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Daniel ALLCOCK & James A. CARLSON & Domingo TOLEDO

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HYPERBOLIC GEOMETRY AND MODULI OF REAL CUBIC SURFACES

BY Daniel ALLCOCK, James A. CARLSON AND Domingo TOLEDO

ABSTRACT. – Let $\mathcal{M}_0^{\mathbb{R}}$ be the moduli space of smooth real cubic surfaces. We show that each of its components admits a real hyperbolic structure. More precisely, one can remove some lower-dimensional geodesic subspaces from a real hyperbolic space H^4 and form the quotient by an arithmetic group to obtain an orbifold isomorphic to a component of the moduli space. There are five components. For each we describe the corresponding lattices in $\mathrm{PO}(4,1)$. We also derive several new and several old results on the topology of $\mathcal{M}_0^{\mathbb{R}}$. Let $\mathcal{M}_s^{\mathbb{R}}$ be the moduli space of real cubic surfaces that are stable in the sense of geometric invariant theory. We show that this space carries a hyperbolic structure whose restriction to $\mathcal{M}_0^{\mathbb{R}}$ is that just mentioned. The corresponding lattice in $\mathrm{PO}(4,1)$, for which we find an explicit fundamental domain, is nonarithmetic.

RÉSUMÉ. – On note $\mathcal{M}_0^{\mathbb{R}}$ l'espace des modules des surfaces cubiques réelles lisses. Nous montrons que chacune de ses composantes admet une structure hyperbolique réelle. Plus précisément, en enlevant de l'espace hyperbolique réel H^4 certaines sous-variétés totalement géodésiques de dimension inférieure, puis en prenant le quotient par un groupe arithmétique, on obtient une orbifold isomorphe à une composante de l'espace des modules. Il y a cinq composantes. Nous décrivons le réseau de $PO(4,1)$ qui correspond à chacune d'entre elles. Nous démontrons également quelques résultats sur la topologie de $\mathcal{M}_0^{\mathbb{R}}$, dont certains sont nouveaux. On note $\mathcal{M}_s^{\mathbb{R}}$ l'espace des modules des surfaces cubiques réelles qui sont stables au sens de la théorie géométrique des invariants. Nous montrons que cet espace admet une structure hyperbolique dont la restriction à $\mathcal{M}_0^{\mathbb{R}}$ est celle évoquée ci-dessus. Nous décrivons un domaine fondamental pour le réseau correspondant de $PO(4,1)$, qui s'avère être non arithmétique.

1. Introduction

The purpose of this paper is to study the geometry and topology of the moduli space of real cubic surfaces in $\mathbb{R}P^3$. It is a classical fact, going back to Schläfli [31, 33] and Klein [21], that the moduli space of smooth real cubic surfaces has five connected components.

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We show in this paper that each of these components has a real hyperbolic structure that we compute explicitly, both in arithmetic and in geometric terms. We use this geometric structure to compute, to a large extent, the topology of each component. These structures are not complete. We also prove a more subtle result, that the moduli space of stable real cubic surfaces has a real hyperbolic structure, which is complete, and that restricts, on each component of the moduli space of smooth surfaces, to the (incomplete) structures just mentioned. The most surprising fact to us is that the resulting discrete group of isometries of hyperbolic space is not arithmetic.

To describe our results, we use the following notation. We write $\mathscr{C}$ for the space of non-zero cubic forms with complex coefficients in 4 variables, Δ for the discriminant locus (forms where all partial derivatives have a common zero), $\mathscr{C}_0$ for the space $\mathscr{C}-\Delta$ of forms that define a smooth hypersurface in $\mathbb{C}P^3$, and $\mathscr{C}_s$ for the space of forms that are stable in the sense of geometric invariant theory for the action of $\mathrm{GL}(4,\mathbb{C})$ on $\mathscr{C}$. It is classical that these are the forms that define a cubic surface which is either smooth or has only nodal singularities [18, §19].

We denote all the corresponding real objects with a superscript $\mathbb{R}$. Thus $\mathscr{C}^{\mathbb{R}}$ denotes the space of non-zero cubic forms with real coefficients, and $\Delta^{\mathbb{R}}$, $\mathscr{C}_0^{\mathbb{R}}$ and $\mathscr{C}_s^{\mathbb{R}}$ the intersection with $\mathscr{C}^{\mathbb{R}}$ of the corresponding subspaces of $\mathscr{C}$. We will also use the prefix P for the corresponding projective objects, thus $P\mathscr{C}^{\mathbb{R}} \cong \mathbb{R}P^{19}$ is the projective space of cubic forms with real coefficients, and $P\Delta^{R}$, $P\mathscr{C}_0^{\mathbb{R}}$, $P\mathscr{C}_s^{\mathbb{R}}$ are the images of the objects just defined. The group $\mathrm{GL}(4,\mathbb{R})$ acts properly on $\mathscr{C}_0^{\mathbb{R}}$ and $\mathscr{C}_s^{\mathbb{R}}$ (equivalently, $\mathrm{PGL}(4,\mathbb{R})$ acts properly on $P\mathscr{C}_0^{\mathbb{R}}$ and $P\mathscr{C}_s^{\mathbb{R}}$) and we write $\mathscr{M}_0^{\mathbb{R}}$ and $\mathscr{M}_s^{\mathbb{R}}$ for the corresponding quotient spaces, namely the moduli spaces of smooth and of stable real cubic surfaces.

