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ON THE ASYMPTOTIC GEOMETRY OF GRAVITATIONAL INSTANTONS

BY Vincent MINERBE

Abstract. – We investigate the geometry at infinity of the so-called "gravitational instantons", i.e. asymptotically flat hyperkähler four-manifolds, in relation with their volume growth. In particular, we prove that gravitational instantons with cubic volume growth are ALF, namely asymptotic to a circle fibration over a Euclidean three-space, with fibers of asymptotically constant length.

Résumé. – Nous étudions la géométrie à l'infini des instantons gravitationnels, i.e. des variétés hyperkählériennes, asymptotiquement plates et de dimension quatre. En particulier, nous prouvons que les instantons gravitationnels dont la croissance de volume est cubique sont asymptotiques à une fibration en cercles au-dessus d'un espace euclidien à trois dimensions, avec des fibres de longueur asymptotiquement constante ; autrement dit, ils sont ALF (*asymptotically locally flat*).

Introduction

Gravitational instantons are non-compact hyperkähler four-manifolds with decaying curvature at infinity. "Hyperkähler" means the manifold carries three complex structures I, J, K that are parallel with respect to a single Riemannian metric and satisfy the quaternionic relations ($IJ = -JI = K$, etc.). In other words, the holonomy group of the metric reduces to $Sp(1) = SU(2)$. As a consequence, hyperkähler four-manifolds are Ricci flat and anti-self-dual [3]; the converse is true for simply connected manifolds.

Gravitational instantons were introduced in the late seventies by Stephen Hawking [19], as building blocks for his Euclidean quantum gravity theory. Very roughly, the idea consists in modeling gravitation by drawing an analogy with gauge theories, which are so efficient for the other fundamental interactions. The Universe is represented by a *Riemannian* manifold (equivalent in gauge theory: a connection on a principal bundle) which is assumed to be Ricci flat, as a counterpart of the vacuum Einstein equation in Relativity (in gauge theory: the Yang-Mills equation). Curvature decay is a "finite action" assumption: the curvature tensor, which measures the strength of the gravitational field, should typically be in L^2 (we will further discuss this decay issue below). Finally, the jump to “hyperkähler” is explained by the analogy with gauge theory: it can be thought of as an anti-self-duality assumption.

More recently, gravitational instantons also appeared in string theory and it triggered some interest from both mathematicians and physicists (cf. [8, 9, 10, 11, 13, 14, 18, 20]…). For instance, their L^2 cohomology was computed ([18], [20]) so as to test Sen's S-duality conjecture in string theory. New examples were built ([8, 9, 11]) and, from string theory arguments, S. Cherkis and A. Kapustin conjectured a classification scheme [14], with four families.

– The first one consists of Asymptotically Locally Euclidean (ALE for short) gravitational instantons. ALE means that, outside a compact set, they are diffeomorphic to the quotient of $\mathbb{R}^4$ (minus a ball) by a finite subgroup of $O(4)$ and the metric is asymptotic to the Euclidean metric $g_{\mathbb{R}^4}$. Indeed, this family is very well understood, since P. Kronheimer ([23, 24]) classified ALE gravitational instanton in 1989. In particular, he proved the underlying manifold is the minimal resolution of the quotient of $\mathbb{C}^2$ by a finite subgroup of $SU(2)$ (i.e. cyclic, binary dihedral, tetrahedral, octahedral or icosahedral group).
– The second family consists of the so called ALF (“Asymptotically Locally Flat”) gravitational instantons: outside a compact set, they are diffeomorphic to the total space of a circle fibration π over $\mathbb{R}^3$ or $\mathbb{R}^3/\{\pm\,\mathrm{id}\}$ (minus a ball); moreover, the fibers have asymptotically constant length and the metric is asymptotic to $\pi^* g_{\mathbb{R}^3} + \eta^2$, where η is a (local) connection one-form on the circle fibration. Some examples are discussed below (Section 1.2). A Kronheimer-like classification is conjectured, but involving only cyclic or dihedral groups in $SU(2)$ (see Section 1.2 for concrete examples).
– The third and fourth families, called ALG and ALH (by induction !) have a similar fibration structure at infinity. In the ALG case, the fibers are tori and the base is $\mathbb{R}^2$. For ALH gravitational instantons, the fibers are compact orientable flat three-manifolds (there are six possibilities) and the base is $\mathbb{R}$.

