ISSN 0012-9593

quatrième série - tome 43 fascicule 6 novembre-décembre 2010

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4e série, t. 43, 2010, p. 925 à 955

PURITY OF LEVEL m STRATIFICATIONS

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ABSTRACT. – Let k be a field of characteristic $p > 0$. Let D_m be a BT_m over k (i.e., an m-truncated Barsotti–Tate group over k). Let S be a k-scheme and let X be a BT_m over S. Let $S_{D_m}(X)$ be the subscheme of S which describes the locus where X is locally for the fppf topology isomorphic to D_m. If $p \geq 5$, we show that $S_{D_m}(X)$ is pure in S, i.e. the immersion $S_{D_m}(X) \hookrightarrow S$ is affine. For $p \in \{2, 3\}$, we prove purity if D_m satisfies a certain technical property depending only on its p-torsion $D_m[p]$. For $p \geq 5$, we apply the developed techniques to show that all level m stratifications associated to Shimura varieties of Hodge type are pure.

RÉSUMÉ. – Soit k un corps de caractéristique $p > 0$. Soit D_m un BT_m sur k (i.e., un groupe de Barsotti–Tate tronqué en échelon m sur k). Soient S un k-schéma et X un BT_m sur S. Soit $S_{D_m}(X)$ le sous-schéma de S correspondant au lieu où X est isomorphe à D_m localement pour la topologie fppf. Si $p \geq 5$, nous montrons que $S_{D_m}(X)$ est pur dans S, i.e. l'immersion $S_{D_m}(X) \hookrightarrow S$ est affine. Pour $p \in \{2, 3\}$, nous prouvons la pureté pour D_m satisfaisant une certaine propriété technique dépendant uniquement de la p-torsion $D_m[p]$. Pour $p \geq 5$, nous utilisons les techniques développées pour montrer que toutes les stratifications par l'échelon associées aux variétés de Shimura de type Hodge sont pures.

1. Introduction

Let p be a prime number. Let k be a field of characteristic p. Let c, d, and m be positive integers. In this paper, a BT_m is an m-truncated Barsotti–Tate group of codimension c and dimension d. Let D_m be a fixed BT_m over k.

Let S be an arbitrary k-scheme and let X_m be a BT_m over S. Let $S_{D_m}(X_m)$ be the (necessarily unique) locally closed subscheme of S that satisfies the following property. A morphism $f\colon S' \to S$ of k-schemes factors through $S_{D_m}(X_m)$ if and only if $f^*(X_m)$ and $D_m \times_{\mathrm{Spec}\, k} S'$ are locally for the fppf topology isomorphic as BT_m's over S' (see Subsection 2.2 for the existence of $S_{D_m}(X_m)$). If D is either a $\mathrm{BT}_{m'}$ for some $m' \geq m$, or a p-divisible group over k, we will also write $S_D(X_m)$ instead of $S_{D[p^m]}(X_m)$.

The following notion of purity (that has already been considered in [24], Section 2.1.1) will be central.

Definition 1.1. – *A subscheme T of a scheme S is called* pure in S *if the immersion $T \hookrightarrow S$ is affine.*

We remark that the purity of T in a locally noetherian scheme S implies the following weaker variant of purity: If Y is an irreducible component of the Zariski closure $\bar{T}$ of T in S, then the complement of $Y \cap T$ in Y is either empty or of pure codimension 1. On the other hand, if S is separated and T is (globally) an affine scheme, then T is pure in S.

Purity results for strata defined by p-divisible groups have a long history. The earliest hints of purity are probably the computations mentioned by Y. Manin in [15], at the bottom of p. 44. For Newton polygon strata, J. de Jong and F. Oort have shown the above mentioned weaker version of purity in [11] and one of us has shown in [24] that these strata are even pure in the sense of Definition 1.1. For p-rank strata, Th. Zink proved in [29] the weaker version of purity. Moreover, T. Itō proved in [10] the existence of generalized Hasse–Witt invariants for PEL unitary Shimura varieties of signature $(n-1,1)$ at primes p where the unitary group is split. This result implies in fact a stronger kind of purity (see below).

