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MULTI-HARNACK SMOOTHINGS OF REAL PLANE BRANCHES

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ABSTRACT. – Let $\Delta \subset \mathbf{R}^2$ be an integral convex polygon. G. Mikhalkin introduced the notion of *Harnack curves*, a class of real algebraic curves, defined by polynomials supported on Δ and contained in the corresponding toric surface. He proved their existence, via *Viro's patchworking* method, and that the topological type of their real parts is unique (and determined by Δ). This paper is concerned with the description of the analogous statement in the case of a smoothing of a real plane branch $(C, 0)$. We introduce the class of *multi-Harnack* smoothings of $(C, 0)$ by passing through a resolution of singularities of $(C, 0)$ consisting of g monomial maps (where g is the number of characteristic pairs of the branch). A multi-Harnack smoothing is a g-parametrical deformation which arises as the result of a sequence, beginning at the last step of the resolution, consisting of a suitable *Harnack smoothing* (in terms of Mikhalkin's definition) followed by the corresponding monomial blow down. We prove then the unicity of the topological type of a multi-Harnack smoothing. In addition, the multi-Harnack smoothings can be seen as *multi-semi-quasi-homogeneous* in terms of the parameters. Using this property we analyze the asymptotic *multi-scales* of the ovals of a multi-Harnack smoothing. We prove that these scales characterize and are characterized by the equisingularity class of the branch.

RÉSUMÉ. – Soit $\Delta \subset \mathbf{R}^2$ un polygone convexe à sommets entiers ; G. Mikhalkin a défini les « courbes de Harnack » (définies par un polynôme de support contenu dans Δ et plongées dans la surface torique correspondante) et montré leur existence (via la « méthode du patchwork de Viro ») ainsi que l'unicité de leur type topologique plongé (qui est determiné par Δ). Le but de cet article est de montrer un résultat analogue pour la lissification (*smoothing*) d'un germe de branche réelle plane (C, O) analytique réelle. On définit pour cela une classe de *smoothings* dite « Multi-Harnack » à l'aide de la résolution des singularités constituée d'une suite de g éclatements toriques, si g est le nombre de paires de Puiseux de la branche (C, O). Un *smoothing* multi-Harnack est réalisé de la manière suivante : à chaque étape de la résolution (en commençant par la dernière) et de manière successive, un *smoothing* « De Harnack » (au sens de Mikhalkin) intermédiaire est obtenu par la méthode de Viro. On montre alors l'unicité du type topologique de tels *smoothings*. De plus, on peut supposer ces *smoothings* « multi-semi-quasi homogènes » ; on montre alors que des propriétés métriques (« multi-taille » des ovales) de tels *smoothings* sont caractérisées en fonction de la classe d'équisingularité de (C, O) et que réciproquement ces tailles caractérisent la classe d'équisingularité de la branche.

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Introduction

The 16^{th} problem of Hilbert addresses the determination and the understanding of the possible topological types of smooth real algebraic curves of a given degree in the projective plane $\mathbf{R}P^2$. This paper is concerned with a local version of this problem: given a germ $(C, 0)$ of real algebraic plane curve singularity, determine the possible topological types of the smoothings of C. A *smoothing* of C is a real analytic family of plane curves C_t for $t \in [0, 1]$, such that $C_0 = C$ and for $0 < t \ll 1$ the curve $C_t \cap B$ is nonsingular and transversal to the boundary of a Milnor ball B of the singularity $(C, 0)$. In this case the real part $\mathbf{R}C_t$ of C_t intersected with the interior of B consists of finitely many ovals and non-compact components.

In the algebraic case it was shown by Harnack that a real plane projective curve of degree d has at most $\frac{1}{2}(d-1)(d-2)+1$ connected components. A smooth curve with this number of components is called a M-curve. In the local case there is a similar bound, which arises from the application of the classical topological theory of Smith. A smoothing which reaches this bound on the number of connected components is called a M-smoothing. It should be noticed that in the local case M-smoothings do not always exist (see [17]). One relevant open problem in the theory is to determine the actual maximal number of components of a smoothing of $(C, 0)$, for C running in a suitable form of equisingularity class refining the classical notion of Zariski of equisingularity class in the complex case (see [19]).

Quite recently Mikhalkin proved a beautiful topological rigidity property of those M-curves in $\mathbf{R}P^2$ which are embedded in *maximal position* with respect to the coordinate lines (see [22] and Section 4). His result, which holds more generally, for those M-curves in projective toric surfaces which are cyclically in maximal position with respect to the toric coordinate lines, is proved by analyzing the topological properties of the associated amoebas. The *amoeba* of a curve $C \subset (\mathbf{C}^*)^2$ is the image of it by the map $\mathrm{Log} : (\mathbf{C}^*)^2 \to \mathbf{R}^2$, given by $(x, y) \mapsto (\log|x|, \log|y|)$. Conceptually, the amoebas are intermediate objects which lay in between classical algebraic curves and tropical curves. See [5, 8, 14, 15, 22, 23, 28] for more on this notion and its applications.

