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DYNAMICS OF MEROMORPHIC MAPS WITH SMALL TOPOLOGICAL DEGREE III: GEOMETRIC CURRENTS AND ERGODIC THEORY

BY Jeffrey DILLER, Romain DUJARDIN AND Vincent GUEDJ

ABSTRACT. – We continue our study of the dynamics of mappings with small topological degree on projective complex surfaces. Previously, under mild hypotheses, we have constructed an ergodic "equilibrium" measure for each such mapping. Here we study the dynamical properties of this measure in detail: we give optimal bounds for its Lyapunov exponents, prove that it has maximal entropy, and show that it has product structure in the natural extension. Under a natural further assumption, we show that saddle points are equidistributed towards this measure. This generalizes results that were known in the invertible case and adds to the small number of situations in which a natural invariant measure for a non-invertible dynamical system is well-understood.

RÉSUMÉ. – Nous poursuivons notre étude de la dynamique des applications rationnelles de petit degré topologique sur les surfaces complexes projectives. Dans un travail précédent nous avons construit une mesure ergodique naturelle, dite « d'équilibre », sous des hypothèses très générales. Nous étudions maintenant en détail les propriétés dynamiques de cette mesure: nous donnons des bornes optimales pour ses exposants de Lyapounov, montrons qu'elle est d'entropie maximale et qu'elle a une structure produit dans l'extension naturelle. Sous une hypothèse supplémentaire naturelle, nous montrons que cette mesure décrit la répartition des points selles. Ceci généralise des résultats qui étaient auparavant connus dans le cas inversible et vient ainsi s'ajouter au petit nombre de situations où une mesure invariante naturelle pour un système dynamique non inversible est vraiment bien comprise.

Introduction

In this article we continue our investigation, begun in [7, 8], of dynamics on complex surfaces for rational transformations with small topological degree. Our previous work culminated in the construction of a canonical mixing invariant measure for a very broad class of such mappings. We intend now to study in detail the nature of this measure. As we will show,

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the measure meets conjectural expectations concerning, among other things, Lyapunov exponents, entropy, product structure in the natural extension, and equidistribution of saddle orbits.

Before entering into the details of our results, let us recall our setting. Let X be a complex projective surface (always compact and connected), and $f : X \to X$ be a rational mapping. Our main requirement is that f has *small topological degree*:

$$\lambda_2(f) < \lambda_1(f). \tag{1}$$

Here the topological (or second dynamical) degree $\lambda_2(f)$ is the number of preimages of a generic point, whereas the first dynamical degree $\lambda_1(f) := \lim \left\|(f^n)^*|_{H^{1,1}(X)}\right\|^{1/n}$ measures the asymptotic volume growth of preimages of curves under iteration of f. We refer the reader to [7] for a more precise discussion of dynamical degrees. In particular it was observed there that the existence of maps with small topological degree imposes some restrictions on the ambient surface: either X is rational (in particular, projective), or X has Kodaira dimension zero.

Let us recall that the ergodic theory of mappings with large topological degree ($\lambda_2 > \lambda_1$) has been extensively studied, and that results analogous to our Theorems B and C are true in this context [5, 11, 24]. In dimension 1, all rational maps have large topological degree, and in this setting these results are due to [19, 28]. We note also that in the birational case $\lambda_2 = 1$, the main results of this paper are obtained in [2, 3] (for polynomial automorphisms of $\mathbf{C}^2$) in [6] (for automorphisms of projective surfaces) and in [15] (for general birational maps). Hence the focus here is on noninvertible mappings which, as the reader will see, present substantial additional difficulties.

We will work under two additional assumptions, which we now introduce.

Good birational model

We need to assume that the linear actions $(f^n)^*$ induced by f^n on cohomology are compatible with the dynamics, i.e.

(H1) $$(f^n)^* = (f^*)^n, \text{ for all } n \in \mathbf{N}.$$

This condition, often called "algebraic stability" in the literature, was first considered by Fornaess and Sibony [18]. There is some evidence that for any mapping (X, f) with small topological degree, there should exist a birationally conjugate mapping $(\tilde{X}, \tilde{f})$ that satisfies (H1). Birational conjugacy does not affect dynamical degrees, so in this case we simply replace the given system (X, f) with the "good birational model" $(\tilde{X}, \tilde{f})$.

We observed in [7] that the minimal model for X is a good birational model when X has Kodaira dimension zero. For rational X, there is a fairly explicit blowing up procedure [9] that produces a good model when $\lambda_2 = 1$. More recently, Favre and Jonsson [17] have proven that each polynomial mapping of $\mathbf{C}^2$ with small topological degree admits a good model on passing to an iterate.

Under assumption (H1), in [7], we have constructed and studied canonical invariant currents T^+ and T^-. These are defined by

$$T^+ = \lim \frac{c^+}{\lambda^n}(f^n)^*\omega \text{ and } T^- = \lim \frac{c^-}{\lambda^n}(f^n)_*\omega$$

where ω is a fixed Kähler form on X and $c^{\pm}$ are normalizing constants chosen so that in cohomology $\{T^+\}\cdot\{T^-\} = \{\omega\}\cdot\{T^-\} = \{\omega\}\cdot\{T^+\} = 1$. A fact of central importance to us is that these currents have additional geometric structures: T^+ is laminar, while T^- is woven (see §1 below for definitions).

