ISSN 0012-9593

quatrième série - tome 43 fascicule 5 septembre-octobre 2010

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 43, 2010, p. 719 à 760

KINETIC EQUATIONS WITH MAXWELL BOUNDARY CONDITIONS

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ABSTRACT. – We prove global stability results of *DiPerna-Lions* renormalized solutions for the initial boundary value problem associated to some kinetic equations, from which existence results classically follow. The (possibly nonlinear) boundary conditions are completely or partially diffuse, which includes the so-called Maxwell boundary conditions, and we prove that it is realized (it is not only a boundary inequality condition as it has been established in previous works). We are able to deal with Boltzmann, Vlasov-Poisson and Fokker-Planck type models. The proofs use some trace theorems of the kind previously introduced by the author for the Vlasov equations, new results concerning weak-weak convergence (the renormalized convergence and the biting L^1-weak convergence), as well as the Darrozès-Guiraud information in a crucial way.

RÉSUMÉ. – Nous montrons la stabilité des solutions renormalisées au sens de *DiPerna-Lions* pour des équations cinétiques avec conditions initiale et aux limites. La condition aux limites (qui peut être non linéaire) est partiellement diffuse et est réalisée (c'est-à-dire qu'elle n'est pas relaxée). Les techniques que nous introduisons sont illustrées sur l'équation de Fokker-Planck-Boltzmann et le système de Vlasov-Poisson-Fokker-Planck ainsi que pour des conditions aux limites linéaires sur l'équation de Boltzmann et le système de Vlasov-Poisson. Les démonstrations utilisent des théorèmes de trace du type de ceux introduits par l'auteur pour les équations de Vlasov, des résultats d'analyse fonctionnelle sur les convergences faible-faible (la convergence renormalisée et la convergence au sens du *biting lemma*), ainsi que l'information de Darrozès-Guiraud d'une manière essentielle.

1. Introduction and main results

Let Ω be an open and bounded subset of $\mathbb{R}^N$ and set $\mathcal{O} = \Omega \times \mathbb{R}^N$. We consider a gas confined in $\Omega \subset \mathbb{R}^N$. The state of the gas is given by the distribution function $f = f(t,x,v) \geq 0$ of particles, which at time $t \geq 0$ and at position $x \in \Omega$, move with the velocity $v \in \mathbb{R}^N$. The evolution of f is governed by a kinetic equation written in the domain $(0,\infty) \times \mathcal{O}$ and it is complemented with a boundary condition that we describe now.

We assume that the boundary $\partial\Omega$ is sufficiently smooth. The regularity that we need is that there exists a vector field $n \in W^{2,\infty}(\Omega;\mathbb{R}^N)$ such that $n(x)$ coincides with the outward unit normal vector at $x \in \partial\Omega$. We then define $\Sigma^x_\pm := \{v \in \mathbb{R}^N; \pm v \cdot n(x) > 0\}$ the sets of outgoing (Σ^x_+) and incoming (Σ^x_-) velocities at point $x \in \partial\Omega$ as well as $\Sigma = \partial\Omega \times \mathbb{R}^N$ and

$$\Sigma_\pm = \{(x,v) \in \Sigma; \pm n(x) \cdot v > 0\} = \{(x,v);\ x \in \partial\Omega,\ v \in \Sigma^x_\pm\}.$$

We also denote by $d\sigma_x$ the Lebesgue surface measure on $\partial\Omega$ and by $d\lambda_k$ the measure on $(0,\infty) \times \Sigma$ defined by $d\lambda_k = |n(x) \cdot v|^k \, dt d\sigma_x dv$, $k = 1$ or 2.

The boundary condition takes into account how the particles are reflected by the wall and thus takes the form of a balance between the values of the trace γf of f on the outgoing and incoming velocities subsets of the boundary:

$$(\gamma_- f)(t,x,v) = \mathscr{R}_x(\gamma_+ f(t,x,.))(v) \quad \text{on} \quad (0,\infty) \times \Sigma_-, \tag{1.1}$$

where $\gamma_\pm f := \mathbf{1}_{(0,\infty)\times\Sigma_\pm} \gamma f$. The reflection operator is time independent, local in position but can be local or nonlocal in the velocity variable. In order to describe the interaction between particles and wall by means of the reflection operator $\mathscr{R}$, J. C. Maxwell [54] proposed in 1879 the following phenomenological law by splitting the reflection operator into a local reflection operator and a diffuse (also denominated as *Maxwell*) reflection operator (which is nonlocal in the velocity variable):