In this paper we study smoothings of a *real plane branch* singularity $(C, 0)$, *i.e.*, C is a real algebraic plane curve such that the germ $(C, 0)$ is analytically irreducible in $(\mathbf{C}^2, 0)$. Risler proved that any such germ $(C, 0)$ admits an M-smoothing with the maximal number ovals, namely $\frac{1}{2}\mu(C)_0$, where μ denotes the Milnor number. The technique used, called nowadays the blow-up method, is a generalization of the classical Harnack construction of M-curves by small perturbations, which uses the components of the exceptional divisor as a basis of rank one (see [30], [18] and [19]). One of our motivations was to study to which extent Mikhalkin's result holds for smoothings of singular points of real algebraic plane curves, particularly for *Harnack smoothings*, those M-smoothings which are in maximal position with good oscillation with respect to two coordinates lines through the singular point.

We develop a new construction of smoothings of a real plane branch $(C, 0)$ by using Viro's patchworking method (see [16, 35, 36, 37, 38], see also [8, 15, 31] for an exposition and [3, 32, 33, 34] for some generalizations). Since real plane branches are Newton degenerate in general, we cannot apply patchworking method directly. Instead we apply the patchworking method for certain *Newton non-degenerate* curve singularities with several branches which are defined by *semi-quasi-homogeneous* (sqh) power series. These singularities appear as a result of iterating deformations of the strict transforms $(C^{(j)}, o_j)$ of the branch at certain infinitely near points o_j of the embedded resolution of singularities of $(C, 0)$. Our method yields multi-parametric deformations, which we call *multi-semi-quasi-homogeneous* (msqh), and provides simultaneously msqh-smoothings of the strict transforms $(C^{(j)}, o_j)$. We exhibit suitable hypotheses which characterize M-smoothings for this class of deformations in terms of the existence of certain M-curves and Harnack curves in projective toric surfaces (see Theorem 9.1).

We introduce the notion of *multi-Harnack smoothings*, those Harnack smoothings such that the msqh-M-smoothings of the strict transforms $C^{(j)}$ appearing in the process are again Harnack. We prove that any real plane branch C admits a multi-Harnack smoothing. For this purpose we prove first the existence of Harnack smoothings of singularities defined by certain sqh-series (see Proposition 5.8). One of our main results states that multi-Harnack smoothings of a real plane branch $(C, 0)$ have a unique topological type which depends only on the complex equisingular class of $(C, 0)$ (see Theorem 9.4). In particular multi-Harnack smoothings do not have nested ovals. Theorem 9.4 can be understood as a local version of Mikhalkin's Theorem (see Theorem 4.1). The proof is based on Theorem 9.1 and on an extension of Mikhalkin's result for Harnack smoothings of certain non-degenerate singular points (Theorem 5.2). We also analyze certain multi-scaled regions containing the ovals (see Theorem 10.1). The phenomena is quite analog to the analysis of the asymptotic concentration of the curvature of the Milnor fibers in the complex case, due to García Barroso and Teissier [7].

It is a challenge for the future to extend as possible the techniques and results of this paper to the constructions of smoothings of other singular points of real plane curves.

The paper is organized as follows. In Section 1 we introduce some definitions and notations; in Sections 2 and 3 we introduce the patchworking method, also in the toric context; we recall Mikhalkin's result on Harnack curves in projective toric surfaces in Section 4; in Section 5 we recall the notion of smoothings of real plane curve singular points and we determine the topological types of Harnack smoothings of singularities defined by certain non-degenerate semi-quasi-homogeneous polynomials (see Theorem 5.2). In Section 6 we recall the construction of a toric resolution of a plane branch. In Section 8 we describe some geometrical features of the patchworking method for the sqh-smoothings. In Sections 9 and 10, after introducing the notion of msqh-smoothing, we prove the main results of the paper: the characterization of M-msqh-smoothings and multi-Harnack smoothings in Theorem 9.1 and Corollary 9.2, the characterization of the topological type of multi-Harnack smoothings in Theorem 9.4 and the description of the asymptotic multi-scales of the ovals in Theorem 10.1. Finally, in the last section we analyse two more examples in detail.

