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NORMAL FORMS OF ANALYTIC PERTURBATIONS OF QUASIHOMOGENEOUS VECTOR FIELDS: RIGIDITY, INVARIANT ANALYTIC SETS AND EXPONENTIALLY SMALL APPROXIMATION

BY Eric LOMBARDI AND Laurent STOLOVITCH

This article is dedicated to Bernard Malgrange on the occasion of his 80th birthday

ABSTRACT. – In this article, we study germs of holomorphic vector fields which are "higher order" perturbations of a quasihomogeneous vector field in a neighborhood of the origin of $\mathbb{C}^n$, fixed point of the vector fields. We define a "Diophantine condition" on the quasihomogeneous initial part S which ensures that if such a perturbation of S is formally conjugate to S then it is also holomorphically conjugate to it. We study the normal form problem relatively to S. We give a condition on S that ensures that there always exists an holomorphic transformation to a normal form. If this condition is not satisfied, we also show, that under some reasonable assumptions, each perturbation of S admits a Gevrey formal normalizing transformation to a Gevrey formal normal form. Finally, we give an exponentially good approximation of the dynamic by a partial normal form.

RÉSUMÉ. – Dans cet article, nous étudions des germes de champs de vecteurs holomorphes qui sont des perturbations « d'ordres supérieurs » de champs de vecteurs quasi-homogènes au voisinage de l'origine de $\mathbb{C}^n$, point fixe des champs considérés. Nous définissons une condition « diophantienne » sur le champ quasi-homogène initial S qui assure que si une telle perturbation de S est formellement conjuguée à S alors elle l'est aussi holomorphiquement. Nous étudions le problème de mise sous forme normale relativement à S. Nous donnons une condition suffisante assurant l'existence d'une transformation holomorphe vers une forme normale. Lorsque cette condition n'est pas satisfaite, nous montrons néanmoins, sous une condition raisonnable, l'existence d'une normalisation formelle Gevrey vers une forme normale Gevrey. Enfin, nous montrons l'existence d'une approximation exponentiellement bonne de la dynamique par une forme normale partielle.

1. Introduction

The aim of this article is to study germs of holomorphic vector fields in a neighborhood of a fixed point, say 0, in $\mathbb{C}^n$. Lot of work is devoted to this problem mainly when the vector field is not too degenerate, that is when not all the eigenvalues of the linear part $DX(0)$ of X at the origin are zero. In this situation, the aim is to compare the vector field to its linear part. One way to achieve this, is to transform the vector field "as close as possible", in some sense, to its linear part by mean of a regular change of variables.

In this article we shall focus on germs of vector fields which are degenerate and which may not have a nonzero linear part at the origin. This problem has been widely studied in dimension 2 mostly by mean of desingularizations (blow-ups). Unfortunately, this tool is not available in dimension greater than 3.

We shall be given a "reference" polynomial vector field S to which we would like to compare a suitable perturbation of it. This means that we would like to know if some of the geometric or dynamical properties of the model can survive for the perturbation. For instance, the models $S_1 = y\frac{\partial}{\partial x}$ and $S_2 = y\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$ are quite different although they have the same linear part at the origin of $\mathbb{C}^2$. In fact, for S_1, each point of $\{y = 0\}$ is fixed whereas the "cusp" $\{2x^3 - 3y^2 = 0\}$ is globally invariant by S_2.

In this article, we shall assume that the *unperturbed vector field S is quasihomogeneous* with respect to some weight $p = (p_1, \dots, p_n) \in (\mathbb{N}^*)^n$. This means that each variable x_i has the weight p_i while $\frac{\partial}{\partial x_i}$ has the weight $-p_i$. Hence, the monomial x^Q is quasihomogeneous of quasidegree $(Q, p) := \sum_{i=1}^n q_i p_i$. In particular, the vector field $S = \sum_{i=1}^n S_i(x)\frac{\partial}{\partial x_i}$ is quasihomogeneous of quasidegree s if and only if S_i is a quasihomogeneous polynomial of degree $s + p_i$.

