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$\mathcal{C}^0$-RIGIDITY OF CHARACTERISTICS IN SYMPLECTIC GEOMETRY

BY EMMANUEL OPSHTEIN

ABSTRACT. – The paper concerns a $\mathcal{C}^0$-rigidity result for the characteristic foliations in symplectic geometry. A symplectic homeomorphism (in the sense of Eliashberg-Gromov) which preserves a smooth hypersurface also preserves its characteristic foliation.

RÉSUMÉ. – Cet article porte sur un résultat de rigidité $\mathcal{C}^0$ du feuilletage caractéristique en géométrie symplectique. Un homéomorphisme symplectique (au sens d'Eliashberg-Gromov) qui préserve une hypersurface lisse préserve également son feuilletage caractéristique.

Introduction

Gromov and Eliashberg showed that a $\mathcal{C}^0$-limit of symplectic diffeomorphisms which is itself a diffeomorphism is symplectic ([3, 8], see also [9]). This rigidity result leads to the definition of symplectic homeomorphisms (the $\mathcal{C}^0$-limits of symplectic diffeomorphisms which are homeomorphisms), and shows that they define a proper subset of volume preserving homeomorphisms in dimension at least 4. It also raises the question of the survival of the symplectic invariants to this limit process. Which classical invariants of symplectic geometry remain invariants of this maybe softer $\mathcal{C}^0$-symplectic geometry? This paper shows that the characteristic foliation is one of them.

THEOREM 1. – *Let S and S' be smooth hypersurfaces of some symplectic manifolds (M,ω), (M',ω'). Any symplectic homeomorphism between M and M' which sends S to S' transports the characteristic foliation of S to that of S'.*

The characteristic foliation is a symplectic invariant of a given hypersurface S, which can be defined as the integral foliation of the (one dimensional) null space of the restriction of the symplectic form to S. This definition is intrinsically smooth since it involves the tangent spaces of S. But the roles of this foliation in symplectic geometry are many. In particular, one of its rather folkloric properties concerns non-removable intersection: if two smoothly bounded open sets intersect exactly on their boundaries, and if no symplectic perturbation can separate them, then the boundaries share a common closed invariant subset of the characteristic foliation [10, 11, 13, 14]. This paper proceeds from the remark that this property has a meaning also in the continuous category, so defining this foliation in continuous terms is conceivable.

An application of this theorem is a weak answer to a question by Eliashberg and Hofer about the symplectic characterization of a hypersurface by the open set it bounds: *Under which conditions does the existence of a symplectomorphism between two smoothly bounded open sets in symplectic manifolds imply that their boundaries are symplectomorphic also* [6] *?* Previous works show that the only realistic constraints on the domains and their boundaries alone must be very restrictive [1, 2, 5]. In contrast, Theorem 1 can allow to get rid of these conditions on the expense of only considering a special class of symplectomorphisms.

Theorem 2. – *Let U be a smoothly bounded open set in $\mathbb{R}^4$. Assume that there is a symplectomorphism between $B^4(1)$ and U which extends continuously to a homeomorphism between S^3 and ∂U. Then ∂U is symplectomorphic to S^3.*

The paper is organized as follows. We first define symplectic hammers and explain their roles: Theorem 1 proceeds from a localization of their actions along characteristics (Section 1). This localization is proved in section 2. We then present the application in the last section.

Acknowledgments. I wish to thank Leonid Polterovich for making me aware of a serious mistake in the first version of the proof.

1. Symplectic hammers

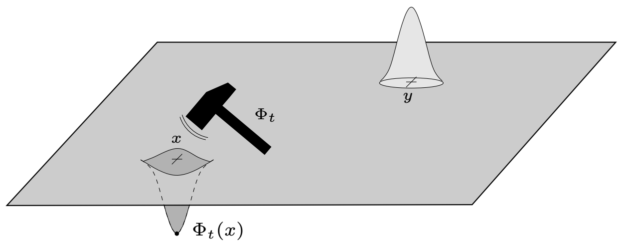
Let S be a hypersurface in a symplectic manifold M. We say that B is a small ball centered on S if it is a symplectic embedding of an euclidean ball centered at the origin into M which sends $\mathbb{R}^{2n-1} := \mathbb{R} \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$ to S. Such a ball is disconnected by S into two components denoted by S_+ and S_-. By a classical result, any point of S is the center of such a ball. Fix also a metric on M in order to refer to small sets.