The space $P\Delta^{\mathbb{R}}$ has real codimension one in $P\mathscr{C}^{\mathbb{R}}$, its complement $P\mathscr{C}_0^{\mathbb{R}}$ has five connected components, and the topology of a surface in each component is classically known [21, 33, 34]. We label the components $P\mathscr{C}_{0,j}^{\mathbb{R}}$, for $j = 0,1,2,3,4$, choosing the indexing so that a surface in $P\mathscr{C}_{0,j}^{\mathbb{R}}$ is topologically a real projective plane with $3-j$ handles attached (see table 1.1; the case of -1 many handles means the disjoint union $\mathbb{R}P^2 \sqcup S^2$). It follows that the moduli space $\mathscr{M}_0^{\mathbb{R}}$ has five connected components, $\mathscr{M}_{0,j}^{\mathbb{R}}$, for $j = 0,1,2,3,4$. We can now state our first theorem:

Theorem 1.1. – *For each $j = 0,\ldots,4$ there is a union $\mathscr{H}_j$ of two- and three-dimensional geodesic subspaces of the four-dimensional real hyperbolic space H^4 and an isomorphism of real analytic orbifolds*

$$\mathscr{M}_{0,j}^{\mathbb{R}} \cong P\Gamma_j^{\mathbb{R}} \setminus (H^4 - \mathscr{H}_j).$$

Here $P\Gamma_j^{\mathbb{R}}$ is the projectivized group of integer matrices which are orthogonal with respect to the quadratic form obtained from the diagonal form $[-1,1,1,1,1]$ by replacing the last j of the 1's by 3's.

The real hyperbolic structure on the component $\mathscr{M}_{0,0}^{\mathbb{R}}$ has been studied by Yoshida [41]. The other cases are new.

The space $P\mathscr{C}_s^{\mathbb{R}}$ is connected, since it is obtained from the manifold $P\mathscr{C}^{\mathbb{R}}$ by removing a subspace of codimension two (part of the singular set of $P\Delta^{\mathbb{R}}$). Thus the moduli space $\mathscr{M}_s^{\mathbb{R}}$ is connected. We have the following uniformization theorem for this space:

THEOREM 1.2. – *There are a nonarithmetic lattice $P\Gamma^{\mathbb{R}} \subset PO(4,1)$ and a homeomorphism*

$$\mathcal{M}_s^{\mathbb{R}} \cong P\Gamma^{\mathbb{R}} \setminus H^4.$$

Moreover, there is a $P\Gamma^{\mathbb{R}}$-invariant union of two- and three-dimensional geodesic subspaces $\mathcal{H}'$ of H^4 so that this homeomorphism restricts to an isomorphism of real analytic orbifolds,

$$\mathcal{M}_0^{\mathbb{R}} \cong P\Gamma^{\mathbb{R}} \setminus (H^4 - \mathcal{H}').$$

To our knowledge this is the first appearance of a non-arithmetic lattice in a moduli problem for real varieties. Observe that the group $P\Gamma^{\mathbb{R}}$ uniformizes a space assembled from arithmetic pieces much in the spirit of the construction by Gromov and Piatetskii-Shapiro of non-arithmetic lattices in real hyperbolic space. We thus view this theorem as an appearance "in nature" of their construction.

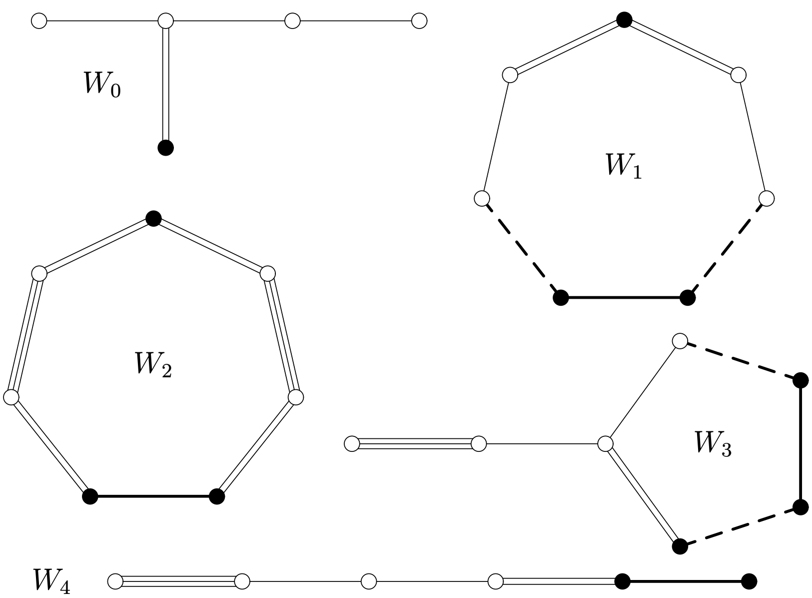

FIGURE 1.1. Coxeter polyhedra for the reflection subgroups W_j of $P\Gamma_j^{\mathbb{R}}$. The blackened nodes and triple bonds correspond to faces of the polyhedra that represent singular cubic surfaces. See the text for the explanation of the edges.

