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THE CUBIC SZEGŐ EQUATION

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ABSTRACT. – We consider the following Hamiltonian equation on the L^2 Hardy space on the circle,

$$i\partial_t u = \Pi(|u|^2 u) ,$$

where Π is the Szegő projector. This equation can be seen as a toy model for totally non dispersive evolution equations. We display a Lax pair structure for this equation. We prove that it admits an infinite sequence of conservation laws in involution, and that it can be approximated by a sequence of finite dimensional completely integrable Hamiltonian systems. We establish several instability phenomena illustrating the degeneracy of this completely integrable structure. We also classify the traveling waves for this system.

RÉSUMÉ. – On considère l'équation hamiltonienne suivante sur l'espace de Hardy du cercle

$$i\partial_t u = \Pi(|u|^2 u) ,$$

où Π désigne le projecteur de Szegő. Cette équation est un cas modèle d'équation sans aucune propriété dispersive. On établit qu'elle admet une paire de Lax et une infinité de lois de conservation en involution, et qu'elle peut être approchée par une suite de systèmes hamiltoniens de dimension finie complètement intégrables. Néanmoins, on met en évidence des phénomènes d'instabilité illustrant la dégénérescence de cette structure complètement intégrable. Enfin, on caractérise les ondes progressives de ce système.

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1. Introduction

1.1. Motivation

This work can be seen as a continuation of a series of papers due to N. Burq, N. Tzvetkov and the first author [4, 5, 6, 7] — see also [10] for a survey— , devoted to the influence of the geometry of a Riemannian manifold M onto the qualitative properties of solutions to the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = |u|^2 u \, , (t,x) \in \mathbb{R} \times M \, . \tag{1}$$

The usual strategy for finding global solutions to the Cauchy problem for (1) is to solve it locally in time in the energy space $H^1 \cap L^4$ using a fixed point argument and then to globalize in time, by means of conservation of energy and of L^2 norm. In most of the cases, the fixed point strategy leads to define a smooth local in time flow map, in the sense of regular well-posedness defined in [10], Definition 2.3, and recalled in Appendix 1 (Section 10). As a corollary of the work of Burq, Gérard, Tzvetkov — see [6], Remark 2.12, p. 205, or [10], sketch of the proof of Theorem 5.2— one obtains, whatever the geometry is, the following general result. If there exists a smooth local in time flow map on the Sobolev space $H^s(M)$, then the following Strichartz–type estimate must hold

$$\|\mathrm{e}^{it\Delta} f\|_{L^4([0,1]\times M)} \lesssim \|f\|_{H^{s/2}(M)} \, . \tag{2}$$

Inequality (2) is valid for instance if $M = \mathbb{R}^d$, $d = 1, 2, 3, 4$ and Δ is the Euclidean Laplacian, where s is given by the scaling formula

$$s = \max\left(0, \frac{d}{2} - 1\right) \, .$$

In [4, 6], it is observed that, on the two-dimensional sphere, the infimum of the numbers s such that (2) holds is $1/4$, hence is larger than the regularity given by the latter formula. This can be interpreted as a lack of dispersion properties for the spherical geometry. It is therefore natural to ask whether there exist some geometries for which these dispersion properties totally disappear. Such an example arises in sub-Riemannian geometry, more precisely for radial solutions of the Schrödinger equation associated to the sub-Laplacian on the Heisenberg group, as observed in [11], where part of the results of this paper are announced. Here we present a more elementary example of such a situation. Let us choose $M = \mathbb{R}^2_{x,y}$ and replace the Laplacian by the Grushin operator $G := \partial_x^2 + x^2\partial_y^2$, so that our equation is

$$i\partial_t u + \partial_x^2 u + x^2 \partial_y^2 u = |u|^2 u \, . \tag{3}$$

Notice that this equation enjoys the following scaling invariance: if $u(t, x, y)$ is a solution, then

$$\lambda u(\lambda^2 t, \lambda x, \lambda^2 y)$$

is also a solution. In this context, it is natural to replace the standard Sobolev space $H^s(M)$ by the Grushin Sobolev space $H^s_G(M)$, defined as the domain of $\sqrt{(-G)^s}$. Observe that the above scaling transformation leaves invariant the homogeneous norm of $H^{1/2}_G(M)$, which suggests that Equation (3) is *subcritical* with respect to the energy regularity $H^1_G(M)$. However, we are going to see that the corresponding version of (2),

$$\|\mathrm{e}^{itG}f\|_{L^4([0,1]\times M)} \le C\|f\|_{H^{s/2}_G(M)} \ , \tag{4}$$

cannot hold if $s < \frac{3}{2}$, which means, in view of Proposition 10 proved in Appendix 1, that no smooth flow can exist on the energy space, hence Equation (3) should rather be regarded as *supercritical* with respect to the energy regularity. In fact, the critical regularity $s_c = \frac{3}{2}$ is the regularity which corresponds to the Sobolev embedding in M, since x has homogeneity 1 and y has homogeneity 2. This is an illustration of a total lack of dispersion for Equation (3).

