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BRUHAT-TITS THEORY FROM BERKOVICH'S POINT OF VIEW. I. REALIZATIONS AND COMPACTIFICATIONS OF BUILDINGS

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ABSTRACT. – We investigate Bruhat-Tits buildings and their compactifications by means of Berkovich analytic geometry over complete non-Archimedean fields. For every reductive group G over a suitable non-Archimedean field k we define a map from the Bruhat-Tits building $\mathcal{B}(\mathrm{G}, k)$ to the Berkovich analytic space G^{an} associated with G. Composing this map with the projection of G^{an} to its flag varieties, we define a family of compactifications of $\mathcal{B}(\mathrm{G}, k)$. This generalizes results by Berkovich in the case of split groups.

Moreover, we show that the boundary strata of the compactified buildings are precisely the Bruhat-Tits buildings associated with a certain class of parabolics. We also investigate the stabilizers of boundary points and prove a mixed Bruhat decomposition theorem for them.

RÉSUMÉ. – Nous étudions les immeubles de Bruhat-Tits et leurs compactifications au moyen de la géométrie analytique sur les corps complets non archimédiens au sens de Berkovich. Pour tout groupe réductif G sur un corps non archimédien convenable k, nous définissons une application de l'immeuble de Bruhat-Tits $\mathcal{B}(\mathrm{G}, k)$ vers l'espace analytique de Berkovich G^{an} associé à G. En composant cette application avec la projection sur les variétés de drapeaux, nous obtenons une famille de compactifications de $\mathcal{B}(\mathrm{G}, k)$. Ceci généralise des résultats de Berkovich sur le cas déployé.

En outre, nous démontrons que les strates au bord des immeubles compactifiés sont précisément les immeubles de Bruhat-Tits associés à certaines classes de sous-groupes paraboliques. Nous étudions également les stabilisateurs des points au bord et démontrons un théorème de décomposition de Bruhat mixte pour ces groupes.

Introduction

1. – In the mid sixties, F. Bruhat and J. Tits initiated a theory which led to a deep understanding of reductive algebraic groups over valued fields [19], [20]. The main tool (and a concise way to express the achievements) of this long-standing work is the notion of a *building*. Generally speaking, a building is a gluing of (poly)simplicial subcomplexes, all isomorphic to a given tiling naturally acted upon by a Coxeter group [2]. The copies of this tiling in the building are called *apartments* and must satisfy, by definition, strong incidence properties which make the whole space very symmetric. The buildings considered by F. Bruhat and J. Tits are Euclidean ones, meaning that their apartments are Euclidean tilings (in fact, to cover the case of non-discretely valued fields, one has to replace Euclidean tilings by affine spaces acted upon by a Euclidean reflection group with a non-discrete, finite index, translation subgroup [47]). A Euclidean building carries a natural non-positively curved metric, which allows one to classify in a geometric way maximal bounded subgroups in the rational points of a given non-Archimedean semisimple algebraic group. This is only an instance of the strong analogy between the Riemannian symmetric spaces associated with semisimple real Lie groups and Bruhat-Tits buildings [45]. This analogy is our guideline here.

Indeed, in this paper we investigate Bruhat-Tits buildings and their compactification by means of analytic geometry over non-Archimedean valued fields, as developed by V. Berkovich—see [8] for a survey. Compactifications of symmetric spaces is now a very classical topic, with well-known applications to group theory (e.g., group cohomology [11]) and to number theory (via the study of some relevant moduli spaces modeled on Hermitian symmetric spaces [23]). For deeper motivation and a broader scope on compactifications of symmetric spaces, we refer to the recent book [10], in which the case of locally symmetric varieties is also covered. One of our main results is to construct for each semisimple group G over a suitable non-Archimedean valued field k, a family of compactifications of the Bruhat-Tits building $\mathcal{B}(\mathrm{G}, k)$ of G over k. This family is finite, actually indexed by the conjugacy classes of proper parabolic k-subgroups in G. Such a family is of course the analogue of the family of Satake [42] or Furstenberg [27] compactifications of a given Riemannian non-compact symmetric space—see [32] for a general exposition.

In fact, the third author had previously associated, with each Bruhat-Tits building, a family of compactifications also indexed by the conjugacy classes of proper parabolic k-subgroups [50] and generalizing the "maximal" version constructed before by E. Landvogt [35]. The Bruhat-Tits building $\mathcal{B}(\mathrm{G}, k)$ of G over k is defined as the quotient for a suitable equivalence relation, say $\sim$, of the product of the rational points $\mathrm{G}(k)$ by a natural model, say Λ, of the apartment; we will refer to this kind of construction as a *gluing procedure*. The family of compactifications of [50] was obtained by suitably compactifying Λ to obtain a compact space $\overline{\Lambda}$ and extending $\sim$ to an equivalence relation on $\mathrm{G}(k) \times \overline{\Lambda}$. As expected, for a given group G we eventually identify the latter family of compactifications with the one we construct here, see [39].