1. Basic notations and definitions

A *real algebraic* (resp. *analytic*) *variety* is a complex algebraic (resp. analytic) variety V equipped with an anti-holomorphic involution η; we denote by $\mathbf{R}V$ its real part *i.e.*, the set of points of V fixed by η. For instance, a real algebraic plane curve $C \subset \mathbf{C}^2$ is a complex plane curve which is invariant under complex conjugation.

In this paper the term *polygon* (resp. *polyhedron*) means a convex polygon (resp. polyhedron) with integral vertices. We use the following notations and definitions.

If $F = \sum_\alpha c_\alpha x^\alpha \in \mathbf{C}[x]$ (resp. $F \in \mathbf{C}\{x\}$) for $x = (x_1, \dots, x_n)$ the *Newton polygon* of F (resp. the *local Newton polygon*) is the convex hull in $\mathbf{R}^n$ of the set $\{\alpha \mid c_\alpha \neq 0\}$ (resp. of $\cup_{c_\alpha \neq 0} \alpha + \mathbf{R}^n_{\geq 0}$); if the local Newton polyhedron of F meets all the coordinate axes, the *Newton diagram* of F is the region bounded by the local Newton polyhedron of F and the coordinate hyperplanes. If $\Lambda \subset \mathbf{R}^n$ we denote by F^Λ the *symbolic restriction* $F^\Lambda := \sum_{\alpha \in \Lambda \cap \mathbf{Z}^n} c_\alpha x^\alpha$.

If $P \in \mathbf{C}\{x, y\}$ and Λ is an edge of the local Newton polygon of P then P^Λ is of the form:

$$P^\Lambda = c x^a y^b \prod_{i=1}^{e} (y^n - \alpha_i x^m), \tag{1}$$

where $c \neq 0$, $a, b \in \mathbf{Z}_{\geq 0}$, and the integers $n, m \geq 0$ are coprime. The numbers $\alpha_1, \dots, \alpha_e \in \mathbf{C}^*$ are called *peripheral roots of P along the edge Λ* or *peripheral roots of P^Λ*.

The polynomial $P \in \mathbf{R}[x, y]$ is *non-degenerate* (resp. *real non-degenerate*) with respect to its Newton polygon if for any compact face Λ of it we have that $P^\Lambda = 0$ defines a nonsingular subset of $(\mathbf{C}^*)^2$ (resp. of $(\mathbf{R}^*)^2$). In particular, if Λ is an edge of the Newton polygon of P the peripheral roots (resp. the real peripheral roots) of P^Λ are distinct. Notice that the non-degeneracy (resp. real non-degeneracy) of P implies that $P = 0$ defines a non-singular subset of $(\mathbf{C}^*)^2$ (resp. of $(\mathbf{R}^*)^2$). We say that $P \in \mathbf{R}\{x, y\}$ is *non-degenerate* (resp. *real non-degenerate*) with respect to its local Newton polygon if for any edge Λ of it the equation $P^\Lambda = 0$ defines a nonsingular subset of $(\mathbf{C}^*)^2$ (resp. of $(\mathbf{R}^*)^2$). The notion of non-degeneracy with respect to the Newton polyhedra extends for polynomials of more than two variables (see [20]).

A series $H \in \mathbf{R}\{x, y\}$, with $H(0) = 0$ is *semi-quasi-homogeneous* (sqh) if its local Newton polygon has only one compact edge.

If $Q \in \mathbf{R}\{t, x, y\}$ we abuse of notation by denoting Q_t the series in x, y obtained by specializing Q at t in a neighborhood of the origin.

2. The real part of a projective toric variety

We introduce basic notations and facts on the geometry of toric varieties. We refer the reader to [8], [25] and [6] for proofs and more general statements. For simplicity we state the notations only for surfaces.

To a two dimensional polygon Θ is associated a projective toric variety $Z(\Theta)$. The algebraic torus $(\mathbf{C}^*)^2$ is embedded as an open dense subset of $Z(\Theta)$, and acts on $Z(\Theta)$ in a way which extends the group operation on the torus. There is a one to one correspondence between the faces of Θ and the orbits of the torus action which preserves the dimensions and the inclusions of the closures. If Λ is a one dimensional face of Θ we denote by $Z(\Lambda) \subset Z(\Theta)$ the associated orbit closure. The set $Z(\Lambda)$ is a complex projective line embedded in $Z(\Theta)$. These lines are called the *coordinate lines* of $Z(\Theta)$. The intersection of two coordinate lines $Z(\Lambda_1)$ and $Z(\Lambda_2)$ reduces to a point which is a zero-dimensional orbit (resp. is empty) if and only if the edges Λ_1 and Λ_2 intersect (resp. otherwise). The surface $Z(\Theta)$ may have singular points only at the zero-dimensional orbits. The algebraic real torus $(\mathbf{R}^*)^2$ is embedded as an