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Carel FABER & Sergey SHADRIN & Dimitri ZVONKINE

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TAUTOLOGICAL RELATIONS AND THE r-SPIN WITTEN CONJECTURE

BY CAREL FABER, SERGEY SHADRIN AND DIMITRI ZVONKINE

ABSTRACT. – In [11], A. Givental introduced a group action on the space of Gromov–Witten potentials and proved its transitivity on the semi-simple potentials. In [24, 25], Y.-P. Lee showed, modulo certain results announced by C. Teleman, that this action respects the tautological relations in the cohomology ring of the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable pointed curves.

Here we give a simpler proof of this result. In particular, it implies that in any semi-simple Gromov–Witten theory where arbitrary correlators can be expressed in genus 0 correlators using only tautological relations, the geometric Gromov–Witten potential coincides with the potential constructed via Givental's group action.

As the most important application we show that our results suffice to deduce the statement of a 1991 Witten conjecture relating the r-KdV hierarchy to the intersection theory on the space of r-spin structures on stable curves. We use the fact that Givental's construction is, in this case, compatible with Witten's conjecture, as Givental himself showed in [10].

RÉSUMÉ. – Dans [11], A. Givental a introduit une action de groupe sur l'espace des potentiels de Gromov–Witten et a prouvé sa transitivité sur les potentiels semi-simples. Dans [24, 25], Y.-P. Lee a montré, modulo certains résultats annoncés par C. Teleman, que cette action préserve les relations tautologiques dans l'anneau de cohomologie de l'espace des modules $\overline{\mathcal{M}}_{g,n}$ des courbes stables épointées.

Ici nous donnons une démonstration plus simple de ce résultat. Il en découle, entre autres, que si dans une théorie de Gromov–Witten semi-simple on peut exprimer n'importe quel corrélateur en fonction des corrélateurs de genre 0 en utilisant uniquement des relations tautologiques, alors le potentiel de Gromov–Witten géométrique coïncide avec le potentiel construit via l'action du groupe de Givental.

Ces résultats suffisent pour démontrer une conjecture de Witten de 1991 qui relie la hiérarchie r-KdV à la théorie de l'intersection sur l'espace des structures r-spin sur les courbes stables. Nous utilisons pour cela la compatibilité entre la construction de Givental dans ce cas et la conjecture de Witten, compatibilité établie dans [10] par Givental lui-même.

1. Introduction

In Sections 1.1 to 1.4 we give some background information on moduli spaces and tautological relations; the Gromov–Witten potentials, Frobenius manifolds and semi-simplicity; r-spin structures and Witten's conjecture; Givental's group action and Y.-P. Lee's universal relations. Finally, in Section 1.5 we formulate our main results.

1.1. Moduli spaces and tautological relations

More information on this subject can be found in [13].

1.1.1. *Moduli spaces*. – For $g, n \in \mathbb{N}$, $2-2g-n<0$, let $\mathcal{M}_{g,n}$ be the moduli space of smooth genus g curves with n distinct marked and numbered points. Let $\overline{\mathcal{M}}_{g,n}$ be its Deligne-Mumford compactification or, in other words, the moduli space of stable genus g curves with n marked and numbered points.

1.1.2. *The ψ- and κ-classes*. – There are n naturally defined line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ over $\overline{\mathcal{M}}_{g,n}$. The fiber of $\mathcal{L}_i$ over a point $t \in \overline{\mathcal{M}}_{g,n}$ is the *cotangent line* to the i-th marked point on the corresponding stable curve C_t.

Definition 1.1. – The *ψ-classes* $\psi_1, \ldots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ are the first Chern classes $\psi_i = c_1(\mathcal{L}_i)$ of the line bundles $\mathcal{L}_i$.

Let $\pi : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map; more precisely, π forgets the marked points from $n+1$ to $n+m$ and stabilizes the curve by contracting the genus 0 irreducible components with less than three nodes and marked points.

