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ARITHMETIC FUJITA APPROXIMATION

BY HUAYI CHEN

ABSTRACT. – We prove an arithmetic analogue of Fujita's approximation theorem in Arakelov geometry, conjectured by Moriwaki, by using measures associated to $\mathbb{R}$-filtrations.

RÉSUMÉ. – On démontre un analogue arithmétique du théorème d'approximation de Fujita en géométrie d'Arakelov — conjecturé par Moriwaki — par les mesures associées aux $\mathbb{R}$-filtrations.

1. Introduction

Fujita approximation is an approximative version of Zariski decomposition of pseudo-effective divisors [20]. Let X be a projective variety defined over a field K and L be a *big* line bundle on X, i.e., the *volume* of L, defined as

$$\mathrm{vol}(L) := \limsup_{n\to\infty} \frac{\mathrm{rk}_K H^0(X, L^{\otimes n})}{n^{\dim X}/(\dim X)!},$$

is strictly positive. Fujita's approximation theorem asserts that, for any $\varepsilon > 0$, there exist a projective birational morphism $\nu : X' \to X$, an integer $p > 0$, together with a decomposition $\nu^*(L^{\otimes p}) \cong A \otimes E$, where A is an ample line bundle, E is effective, such that $p^{-\dim X}\mathrm{vol}(A) \geqslant \mathrm{vol}(L) - \varepsilon$. This theorem had been proved by Fujita himself [7] in characteristic 0 case (see also [4]) before its generalization to any characteristic case by Takagi [17]. It is the source of many important results concerning big divisors and the volume function in algebraic geometry, such as the volume function as a limit, its log-concavity and differentiability, etc. We refer readers to [10, 11.4] for a survey, see also [1, 5, 6, 11].

The arithmetic analogue of the volume function and the arithmetic bigness in Arakelov geometry have been introduced by Moriwaki [12, 13]. Let K be a number field and $\mathcal{O}_K$ be its integer ring. Let $\mathscr{X}$ be a projective arithmetic variety of total dimension d over $\mathrm{Spec}\,\mathcal{O}_K$. For any Hermitian line bundle $\overline{\mathscr{L}}$ on $\mathscr{X}$, the *arithmetic volume* of $\overline{\mathscr{L}}$ is defined as

$$\widehat{\mathrm{vol}}(\overline{\mathscr{L}}) := \limsup_{n\to\infty} \frac{\widehat{h}^0(\mathscr{X}, \overline{\mathscr{L}}^{\otimes n})}{n^d/d!}, \tag{1}$$

where

$$\widehat{h}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) := \log \left|\{s \in H^0(\mathcal{X}, \mathcal{L}^{\otimes n}) \mid \|s\|_{\sup} \leqslant 1\}\right|.$$

Similarly, $\overline{\mathcal{L}}$ is said to be (arithmetically) *big* if $\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) > 0$. In [13, 14], Moriwaki has proved that the arithmetic volume function is continuous with respect to $\overline{\mathcal{L}}$, and admits a unique continuous extension to $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{R}}$. In [13], he asked the following question (Remark 5.9 *loc. cit.*): *does the Fujita approximation hold in the arithmetic case?* A consequence of this conjecture is that the right-hand side of (1) is actually a limit (see [13, Remark 4.1]), which is similar to a result of Rumely, Lau and Varley [16] on the existence of the sectional capacity of Hermitian line bundles.

Recall that in [3], the author has proved that, by slope method, one can associate naturally a sequence of Borel probability measures $(\eta_n)_{n\geqslant 1}$ on $\mathbb{R}$ to the Hermitian line bundle $\overline{\mathcal{L}}$ such that

$$\widehat{\deg}(\pi_*(\overline{\mathcal{L}}^{\otimes n})) = [K:\mathbb{Q}]\,\mathrm{rk}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))\int_{\mathbb{R}} x\,\eta_n(\mathrm{d}x).$$

In this probabilistic framework, the existence of sectional capacity is interpreted as the convergence of the sequence of expectations $(\int_{\mathbb{R}} x\,\eta_n(\mathrm{d}x))_{n\geqslant 1}$. The author has actually proved the vague convergence of the sequence $(\eta_n)_{n\geqslant 1}$ to a Borel probability measure, under the ampleness hypothesis of $\mathcal{L}_K$.

