ISSN 0012-9593

quatrième série - tome 43 fascicule 6 novembre-décembre 2010

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 43, 2010, p. 1017 à 1038

ON A CONJECTURE OF KOTTWITZ AND RAPOPORT

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To Bob Kottwitz, my dedicated teacher and mentor, with profound gratitude and admiration.

ABSTRACT. – We prove a conjecture of Kottwitz and Rapoport which implies a converse to Mazur's Inequality for all (connected) split and quasi-split unramified reductive groups. Our results are related to the non-emptiness of certain affine Deligne-Lusztig varieties.

RÉSUMÉ. – On démontre une conjecture de Kottwitz et Rapoport sur une réciproque à l'inégalité de Mazur pour tout groupe (connexe) réductif, déployé ou quasi-déployé non-ramifié. Nos résultats sont liés à la non-vacuité de certaines variétés de Deligne-Lusztig affines.

1. Introduction

Mazur's Inequality ([14], [15]) is related to the study of p-adic estimates of the number of points of certain algebraic varieties over a finite field of characteristic p. It is most easily stated using isocrystals. Before stating the precise inequality, we recall the definition of an isocrystal: it is a pair (V, Φ), where V is a finite-dimensional vector space over the fraction field K of the ring of Witt vectors $W(\overline{\mathbb{F}}_p)$, and Φ is a σ-linear bijective endomorphism of V, where σ is the automorphism of K induced by the Frobenius automorphism of $\overline{\mathbb{F}}_p$. Next, we recall Mazur's inequality.

Suppose that (V, Φ) is an isocrystal of dimension n. By Dieudonné-Manin theory, we can associate to V its Newton vector

$$\nu(V, \Phi) \in (\mathbb{Q}^n)_+ := \{(\nu_1, \ldots, \nu_n) \in \mathbb{Q}^n : \nu_1 \geq \nu_2 \geq \cdots \geq \nu_n\},$$

which classifies isocrystals of dimension n up to isomorphism. If Λ is a $W(\overline{\mathbb{F}}_p)$-lattice in V, then we can associate to Λ the Hodge vector $\mu(\Lambda) \in (\mathbb{Z}^n)_+ := (\mathbb{Q}^n)_+ \cap \mathbb{Z}^n$, which measures the relative position of the lattices Λ and $\Phi(\Lambda)$. Let $\nu(V, \Phi) := (\nu_1, \ldots, \nu_n)$ and $\mu(\Lambda) := (\mu_1, \ldots, \mu_n)$. Mazur's Inequality asserts that $\mu(\Lambda) \geq \nu(V, \Phi)$, where $\geq$ is *the* *dominance order*, i.e., $\mu_1 \geq \nu_1$, $\mu_1 + \mu_2 \geq \nu_1 + \nu_2, \cdots, \mu_1 + \ldots \mu_{n-1} \geq \nu_1 + \ldots \nu_{n-1}$, and $\mu_1 + \ldots \mu_n = \nu_1 + \ldots \nu_n$.

A converse to this inequality is proved by Kottwitz and Rapoport in [11], where they show that if (V, Φ) is an isocrystal of dimension n and $\mu \in (\mathbb{Z}^n)_+$ is such that $\mu \geq \nu(V, \Phi)$, then there exists a $W(\overline{\mathbb{F}}_p)$-lattice Λ in V satisfying $\mu = \mu(\Lambda)$.

Both Mazur's Inequality and its converse can be regarded as statements for the group GL_n, since the dominance order arises naturally in the context of the root system for GL_n. In fact, there is a bijection (see [9]) between isomorphism classes of isocrystals of dimension n and the set of σ-conjugacy classes in $GL_n(K)$. Kottwitz studies in ibid. the set $B(G)$ of the σ-conjugacy classes in $G(K)$, for a connected reductive group G over $\mathbb{Q}_p$, and, as he notes, there is a bijection between $B(G)$ and the isomorphism classes of isocrystals with "G-structure" of a certain dimension (for $G = GL_n$ these are simply the above isocrystals). Thus, results on isocrystals, and more generally isocrystals with additional structure, are related to those on the σ-conjugacy classes of certain reductive groups.

With this viewpoint in mind, we are interested in the group-theoretic generalizations of Mazur's Inequality and its converse, especially since they appear naturally in the study of the non-emptiness of certain affine Deligne-Lusztig varieties. To make these statements more precise, we introduce some notation.

Let F be a finite extension of $\mathbb{Q}_p$, with uniformizing element π, and let $\mathfrak{o}_F$ be the ring of integers of F. Suppose that G is a split connected reductive group over F (unramified quasi-split groups are treated in the last section of the paper). Let B be a Borel subgroup in G and T a maximal torus in B, both defined over $\mathfrak{o}_F$. Let L be the completion of the maximal unramified extension of F in some algebraic closure of F, and σ the Frobenius automorphism of L over F. The valuation ring of L is denoted by $\mathfrak{o}_L$.

