ISSN 0012-9593

quatrième série - tome 42　　*fascicule 6*　　*novembre-décembre 2009*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Artur AVILA & Jairo BOCHI & Amie WILKINSON

Nonuniform center bunching and the genericity of ergodicity among C^1 partially hyperbolic symplectomorphisms

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 42, 2009, p. 931 à 979

NONUNIFORM CENTER BUNCHING AND THE GENERICITY OF ERGODICITY AMONG C^1 PARTIALLY HYPERBOLIC SYMPLECTOMORPHISMS

BY Artur AVILA, Jairo BOCHI AND Amie WILKINSON

ABSTRACT. – We introduce the notion of nonuniform center bunching for partially hyperbolic diffeomorphims, and extend previous results by Burns–Wilkinson and Avila–Santamaria–Viana. Combining this new technique with other constructions we prove that C^1-generic partially hyperbolic symplectomorphisms are ergodic. We also construct new examples of stably ergodic partially hyperbolic diffeomorphisms.

RÉSUMÉ. – Nous introduisons une notion non-uniforme de resserrement central pour les difféomorphismes partiellement hyperboliques qui nous permet de généraliser quelques résultats de Burns–Wilkinson et Avila–Santamaria–Viana. Cette nouvelle technique est utilisée, en combinaison avec d'autres constructions, pour démontrer la généricité de l'ergodicité parmi les difféomorphismes symplectiques partiellement hyperboliques de classe C^1. De plus, nous obtenons de nouveaux exemples de dynamiques stablement ergodiques.

1. Introduction

1.1. Abundance of ergodicity

Let (M,ω) be a closed (i.e., compact without boundary) symplectic C^∞ manifold of dimension $2N$. Let $\mathrm{Diff}^1_\omega(M)$ be the space of ω-preserving C^1 diffeomorphisms, endowed with the C^1 topology. Let m be the measure induced by the volume form $\omega^{\wedge N}$, normalized so that $m(M)=1$.

Let $PH^1_\omega(M)$ be the set of diffeomorphisms $f \in \mathrm{Diff}^1_\omega(M)$ that are partially hyperbolic, i.e., there exist an invariant splitting $T_xM = E^u(x)\oplus E^c(x)\oplus E^s(x)$, into nonzero bundles, and a positive integer k such that for every $x\in M$,

$$\|(Df^k|E^u(x))^{-1}\|^{-1} > 1 > \|Df^k|E^s(x)\|\,,$$
$$\|(Df^k|E^u(x))^{-1}\|^{-1} > \|Df^k|E^c(x)\| \geq \|(Df^k|E^c(x))^{-1}\|^{-1} > \|Df^k|E^s(x)\|\,. \tag{1.1}$$

Such a splitting is automatically continuous.

THEOREM A. – *The set of ergodic diffeomorphisms is residual in $PH^1_\omega(M)$.*

Our result is motivated by the following well-known conjecture of Pugh and Shub [26]: *There is a C^2 open and dense subset of the space of C^2 volume-preserving partially hyperbolic diffeomorphisms formed by ergodic maps*. Among the known results in this direction, we have:

- F. and M. A. Rodriguez-Hertz, and Ures [29] proved that C^r-stable ergodicity is dense among C^r volume-preserving partially hyperbolic diffeomorphisms with one-dimensional center bundle, for all $r \geq 2$. (See also [14] for an earlier result.)
- F. and M. A. Rodriguez-Hertz, Tahzibi, and Ures [28] proved that ergodicity holds on a C^1 open and dense subset of the C^2 volume-preserving partially hyperbolic diffeomorphisms with two-dimensional center bundle.

Together with the result from Avila [7], it follows that ergodicity is C^1 generic among volume-preserving partially hyperbolic diffeomorphisms with center dimension at most 2. On the other hand, the techniques yielding the results above seem less effective for the understanding of the case of symplectic maps, and indeed Theorem A is the first result on denseness of ergodicity for non-Anosov partially hyperbolic symplectomorphisms, even allowing for constraints on the center dimension. Our approach develops some new tools of independent interest, as we explain next.

1.2. Center bunching properties

To support their conjecture, Pugh and Shub [26] provided a criterion for a volume-preserving partially hyperbolic map to be ergodic, based on the property of *accessibility*, together with some technical hypotheses. A significantly improved version of this criterion was obtained by Burns and Wilkinson [18]: accessibility and *center bunching* imply ergodicity. Dolgopyat and Wilkinson [19] showed that accessibility is open and dense in the C^1 topology, but center bunching is not a dense condition unless the center dimension is 1 (which cannot happen for symplectic maps). In this paper we introduce and exploit a weaker condition, called *nonuniform center bunching*.

In the context of general (not necessarily volume-preserving) partially hyperbolic diffeomorphisms, the center bunching hypothesis in [18] is a global, uniform property, requiring that at every point in the manifold, the nonconformality of the action on the center bundle be dominated by the hyperbolicity in both the stable and unstable bundles. By contrast, the nonuniform center bunching property introduced here is a property of asymptotic nature about the orbit of a single point; it is the intersection of a forward bunching property of the forward orbit and a backward bunching property of the backward orbit. The precise definitions are slightly technical (see Section 2). However, for Lyapunov regular points (which by Oseledets' theorem have full probability), forward (resp. backward) center bunching means that the biggest difference between the Lyapunov exponents in the center bundle is smaller than the absolute value of the exponents in the stable (resp. unstable) bundle. The set CB^+ of forward center bunched points for a partially hyperbolic diffeomorphism f has the useful property of being $\mathcal{W}^s$-*saturated*, meaning that it is a union of entire stable manifolds of f; similarly the set CB^- of backward center bunched points is $\mathcal{W}^u$-saturated, i.e. a union of unstable manifolds.