In this article, we consider another sequence $(\nu_n)_{n\geqslant 1}$ of Borel probability measures defined by the successive minima of $\pi_*(\overline{\mathcal{L}}^{\otimes n})$ and establish its vague convergence under the bigness hypothesis of $\mathcal{L}_K$. By the arithmetic Riemann-Roch theorem of Gillet and Soulé [9], $\widehat{h}^0(\pi_*(\overline{\mathcal{L}})^{\otimes n})$ is compared to $[K:\mathbb{Q}]\,\mathrm{rk}(\pi_*(\overline{\mathcal{L}}^{\otimes n}))\int_0^\infty x\,\nu_n(\mathrm{d}x)$ and it follows that

$$\widehat{\mathrm{vol}}(\overline{\mathcal{L}}) = \lim_{n\to\infty}\frac{\widehat{h}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})}{n^d/d!} = [K:\mathbb{Q}]d\mathrm{vol}(\mathcal{L}_K)\int_0^\infty x\,\nu(\mathrm{d}x),$$

where ν denotes the vague limit of ν_n.

By developing a variant of the convergence result, we prove the arithmetic Fujita approximation. One difficulty is that, if $\overline{\mathcal{A}}$ is an ample Hermitian line subbundle of $\overline{\mathcal{L}}$ which approximates well $\overline{\mathcal{L}}$, then in general the section algebra of $\mathcal{A}_K$ does not approximate that of $\mathcal{L}_K$ at all. In fact, it approximates only the graded linear series of $\mathcal{L}_K$ generated by small sections (see §4.3). To overcome this difficulty, we need a recent result of Lazarsfeld and Mustaţă [11] on a very general approximation theorem for graded linear series of a big line bundle on a projective variety. It permits to approximate the graded linear series of the generic fiber generated by small sections.

Shortly after the first version of this article had been written, X. Yuan told me that he was working on the same subject and had obtained (see [19]) the arithmetic Fujita approximation independently by using a different method inspired by [11].

The organization of this article is as follows. In the second section, we introduce the notion of approximable graded algebras and study their asymptotic properties. We then recall the notion of Borel measures associated to filtered vector spaces. At the end of the section, we establish a convergence result for filtered approximable algebras. In the third section, we recall the theorem of Lazarsfeld and Mustaţă on the approximability of certain graded linear series. We then describe some approximable graded linear series which come from the arithmetic of a Hermitian line bundle on an arithmetic projective variety. The main theorem of the article is established in the fourth section. We prove that the arithmetic volume of a big Hermitian line bundle can be approximated by the arithmetic volume of its graded linear series of finite type, which implies the Moriwaki's conjecture. We also prove that, if a graded linear series generated by small sections approximates well a big Hermitian line bundle $\overline{\mathscr{L}}$, then it also approximates well the asymptotic measure of $\overline{\mathscr{L}}$ truncated at 0.

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2. Approximable algebras and asymptotic measures

In [3], the author has used the measures associated to filtered vector spaces to study asymptotic invariants of Hermitian line bundles. Several convergence results have been established for graded algebras equipped with $\mathbb{R}$-filtrations under the finite generation condition on the underlying graded algebra [3, Theorem 3.4.3]. However, some graded algebras coming naturally from the arithmetic do not satisfy this condition. In this section, we generalize the convergence result to a so-called *approximable graded algebra* case.

2.1. Approximable graded algebras

In the study of projective varieties, graded algebras are natural objects which often appear as graded linear series of a line bundle. In general, such graded algebras are not always finitely generated. However, according to approximation theorems due to Fujita [7], Takagi [17], Lazarsfeld and Mustaţă [11] etc., they can often be approximated arbitrarily closely by their graded subalgebras of finite type. Inspired by [11], we formalize this observation as a notion. In this section, K denotes an arbitrary field. All algebras and all vector spaces are supposed to be over K.

