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HODGE-TATE AND DE RHAM REPRESENTATIONS IN THE IMPERFECT RESIDUE FIELD CASE

BY KAZUMA MORITA

ABSTRACT. – Let K be a p-adic local field with residue field k such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of $\mathrm{Gal}(\overline{K}/K)$. Then, by using the theory of p-adic differential modules, we show that V is a Hodge-Tate (resp. de Rham) representation of $\mathrm{Gal}(\overline{K}/K)$ if and only if V is a Hodge-Tate (resp. de Rham) representation of $\mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ where K^{pf}/K is a certain p-adic local field with residue field the smallest perfect field k^{pf} containing k.

RÉSUMÉ. – Soit K un corps local p-adique de corps résiduel k tel que $[k : k^p] = p^e < +\infty$ et soit V une représentation p-adique de $\mathrm{Gal}(\overline{K}/K)$. Nous utilisons la théorie des modules différentiels p-adiques pour montrer que V est une représentation de Hodge-Tate (resp. de Rham) de $\mathrm{Gal}(\overline{K}/K)$ si et seulement si V est une représentation de Hodge-Tate (resp. de Rham) de $\mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ où K^{pf}/K est un certain corps local p-adique de corps résiduel le plus petit corps parfait k^{pf} contenant k.

1. Introduction

Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$. Choose an algebraic closure $\overline{K}$ of K and put $G_K = \mathrm{Gal}(\overline{K}/K)$. By a p-adic representation of G_K, we mean a finite dimensional vector space V over $\mathbb{Q}_p$ endowed with a continuous action of G_K. In the case $e = 0$ (i.e. k is perfect), following Fontaine, we can classify p-adic representations of G_K by using the p-adic periods rings B_{HT}, B_{dR}, B_{st} and B_{cris} (Hodge-Tate, de Rham, semi-stable and crystalline representations). In the general case (i.e. k is not necessarily perfect), Hyodo constructed the imperfect residue field version of the ring B_{HT} and Tsuzuki and several authors constructed that of the ring B_{dR}. By using these rings, we can define the imperfect residue field version of Hodge-Tate and de Rham representations of G_K in the evident way ([3], [7], [8], [9], [12]).

Now, we shall state the main result of this article. Let us fix some notations. Fix a lifting $(b_i)_{1\leq i\leq e}$ of a p-basis of k in $\mathscr{O}_K$ (the ring of integers of K) and for each $m \geq 1$, fix a p^m -th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$. Put $K^{(\mathrm{pf})} = \cup_{m\geq 1} K(b_i^{1/p^m}, 1 \leq i \leq e)$

and K^{pf} = the p-adic completion of $K^{(\mathrm{pf})}$. These fields depend on the choice of a lifting of a p-basis of k in $\mathscr{O}_K$. Since K^{pf} becomes a complete discrete valuation field with perfect residue field, we can apply theories in the perfect residue field case to p-adic representations of $G_{K^{\mathrm{pf}}} = \mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})$ where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of K^{pf} containing $\overline{K}$. Note that, if V is a p-adic representation of G_K, it can be also regarded as a p-adic representation of $G_{K^{\mathrm{pf}}}$ (see Section 2.2 for details). Our main result is the following.

THEOREM 1.1. – *Let K be a complete discrete valuation field of characteristic* 0 *with residue field k of characteristic $p > 0$ such that $[k : k^p] = p^e < +\infty$ and V be a p-adic representation of G_K. Let K^{pf} be the field extension of K defined as above. Then, we have the following equivalences*

1. *V is a Hodge-Tate representation of G_K if and only if V is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$,*
2. *V is a de Rham representation of G_K if and only if V is a de Rham representation of $G_{K^{\mathrm{pf}}}$.*

In the case of Hodge-Tate representations, Tsuji [11] had proved a more refined theorem based on this article. This paper is organized as follows. In Section 2, we shall review the definitions and basic known facts on Hodge-Tate and de Rham representations, first in the perfect residue field case and then in the imperfect residue field case. In Section 3, we shall review the theory of p-adic differential modules which play a central role in this article. In Section 4, by using the theory of p-adic differential modules, we shall prove the main theorem, first for Hodge-Tate representations and then for de Rham representations.

