ISSN 0012-9593

quatrième série - tome 42 fascicule 5 septembre-octobre 2009

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 42, 2009, p. 837 à 856

THE CALABI FUNCTIONAL ON A RULED SURFACE

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ABSTRACT. – We study the Calabi functional on a ruled surface over a genus two curve. For polarizations which do not admit an extremal metric we describe the behavior of a minimizing sequence splitting the manifold into pieces. We also show that the Calabi flow starting from a metric with suitable symmetry gives such a minimizing sequence.

RÉSUMÉ. – On étudie la fonctionnelle de Calabi sur une surface réglée au-dessus d'une courbe de genre deux. Pour les polarizations qui n'admettent pas de métrique extrémale, on décrit le comportement d'une suite minimisante partitionnant la variété. On montre aussi que le flot de Calabi partant d'une métrique avec une symétrie appropriée produit une telle suite minimisante.

1. Introduction

In [3] Calabi introduced the problem of minimizing the L^2-norm of the scalar curvature (this is called the *Calabi functional*) over metrics in a fixed Kähler class on a compact Kähler manifold. A critical point of the Calabi functional is called an extremal metric. The Euler-Lagrange equation is that the gradient of the scalar curvature is a holomorphic vector field. It is known that extremal metrics in fact minimize the Calabi functional (see [15], [5], [12]). Recently much progress has been made in understanding when extremal metrics exist, at least on a conjectural level. Kähler-Einstein metrics are a special case and when the first Chern class of the manifold is positive (the manifold is called Fano in this case), Yau conjectured that the existence of Kähler-Einstein metrics is related to the stability of the manifold in the sense of geometric invariant theory. In the case of negative or zero first Chern class Yau [26] and Aubin [2] have shown that Kähler-Einstein metrics always exist, answering a conjecture of Calabi. Tian [24] made significant progress towards understanding the Fano case, solving it completely in the case of surfaces in [23]. Donaldson [9] showed that the scalar curvature can be interpreted as a moment map (this was also observed by Fujiki [13]) and this enabled extending the conjectures about the existence of Kähler-Einstein metrics to more general constant scalar curvature and extremal metrics (see [10], [17], [22]).

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In this paper we look at what we can say about minimizing the Calabi functional in a Kähler class which admits no extremal metric, concentrating on a concrete example. Let Σ be a genus 2 curve and $\mathcal{M}$ a degree -1 line bundle on it. We consider the ruled surface $X = \mathbf{P}(\mathcal{M} \oplus \mathcal{O})$ with a family of polarizations $L_m = C + mS_\infty$, where C is the class of a fibre, S_∞ is the infinity section (with self-intersection 1), and $m > 0$. Technically we should take m to be rational, especially when discussing test-configurations, but by an approximation and continuity argument we can take m to be real. The aim is to study the problem of minimizing the Calabi functional in these Kähler classes. Our main result is the following.

THEOREM 1. – *There exist constants $k_1 \simeq 18.9, k_2 \simeq 5.03$, such that*

1. *If $0 < m < k_1$ then X admits an extremal metric (this is due to Tønnesen-Friedman [25]).*
2. *If $k_1 \leqslant m \leqslant k_2(k_2+2)$ then there exists a minimizing sequence of metrics which breaks X into two pieces and converges to complete extremal metrics on both.*
3. *If $m > k_2(k_2+2)$ then there exists a minimizing sequence of metrics which breaks X into three pieces. It converges to complete extremal metrics on two of these and the third degenerates into a fibration of infinitely long and infinitely thin cylinders.*

Note that the fact that extremal metrics do not exist in the case $m \geqslant k_1$ follows from the work [1] where such existence issues are studied for a large class of ruled manifolds. Alternatively it also follows from Donaldson's theorem [12] on the lower bound of the Calabi functional, and the computations in [22] (see also [21], Section 3.3). Here we go further in that we compute the infimum of the Calabi functional and describe the behavior of a minimizing sequence.