A striking feature of this conjectured classification is the quantification it imposes on the volume growth: the volume of a ball of large radius t is of order t^4 in the ALE case, t^3 in the ALF case, etc. Why not $t^{3.5}$? And then, how can one explain this fibration structure at infinity? The aim of this paper is to answer these questions.

Basically, the volume growth of asymptotically flat manifolds is at most Euclidean: on a complete noncompact Riemannian manifold (M^n, g) whose curvature tensor Rm_g obeys

$$|\mathrm{Rm}|_g = \mathcal{O}(r^{-2-\epsilon}) \quad \text{with } \epsilon > 0 \tag{1}$$

(r is the distance function to some point), there is a constant B such that

$$\forall\, x \in M,\ \forall\, t \geq 1,\ \mathrm{vol}\, B(x,t) \leq B t^n.$$

Note the “faster-than-quadratic” decay rate is not anecdotic. U. Abresh proved such manifolds have finite topological type [1]: there is a compact subset K of M such that $M \backslash K$ has the *topology* of $\partial K \times \mathbb{R}_+^*$. In contrast, M. Gromov observed any (connected) manifold carries a complete metric with quadratic curvature decay ($|\mathrm{Rm}|_g = \mathcal{O}(r^{-2})$, see [27]).

A fundamental geometric result was proved by S. Bando, A. Kasue and H. Nakajima [2] in 1989: if (M^n, g) satisfies (1) and has maximal volume growth, i.e.

$$\forall\, x \in M,\ \forall\, t \geq 1,\ \operatorname{vol} B(x,t) \geq At^n,$$

then M is indeed ALE: there is a compact set K in M, a ball B in $\mathbb{R}^n$, a finite subgroup G of $O(n)$ and a diffeomorphism ϕ between $\mathbb{R}^n \backslash B$ and $M \backslash K$ such that $\phi^* g$ tends to the standard metric $g_{\mathbb{R}^n}$ at infinity. It is also proved in [2] that a complete Ricci flat manifold with maximal volume growth and curvature in $L^{\frac{n}{2}}(\mathrm{dvol})$ is ALE. In particular, gravitational instantons with maximal volume growth are ALE and thus belong to Kronheimer's list. The authors of the paper [2] raise the following natural question: can one understand the geometry at infinity of asymptotically flat manifolds whose volume growth is *not* maximal? No answer has been given since then.

Let us state our main theorem. Here and in the sequel, we will denote by r the distance to some fixed point o, without mentioning it. We will also use the measure $d\mu = \frac{r^n}{\operatorname{vol} B(o,r)}\mathrm{dvol}$. It was shown in [28] that this measure has interesting properties on manifolds with nonnegative Ricci curvature. Note that in maximal volume growth, it is equivalent to the Riemannian measure dvol.

THEOREM 0.1. – *Let (M^4, g) be a connected complete hyperkähler manifold with curvature in $L^2(d\mu)$. Suppose there are positive constants A and B such that*

$$\forall\, x \in M,\ \forall\, t \geq 1,\ At^\nu \leq \operatorname{vol} B(x,t) \leq Bt^\nu$$

with $3 \leq \nu < 4$. Then $\nu = 3$ and M is ALF: there is a compact set K in M such that $M \backslash K$ is the total space of a circle fibration π over $\mathbb{R}^3$ or $\mathbb{R}^3/\{\pm \mathrm{id}\}$ minus a ball and the metric g can be written

$$g = \pi^* g_{\mathbb{R}^3} + \eta^2 + \mathcal{O}(r^{-\tau}) \quad \textit{for any } \tau < 1,$$

where η is a (local) connection one-form for π; moreover, the length of the fibers goes to a finite positive limit at infinity.