Finite energy

Let I^+ denote the indeterminacy set of f, i.e. the collection of those points that f "blows up" to curves; and let I^- denote the analogous set of points which are images of curves under f. The invariant current T^+ (resp. T^-) typically has positive Lelong number at each point of the extended indeterminacy set $I^+_\infty = \bigcup_{n\geq 0} f^{-n}I^+$ (resp. $I^-_\infty = \bigcup_{n\geq 0} f^n I^-$). Condition (H1) is equivalent to asking that the sets I^+_∞ and I^-_∞ be disjoint.

In order to give meaning to and study the wedge product $T^+\wedge T^-$, it is desirable to have more quantitative control on how fast these sets approach one another. This is how our next hypothesis should be understood:

(H2) f has finite dynamical energy.

We refer the reader to [8] for a precise definition of finite energy and its relationship with recurrence properties of indeterminacy points. In that article we proved the following theorem.

Theorem A ([8]). – *Let f be a meromorphic map with small topological degree on a projective surface, satisfying hypotheses (H1) and (H2). Then the wedge product $\mu := T^+\wedge T^-$ is a well-defined probability invariant measure that is f-invariant and mixing. Furthermore the wedge product is described by the geometric intersection of the laminar/woven structures of T^+ and T^-.*

The notion of "geometric intersection" will be described at length in §1.

We can now state the main results of this article. Let us emphasize that they rely on the hypotheses (H1) and (H2) only through the conclusions of Theorem A. Taking these conclusions as a starting point, one can read the proofs given here independently of [7, 8].

Theorem B. – *Let X be a complex projective surface and $f : X \to X$ be a rational map with small topological degree. Assume that f satisfies the conclusions of Theorem A. Then the canonical invariant measure $\mu = T^+\wedge T^-$ has the following properties:*

i. *For μ-a.e. p there exists a nonzero tangent vector e^s at p, such that*

$$\limsup_{n\to\infty} \frac{1}{n}\log|df^n(e^s(p))| \leq -\frac{\log(\lambda_1/\lambda_2)}{2}. \tag{2}$$

ii. *Likewise, for μ-a.e. p there exist a tangent vector e^u at p, and a set of integers $\mathbf{N}'\subset\mathbf{N}$ of density 1 such that*

$$\liminf_{\mathbf{N}'\ni n\to\infty} \frac{1}{n}\log|df^n(e^u(p))| \geq \frac{\log\lambda_1}{2}. \tag{3}$$

iii. *μ has entropy $\log\lambda_1$; thus it has maximal entropy and $h_{\mathrm{top}}(f) = \log\lambda_1$.*
iv. *The natural extension of μ has local product structure.*

In particular it follows from *iv.* and the work of Ornstein and Weiss [31] (see also Briend [4] for useful remarks on the adaptation to the noninvertible case) that the natural extension of μ has the Bernoulli property, hence μ is mixing to all orders and has the K property. A precise definition of local product structure will be given below in §8. This is the analogue of the balanced property of the maximal measure in the large topological degree case.

Let us stress that we do not assume that $\log \operatorname{dist}(\cdot, I^+ \cup C_f)$ is μ integrable (C_f denotes the critical set). This condition is usually imposed to guarantee the existence of Lyapunov exponents and applicability of the Pesin theory of non-uniformy hyperbolic dynamical systems. However, for mappings with small topological degree, it is known to fail in general (see [8, §4.4]). This contrasts with the large topological degree case, in which the maximal entropy measure integrates all quasi-psh functions.

When the Lyapunov exponents $\chi^+(\mu) \geq \chi^-(\mu)$ are well defined, then *(i)* and *(ii)* imply that

$$\chi^+(\mu) \geq \frac{1}{2}\log \lambda_1(f) > 0 > -\frac{1}{2}\log \lambda_1(f)/\lambda_2(f) \geq \chi^-(\mu),$$

hence the measure μ is hyperbolic. These bounds are optimal and were conjectured in [23].

In order to go further and relate μ to the distribution of saddle periodic points, we use Pesin theory and must therefore invoke the above integrability hypothesis.

THEOREM C. – *Under the assumptions of Theorem B, assume further that*

(H3) $$p \mapsto \log \operatorname{dist}(p, I^+ \cup C_f) \in L^1(\mu),$$

where C_f is the critical set.

Then, for every n there exists a set $\mathcal{P}_n \subset \operatorname{Supp}(\mu)$ of saddle periodic points of period n, with $\#\mathcal{P}_n \sim \lambda_1^n$, and such that

$$\frac{1}{\lambda_1^n}\sum_{q\in\mathcal{P}_n}\delta_q \longrightarrow \mu.$$

Let Per_n be the set of all isolated periodic points of f of period n. If furthermore

- *f has no curves of periodic points,*
- *or $X = \mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$,*

then $\#\operatorname{Per}_n \sim \lambda_1^n$, so that asymptotically nearly all periodic points are saddles.

This theorem was proved for birational maps by the second author in [15] (though the possibility of a curve of periodic points was overlooked there). It would be interesting to prove a similar result without using Pesin Theory (i.e. without assumption (H3)).

It would also be interesting to know when saddle points might lie outside $\operatorname{Supp}(\mu)$. One can easily create isolated saddle points by blowing up an attracting fixed point with unequal eigenvalues. We then get an infinitely near saddle point in the direction corresponding to the larger multiplier, whose unstable manifold is contained in the exceptional divisor of the blow-up. We do not know any example of a saddle point outside $\operatorname{Supp}(\mu)$ whose stable and unstable manifolds are both Zariski dense.

While the results in this paper parallel those in [15], new and more elaborate arguments are needed for non-invertible maps. In particular, we are led to work in the natural extension (e.g. for establishing *iii.* and *iv.* of Theorem B), in a situation where there is no symmetry between the preimages along μ (see the examples in §3). An interesting thing to note is that