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2. Preliminaries on Hodge-Tate and de Rham representations

2.1. Hodge-Tate and de Rham representations in the perfect residue field case

(See [4] and [5] for details.) Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic $p > 0$. Choose an algebraic closure $\overline{K}$ of K and consider its p-adic completion $\mathbb{C}_p$. Put

$$\widetilde{\mathbb{E}} = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \{(x^{(0)}, x^{(1)}, \dots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p\}$$

and let $\widetilde{\mathbb{E}}^+$ denote the set of $x = (x^{(i)}) \in \widetilde{\mathbb{E}}$ such that $x^{(0)} \in \mathscr{O}_{\mathbb{C}_p}$ where $\mathscr{O}_{\mathbb{C}_p}$ denotes the ring of integers of $\mathbb{C}_p$. For two elements $x = (x^{(i)})$ and $y = (y^{(i)})$ of $\widetilde{\mathbb{E}}$, their sum and product are defined by $(x + y)^{(i)} = \lim_{j\to+\infty}(x^{(i+j)} + y^{(i+j)})^{p^j}$ and $(xy)^{(i)} = x^{(i)}y^{(i)}$. These sum and product make $\widetilde{\mathbb{E}}$ a perfect field of characteristic $p > 0$ ($\widetilde{\mathbb{E}}^+$ is a subring of $\widetilde{\mathbb{E}}$). Let $\epsilon = (\epsilon^{(n)})$ be an element of $\widetilde{\mathbb{E}}$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$. Then, $\widetilde{\mathbb{E}}$ is the completion of an algebraic closure of $k((\epsilon - 1))$ for the valuation defined by $v_{\mathbb{E}}(x) = v_p(x^{(0)})$ where v_p denotes the p-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$. The field $\widetilde{\mathbb{E}}$ is equipped with a continuous action of the Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ with respect to the topology defined by the valuation $v_{\mathbb{E}}$. Put $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$ (the ring of Witt vectors with coefficients in $\widetilde{\mathbb{E}}^+$) and $\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[1/p] = \{\sum_{k\gg-\infty} p^k[x_k] \mid x_k \in \widetilde{\mathbb{E}}^+\}$ where $[*]$ denotes the Teichmüller lift of $* \in \widetilde{\mathbb{E}}^+$. This ring $\widetilde{\mathbb{B}}^+$ is equipped with a surjective homomorphism

$$\theta : \widetilde{\mathbb{B}}^+ \twoheadrightarrow \mathbb{C}_p : \sum p^k[x_k] \mapsto \sum p^k x_k^{(0)}.$$

If $\tilde{p} = (p^{(n)})$ denotes an element of $\widetilde{\mathbb{E}}^+$ such that $p^{(0)} = p$, we can show that $\mathrm{Ker}\,(\theta)$ is the principal ideal generated by $\omega = [\tilde{p}] - p$. The ring $B^+_{\mathrm{dR},K}$ is defined to be the $\mathrm{Ker}\,(\theta)$-adic completion of $\widetilde{\mathbb{B}}^+$

$$B^+_{\mathrm{dR},K} = \varprojlim_{n\geq 0}\widetilde{\mathbb{B}}^+/(\mathrm{Ker}\,(\theta)^n).$$

This is a discrete valuation ring and $t = \log([\epsilon])$ which converges in $B^+_{\mathrm{dR},K}$ is a generator of the maximal ideal. Put $B_{\mathrm{dR},K} = B^+_{\mathrm{dR},K}[1/t]$. This ring $B_{\mathrm{dR},K}$ becomes a field and is equipped with an action of the Galois group G_K and a filtration defined by $\mathrm{Fil}^i B_{\mathrm{dR},K} = t^i B^+_{\mathrm{dR},K}$ ($i \in \mathbb{Z}$). Then, $(B_{\mathrm{dR},K})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{dR},K}(V) = (B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a de Rham representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{dR},K}(V)\text{)}.$$

Furthermore, we say that a p-adic representation V of G_K is a potentially de Rham representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a de Rham representation of G_L. It is known that a potentially de Rham representation V of G_K is a de Rham representation of G_K (see [5, 3.9]).

