}_X(D)\big) - \{0\}\big) \xrightarrow{\nu_Y} \mathbf{Z}^d\big)$$

to be the set of valuation vectors of non-zero sections of $\mathcal{O}_X(D)$. It is not hard to check that

$$\#\,\mathrm{vect}(|D|) \;=\; h^0(X, \mathcal{O}_X(D)).$$

Then finally set

$$\Delta(D) \;=\; \Delta_{Y_\bullet}(D) \;=\; \text{ closed convex hull}\Big(\bigcup_{m \geq 1} \tfrac{1}{m} \cdot \mathrm{vect}(|mD|)\Big).$$

Thus $\Delta(D)$ is a convex body in $\mathbf{R}^d = \mathbf{Z}^d \otimes \mathbf{R}$, which we call the *Okounkov body* of D (with respect to the fixed flag $Y_\bullet$).

One can view Okounkov's construction as a generalization of a correspondence familiar from toric geometry, where a torus-invariant divisor D on a toric variety X determines a rational polytope P_D. In this case, working with respect to a flag of invariant subvarieties of X, $\Delta(D)$ is a translate of P_D. An analogous polyhedron on spherical varieties has been studied in [10], [35], [1], [26]. On the other hand, the convex bodies $\Delta(D)$ typically have a less classical flavor even when D is ample. For instance, let X be an abelian surface having Picard number $\rho(X) \geq 3$, and choose an ample curve $C \subseteq X$ together with a smooth point $x \in C$, yielding the flag

$$X \;\supseteq\; C \;\supseteq\; \{x\}.$$

Given an ample divisor D on X, denote by $\mu = \mu(D) \in \mathbf{R}$ the smallest root of the quadratic polynomial $p(t) = (D - tC)^2$: for most choices of D, $\mu(D)$ is irrational. Here the Okounkov

(1) In the body of the paper, we will relax many of the hypotheses appearing here in the introduction.

(2) Recall that by definition a divisor D is *big* if $h^0\big(X, \mathcal{O}_X(mD)\big)$ grows like m^d.

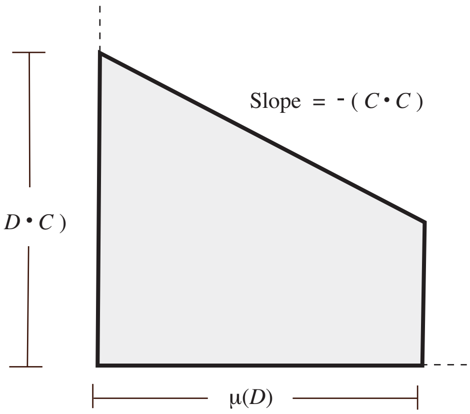

FIGURE 1. Okounkov body of a divisor on an abelian surface

body of D is the trapezoidal region in $\mathbf{R}^2$ shown in Figure 1. Note that in this case $\Delta(D)$, although polyhedral, is usually not rational. We give in §6.3 a four-dimensional example where $\Delta(D)$ is not even polyhedral.

As one might suspect, the standard Euclidean volume of $\Delta(D)$ in $\mathbf{R}^d$ is related to the rate of growth of the dimensions $h^0\big(X, \mathcal{O}_X(mD)\big)$. In fact, Okounkov's arguments in [36, §3] – which are based on results [27] of Khovanskii – go through without change to prove

THEOREM A. – *If D is any big divisor on X, then*

$$\mathrm{vol}_{\mathbf{R}^d}\big(\Delta(D)\big) \;=\; \frac{1}{d!}\cdot \mathrm{vol}_X(D).$$

The quantity on the right is the *volume* of D, defined as the limit

$$\mathrm{vol}_X(D) \;=_{\mathrm{def}}\; \lim_{m\to\infty} \frac{h^0\big(X, \mathcal{O}_X(mD)\big)}{m^d/d!}.$$

In the classical case, when D is ample, $\mathrm{vol}_X(D) = \int c_1(\mathcal{O}_X(D))^d$ is just the top self-intersection number of D. In general, the volume is an interesting and delicate invariant of a big divisor, which has lately been the focus of considerable work (cf. [29, Chapt. 2.2], [6], [15]). It plays a pivotal role in several important recent developments in higher dimensional geometry, e.g. [8], [41], [23], [40].