Definition 1.2. – Let $k_1, \ldots, k_m$ be positive integers, $K = \sum k_i$. The *κ-class* $\kappa_{k_1,\ldots,k_m} \in H^{2K}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is defined by

$$\kappa_{k_1,\ldots,k_m} = \pi_*(\psi_{n+1}^{k_1+1} \ldots \psi_{n+m}^{k_m+1}).$$

1.1.3. *The dual graphs*. – Consider a stable curve C of genus g with n marked points. The topological type of C can be described by a graph G obtained by replacing every irreducible component of the curve by a vertex and every node of the curve by an edge. Every marked point is replaced by a *tail* (an edge that does not lead to any vertex) retaining the same number as the marked point. Each vertex v is labeled by an integer g_v: the geometric genus of the corresponding component of C. The g_v's and the first Betti number of G add up to g.

To a vertex v of G we assign its *valency* n_v: the number of half-edges and tails adjacent to it. The stability of the curve C is equivalent to the *stability condition* on the graph: $2-2g_v-n_v<0$ for every vertex v.

In order to avoid problems with automorphisms, we will label all the half-edges of G. To G we assign the space

$$\overline{\mathcal{M}}_G = \prod_v \overline{\mathcal{M}}_{g_v,n_v},$$

where the product goes over the set of vertices of G. The space $\overline{\mathcal{M}}_G$ comes with a natural map $p : \overline{\mathcal{M}}_G \to \overline{\mathcal{M}}_{g,n}$ whose image is the closure of the set of stable curves homeomorphic to C (relative to the marked points). Note that $p_*[\overline{\mathcal{M}}_G] = |\mathrm{Aut}(G)| \cdot [p(\overline{\mathcal{M}}_G)]$.

We can define a cohomology class on $\overline{\mathcal{M}}_G$ (and hence on $\overline{\mathcal{M}}_{g,n}$ by taking the push-forward of this class under p) by assigning a class $\kappa_{k_1,\dots,k_m}$ to each vertex of G and a power ψ^d of the ψ-class to each half-edge and each tail of G. The class on $\overline{\mathcal{M}}_{g,n}$ does not depend on the labeling of the half-edges of G.

Definition 1.3. – A *stable dual graph* or just a *dual graph* is a graph G with labeled half-edges and numbered tails, satisfying the stability condition, with a set $\{k_1,\dots,k_m\}$ of positive integers assigned to each vertex v and a nonnegative integer d assigned to each half-edge and each tail. The corresponding cohomology class in $H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ is called the *tautological class* assigned to G and denoted by $[G]$. The *genus* of a dual graph is the genus of the corresponding stable curves, its *degree* is the algebraic degree of the corresponding cohomology class, and its *dimension* is $\dim\overline{\mathcal{M}}_{g,n} - \text{degree} = 3g-3+n-\text{degree}$.

In the pictures the sets $\{k_1,\dots,k_m\}$ and the integers d will be represented by $\kappa_{k_1,\dots,k_m}$ and ψ^d respectively, to recall their meanings. The empty sets (corresponding to the cohomology class equal to 1) and the integers $d=0$ will be omitted.

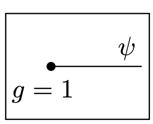
1.1.4. *Tautological and Gorenstein relations*. – Let $G_{g,n}$ be the space of formal rational linear combinations of stable dual graphs of genus g with n tails. Then, extending by linearity the map $G\mapsto[G]$ of Definition 1.3, we obtain a map

$$\begin{aligned}\varphi : G_{g,n} &\to H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})\\ L &\mapsto [L].\end{aligned}$$

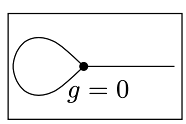
Definition 1.4. – The image of φ is called the *tautological ring* of $\overline{\mathcal{M}}_{g,n}$. A linear combination of stable dual graphs is called a *tautological relation* if it is in the kernel of φ.

Remark 1.5. – The tautological ring can also be viewed as a subring of the rational Chow ring. In this paper we *only* consider the tautological ring as a subring of the cohomology ring. The terms *tautological relation* and *Gorenstein conjecture* (see below) are used correspondingly.