Definition 1.1. – *Given two points x, y on $S \cap B$ and a (small) positive real ε, an ε-symplectic hammer between x and y with support in B is a continuous path of symplectic homeomorphisms Φ_t ($t \in [0,1]$) with common supports in B, and for which there exist two open sets $U_\varepsilon(x)$ and $U_\varepsilon(y)$ contained in the ε-balls around x and y respectively such that:*

1. $\Phi_0 = Id$,
2. $\Phi_t(z) \in S_+$ *for all* $t \in]0,1]$ *and* $z \in S \cap U_\varepsilon(x)$,
3. $\Phi_t(z) \in S_-$ *for all* $t \in]0,1]$ *and* $z \in S \cap U_\varepsilon(y)$,
4. $\Phi_t(z) \in S$ *for all* $t \in [0,1]$ *and* $z \in S \backslash \big(U_\varepsilon(x) \cup U_\varepsilon(y)\big)$.

A smooth hammer will refer to a smooth isotopy of smooth symplectomorphisms verifying the four conditions above.

In other terms, Φ_t preserves the hypersurface S except for two bumps in opposite sides (a symmetry is necessary in view of the volume preservation).



One can easily construct examples of symplectic hammers.

Proposition 1.2. – *If $x, y \in B \cap S$ lie in the same characteristic, there exist ε-symplectic hammers between x and y for all $\varepsilon > 0$.*

Proof. – Since all hypersurfaces are locally symplectically the same, it is enough to produce a symplectic hammer for $\mathbb{R}^{2n-1} = \{\operatorname{Im} z_1 = 0\} \subset \mathbb{C}^n$ between the points $p = 0$ and $q = (1/2, 0, \ldots, 0)$. Putting $x_1 = \operatorname{Re} z_1$, $y_1 = \operatorname{Im} z_1$ and $r_i = |z_i|$, consider a Hamiltonian of the following type.

$$H(z_1, \ldots, z_n) := \chi(y_1)\rho(x_1)\Pi_{i=2}^n f(r_i).$$

If χ, ρ and f are the bell functions represented in Figure 2, and maybe multiplying H by a small constant in order to slow the flow down produces a symplectic hammer between x and y. $\square$

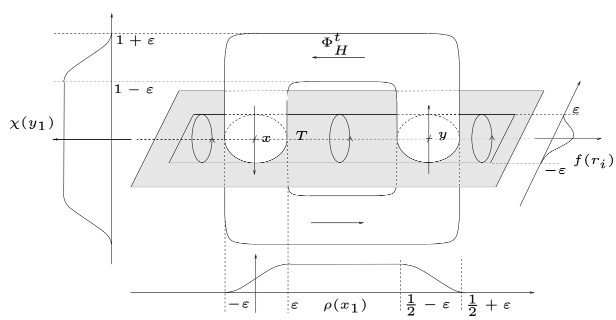


Figure 2. The Hamiltonian flow of H in the proof of Proposition 1.2.

Proposition 1.2 can easily be reversed in the smooth category: two points lie in the same characteristic leaf of $S \cap B$ if and only if there exist smooth symplectic hammers between them. It is less obvious, but still true that *all* the symplectic hammers also meet this constraint. Theorem 1 obviously follows because the class of symplectic hammers is preserved by symplectic homeomorphisms.