Our compactification procedure makes use of embeddings of Bruhat-Tits buildings in the analytic versions of some well-known homogeneous varieties (in the context of algebraic transformation groups), namely flag manifolds. The idea goes back to V. Berkovich in the case when G splits over its ground field k [4, §5]. One aesthetic advantage of the embedding procedure is that it is similar to the historical ways to compactify symmetric spaces, e.g., by seeing them as topological subspaces of some projective spaces of Hermitian matrices or inside spaces of probability measures on a flag manifold. More usefully (as we hope), the fact that we specifically embed buildings into compact spaces from Berkovich's theory may make these compactifications useful for a better understanding of non-Archimedean spaces relevant to number theory (in the case of Hermitian symmetric spaces). For instance, the building of GL_n over a valued field k is the "combinatorial skeleton" of the Drinfel'd half-space Ω^{n-1} over k [18], and it would be interesting to know whether the precise combinatorial description we obtain for our compactifications might be useful to describe other moduli spaces for suitable choices of groups and parabolic subgroups. One other question about these compactifications was raised by V. Berkovich himself [4, 5.5.2] and deals with the potential generalization of Drinfel'd half-spaces to non-Archimedean semisimple algebraic groups of arbitrary type.

2. – Let us now turn to the definition of the embedding maps that allow us to compactify Bruhat-Tits buildings. Let G be a k-isotropic semisimple algebraic group defined over the non-Archimedean valued field k and let $\mathcal{B}(\mathrm{G}, k)$ denote the Euclidean building provided by Bruhat-Tits theory [46]. We prove the following statement (see 2.4 and Prop. 3.34): *assume that the valued field k is a local field (i.e., is locally compact) and (for simplicity) that* G *is almost k-simple; then for any conjugacy class of proper parabolic k-subgroup, say t, there exists a continuous,* $\mathrm{G}(k)$*-equivariant map $\vartheta_t : \mathcal{B}(\mathrm{G}, k) \to \mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ which is a homeomorphism onto its image.* Here $\mathrm{Par}_t(\mathrm{G})$ denotes the connected component of type t in the proper variety $\mathrm{Par}(\mathrm{G})$ of all parabolic subgroups in G (on which G acts by conjugation) [1, Exposé XXVI, Sect. 3]. The superscript $^{\mathrm{an}}$ means that we pass from the k-variety $\mathrm{Par}_t(\mathrm{G})$ to the Berkovich k-analytic space associated with it [4, 3.4.1-2]; the space $\mathrm{Par}(\mathrm{G})^{\mathrm{an}}$ is compact since $\mathrm{Par}(\mathrm{G})$ is projective. We denote by $\overline{\mathcal{B}}_t(\mathrm{G}, k)$ the closure of the image of ϑ_t and call it the *Berkovich compactification* of type t of the Bruhat-Tits building $\mathcal{B}(\mathrm{G}, k)$.

Roughly speaking, the definition of the maps ϑ_t takes up the first half of this paper, so let us provide some further information about it. As a preliminary, we recall some basic but helpful analogies between (scheme-theoretic) algebraic geometry and k-analytic geometry (in the sense of Berkovich). Firstly, the elementary blocks of k-analytic spaces in the latter theory are the so-called *affinoid* spaces; they, by and large, correspond to affine schemes in algebraic geometry. Affinoid spaces can be glued together to define k-analytic spaces, examples of which are provided by analytifications of affine schemes: if $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ is given by a finitely generated k-algebra A, then the set underlying the analytic space X^{an} consists of multiplicative seminorms on A extending the given absolute value on k. Let us simply add that it follows from the "spectral analytic side" of Berkovich theory that each affinoid space X admits a *Shilov boundary*, namely a (finite) subset on which any element of the Banach k-algebra defining X achieves its maximum. We have enough now to give a construction of the maps ϑ_t in three steps:

- Step 1**:** we attach to any point $x \in \mathcal{B}(\mathrm{G}, k)$ an affinoid subgroup G_x whose k-rational points coincide with the parahoric subgroup $\mathrm{G}_x(k)$ associated with x by Bruhat-Tits theory (Th. 2.1).
- Step 2**:** we attach to any so-obtained analytic subgroup G_x a point $\vartheta(x)$ in G^{an} (in fact the unique point in the Shilov boundary of G_x), which defines a map $\vartheta : \mathcal{B}(\mathrm{G}, k) \to \mathrm{G}^{\mathrm{an}}$ (Prop 2.4).
- Step 3**:** we finally compose the map ϑ with an "orbit map" to the flag variety $\mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ of type t (Def. 2.15).

