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EQUIVALENCE PROBLEM FOR MINIMAL RATIONAL CURVES WITH ISOTRIVIAL VARIETIES OF MINIMAL RATIONAL TANGENTS

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ABSTRACT. – We formulate the equivalence problem, in the sense of É. Cartan, for families of minimal rational curves on uniruled projective manifolds. An important invariant of this equivalence problem is the variety of minimal rational tangents. We study the case when varieties of minimal rational tangents at general points form an isotrivial family. The main question in this case is for which projective variety Z, a family of minimal rational curves with Z-isotrivial varieties of minimal rational tangents is locally equivalent to the flat model. We show that this is the case when Z satisfies certain projective-geometric conditions, which hold for a non-singular hypersurface of degree ≥ 4.

RÉSUMÉ. – Nous énonçons le problème d'équivalence, au sens de É. Cartan, pour des familles de courbes rationnelles minimales sur des variétés projectives uniréglées. Un invariant important de ce problème d'équivalence est la variété des tangentes rationnelles minimales. Nous étudions le cas où les variétés de tangentes rationnelles minimales aux points génériques forment une famille isotriviale. La question principale dans ce cas est : pour quelle variété projective Z une famille de courbes rationnelles minimales, dont les variétés de tangentes rationnelles minimales sont Z-isotriviales, est-elle localement équivalente au modèle plat ? Nous montrons que c'est le cas lorsque Z vérifie certaines conditions de géométrie projective qui sont satisfaites pour une hypersurface non singulière de degré ≥ 4.

1. Introduction

We will work over the complex numbers. For a uniruled projective manifold X, an irreducible component $\mathcal{K}$ of the space of rational curves on X is a *family of minimal rational curves* on X if the subvariety $\mathcal{K}_x$ consisting of members of $\mathcal{K}$ through a general point $x \in X$ is projective and non-empty. Minimal rational curves play an important role in the geometry of uniruled projective manifolds (cf. [6], [4]). We are interested in the following 'equivalence problems' in the sense of É. Cartan (cf. [3]) for families of minimal rational curves.

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Question 1.1. – Let X and X' be two uniruled projective manifolds with families of minimal rational curves $\mathcal{K}$ on X and $\mathcal{K}'$ on X'. Given two points $x \in X$ and $x' \in X'$, can we find open neighborhoods $x \in U \subset X$ and $x' \in U' \subset X'$ with a biholomorphic map $\varphi : U \to U'$ such that for each member C of $\mathcal{K}$ (resp. C' of $\mathcal{K}'$) there exists a member C' of $\mathcal{K}'$ (resp. C of $\mathcal{K}$) satisfying

$$\varphi(C \cap U) = C' \cap U' \quad (\text{resp. } \varphi^{-1}(C' \cap U') = C \cap U)?$$

If such a biholomorphic map φ exists, we will say that $(X, \mathcal{K}, x)$ is *equivalent* to $(X', \mathcal{K}', x')$. One motivation for studying this problem is the following theorem.

Theorem 1.2. – *Let X (resp. X') be a Fano manifold with second Betti number 1 and let $\mathcal{K}$ (resp. $\mathcal{K}'$) be a family of minimal rational curves on X (resp. X'). Assume that* $\dim \mathcal{K} = \dim \mathcal{K}' \geq \dim X = \dim X'$. *Suppose for some $x \in X$ and $x' \in X'$, that $(X, \mathcal{K}, x)$ is equivalent to $(X', \mathcal{K}', x')$. Then the equivalence map $\varphi : U \to U'$ extends to a biregular morphism from X to X' sending x to x'.*

Theorem 1.2 follows from the argument in [7] although it was not explicitly stated there. Theorem 1.2 and its variations are useful in proving two Fano manifolds of second Betti number 1 are biregular (cf. [6], [4]). Thus Question 1.1 has interesting applications in algebraic geometry.