$$\mathscr{R} = (1-\alpha)\, L + \alpha\, D. \tag{1.2}$$

Here $\alpha \in (0,1]$ is a constant, called the *accommodation coefficient*. The local reflection operator L is defined by

$$(L_x\, \phi)\,(v) = \phi(R_x\, v),$$

with $R_x\, v = -v$ (inverse reflection) or $R_x\, v = v - 2\,(v \cdot n(x))\, n(x)$ (specular reflection). The diffuse reflection operator $D = (D_x)_{x\in\partial\Omega}$ according to the Maxwellian profile M with temperature (of the wall) $\Theta > 0$ is defined at the boundary point $x \in \partial\Omega$ for any measurable function ϕ on Σ^x_+ by

$$(D_x\, \phi)(v) = M(v)\ \tilde\phi(x),$$

where the normalized Maxwellian M is

$$M(v) = (2\,\pi)^{\frac{1-N}{2}}\ \Theta^{-\frac{N+1}{2}}\ e^{-\frac{|v|^2}{2\,\Theta}}, \tag{1.3}$$

and the outcoming flux of mass of particles $\tilde\phi(x)$ is

$$\tilde\phi(x) = \int_{v'\cdot n(x)>0} \phi(v')\, v' \cdot n(x)\, dv' = \int_{\Sigma^x_+} \frac{\phi}{M}\, d\mu_x. \tag{1.4}$$

It is worth emphasizing that the normalization condition (1.3) is made in order that the measure $d\mu_x(v) := M(v)\,|n(x) \cdot v|\, dv$ is a probability measure on $\Sigma^x_\pm$ for any $x \in \partial\Omega$. Moreover, for any measurable function ϕ on Σ^x_+ there holds

$$\int_{\Sigma^x_-} \mathscr{R}_x \phi\, |n(x)\cdot v|\, dv = \int_{\Sigma^x_-} L_x \phi\, |n(x)\cdot v|\, dv = \int_{\Sigma^x_-} D_x \phi\, |n(x)\cdot v|\, dv = \int_{\Sigma^x_+} \phi\, n(x)\cdot v\, dv, \tag{1.5}$$

which means that all the particles which reach the boundary are reflected (no particle goes out of the domain nor enters in the domain).

The reflection law (1.2) was the only model for the gas/surface interaction that appeared in the literature before the late 1960s. In order to describe with more accuracy the interaction between molecules and wall, other models have been proposed in [25, 26, 51] where the reflection operator $\mathscr{R}$ is a general integral operator satisfying the so-called non-negative, normalization and reciprocity conditions, see [29] and Remark 6.4. We do not know whether our analysis can be adapted to such a general kernel. However, the boundary condition can be generalized in another direction, see [12, 30], and we will sometimes assume that the following nonlinear boundary condition holds

(1.6) $$\mathscr{R}\,\phi = (1-\tilde{\alpha})\,L\,\phi + \tilde{\alpha}\,D\,\phi, \qquad \tilde{\alpha} = \alpha(\tilde{\phi}),$$

where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function which satisfies $0 < \bar{\alpha} \le \alpha(s) \le 1$ for any $s \in \mathbb{R}_+$.

In the domain, the evolution of f is governed by a kinetic equation

(1.7) $$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathscr{I}(f) \quad \text{in} \quad (0,\infty) \times \mathcal{O},$$

where $\mathscr{I}(f)$ models the interactions of particles each one with each other and with the environment. Typically, it may be a combination of the quadratic Boltzmann collision operator (describing the collision interactions of particles by binary elastic shocks), the Vlasov-Poisson operator (describing the fact that particles interact by the way of the two-body long range Coulomb force) or the Fokker-Planck operator (which takes into account the fact that particles are submitted to a heat bath). More precisely, for the nonlinear boundary condition (1.6) we are able to deal with Fokker-Planck type equations, in particular the Fokker-Planck-Boltzmann equation (FPB in short) and the Vlasov-Poisson-Fokker-Planck system (VPFP in short), while for a constant accommodation coefficient we are able to deal with Vlasov type equations such as the Boltzmann equation and the Vlasov-Poisson system (VP in short). We refer to Section 6 where these models are presented. It is worth mentioning that the method presented in this paper seems to fail for the Vlasov-Maxwell system.