Definition 2.1. – Let $B = \bigoplus_{n\geqslant 0} B_n$ be an integral graded K-algebra. We say that B is *approximable* if the following conditions are verified:

(a) all vector spaces B_n are finite dimensional and $B_n \neq 0$ for sufficiently large n;

(b) for any $\varepsilon \in (0,1)$, there exists an integer $p_0 \geqslant 1$ such that, for any integer $p \geqslant p_0$, one has
$$\liminf_{n\to\infty} \frac{\mathrm{rk}(\mathrm{Im}(S^n B_p \to B_{np}))}{\mathrm{rk}(B_{np})} > 1-\varepsilon,$$
where $S^n B_p \to B_{np}$ is the canonical homomorphism defined by the algebra structure on B.

REMARK 2.2. – The condition (a) serves to exclude the degenerate case so that the presentation becomes simpler. In fact, if an integral graded algebra B is not concentrated on B_0, then by choosing an integer $q \geqslant 1$ such that $B_q \neq 0$, we obtain a new graded algebra $\bigoplus_{n\geqslant 0} B_{nq}$ which verifies (a). This new algebra often contains the information about B in which we are interested.

EXAMPLE 2.3. – The following are some examples of approximable graded algebras.

1) If B is an integral graded algebra of finite type such that $B_n \neq 0$ for sufficiently large n, then it is clearly approximable.
2) Let X be a projective variety over $\operatorname{Spec} K$ and L be a big line bundle on X. Then by Fujita's approximation theorem, the total graded linear series $\bigoplus_{n\geqslant 0} H^0(X, L^{\otimes n})$ of L is approximable.
3) More generally, Lazarsfeld and Mustaţă have shown that, with the notation of 2), any graded subalgebra of $\bigoplus_{n\geqslant 0} H^0(X, L^{\otimes n})$ containing an ample divisor (see Definition 3.1) and verifying the condition (a) above is approximable.

We shall revisit the examples 2) and 3) in §3.1.

The following properties of approximable graded algebras are quite similar to some classical results on big line bundles.

PROPOSITION 2.4. – *Let $B = \bigoplus_{n\geqslant 0} B_n$ be an integral graded algebra which is approximable. Then there exists a constant $a \in \mathbb{N} \setminus \{0\}$ such that, for any sufficiently large integer p, the algebra $\bigoplus_{n\geqslant 0} \operatorname{Im}(S^n B_p \to B_{np})$ has Krull dimension a. Furthermore, set $d(B) := a - 1$. The sequence*

$$\left(\frac{\operatorname{rk}(B_n)}{n^{d(B)}/d(B)!}\right)_{n\geqslant 1} \tag{2}$$

converges in $\mathbb{R}_+$.

Proof. – Assume that $B_m \neq 0$ for all $m \geqslant m_0$, where $m_0 \in \mathbb{N}$. Since B is integral, for any integer $n \geqslant 1$ and any integer $m \geqslant m_0$, one has

$$\operatorname{rk}(B_{n+m}) \geqslant \operatorname{rk}(B_n). \tag{3}$$

For any integer $p \geqslant m_0$, denote by $a(p)$ the Krull dimension of $\bigoplus_{n\geqslant 0} \operatorname{Im}(S^{np} B_p \to B_{np})$, and define

$$f(p) := \liminf_{n\to\infty} \frac{\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np}))}{\operatorname{rk}(B_{np})}. \tag{4}$$

The approximable condition shows that $\lim_{p\to\infty} f(p) = 1$. Recall that the classical result on Hilbert polynomials implies

$$\operatorname{rk}(\operatorname{Im}(S^n B_p \to B_{np})) \asymp n^{a(p)-1} \qquad (n \to \infty).$$

Thus, if $f(p) > 0$, then $\operatorname{rk}(B_{np}) \asymp n^{a(p)-1}$, and hence by (3), one has $\operatorname{rk}(B_n) \asymp n^{a(p)-1}$ $(n \to \infty)$. So $a(p)$ is constant if $f(p) > 0$. In particular, $a(p)$ is constant when p is sufficiently large. Denote by $d(B)$ this constant minus 1.