To construct metrics on our ruled surface, we use the momentum construction given in Hwang-Singer [16]. This construction has been used repeatedly in the past to find special metrics on ruled manifolds, in particular extremal metrics. See [1] for a unified treatment of these constructions or [16] for a historical overview and more references. The momentum construction allows us to construct circle invariant metrics from functions on an interval and it gives a convenient expression for the scalar curvature. More precisely, let $\phi : [0, m] \to \mathbf{R}$ be a smooth function, positive on the interior $(0, m)$, vanishing at the endpoints, and such that $\phi'(0) = 2$, $\phi'(m) = -2$. The momentum construction gives a metric ω_ϕ in the Kähler class L_m, with scalar curvature

$$S(\omega_\phi) = \frac{-2}{1+\tau} - \frac{1}{2(1+\tau)}\big[(1+\tau)\phi\big]''.$$

Here τ is the moment map for the S^1-action on the fibres and working with this coordinate is the central idea of the momentum construction. We will recall this construction in Section 2. Of particular importance to us is the fact that we can consider momentum profiles which vanish on a subset of $(0, m)$. These correspond to degenerate metrics and they arise as the limits of the minimizing sequences in Theorem 1.

In Section 3 we consider the problem of directly minimizing the Calabi functional on the set of metrics obtained by the momentum construction. Since the L^2-norm of the scalar curvature is equivalent to the H^2-norm of the momentum profiles, this is straightforward. We find that the Euler-Lagrange equation for a minimizer ϕ is $\phi S(\phi)'' = 0$ and $S(\phi)''$ must be a negative distribution, i.e., $S(\phi)$ is concave. We show that a unique minimizer exists in each

Kähler class and its momentum profile is in C^2. Note that $S(\phi)'' = 0$ is the equation for ϕ to define an extremal metric.

In Section 4 we explicitly construct the minimizers, which can be degenerate in the sense that the momentum profiles can vanish on a subset of $(0, m)$. Here we will see the three different kinds of behavior stated in Theorem 1. In Section 5 we construct test-configurations for X and calculate their Futaki invariants. This will clarify the role of the concavity of $S(\phi)$ for minimizers of the Calabi functional. In fact, rational, piecewise-linear convex functions on $[0, m]$ give test-configurations essentially by the construction in [10] as generalized to bundles of toric varieties in [21]. We can approximate $-S(\phi)$ by such functions, and Donaldson's theorem on lower bounds for the Calabi functional in [12] shows that ω_ϕ actually achieves the infimum of the Calabi functional on the whole Kähler class, not just the metrics arising from the momentum construction. This will complete the proof of Theorem 1.

An alternative approach to minimizing the Calabi functional is using the Calabi flow introduced in [3]. This is the flow which deforms the Kähler potential in the direction of the scalar curvature. It is expected (see [10], [11]) that the Calabi flow should minimize the Calabi functional and if there is no extremal metric in a given Kähler class, then it should break up the manifold into pieces which admit complete extremal metrics or collapse in some way. In Sections 6 and 7 we will verify this, showing

THEOREM 2. – *If the initial metric is given by the momentum construction then the Calabi flow exists for all time and the momentum profiles converge in H^2 to the minimizer of the Calabi functional.*

Not much is known about the long time behavior of the flow in general, except in the case of Riemann surfaces, where existence and convergence to a uniformizing metric has been shown by Chruściel [8] (see also [4] and [20]). More recently Chen and He [7] have studied the flow on toric Fano surfaces. The Calabi flow on ruled manifolds has been previously studied in [14], where the long time existence and convergence is proved for the Kähler classes which admit an extremal metric. We use similar techniques, the main difference being that we introduce some variants of the Mabuchi functional when no extremal metric exists. In particular in the unstable case where $k_1 \leqslant m \leqslant k_2(k_2+2)$ we define a functional which decreases along the Calabi flow, is bounded below, and whose derivative is given by the difference between the Calabi functional and its infimum. This leads to the convergence result. The case $m > k_2(k_2+2)$ is more delicate since the analogous Mabuchi-type functional is not bounded from below. Nevertheless it has at worst logarithmic decay along the Calabi flow and this is enough to show that the flow minimizes the Calabi functional. This is discussed in Section 6.