Define $B_{\mathrm{HT},K}$ to be the associated graded algebra to the filtration $\mathrm{Fil}^i B_{\mathrm{dR},K}$. The quotient $\mathrm{gr}^i B_{\mathrm{HT},K} = \mathrm{Fil}^i B_{\mathrm{dR},K}/\mathrm{Fil}^{i+1} B_{\mathrm{dR},K}$ ($i \in \mathbb{Z}$) is a one-dimensional $\mathbb{C}_p$-vector space spanned by the image of t^i. Thus, we obtain the presentation

$$B_{\mathrm{HT},K} = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i)$$

where $\mathbb{C}_p(i) = \mathbb{C}_p \otimes \mathbb{Z}_p(i)$ is the Tate twist. Then, $(B_{\mathrm{HT},K})^{G_K}$ is canonically isomorphic to K. Thus, for a p-adic representation V of G_K, $D_{\mathrm{HT},K}(V) = (B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$ is naturally a K-vector space. We say that a p-adic representation V of G_K is a Hodge-Tate representation of G_K if we have

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{HT},K}(V) \quad \text{(we always have } \dim_{\mathbb{Q}_p} V \geq \dim_K D_{\mathrm{HT},K}(V)\text{)}.$$

Furthermore, we say that a p-adic representation V of G_K is a potentially Hodge-Tate representation of G_K if there exists a finite field extension L/K in $\overline{K}$ such that V is a Hodge-Tate representation of G_L. It is known that a potentially Hodge-Tate representation V of G_K is a Hodge-Tate representation of G_K (see [5, 3.9]). Since we have $\mathrm{gr}B_{\mathrm{dR},K} \simeq \bigoplus_{i\in\mathbb{Z}}\mathbb{C}_p(i)$, if V is a de Rham representation of G_K, there exists a G_K-equivariant isomorphism $\mathbb{C}_p\otimes_{\mathbb{Q}_p}V \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p}V}\mathbb{C}_p(n_j)$ $(n_j\in\mathbb{Z})$. Thus, it follows that a de Rham representation V of G_K is a Hodge-Tate representation of G_K.

2.2. Hodge-Tate and de Rham representations in the imperfect residue field case

Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic $p>0$ such that $[k:k^p]=p^e<+\infty$. Choose an algebraic closure $\overline{K}$ of K and put $G_K=\mathrm{Gal}(\overline{K}/K)$. As in the introduction, fix a lifting $(b_i)_{1\leq i\leq e}$ of a p-basis of k in $\mathscr{O}_K$ (the ring of integers of K) and for each $m\geq 1$, fix a p^m -th root b_i^{1/p^m} of b_i in $\overline{K}$ satisfying $(b_i^{1/p^{m+1}})^p=b_i^{1/p^m}$. Put

$$K^{(\mathrm{pf})}=\cup_{m\geq 0}K(b_i^{1/p^m},1\leq i\leq e)\quad\text{and}\quad K^{\mathrm{pf}}=\text{the }p\text{-adic completion of }K^{(\mathrm{pf})}.$$