Up to a finite covering, the topology at infinity (i.e. modulo a compact set) is therefore either that of $\mathbb{R}^3 \times \mathbb{S}^1$ (trivial fibration over $\mathbb{R}^3$) or that of $\mathbb{R}^4$ (Hopf fibration).

Our integral assumption on the curvature might be surprising at first sight. Its relevance follows from [28]. Indeed, it turns out to imply $\mathrm{Rm} = \mathcal{O}(r^{-2-\epsilon})$ and even more: a little analysis (cf. Appendix A) provides $\nabla^k \mathrm{Rm} = \mathcal{O}(r^{-3-k})$, for any k in $\mathbb{N}$!

Our volume growth assumption is uniform: the constants A and B are assumed to hold at any point x. This is not anecdotic. By looking at flat examples, we will see the importance of this uniformity. This feature is not present in the maximal volume growth case, where the uniform estimate

$$\exists\, A,\ B \in \mathbb{R}_+^*,\ \forall\, x \in M,\ \forall\, t \geq 1,\ At^n \leq \operatorname{vol} B(x,t) \leq Bt^n$$

is equivalent to

$$\exists\, A,\ B \in \mathbb{R}_+^*, \exists\, x \in M,\ \forall\, t \geq 1,\ At^n \leq \operatorname{vol} B(x,t) \leq Bt^n.$$

The idea of the proof is purely Riemannian. The point is the geometry at infinity collapses, the injectivity radius remains bounded while the curvature gets very small, so Cheeger-Fukaya-Gromov theory [6], [5] applies. The fibers of the circle fibration will come from suitable regularizations of short loops based at each point. The hyperkähler assumption will be used to control the holonomy of these short loops, which is crucial in the proof.

The structure of this paper is the following.

In a first section, we will consider examples, with three goals: first, we want to explain our volume growth assumption through the study of flat manifolds; second, these flat examples will also provide some ideas about the techniques we will develop later; third, we will describe the Taub-NUT metric, so as to provide the reader with a concrete example to think of.

In a second section, we will try to analyze some relations between three Riemannian notions: curvature, injectivity radius, volume growth. We will introduce the "fundamental pseudo-group". This object, due to M. Gromov [16], encodes the Riemannian geometry at a fixed scale. It is our basic tool and its study will explain for instance the volume growth self-improvement phenomenon in our theorem (from $3 \leq \nu < 4$ to $\nu = 3$).

In the third section, we completely describe the fundamental pseudo-group at a convenient scale, for gravitational instantons. This enables us to build the fibration at infinity, first locally, and then globally. Then we make a number of estimates to obtain the description of the geometry at infinity that we announced in the theorem. This part requires a good control on the covariant derivatives of the curvature tensor and the distance functions. This is provided by the appendices.

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1. Examples

1.1. Flat plane bundles over the circle

To have a clear picture in mind, it is useful to understand flat manifolds obtained as quotients of the Euclidean space $\mathbb{R}^3$ by the action of a screw operation ρ. Let us suppose this rigid motion is the composition of a rotation of angle θ and of a unit translation along the rotation axis. The quotient manifold is always diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$, but its Riemannian structure depends on θ: one obtains a flat plane bundle over the circle whose holonomy is the rotation of angle θ. These very simple examples conceal interesting features, which shed light on the link between injectivity radius, volume growth and holonomy. In this paragraph, we stick to dimension 3 for the sake of simplicity, but what we will observe remains relevant in higher dimension.

When the holonomy is trivial, i.e. $\theta = 0$, the Riemannian manifold is nothing but the standard $\mathbb{R}^2 \times \mathbb{S}^1$. The volume growth is uniformly comparable to that of the Euclidean $\mathbb{R}^2$:

$$\exists\, A,\, B \in \mathbb{R}_+^*,\ \forall\, x \in M,\ \forall\, t \geq 1,\ At^2 \leq \operatorname{vol} B(x,t) \leq Bt^2.$$

The injectivity radius is $1/2$ at each point, because of the lift of the base circle, which is even a closed geodesic; the iterates of these loops yield closed geodesics whose lengths describe all the natural integers, at each point.