Finally, we complement these equations with a given initial condition

(1.8) $$f(0,.) = f_{in} \ge 0 \ \text{ on } \mathcal{O},$$

which satisfies the natural physical bounds of finite mass, energy and entropy

(1.9) $$\iint_{\mathcal{O}} f_{in}\,(1+|v|^2+|\log\, f_{in}|)\,dxdv =: C_0 < \infty.$$

We begin with a general existence result that we state deliberately in an imprecise way and we refer to Section 6 (and Theorem 6.2) for a more precise statement.

THEOREM 1.1. – *Consider the initial boundary value problem* (1.1)-(1.7)-(1.8) *associated to the FPB equation or the VPFP system with possibly mass flux depending accommodation coefficient* (1.6) *or the boundary value problem associated to the Boltzmann equation or the VP system with constant accommodation coefficient* (1.2)*. For any non-negative initial datum f_{in} with finite mass, energy and entropy (*(1.9) *holds) there exists at least one (renormalized) solution $f \in C([0,\infty); L^1(\mathcal{O}))$ with finite mass, energy and entropy to the kinetic Equation* (1.7) *associated to the initial datum f_{in} and such that the trace function γf fulfills the boundary condition* (1.1).

The Boltzmann equation and the FPB equation for initial data satisfying the natural bound (1.9) was first studied by R. DiPerna and P.-L. Lions [35, 37, 39] who proved stability and existence results for global renormalized solutions in the case of the whole space ($\Omega = \mathbb{R}^N$). Afterwards, the corresponding boundary value problem with reflection boundary conditions (1.1) and constant accommodation coefficient has been extensively studied in the case of the Boltzmann model [5, 6, 7, 8, 47], [27, 28, 44, 48, 55]. It has been proved, in the partial absorption case $\gamma_- f = \theta\, \mathscr{R}\gamma_+ f$ with $\theta \in [0, 1)$ and in the completely local reflection case (i.e. (1.1) holds with $\alpha \equiv 0$), that there exists a global renormalized solution. But in the most interesting physical case (when $\theta \equiv 1$ and $\alpha \in (0, 1]$), it has only been proved in [7] that the following boundary inequality condition

$$\gamma_- f \geq \mathscr{R}(\gamma_+ f) \quad \text{on } (0, \infty) \times \Sigma_- \tag{1.10}$$

holds, instead of the boundary equality condition (1.1). However, it is worth mentioning that if the renormalized solution built in [7] is in fact a solution to the Boltzmann equation in the sense of distributions, then that solution satisfies the boundary equality condition (1.1) (a result that one deduces thanks to the Green formula by gathering the fact that the solution is mass preserving and the fact that the solution already satisfies the boundary inequality condition (1.10)). Also, the Boltzmann equation with nonlinear boundary conditions has been treated in the setting of a strong but non global solution framework in [43].

With regard to existence results for the initial value problem for the VPFP system set in the whole space, we refer to [14, 15, 16, 20, 21, 22, 23, 36, 59, 61, 65] as well as [32] for physical motivations. The initial boundary value problem has been addressed in [13, 19]. We also refer to [1, 4, 46, 58, 68] for the initial boundary value problem for the VP system and to [58] for the corresponding stationary problem. We emphasize that in all these works only local reflection or prescribed incoming data are treated, and to our knowledge, there is no result concerning the diffuse boundary condition for the VP system or for the VPFP system.

We also mention that there is a great deal of information for the boundary value problem in an abstract setting in [45, 67] with possibly nonlinear boundary conditions [11, 57].

In short, the present work improves the already known existence results for kinetic equations with diffusive boundary reflection into three directions.

• On the one hand, we prove that (1.1) is fulfilled, while only the boundary inequality condition (1.10) was previously established.

• On the other hand, we are able to consider a large class of kinetic models (including Vlasov-Poisson term) while only the Boltzmann equation (or linear equations) could be handled with earlier techniques.

• Finally, we are able to handle some nonlinear boundary condition in the case of Fokker-Planck type equation.

We do not present the proof of Theorem 1.1 (nor the proof of its accurate version Theorem 6.2) because it classically follows from a sequential stability or sequential compactness result that we present below and a standard (but tedious) approximation procedure, see for instance [55] or the above quoted references. We deliberately state again the sequential stability result in an imprecise way, referring to Section 6 for a more accurate version.