Example 1.6. – The following linear combination of dual graphs in $G_{1,1}$ is a tautological relation:



$-\frac{1}{24}$



$= 0.$

Remark 1.7. – The tautological ring is indeed a ring. Given two linear combinations L_1 and L_2 of dual graphs, the class $[L_1]\cup[L_2]$ can be algorithmically expressed as $[L]$ for some linear combination L of dual graphs.

Remark 1.8. – Although for large g and n the dimension of the tautological ring is much smaller than that of the total cohomology ring of $\overline{\mathcal{M}}_{g,n}$, it is quite hard to construct a nontautological cohomology class (see [12]).

Remark 1.9. – The so-called *Gorenstein conjecture* ([14], Question 5.5, [6], Section 5.1) states that the Poincaré duality is a perfect pairing on the tautological ring.

Definition 1.10. – A linear combination $L \in G_{g,n}$ of dual graphs is called a *Gorenstein relation* if the intersection of the class $[L]$ with all tautological classes of complementary dimension vanishes.

Thus a tautological relation is always a Gorenstein relation, while the converse is equivalent to the Gorenstein conjecture. Note also that it is possible to check by an algorithm whether L is a Gorenstein relation or not, while (because the Gorenstein conjecture is open) no such algorithm is known for tautological relations.

1.2. Gromov–Witten potentials and semi-simplicity

1.2.1. *The Gromov–Witten potential of a point*. –

Definition 1.11. – The *genus g Gromov–Witten descendant potential of a point* is the formal power series

$$F_g^{\mathrm{pt}}(t_0, t_1, \dots) = \sum_{n \geq 0} \sum_{d_1, \dots, d_n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g \frac{t_{d_1} \dots t_{d_n}}{n!},$$

where

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}.$$

This bracket vanishes unless $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n = \sum d_i$. The *total Gromov–Witten descendant potential of a point* is $F^{\mathrm{pt}} = \sum F_g^{\mathrm{pt}} \hbar^{g-1}$ and its exponential $Z^{\mathrm{pt}} = \exp F^{\mathrm{pt}}$ is called the *Gromov–Witten partition function* of a point.

The partition function Z^{pt} lies in the space $\mathbb{Q}[[\hbar^{\pm 1}, t_0, t_1, \dots]]$. The stability condition implies that for every monomial M of Z^{pt} we have $2 \deg_\hbar M + \deg_t M > 0$.

1.2.2. *An axiomatization of genus 0 Gromov–Witten potentials*. – A formal genus 0 Gromov–Witten potential is defined to model certain properties of F_0^{pt} and those of genus 0 Gromov–Witten potentials of more general target Kähler manifolds X (see, for instance, [11], [9] or [8]). We restrict our considerations to the even part of the cohomology of X. In our description we explain in brackets the geometric aspects that motivate the axiomatic definitions.

Let V be a complex vector space [the space $H^{\mathrm{even}}(X, \mathbb{C})$] in which we choose for convenience a basis A. The space V is endowed with a distinguished element 1 [the cohomology class 1], which, unless otherwise stated, is chosen to be the first vector of the basis. The space V is also endowed with a nondegenerate symmetric bilinear form η [the Poincaré pairing]. The coefficients of η in the basis will be denoted by $\eta_{\mu\nu}$ and the coefficients of the inverse matrix by $\eta^{\mu\nu}$. Given a triple $(V, 1, \eta)$ we can define a genus 0 Gromov-Witten potential.

Let M be a neighborhood of the origin in V. Let F_0 be a power series in variables t_d^μ, $d = 1, 2, 3, \dots$, $\mu \in A$, whose coefficients are analytic functions on M in variables t_0^μ. The coefficients of F_0 are denoted by

$$F_0 = \sum_{n \geq 0} \sum_{\substack{d_1, \dots, d_n \geq 0 \\ \mu_1, \dots, \mu_n \in A}} \langle \tau_{d_1, \mu_1} \dots \tau_{d_n, \mu_n} \rangle \frac{t_{d_1}^{\mu_1} \dots t_{d_n}^{\mu_n}}{n!}.$$