A natural approach to Question 1.1 is to find local properties of the family $\mathcal{K}$ near x which are invariant under the equivalence, i.e., *local invariants* of the family. An important invariant is provided by the variety of minimal rational tangents. Recall that given a general point $x \in X$, the *variety of minimal rational tangents* at x is the subvariety $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ defined as the union of the tangent directions of members of $\mathcal{K}$ through x. A great advantage of variety of minimal rational tangents $\mathcal{C}_x$ is that it is equipped with a projective embedding $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ and consequently all projective geometric invariants of the projective variety $\mathcal{C}_x$ give rise to invariants of the equivalence problem.

Throughout the paper we will consider only those $(X, \mathcal{K})$ for which the following condition holds.

Assumption 1.3. – $\dim X \geq 3$ and $\mathcal{C}_x$ at general point $x \in X$ is an irreducible non-singular variety and is not a linear subvariety in $\mathbb{P}T_x(X)$. In particular, it has positive dimension.

What happens if $\mathcal{C}_x$ is reducible is a very important and difficult issue requiring ideas and methods different from those considered below. One justification of making the assumption that $\mathcal{C}_x$ is irreducible is that there is a large class of examples satisfying it. As a matter of fact, all known examples with $\dim \mathcal{C}_x > 0$ satisfy the irreducibility assumption. The non-singularity assumption is not really restrictive. It is believed to be always true. Finally, the non-linearity assumption is harmless. When $\mathcal{C}_x$ is linear and irreducible, we can foliate X by projective spaces (e.g. [1, Theorem 3.1]) and the equivalence problem becomes trivial.

The main question in the equivalence problem for minimal rational curves is to study to what extent the equivalence is decided by the information of varieties of minimal rational tangents. More precisely, the main question is the following.

QUESTION 1.4. – Let X and X' be two projective manifolds with families of minimal rational curves $\mathscr{K}$ on X and $\mathscr{K}'$ on X' satisfying Assumption 1.3. Let $x \in X$ and $x' \in X'$ be general points in the sense of Assumption 1.3. Suppose there exist open neighborhoods $U \subset X$, $U' \subset X'$ and a commuting diagram

$$\begin{array}{ccc} \mathbb{P}T(U) & \xrightarrow{\psi} & \mathbb{P}T(U') \\ \downarrow & & \downarrow \\ U & \xrightarrow{\varphi} & U' \end{array}$$

where the vertical maps are natural projections and the horizontal maps are biholomorphisms satisfying

$$\psi_x(\mathscr{C}_x) = \mathscr{C}_{\varphi(x)} \text{ for each } x \in U.$$

Is $(X, \mathscr{K}, x)$ equivalent to $(X', \mathscr{K}', x')$?

We will see below that the answer is not affirmative in general. A general result toward Question 1.4 is provided by the following result, which is just a restatement of Theorem 3.1.4 of [6].

THEOREM 1.5. – *Let X and X' be two projective manifolds with families of minimal rational curves $\mathscr{K}$ on X and $\mathscr{K}'$ on X' satisfying Assumption 1.3. Let $x \in X$ and $x' \in X'$ be general points in the sense of Assumption 1.3. Suppose there exist open neighborhoods $U \subset X$, $U' \subset X'$ and a commuting diagram*

$$\begin{array}{ccc} \mathbb{P}T(U) & \xrightarrow{\psi} & \mathbb{P}T(U') \\ \downarrow & & \downarrow \\ U & \xrightarrow{\varphi} & U' \end{array}$$

where the vertical maps are natural projections and the horizontal maps are biholomorphisms satisfying

$$\psi_x(\mathscr{C}_x) = \mathscr{C}_{\varphi(x)} \textit{ for each } x \in U$$

and $\psi = d\varphi$, the derivative of φ. Then $(X, \mathscr{K}, x)$ is equivalent to $(X', \mathscr{K}', x')$.

In comparison to Question 1.4, the crucial additional assumption in Theorem 1.5 is that the holomorphic map ψ comes from the derivative of φ. In this sense, the condition for Theorem 1.5 is *differential-geometric*. A central question is under what *algebraic-geometric* conditions on the varieties of minimal rational tangents, we can get this differential geometric condition. In this paper, we will concentrate on the following special case.