We study the variation of these bodies as functions of D. It is not hard to check that $\Delta(D)$ depends only on the numerical equivalence class of D, and that $\Delta(pD) = p\cdot\Delta(D)$ for every positive integer p. It follows that there is a naturally defined Okounkov body $\Delta(\xi) \subseteq \mathbf{R}^d$ associated to every big rational numerical equivalence class $\xi \in N^1(X)_{\mathbf{Q}}$, and as before $\mathrm{vol}_{\mathbf{R}^d}(\Delta(\xi)) = \frac{1}{d!}\cdot \mathrm{vol}_X(\xi)$. We prove:

THEOREM B. – *There exists a closed convex cone*

$$\Delta(X) \;\subseteq\; \mathbf{R}^d \times N^1(X)_{\mathbf{R}}$$

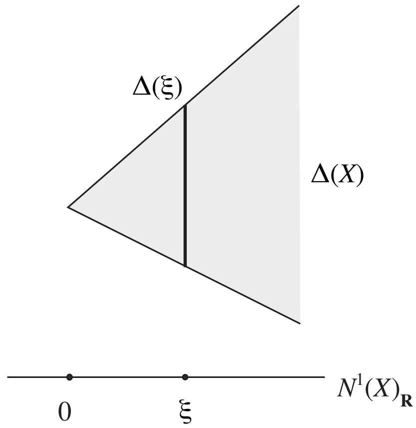

FIGURE 2. Global Okounkov body

characterized by the property that in the diagram

$$\begin{array}{ccc} \Delta(X) & \subseteq & \mathbf{R}^d \times N^1(X)_{\mathbf{R}} \\ & \searrow \quad \swarrow_{\mathrm{pr}_2} & \\ & N^1(X)_{\mathbf{R}}, & \end{array}$$

the fibre $\Delta(X)_\xi \subseteq \mathbf{R}^d \times \{\xi\} = \mathbf{R}^d$ of $\Delta(X)$ over any big class $\xi \in N^1(X)_{\mathbf{Q}}$ is $\Delta(\xi)$.

This is illustrated schematically in Figure 2. The image of $\Delta(X)$ in $N^1(X)_{\mathbf{R}}$ is the so-called pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ of X, i.e. the closure of the cone spanned by all effective divisors: its interior is the big cone $\mathrm{Big}(X)$ of X. Thus the theorem yields a natural definition of $\Delta(\xi) \subseteq \mathbf{R}^d$ for any big class $\xi \in N^1(X)_{\mathbf{R}}$, viz. $\Delta(\xi) = \Delta(X)_\xi$. It is amusing to note that already in the example of an abelian surface considered above, the cone $\Delta(X)$ is non-polyhedral. [(3)]

Theorem B renders transparent several basic properties of the volume function vol_X established by the first author in [29, 2.2C, 11.4.A]. First, since the volumes of the fibres $\Delta(\xi) = \Delta(X)_\xi$ vary continuously for ξ in the interior of $\mathrm{pr}_2(\Delta(X)) \subseteq \mathbf{N}^1(X)_{\mathbf{R}}$, one deduces that the volume of a big class is computed by a continuous function

$$\mathrm{vol}_X : \mathrm{Big}(X) \longrightarrow \mathbf{R}.$$

Moreover $\Delta(\xi) + \Delta(\xi') \subseteq \Delta(\xi + \xi')$ for any two big classes $\xi, \xi' \in N^1(X)_{\mathbf{R}}$, and so the Brunn-Minkowski theorem yields the log-concavity relation

$$\mathrm{vol}_X(\xi + \xi')^{1/d} \geq \mathrm{vol}_X(\xi)^{1/d} + \mathrm{vol}_X(\xi')^{1/d}$$

for any two such classes. [(4)]

(3) This follows for instance from the observation that $\mu(D)$ varies non-linearly in D.

(4) In the classical setting, it was this application of Brunn-Minkowski that motivated Okounkov's construction in [36]. We remark that it was established in [29] that vol_X is actually continuous on all of $N^1(X)_{\mathbf{R}}$ – i.e. that