We write X for the group of co-characters $X_*(T)$. Let $\mu \in X$ be a dominant element and $b \in G(L)$. The affine Deligne-Lusztig variety $X^G_\mu(b)$ is defined by

$$X^G_\mu(b) := \{x \in G(L)/G(\mathfrak{o}_L) : x^{-1} b \sigma(x) \in G(\mathfrak{o}_L)\mu(\pi)G(\mathfrak{o}_L)\}.$$

These p-adic "counterparts" of the classical Deligne-Lusztig varieties get their name by virtue of being defined in a similar way as the latter, and have been studied by a number of authors (see, for example, [8], [7], [23], and references therein). For the relevance of affine Deligne-Lusztig varieties to Shimura varieties, the reader may wish to consult [18].

We need some more notation to be able to formulate the group-theoretic generalizations of Mazur's Inequality and its converse. Let $P = MN$ be a parabolic subgroup of G that contains B, where M is the unique Levi subgroup of P containing T. The Weyl group of T in G is denoted by W. We let X_G and X_M be the quotient of X by the coroot lattice for G and M, respectively. Also, we let $\varphi_G : X \to X_G$ and $\varphi_M : X \to X_M$ denote the respective natural projection maps.

Let $B = TU$, with U the unipotent radical. If $g \in G(L)$, then there is a unique element of X, denoted by $r_B(g)$, so that $g \in G(\mathfrak{o}_L) r_B(g)(\pi)U(L)$. We have a well-defined map $w_G : G(L) \to X_G$, the Kottwitz map [9], where for $g \in G(L)$, we write $w_G(g)$ for the image of $r_B(g)$ under the canonical surjection $X \to X_G$. In a completely analogous way, considering M instead of G, one defines the map $w_M : M(L) \to X_M$.

We use the partial ordering $\overset{P}{\leq}$ in X_M, where for $\mu, \nu \in X_M$, we write $\nu \overset{P}{\leq} \mu$ if and only if $\mu - \nu$ is a nonnegative integral linear combination of the images in X_M of the coroots corresponding to the simple roots of T in N.

We will make use of a subset X_M^+ of X_M, which we now define. Let $\mathfrak{a}_P := X_*(T_P) \otimes_{\mathbb{Z}} \mathbb{R}$, where T_P is the identity component of the center of M (and thus T_P is a split torus over F). Note that there is a canonical isomorphism $\varrho : \mathfrak{a}_P \simeq X_M \otimes_{\mathbb{Z}} \mathbb{R}$ obtained by tensoring with $\mathbb{R}$ the composition $X_*(T_P) \hookrightarrow X_*(T) \twoheadrightarrow X_M$. Let $\xi : X_M \to X_M \otimes_{\mathbb{Z}} \mathbb{R}$ be the natural map. The subset $X_M^+ \subset X_M$ is defined as the set of all elements $\nu \in X_M$ such that $(\varrho^{-1} \circ \xi)(\nu)$ lies in the subset

$$\{x \in \mathfrak{a}_P : \langle \alpha, x \rangle > 0, \text{ for every root } \alpha \text{ of } T_P \text{ in } N\} \subset \mathfrak{a}_P.$$

The pairing $\langle\, ,\rangle$ appearing in the last line is induced by the usual one between weights and coweights of T_P.

Next, let $b \in M(L)$. We recall briefly the notion of b being basic (for further details see [9]): In loc. cit., §4, Kottwitz defines a map $\epsilon : M(L) \to \mathrm{Hom}_L(\mathbb{D}, M)$, which he denotes by ν, and where $\mathbb{D}$ is the diagonalizable pro-algebraic group over $\mathbb{Q}_p$ whose character group is $\mathbb{Q}$. An element $b \in M(L)$ is called *basic* if $\epsilon(b) \in \mathrm{Hom}_L(\mathbb{D}, M)$ factors through the center of M. The element $\epsilon(b)$ is linked with the *slopes* of the isocrystal corresponding to b. Let us mention that $\epsilon(b)$ is characterized by the existence of an integer $n > 0$, an element $c \in M(L)$ and a uniformizing element π of F such that the following three conditions hold:

$$n\epsilon(b) \in \mathrm{Hom}_L(\mathbb{G}_m, M),$$

$$\mathrm{Int}(c) \circ (n\epsilon(b)) \text{ is defined over the fixed field of } \sigma^n \text{ on } L, \text{ and}$$

$$c(b\sigma)^n c^{-1} = c \cdot (n\epsilon(b))(\pi) \cdot c^{-1} \cdot \sigma^n,$$

where $\mathrm{Int}(c)$ denotes the inner automorphism $x \mapsto cxc^{-1}$ of $M(L)$, and where we recall that σ is the Frobenius of L over F.

We now state the first main result of this paper.

THEOREM 1.1. – *Let $\mu \in X$ be dominant and let $b \in M(L)$ be a basic element such that $w_M(b)$ lies in X_M^+. Then*

$$X_\mu^G(b) \neq \varnothing \Longleftrightarrow w_M(b) \overset{P}{\leq} \varphi_M(\mu).$$

We prove a similar theorem for quasi-split unramified groups. The precise formulation (Theorem 5.1) and the proof of that result is postponed until the last section of the paper.