We shall then consider a germ of *holomorphic vector field X which is a good perturbation of a quasihomogeneous vector field S*, this means that the smallest quasidegree of nonzero terms in the Taylor expansion of $X - S$ is greater than s. In the homogeneous case ($p = (1, \dots, 1)$), a linear vector field S is quasihomogeneous of degree 0 and a good perturbation is a nonlinear perturbation of S (i.e. the order at 0 of the components of $X - S$ is greater or equal than 2).

We shall develop an approach of these problems through *normal forms*. By this, we mean that the group of germs of holomorphic diffeomorphisms (biholomorphisms) of $(\mathbb{C}^n, 0)$ acts on the space of vector fields by conjugacy: if X (resp. Φ) is a germ of vector field (resp. biholomorphism) at 0 of $\mathbb{C}^n$, then the conjugacy of X by Φ is $\Phi_* X(y) := D\Phi(\Phi^{-1}(y))X(\Phi^{-1}(y))$. A normal form is a special representative of this orbit which satisfies some properties. Although, the formal normal form theory of vector fields which are non-linear perturbations of a semi-simple (resp. nilpotent, general) linear vector field is well known [1] (resp. [3, 12, 29]), it is much more difficult to handle the problem when the vector field does not have a nonzero linear part. It might also be useful in problems with parameters to consider some of the parameters as a variable with a prescribed weight.

First of all, we shall define a special Hermitian product $\langle ., . \rangle_{p,\delta}$ on each space $\mathcal{H}_\delta$ of quasihomogeneous vector fields of quasidegree δ (see (5)). Its main property is that the associated norm of a product is less than or equal to the product of the norms. Let us consider the *cohomological operator*:

$$\begin{aligned} d_0 : \mathcal{H}_\delta &\to \mathcal{H}_{s+\delta} \\ U &\mapsto [S, U] \end{aligned}$$

where $[., .]$ denotes the usual Lie bracket of vector fields. We emphasize that, contrary to the case where S is linear ($s = 0$), d_0 does not leave $\mathcal{H}_\delta$ invariant. Let $d_0^* : \mathcal{H}_{\delta+s} \to \mathcal{H}_\delta$ be the *adjoint* of d_0 with respect to the Hermitian product. An element of the kernel of this operator will be called *resonant or harmonic*. The first result we have is the following:

FORMAL NORMAL FORM TRANSFORMATION (see Proposition 4.4)
There exists a formal change of coordinates tangent to Id at the origin, such that, in the new coordinates, $X - S$ is resonant.

This means that there exists $\hat{\Phi} \in (\mathbb{C}[[x_1, \ldots, x_n]])^n$ such that $\hat{\Phi}(0) = 0$ and $D\hat{\Phi}(0) = \mathrm{Id}$ and $d_0^*(\hat{\Phi}_* X - S) = 0$. When S is linear, this corresponds to classical normal forms [1, 29]. In the homogeneous case, the first result in this direction is due to G. Belitskii [3, 4] using a renormalized scalar product. In the quasihomogeneous case, a general scheme has been devised by H. Kokubu and al. [22] in order to obtain a unique normal form. This scheme can be combined with our definition. For instance, a formal normal form of a nonlinear perturbation of

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= 0 \end{aligned} \tag{1}$$

is of the form

$$\begin{aligned} \dot{x} &= y + xP_1(x,u) \\ \dot{y} &= z + yP_1(x,u) + xP_2(x,u) \\ \dot{z} &= zP_1(x,u) + yP_2(x,u) + xP_3(x,u) \end{aligned} \tag{2}$$

where $u = y^2 - 2xz$ and where the P_i's are formal power series [21].

One of the main novelties of this article is to consider the *Box operator*

$$\begin{aligned} \Box_\delta : \mathcal{H}_\delta &\to \mathcal{H}_\delta \\ U &\mapsto \Box_\delta(U) := d_0 d_0^*(U) \end{aligned}$$

which is self-adjoint and whose spectrum is non-negative. *Its nonzero spectrum is composed of the (squared) small divisors* of the problem. These are the numbers that we need to control. For instance in the homogeneous case, if $S = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$, then the eigenvalues of $\Box_{k-1}$ are the $|(Q,\lambda) - \lambda_i|^2$, where $Q \in \mathbb{N}^n$, $|Q| = k$ and $1 \leq i \leq n$.