Proposition 1.3. – *A hypersurface S and a small ball B centered on S being given, there exists an ε-hammer between $x, y \in B \cap S$ with support in B for all small ε if and only if x and y are on the same characteristic leaf.*

Proof of Theorem 1 (assuming Proposition 1.3). – Let M, M', S, S' and Φ be as in Theorem 1, and put any metric on M and M'. Consider two points $x, y \in S$ which lie in the same characteristic. Consider a covering $\mathcal{B} = \{B_\alpha\}$ of S by small balls (in the above sense) whose images by Φ are contained in small balls B'_α centered on S'. Let $(x_i)_{i\leq N}$ be a chain between x and y (that is $x_0 = x, x_N = y$) such that x_i and x_{i+1} are always in a same ball B_i. Then there exist ε-hammers $\Phi_t^{(\varepsilon)}$ with supports in B_i between x_i and x_{i+1} for all ε. The isotopies $\Phi \circ \Phi_t^{(\varepsilon)} \circ \Phi^{-1}$ define continuous $\delta(\varepsilon)$-symplectic hammers with support in B'_i between $\Phi(x_i)$ and $\Phi(x_{i+1})$, where $\delta(\varepsilon)$ goes to zero with ε. Therefore by Proposition 1.3, $\Phi(x_i)$ and $\Phi(x_{i+1})$ are on the same characteristic, so $\Phi(x)$ and $\Phi(y)$ are also on the same characteristic. □

2. Proof of Proposition 1.3

The idea is the following. Since preserving a foliation is a local property, and since all hypersurfaces are locally the same in the symplectic world, we could translate the non-preservation of *one* characteristic by a symplectic homeomorphism to the existence of a local, hence universal object (a hammer between points on distinct characteristics) which would exist on all hypersurfaces. These continuous hammers would allow to break intersections between open sets as long as these intersections only consist of one characteristic. But some such intersections are known to be non-removable: the most famous one being the intersection between the complement of the cylinder $Z(1)$ and the closed ball $B^{2n}(1)$.

LEMMA 2.1. – *If Proposition 1.3 does not hold, then for any point x of the euclidean sphere $S^{2n-1} \subset \mathbb{C}^n$ and for any positive (small) ε, there exists a continuous ε-symplectic hammer between x and a point y which does not lie in the characteristic circle passing through x.*

Proof. – Assume that Proposition 1.3 does not hold. Then there exists a small ball centered on a hypersurface S, two points $p, q \in S \cap B$ which are not in the same characteristic of $S \cap B$ and a family $\Phi^{(\varepsilon)} := (\Phi_t^{(\varepsilon)})_{t\in[0,1]}$ of ε-symplectic hammers with supports in B between p and q. By definition of a small ball, there is a symplectic diffeomorphism Ψ_1 between B and an euclidean ball $B_1 \subset \mathbb{C}^n$ around the origin with $\Psi(S \cap B) = \mathbb{R}^{2n-1} \cap B_1$. Then Ψ_1 takes $\Phi(\varepsilon)$ to an ε-hammer between $\Psi_1(p)$ and $\Psi_1(q)$ which are not on the same characteristic. By use of translation and rescaling, we can assume that $\Psi_1(p)$ is the origin and B_1 is as small a neighbourhood of 0 as wished.

Now given the point $x \in S^{2n-1}$, and if B_1 is small enough, there exists a symplectic diffeomorphism $\Psi_2 : B_1 \longrightarrow \mathbb{C}^n$ with $\Psi_2(B_1 \cap \mathbb{R}^{2n-1}) \subset S^{2n-1}$, $\Psi_2(0) = x$ and such that different characteristics of $\mathbb{R}^{2n-1} \cap B_1$ are sent by Ψ_2 not only to different characteristics of $S^{2n-1} \cap \Psi_2(B_1)$ but even of S^{2n-1} (this means that we do not allow Ψ_2 to "bend" B_1 so as to take two different characteristics to two different segments of a same characteristic circle of S^{2n-1}). The continuous symplectic isotopies obtained by transporting $\Phi^{(\varepsilon)}$ by $\Psi_2 \circ \Psi_1$ are ε-hammers between x and the point $y := \Psi_2(\Psi_1(q))$ which is not on the characteristic through x. □