As far as the author is aware, Theorem 2 is the first case where the Calabi flow has been successfully analyzed on a manifold which does not admit an extremal metric. We hope that this example will be useful in studying the Calabi flow in general. For example in the Kähler classes which do not admit an extremal metric, we can see that the Sobolev constant and diameter do not remain bounded along the flow. This is in stark contrast to the (normalized) Kähler-Ricci flow, for which the diameter and Sobolev constant remain uniformly bounded for all time (see [19], [27] and [28]).

Note that throughout the paper we have ignored factors of 2π, for example in the definition of the Calabi functional. Also we normalize the Futaki invariant slightly differently from usual in Section 5. Hopefully this will lead to no confusion.

Acknowledgements. – Part of this work has appeared in the author's PhD thesis [21]. I would like to thank my supervisor Simon Donaldson for his encouragement and for sharing his insights.

2. Metrics on the ruled surface

In this section we describe the momentum construction for metrics on the ruled surface (see Hwang-Singer [16]). Let X be the ruled surface as above, so that $X = \mathbf{P}(\mathcal{M} \oplus \mathcal{O}) \to \Sigma$, where Σ is a genus 2 curve, and $\mathcal{M}$ is a degree -1 line bundle over Σ. Let ω_Σ be a metric on Σ with area 2π and constant scalar curvature -2 (we use the "complex" scalar curvature, which is half of the usual Riemannian one). Also, let h be a Hermitian metric on $\mathcal{M}$ with curvature form $i\omega_\Sigma$. We consider metrics on $\mathcal{M} \setminus \{0\}$, the complement of the zero section in the total space of $\mathcal{M}$, of the form

$$\omega = p^*\omega_\Sigma + 2i\partial\bar{\partial} f(s),$$

where $p : \mathcal{M} \to \Sigma$ is the projection map, $s = \frac{1}{2}\log|z|_h^2$ is the logarithm of the fibrewise norm and $f(s)$ is a suitable strictly convex function that makes ω positive definite. The point of the momentum construction is the change of coordinate from s to $\tau = f'(s)$. The metric ω is invariant under the $U(1)$-action on $\mathcal{M}$, and τ is just the moment map for this action. Let $I \subset \mathbf{R}$ be the image of τ, and let $F : I \to \mathbf{R}$ be the Legendre transform of f. By definition this means that

$$f(s) + F(\tau) = s\tau,$$

and F is a strictly convex function. The *momentum profile* is defined to be the function

$$\phi(\tau) = \frac{1}{F''(\tau)}.$$

We have the following relations:

$$s = F'(\tau), \quad \frac{ds}{d\tau} = F''(\tau), \quad \phi(\tau) = f''(s).$$

The metric in local coordinates. – Let us now see what the metric ω looks like in local coordinates. Choose a local coordinate z on Σ and a fibre coordinate w for $\mathcal{M}$. The fibrewise norm is given by $|(z,w)|_h^2 = |w|^2 h(z)$ for some positive function h, so that

$$s = \frac{1}{2}\log|w|^2 + \frac{1}{2}\log h(z).$$

We can choose the local trivialization of $\mathcal{M}$ in such a way that at a point (z_0, w_0) we have $d\log h(z) = 0$. We can then compute at the point (z_0, w_0)

$$\begin{aligned} 2i\partial\bar{\partial} f(s) &= if'(s)\partial\bar{\partial}\log h(z) + f''(s)\frac{i\, dw \wedge d\bar{w}}{2|w|^2} \\ &= \tau p^*\omega_\Sigma + \phi(\tau)\frac{i\, dw \wedge d\bar{w}}{2|w|^2}. \end{aligned}$$