The justification is as follows. Notice that $u = e^{itG}f$ can be explicitly described by using the Fourier transform in the y variable, and by making an expansion along the Hermite functions h_m in the x variable, leading to the representation

$$u(t,x,y) = (2\pi)^{-1/2}\sum_{m=0}^{\infty}\int_{\mathbb{R}} \mathrm{e}^{-it(2m+1)|\eta|+iy\eta}\hat{f}_m(\eta)h_m(\sqrt{|\eta|}x)\, d\eta \ ,$$

with

$$\|f\|^2_{H^{s/2}_G} = \sum_{m=0}^{\infty}\int_{\mathbb{R}}(1+(2m+1)|\eta|)^{s/2}|\hat{f}_m(\eta)|^2\,\frac{d\eta}{\sqrt{|\eta|}}.$$

Let us focus onto data concentrated on modes $m = 0, \eta \sim N^2$, specifically

$$f(x,y) = \frac{1}{\sqrt{N}}\int_0^{\infty} \mathrm{e}^{iy\eta-\eta\frac{x^2}{2}-\frac{\eta}{N^2}}\, d\eta = N^{\frac{3}{2}}F(Nx,N^2y)$$

with

$$F(x,y) := \frac{1}{1+\frac{x^2}{2}-iy} \ .$$

Then the above formula for u gives

$$u(t,x,y) = f(x,y-t) \ ,$$

so that

$$\|u\|_{L^4([0,1]\times\mathbb{R}^2_{x,y})} = N^{3/4}\|F\|_{L^4} \ .$$

Since $\|f\|_{H^{s/2}_G} \simeq N^{s/2}$ as $N\to\infty$, this proves the claim.

Notice that a total lack of dispersion also occurs for the — trivial— equation with $G = 0$,

$$i\partial_t u = |u|^2 u \ , \ u(0,x) = u_0(x) \ , \tag{5}$$

for which $H^s_G = L^2$ for every $s \ge 0$, hence inequality (4) cannot hold. However, the explicit formula

$$u(t,x) = \mathrm{e}^{-it|u_0(x)|^2}u_0(x)$$

solves explicitly (5), defining a —nonsmooth— flow map on L^2!

These observations invite us to study the *structure of the nonlinear evolution problem* (3) in more detail. Denote by $V^{\pm}_m$ the space of functions of the form

$$v^{\pm}_m(x,y) = \int_0^{\infty} e^{\pm i\eta y}g(\eta)h_m(\sqrt{\eta}x)\, d\eta \ , \ \int_0^{\infty}\eta^{-1/2}|g(\eta)|^2\, d\eta < \infty \ ,$$

so that we have the orthogonal decomposition

$$L^2(M) = \oplus_\pm \oplus_{m=0}^\infty V_m^\pm \ , \ G_{|V_m^\pm} = \pm i(2m+1)\partial_y \ .$$

Denote by $\Pi_m^\pm : L^2(M) \to V_m^\pm$ the orthogonal projection. Expanding the solution as

$$u = \sum_\pm \sum_{m=0}^\infty u_m^\pm \ , \ u_m^\pm = \Pi_m^\pm u \ ,$$

the equation reads as a system of coupled transport equations,

$$i(\partial_t \pm (2m+1)\partial_y)u_m = \Pi_m^\pm(|u|^2 u) \ . \tag{6}$$

Therefore a better understanding of Equation (3) requires to study the interaction between the nonlinearity $|u|^2u$ and the projectors $\Pi_m^\pm$. Notice that similar interactions arise in the literature, see for instance [18] in the study of the Lowest Landau Level for Bose-Einstein condensates, or [8] in the study of critical high frequency regimes of NLS on the sphere. Other examples can be found in the introduction of [11]. The present paper is devoted to a toy model for this kind of interaction.

1.2. A toy model: the cubic Szegő equation

Let

$$\mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$$

be the unit circle in the complex plane. If u is a distribution on $\mathbb{S}^1$, $u \in \mathscr{D}'(\mathbb{S}^1)$, then u admits a Fourier expansion in the distributional sense

$$u = \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ik\theta} \ .$$

For every subspace E of $\mathscr{D}'(\mathbb{S}^1)$, we denote by E_+ the subspace

$$E_+ = \{u \in E \ ; \ \forall k < 0, \hat{u}(k) = 0\} \ .$$

In particular, L^2_+ is the Hardy space of L^2 functions which extend to the unit disc $\{|z| < 1\}$ as holomorphic functions,

$$u(z) = \sum_{k=0}^\infty \hat{u}(k)z^k \ , \ \sum_{k=0}^\infty |\hat{u}(k)|^2 < +\infty \ .$$

Let us endow $L^2(\mathbb{S}^1)$ with the scalar product

$$(u|v) := \int_{\mathbb{S}^1} u\overline{v}\,\frac{d\theta}{2\pi} \ ,$$

and denote by $\Pi : L^2(\mathbb{S}^1) \to L^2_+(\mathbb{S}^1)$ the orthogonal projector on $L^2_+(\mathbb{S}^1)$, the so-called Szegő projector,

$$\Pi\left(\sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ik\theta}\right) = \sum_{k\geq 0} \hat{u}(k)e^{ik\theta}.$$

We consider the following evolution equation on $L^2_+(\mathbb{S}^1)$,

$$i\partial_t u = \Pi(|u|^2u) \ . \tag{7}$$