We remark that since every σ-conjugacy class in $G(L)$ contains an element that is basic in some standard Levi subgroup M (see [9]), Theorem 1.1 proves the non-emptiness of the affine Deligne-Lusztig varieties $X_\mu^G(b)$, where $b \in G(L)$.

One direction in the theorem, namely

$$X_\mu^G(b) \neq \varnothing \Longrightarrow w_M(b) \overset{P}{\leq} \varphi_M(\mu),$$

is the group-theoretic generalization of Mazur's Inequality, and it is proved by Rapoport and Richartz in [19] (see also [10, Theorem 1.1, part (1)]).

The other direction, i.e., the group-theoretic generalization of the converse to Mazur's Inequality, is a conjecture of Kottwitz and Rapoport [11]. Next, we discuss how their conjecture is reduced to one formulated only in terms of root systems. Let

$$\mathscr{P}_\mu := \{\nu \in X : \text{(i) } \varphi_G(\nu) = \varphi_G(\mu); \text{ and } \text{(ii) } \nu \in \mathrm{Conv}\,(W\mu)\},$$

where $\mathrm{Conv}\,(W\mu)$ is the convex hull of the Weyl orbit $W\mu := \{w(\mu) : w \in W\}$ of μ in $\mathfrak{a} := X \otimes_{\mathbb{Z}} \mathbb{R}$. Then we have (cf. [10, Theorem 4.3])

$$X^G_\mu(b) \neq \varnothing \Longleftrightarrow w_M(b) \in \varphi_M(\mathscr{P}_\mu).$$

Thus the other implication in Theorem 1.1 follows if we show that

$$w_M(b) \overset{P}{\leq} \varphi_M(\mu) \Longrightarrow w_M(b) \in \varphi_M(\mathscr{P}_\mu).$$

For this, it suffices to show that for $\nu \in X_M$ we have

$$\nu \overset{P}{\leq} \varphi_M(\mu) \Longrightarrow \nu \in \varphi_M(\mathscr{P}_\mu). \tag{1}$$

(Note that the condition from Theorem 1.1 that $b \in M(L)$ be basic does not appear in the last implication. Also, we do not require that $\nu \in X^+_M$, but only that $\nu \in X_M$.) As can be seen from [10, Section 4.4], we have

$$\nu \overset{P}{\leq} \varphi_M(\mu) \Longleftrightarrow \begin{cases} \text{(i) } \nu \text{ and } \mu \text{ have the same image in } X_G, \text{ and} \\ \text{(ii) the image of } \nu \text{ in } \mathfrak{a}_M \text{ lies in } \mathrm{pr}_M\,(\mathrm{Conv}\,(W\mu))\,. \end{cases} \tag{2}$$

Taking into account (2), the implication (1) can be reformulated:

$$\left.\begin{array}{l} \text{(i) } \nu \text{ and } \mu \text{ have the same image in } X_G, \text{ and} \\ \text{(ii) the image of } \nu \text{ in } \mathfrak{a}_M \text{ lies in } \mathrm{pr}_M\,(\mathrm{Conv}\,(W\mu)) \end{array}\right\} \Longrightarrow \nu \in \varphi_M(\mathscr{P}_\mu). \tag{3}$$

The implication (3) follows from

THEOREM 1.2 (Kottwitz-Rapoport Conjecture; split case). – *We have that*

$$\varphi_M\,(\mathscr{P}_\mu) = \{\nu \in X_M : \text{(i) } \nu, \mu \text{ have the same image in } X_G; \quad \text{(ii) \textit{the image of} } \nu \text{ in } \mathfrak{a}_M \text{ \textit{lies in} } \mathrm{pr}_M\,(\mathrm{Conv}\,(W\mu))\},$$

where $\mathfrak{a}_M := X_M \otimes_{\mathbb{Z}} \mathbb{R}$ *and* $\mathrm{pr}_M : \mathfrak{a} \to \mathfrak{a}_M$ *denotes the natural projection induced by* φ_M.

For the above theorem, it is easily seen that the set on the left-hand side is contained in the set on the right-hand side. The point is to prove the converse, which is equivalent to the implication (3).

A variant of Theorem 1.2, in the case of quasi-split unramified groups, is proved in the last section (see Theorem 5.2). We remark that Theorem 1.2 is a statement that is purely a root-theoretic one, so it remains true when we work over other fields of characteristic zero, not just $\mathbb{Q}_p$.

Theorem 1.2 had been previously proved for GL_n and GSp_{2n} by Kottwitz and Rapoport [11] and then for all classical groups by Lucarelli [12]. In addition, Wintenberger, using different methods, proved this result for μ minuscule (see [24]). A more general version of this theorem for GL_n was proved in [5, Theorem A] using the theory of toric varieties. (For more details about the precise relation between Theorem 1.2 and cohomology-vanishing on toric varieties associated with root systems see [5], [4].)