DEFINITION 1.6. – Let $Z \subset \mathbb{P}^{n-1}$ be a fixed irreducible non-singular non-linear projective variety. For an n-dimension projective manifold X and a family of minimal rational curves $\mathscr{K}$, we say that it has Z-isotrivial varieties of minimal rational tangents, if for a general point $x \in X$, $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ is isomorphic to $Z \subset \mathbb{P}^{n-1}$ as projective varieties.

Note that for any Z, there exists $(X, \mathscr{K})$ with Z-isotrivial varieties of minimal rational tangents:

Example 1.7. – Let $Z \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a non-singular irreducible projective variety contained in a hyperplane. Let $\psi : X_Z \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ with center Z. Let $\mathcal{K}_Z$ be the family of curves which are proper transforms of lines in $\mathbb{P}^n$ intersecting Z. Then $\mathcal{K}_Z$ is a family of minimal rational curves on X_Z with Z-isotrivial varieties of minimal rational tangents. In fact, X_Z is quasi-homogeneous with an open orbit containing $\psi^{-1}(\mathbb{P}^n \setminus \mathbb{P}^{n-1})$.

Now we can formulate the following special case of Question 1.4.

Question 1.8. – Let $Z \subset \mathbb{P}^{n-1}$ be an irreducible non-singular non-linear variety. Let X be an n-dimensional projective manifold and let $\mathcal{K}$ be a family of minimal rational curves on X with Z-isotrivial varieties of minimal rational tangents. Is $(X, \mathcal{K}, x)$ for a general $x \in X$ equivalent to that of Example 1.7?

The answer is not always affirmative:

Example 1.9. – Let W be a 2ℓ-dimensional complex vector space with a symplectic form. Fix an integer k, $1 < k < \ell$ and let S be the variety of all k-dimensional isotropic subspaces of W. S is a uniruled homogeneous projective manifold. There is a unique family $\mathcal{K}$ of minimal rational curves, just the set of all lines on S under the Plücker embedding. The varieties of minimal rational tangents are Z-isotrivial where Z is the projectivization of the vector bundle $\mathcal{O}(-1)^{2\ell-2k} \oplus \mathcal{O}(-2)$ on $\mathbb{P}^{k-1}$ embedded by the dual tautological bundle of the projective bundle (cf. Proposition 3.2.1 of [9]). Let us denote it by $Z \subset \mathbb{P}V$. There is a distinguished hypersurface $R \subset Z$ corresponding to $\mathbb{P}\mathcal{O}(-1)^{2\ell-2k}$. Let D be the linear span of R in V. This D defines a distribution on S which is not integrable (cf. Section 4 of [9]). However, the corresponding distribution on X_Z of Example 1.7 is integrable. Thus $(S, \mathcal{K}, x)$ cannot be equivalent to $(X_Z, \mathcal{K}_Z, y)$ at general points x, y.

Thus the correct formulation of Question 1.8 is to ask for which Z the answer to Question 1.8 is affirmative. Up to now the only result in this line is the following result of Ngaiming Mok in [10]:

Theorem 1.10. – *Let S be an n-dimensional irreducible Hermitian symmetric space of compact type with a base point $o \in S$. If the projective variety $Z \subset \mathbb{P}^{n-1}$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ for the family of minimal rational curves on S, then Question 1.8 has an affirmative answer.*

For example when S is the n-dimensional quadric hypersurface, $Z \subset \mathbb{P}^{n-1}$ is just an $(n-2)$-dimensional non-singular quadric hypersurface. Then $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ in Question 1.8 defines a conformal structure at general points of X. In this case, Theorem 1.10 says that this conformal structure is flat. In general, for each S, we can interpret the condition of Question 1.8 as a certain G-structure at general points of X and Theorem 1.10 says that this G-structure is flat.

It is worth recalling Mok's strategy for the proof of Theorem 1.10. The main point is to show that this G-structure which is defined at general points of X can be extended to a G-structure in a *neighborhood* of a general minimal rational curve. Once this extension is obtained, one can deduce the flatness by applying [5] which shows the vanishing of the