We obtain much more information about the groups $P\Gamma_j^{\mathbb{R}}$ and $P\Gamma^{\mathbb{R}}$ than we have stated here. Section 5 gives an arithmetic description of each $P\Gamma_j^{\mathbb{R}}$ and shows that they are essentially Coxeter groups. (Precisely: they are Coxeter groups for $j = 0, 3, 4$ and contain a Coxeter subgroup of index 2 if $j = 1, 2$.) We use Vinberg's algorithm to derive their Coxeter diagrams and consequently their fundamental domains. So we have a very explicit geometric description of the groups $P\Gamma_j^{\mathbb{R}}$. The results are summarized in Fig. 1.1. In these diagrams the nodes represent facets of the polyhedron, and two facets meet at an angle of $\pi/2$, $\pi/3$, $\pi/4$ or $\pi/6$, or are parallel (meet at infinity) or are ultraparallel, if the number of bonds between the two corresponding nodes is respectively 0, 1, 2, 3, or a heavy or dashed line. See Section 5 for more details.

j	Topology of Surface	Real Lines	Real Tritang. Planes	Monodromy on Lines
0	$\mathbb{R}P^2 \# 3T^2$	27	45	A_5
1	$\mathbb{R}P^2 \# 2T^2$	15	15	$S_3 \times S_3$
2	$\mathbb{R}P^2 \# T^2$	7	5	$(\mathbb{Z}/2)^3 \rtimes \mathbb{Z}/2$
3	$\mathbb{R}P^2$	3	7	S_4
4	$\mathbb{R}P^2 \sqcup S^2$	3	13	S_4

TABLE 1.1. The classical results on the components of the moduli space of real cubic surfaces. The components are indexed by j according to our conventions. The third item in the last column corrects an error of Segre.

j	Euler	Volume	Fraction	$\pi_1^{\mathrm{orb}}(\mathcal{M}^{\mathbb{R}}_{0,j})$
0	1/1920	.00685	2.03%	S_5
1	1/288	.04569	13.51%	$(S_3 \times S_3) \rtimes \mathbb{Z}/2$
2	5/576	.11423	33.78%	$(D_\infty \times D_\infty) \rtimes \mathbb{Z}/2$
3	1/96	.13708	40.54%	
4	1/384	.03427	10.14%	

TABLE 1.2. The orbifold Euler characteristic, volume, fraction of total volume, and orbifold fundamental groups of the moduli spaces $\mathcal{M}^{\mathbb{R}}_{0,j}$. See Theorem 7.1 for the notation.

The group $P\Gamma^{\mathbb{R}}$ is not a Coxeter group (even up to finite index) but we find that a subgroup of index two has a fundamental domain that is a Coxeter polyhedron. We describe this polyhedron explicitly in Section 12, thus we have a concrete geometric description of $P\Gamma^{\mathbb{R}}$, and we also find a representation of this group by matrices with coefficients in $\mathbb{Z}[\sqrt{3}]$.

Much of the classical theory of real cubic surfaces, as well as new results, are encoded in these Coxeter diagrams. The new results are our computation of the groups $\pi_1^{\mathrm{orb}}(\mathcal{M}^{\mathbb{R}}_{0,j})$ (see table 1.2) and our proof that each $\mathcal{M}^{\mathbb{R}}_{0,j}$ has contractible universal cover. These results appear in Section 7, where we describe the topology of the spaces $\mathcal{M}^{\mathbb{R}}_{0,j}$. As an application to the classical theory, we re-compute the monodromy representation of $\pi_1(P\mathcal{C}^{\mathbb{R}}_{0,j})$ on the configuration of lines on a cubic surface, which was first computed by Segre in his treatise [34]. We confirm four of his computations and correct an error in the remaining one (the case $j = 2$). See the last column of table 1.1 and Section 8 for details. We also compute the hyperbolic volume of each component in Section 9. The results are summarized in table 1.2.

Our methods are based on our previous work on the complex hyperbolic structure of the moduli space of complex cubic surfaces [1]. We proved that this moduli space $\mathcal{M}_s$ is isomorphic to the quotient $P\Gamma \setminus \mathbb{C}H^4$ of complex hyperbolic 4-space $\mathbb{C}H^4$ by the lattice