Forgetting provisionally that we wish to compactify the building $\mathcal{B}(\mathrm{G},k)$ (in which case we have to assume that $\mathcal{B}(\mathrm{G},k)$ is locally compact, or equivalently, that k is local), this three-step construction of the map $\vartheta_t : \mathcal{B}(\mathrm{G},k) \to \mathrm{Par}_t(\mathrm{G})^{\mathrm{an}}$ works whenever the ground field k allows the functorial existence of $\mathcal{B}(\mathrm{G},k)$ (see 1.3 for a reminder of these conditions). We note that in Step 2, the uniqueness of the point $\vartheta(x)$ in the Shilov boundary of G_x comes from the use of a field extension splitting G and allowing to see x as a special point (see below) and from the fact that integral structures attached to special points in Bruhat-Tits theory are explicitly described by means of Chevalley bases. At last, the point $\vartheta(x)$ determines G_x because the latter analytic subgroup is the holomorphic envelop of $\vartheta(x)$ in G^{an}. Here is a precise statement for Step 1 (Th. 2.1).

THEOREM 1. – *For any point x in $\mathcal{B}(\mathrm{G},k)$, there is a unique affinoid subgroup G_x of G^{an} satisfying the following condition: for any non-Archimedean extension* K *of k, we have* $\mathrm{G}_x(\mathrm{K}) = \mathrm{Stab}_{\mathrm{G}(\mathrm{K})}(x)$.

This theorem (hence Step 1) improves an idea used for another compactification procedure, namely the one using the map attaching to each point $x \in \mathcal{B}(\mathrm{G},k)$ the biggest parahoric subgroup of $\mathrm{G}(k)$ fixing it [33]. The target space of the map $x \mapsto \mathrm{G}_x(k)$ in [loc. cit.] is the space of closed subgroups of $\mathrm{G}(k)$, which is compact for the Chabauty topology [16, VIII.5]. This idea does not lead to a compactification of $\mathcal{B}(\mathrm{G},k)$ but only of the set of vertices of it: if k is discretely valued and if G is simply connected, any two points in a given facet of the Bruhat-Tits building $\mathcal{B}(\mathrm{G},k)$ have the same stabilizer. Roughly speaking, in the present paper we use Berkovich analytic geometry, among other things, to overcome these difficulties thanks to the fact that we can use arbitrarily large non-Archimedean extensions of the ground field. More precisely, up to taking a suitable non-Archimedean extension K of k, any point $x \in \mathcal{B}(\mathrm{G},k)$ can be seen as a special point in the bigger (split) building $\mathcal{B}(\mathrm{G},\mathrm{K})$, in which case we can attach to x an affinoid subgroup of $(\mathrm{G} \otimes_k \mathrm{K})^{\mathrm{an}}$. As a counterpart, in order to obtain the affinoid subgroup G_x defined over k as in the above theorem, we have to apply a Banach module avatar of Grothendieck's faithfully flat descent formalism [30, VIII] (Appendix 1).

As an example, consider the case where $\mathrm{G} = \mathrm{SL}(3)$ and the field k is discretely valued. The apartments of the building are then tilings of the Euclidean plane by regular triangles (*alcoves* in the Bruhat-Tits terminology). If the valuation v of k is normalized so that $v(k^\times) = \mathbb{Z}$, then in order to define the group G_x when x is the barycenter of a triangle, we have to (provisionally) use a finite ramified extension K such that $v(\mathrm{K}^\times) = \frac{1}{3}\mathbb{Z}$ (the apartments in $\mathcal{B}(\mathrm{G},\mathrm{K})$ have "three times more walls" and x lies at the intersection of three of them). The general case, when the barycentric coordinates of the point x (in the closure of its facet) are not a priori rational, requires an a priori infinite extension.

As already mentioned, when G splits over the ground field k, our compactifications have already been defined by V. Berkovich [4, §5]. His original procedure relies from the very beginning on the explicit construction of reductive group schemes over $\mathbb{Z}$ by means of Chevalley bases [21]. If T denotes a maximal split torus (with character group $\mathrm{X}^*(\mathrm{T})$), then the model for an apartment in $\mathcal{B}(\mathrm{G},k)$ is $\Lambda = \mathrm{Hom}(\mathrm{X}^*(\mathrm{T}), \mathbb{R}_+^\times)$ seen as a real affine space. Choosing a suitable (special) maximal compact subgroup $\mathbf{P}$ in G^{an}, V. Berkovich identifies Λ with the image of T^{an} in the quotient variety $\mathrm{G}^{\mathrm{an}}/\mathbf{P}$. The building $\mathcal{B}(\mathrm{G},k)$ thus appears