Our next main result, Theorem B, generalizes the core result of [18] (Theorem 5.1 of that paper). It states that for any C^2 partially hyperbolic diffeomorphism, *the set of Lebesgue density points of any bi essentially saturated set meets CB^+ in a $\mathcal{W}^s$-saturated set and CB^- in a $\mathcal{W}^u$-saturated set.* (A bi essentially saturated set is one that coincides mod 0 with a $\mathcal{W}^s$-saturated set and mod 0 with a $\mathcal{W}^u$-saturated set.)

Burns and Wilkinson [18] obtain their ergodicity criterion as a simple consequence of their technical core result. Indeed, assuming accessibility (or even essential accessibility), ergodicity in [18] follows in one step from the core result, using a Hopf argument; it is not necessary to establish local ergodicity first (as one does in proving ergodicity for hyperbolic systems). It is unclear to us whether the Burns–Wilkinson criterion for ergodicity can be improved by replacing uniform center bunching by almost everywhere nonuniform center bunching, in part because the uniform version in [18] is by nature *not* a "local ergodicity" result. In reality, it is possible to deduce a new ergodicity criterion (Corollary C) from Theorem B. Namely, ergodicity follows from almost everywhere nonuniform center bunching together with a stronger form of essential accessibility, where we only allow su-paths whose corners are center-bunched points. While this accessibility condition is far from automatic, it can be verified in some interesting classes of examples: see §1.4 below.

1.3. Outline of the proof of Theorem A

Let us explain how nonuniform center bunching combines with other ingredients to yield Theorem A. Take a symplectomorphism with the following C^1 generic properties:

(a) it is stably accessible, by Dolgopyat and Wilkinson [19];
(b) all central Lyapunov exponents vanish at almost every point, by Bochi [9].

Notice that property (b) implies almost every point is center bunched. But Theorem B requires C^2 regularity. This is achieved by taking a perturbation, which still has property (a), but loses property (b). What happens is that each point in some set of measure close to 1 has small center Lyapunov exponents and thus is center bunched.

Before getting useful consequences from Theorem B, we need to provide a local source of ergodicity. This is achieved through a novel application of the Anosov–Katok [2] examples. (By comparison, [28] uses Bonatti–Díaz blenders.) We proceed as follows. By perturbing, we find a periodic point whose center eigenvalues have unit modulus. Perturbing again, we create a disk tangent to the center direction that is invariant by a power of the map. We can choose any dynamics close to the identity on this disk, so we select an ergodic Anosov–Katok map. Ergodicity is spread from the center disk to a ball around the periodic point using Theorem B, and then to the whole manifold by accessibility. (In fact, since the set of center bunched points is not of full measure, a G_δ argument is necessary to conclude ergodicity – see Section 3 for the precise procedure.)

1.4. Further applications of nonuniform center bunching

By means of our ergodicity criterion (Corollary C) we construct an example of a stably ergodic partially hyperbolic diffeomorphism that is almost everywhere nonuniformly center bunched (but not center bunched in the sense of [18]) in a robust way.

We also prove in this paper an extension of Theorem B to sections of bundles over partially hyperbolic diffeomorphisms. This result, Theorem D, brings into the nonuniform setting a recent result of Avila, Santamaria and Viana [8], which they use to show that the generic bunched $SL(n,\mathbb{R})$ cocycle over an accessible, center bunched, volume-preserving partially hyperbolic diffeomorphism has a nonvanishing exponent. The result from [8] has also been used in establishing measurable rigidity of solutions to the cohomological equation over center-bunched systems; see [33]. Theorem D has similar applications in the setting where nonuniform center bunching holds, and we detail some of them in Section 6.

We conceive that our methods may be further extended to apply in certain "singular partially hyperbolic" contexts where partial hyperbolicity holds on an open, noncompact subset of the manifold M but decays in strength near the boundary. Such conditions hold, for example, for geodesic flows on certain nonpositively curved manifolds. Under suitable accessibility hypotheses, these systems should be ergodic with respect to volume.

1.5. Questions

Combining results of [19] and Brin [15], one obtains that topological transitivity holds for a C^1 open and dense set of partially hyperbolic symplectomorphisms. On the other hand, the C^1-interior of the ergodic symplectomorphisms is contained in the partially hyperbolic diffeomorphisms [22, 30]. This suggests the following natural question.

QUESTION 1. – Can Theorem A be improved to an open (and dense) instead of residual set?

Notice that it is not known even whether the set of C^1 Anosov ergodic maps has nonempty interior.

Dropping partial hyperbolicity, recall that C^1 generic symplectic and volume-preserving diffeomorphisms are transitive by [5] and [11], while ergodicity is known to be C^0-generic among volume-preserving homeomorphisms by [24]. So the following well-known question arises:

QUESTION 2. – Is ergodicity generic among C^1 symplectic and volume-preserving diffeomorphisms?

1.6. Organization of the paper

In Section 2 we define nonuniform center bunching, state Theorem B, and derive Corollary C from it.

In Section 3 we prove Theorem A following the outline given in §1.3. As we have explained, the proof uses the existence (after perturbation) of a periodic point with elliptic central behavior. Such a result goes along the lines of [12, 22, 30], but we have not been able to find a precise reference. In Section 4, which can be read independently from the rest of the paper, we provide a proof of this result by reducing it to its ergodic counterpart and applying the Ergodic Closing Lemma. This approach is different from the one taken in the literature. For this reason, we included an appendix explaining how to use it to reobtain some results from [12].

The proof of Theorem B, despite having much in common with [18], is given here in full detail in Section 5. In Section 6 we formulate and prove the more general Theorem D. The new examples of stably ergodic maps are constructed in Section 7.