These fields depend on the choice of a lifting of a p-basis of k in $\mathscr{O}_K$. Since $K^{(\mathrm{pf})}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K^{\mathrm{pf}}}=\mathrm{Gal}(\overline{K^{\mathrm{pf}}}/K^{\mathrm{pf}})\simeq G_{K^{(\mathrm{pf})}}=\mathrm{Gal}(\overline{K}/K^{(\mathrm{pf})})$ $(\subset G_K)$ where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of K^{pf} containing $\overline{K}$. With this isomorphism, we identify $G_{K^{\mathrm{pf}}}$ with a subgroup of G_K. We have a bijective map from the set of finite extensions of $K^{(\mathrm{pf})}$ contained in $\overline{K}$ to the set of finite extensions of K^{pf} contained in $\overline{K^{\mathrm{pf}}}$ defined by $L\to LK^{\mathrm{pf}}$. Furthermore, LK^{pf} is the p-adic completion of L. Hence, we have an isomorphism of rings

$$\mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}\simeq\mathscr{O}_{\overline{K^{\mathrm{pf}}}}/p^n\mathscr{O}_{\overline{K^{\mathrm{pf}}}}$$

where $\mathscr{O}_{\overline{K}}$ and $\mathscr{O}_{\overline{K^{\mathrm{pf}}}}$ denote the rings of integers of $\overline{K}$ and $\overline{K^{\mathrm{pf}}}$. Thus, the p-adic completion of $\overline{K}$ is isomorphic to the p-adic completion of $\overline{K^{\mathrm{pf}}}$, which we will write $\mathbb{C}_p$. As in Subsection 2.1, construct the rings $\widetilde{\mathbb{E}}^+$ and $\widetilde{\mathbb{A}}^+=W(\widetilde{\mathbb{E}}^+)$ from this $\mathbb{C}_p$. Let k^{pf} denote the perfect residue field of K^{pf} and put $\mathscr{O}_{K_0}=\mathscr{O}_K\cap W(k^{\mathrm{pf}})$. Let $\alpha:\mathscr{O}_K\otimes_{\mathscr{O}_{K_0}}\widetilde{\mathbb{A}}^+\twoheadrightarrow\mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}$ be the natural surjection and define $\widetilde{\mathbb{A}}^+_{(K)}$ to be $\widetilde{\mathbb{A}}^+_{(K)}=\varprojlim_{n\geq 0}(\mathscr{O}_K\otimes_{\mathscr{O}_{K_0}}\widetilde{\mathbb{A}}^+)/(\mathrm{Ker}\,(\alpha))^n$. Let $\theta_K:\widetilde{\mathbb{A}}^+_{(K)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\twoheadrightarrow\mathbb{C}_p$ be the natural extension of $\theta:\widetilde{\mathbb{A}}^+[1/p]\twoheadrightarrow\mathbb{C}_p$. Define $B^+_{\mathrm{dR},K}$ to be the $\mathrm{Ker}\,(\theta_K)$-adic completion of $\widetilde{\mathbb{A}}^+_{(K)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$

$$B^+_{\mathrm{dR},K}=\varprojlim_{n\geq 0}(\widetilde{\mathbb{A}}^+_{(K)}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)/(\mathrm{Ker}\,(\theta_K)^n).$$

This is a K-algebra equipped with an action of the Galois group G_K. Let $\widetilde{b_i}$ denote $(b_i^{(n)})\in\widetilde{\mathbb{E}}^+$ such that $b_i^{(0)}=b_i$ and then the series which defines $\log([\widetilde{b_i}]/b_i)$ converges to an element t_i in $B^+_{\mathrm{dR},K}$. Then, the ring $B^+_{\mathrm{dR},K}$ becomes a local ring with the maximal ideal $m_{\mathrm{dR}}=(t,t_1,\dots,t_e)$. Define a filtration on $B^+_{\mathrm{dR},K}$ by $\mathrm{fil}^iB^+_{\mathrm{dR},K}=m^i_{\mathrm{dR}}$. Then, the homomorphism

$$f:B^+_{\mathrm{dR},K^{\mathrm{pf}}}[[t_1,\dots,t_e]]\to B^+_{\mathrm{dR},K}$$

is an isomorphism of filtered algebras (see [3, Proposition 2.9]). From this isomorphism, it follows easily that

$$i:B^+_{\mathrm{dR},K^{\mathrm{pf}}}\hookrightarrow B^+_{\mathrm{dR},K}\quad\text{and}\quad p:B^+_{\mathrm{dR},K}\twoheadrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}}:\ t_i\mapsto 0$$