For each quasidegree $\delta > s$, let us set

$$a_\delta := \min_{\lambda \in \mathrm{Spec}(\Box_\delta)\setminus\{0\}} \sqrt{\lambda}.$$

Then, we shall construct inductively a sequence of positive numbers η_δ from the a_δ's (see (14)). We shall say that S *is Diophantine* if there exist positive constants M, c such that $\eta_\delta \leq Mc^\delta$. Being Diophantine is a quantitative way of saying that the sequence $\{a_\delta\}$ does not accumulate the origin too fast. Hence, we have defined a *small divisors condition for quasihomogeneous vector fields*. For instance in the homogeneous case, $S = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$ is Diophantine if it satisfies Brjuno's small divisors condition [7]:

$$(\omega) \qquad -\sum_{k\geq 1} \frac{\ln \omega_k}{2^k} < +\infty,$$

where

$$\omega_k := \inf\{|(Q,\lambda)-\lambda_i| \neq 0, Q\in\mathbb{N}^n, 2\leq |Q|\leq 2^k, 1,\leq i\leq n\}.$$

RIGIDITY THEOREM (see Theorem 5.8). – *In the general quasihomogeneous case, assume that the quasihomogeneous vector field S is Diophantine. Let X be a good holomorphic deformation of S. If X is formally conjugate to S then it is holomorphically conjugate to it.*

For instance in the homogeneous case and if $S = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$, this is the classical Siegel-Brjuno linearization theorem: if S satisfies the Diophantine condition (ω) and if a holomorphic nonlinear perturbation X is formally linearizable, then X is holomorphically linearizable. For instance, a good holomorphic perturbation of S:

$$\dot{x} = x^2$$
$$\dot{y} = xy$$

which is formally conjugate to it, is also holomorphically conjugate to S. This is due to the fact that $\min_{\lambda\in \mathrm{Spec}(\Box_\delta)\setminus\{0\}} \sqrt{\lambda} \geq M\sqrt{\delta}$. Hence, the "small divisors" are in fact large. The same statement holds for perturbations of (1) since $\min_{\lambda\in \mathrm{Spec}(\Box_\delta)\setminus\{0\}} \sqrt{\lambda}$ is bounded away from 0.

Assume that the ring of polynomial first integrals of S is generated by some quasihomogeneous polynomials $h_1,\dots,h_r$. Let us denote by $\mathcal{I}$ (resp. $\widehat{\mathcal{I}}$) the ideal they generate in the ring of germs of holomorphic functions at the origin (resp. formal power series). The germ of the variety $\Sigma = \{h_1 = \cdots = h_r = 0\}$ at the origin is invariant by the flow of S. Does a good perturbation of S still have an invariant variety of this kind?

INVARIANT VARIETY THEOREM (see Theorem 5.6). – *In the general quasihomogeneous case, assume that the quasihomogeneous vector field S is Diophantine. Let X be a good holomorphic deformation of S. If X is essentially formally conjugate to S modulo $\widehat{\mathcal{I}}$ then it is holomorphically conjugate to S modulo $\mathcal{I}$.*

This means that there exists a germ of holomorphic diffeomorphism Φ such that

$$\Phi_* X = S + \sum_{i=1}^n g_i(x)\frac{\partial}{\partial x_i}, \quad \text{with } g_i \in \mathcal{I}.$$

Hence, in the new holomorphic coordinate system, Σ is an invariant variety of X since $g_{i|\Sigma} = 0$. The Diophantine condition can eventually be relaxed a little bit taking into account the ideal $\mathcal{I}$. This is a first step toward the generalization to any dimension of Camacho-Sad's theorem [8] about the existence of a holomorphic separatrix of a two dimensional foliation with an isolated singularity. If $S = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$, this was proved by L. Stolovitch [35]. Furthermore, for instance, if a formal normal form (2) of a perturbation of (1) satisfies $P_i(x,0) = 0$, $i = 1,2,3$, then in good holomorphic coordinates, $\{y = z = 0\}$ is an invariant analytic set of the perturbation.

What happens if instead of accumulating the origin, the sequence a_δ tends to infinity with δ? Let us set $\nu := \max\left(1, \frac{\max p_i}{2}\right)$.