The weak version of purity is an important tool to estimate and compute the dimensions of strata in the locally noetherian case. Purity itself is an important step towards determining whether a (quasi-affine) stratum is in fact affine, or whether a cohomological sheaf is in fact zero. For instance, a genuine (cohomological) application of purity (and not of global affineness!) to some simple Shimura varieties can be found in [21], Proposition 6.2.

The goal of this paper is to show that $S_{D_m}(X_m)$ is pure for all schemes S and all BT_m's X_m if D_m satisfies a certain condition (C) introduced in Subsection 4.2. Here we remark that condition (C) depends only on $D_m[p]$ and it can be checked easily. Condition (C) is satisfied if any one of the three conditions below holds (cf. Lemma 4.3 (c) and (d) and Example 4.4):

(i) We have $p \geq 5$.
(ii) We have $p = 3$ and $\min(c,d) \leq 6$.
(iii) There exists an integer $a \geq 2$ such that we have a ring monomorphism $\mathbb{F}_{p^a} \hookrightarrow \mathrm{End}(D_m[p])$ with the property that $\mathbb{F}_{p^a}$ acts on the tangent space of $D_m[p]$ via scalar endomorphisms.

For the remainder of the introduction, we will assume that condition (C) holds for D_m. The main result of the paper is the following theorem.

Theorem 1.2. – *The locally closed subscheme $S_{D_m}(X_m)$ is pure in S.*

We obtain the following corollary.

Corollary 1.3. – *Let S be locally noetherian and let Y be an irreducible component of $\overline{S_{D_m}(X_m)}$. Then the complement of $S_{D_m}(X_m) \cap Y$ in Y is either empty or of pure codimension* 1.

Now let D be a p-divisible group over k such that $D[p^m] = D_m$. For every reduced k-scheme S and every p-divisible group X over S denote by $\mathfrak{n}_D(X)$ the (necessarily unique) reduced locally closed subscheme of S such that for each field extension K of k we have

$$\mathfrak{n}_D(X)(K) = \{\, s \in S(K) \mid D \text{ and } s^*(X) \text{ have equal Newton polygons} \,\}.$$

Thus $\mathfrak{n}_D(X)$ is the Newton polygon stratum of S defined by X that corresponds to the Newton polygon of D. The locally closed subscheme $\mathfrak{n}_D(X)$ is pure in S by [24], Theorem 1.6. Thus we get another purity result:

Corollary 1.4. – *For each $m \in \mathbb{N}^*$, the locally closed subscheme $\mathfrak{n}_D(X) \cap S_D(X[p^m])$ is pure in S.*

Moreover, we can use the well known fact that there exists an integer $n_D \geq 1$ with the following property. If C is a p-divisible group over an algebraic closure $\bar{k}$ of k such that $C[p^{n_D}]$ is isomorphic to $D[p^{n_D}]_{\bar{k}}$, then C is isomorphic to $D_{\bar{k}}$ (for instance, see [22], Theorem 1 or [24], Corollary 1.3). We assume that n_D is chosen minimal. Then there exists a (necessarily unique) reduced locally closed subscheme $\mathfrak{u}_D(X)$ of S such that for every algebraically closed field extension K of k we have

$$\mathfrak{u}_D(X)(K) = \{\, s \in S(K) \mid D_K \cong s^*(X) \,\}.$$

Indeed, we have $\mathfrak{u}_D(X) = S_D(X[p^{n_D}])_{\mathrm{red}}$. From Theorem 1.2, we obtain the following purity result:

Corollary 1.5. – *The locally closed subscheme $\mathfrak{u}_D(X)$ is pure in S.*

For special fibres of good integral models in unramified mixed characteristic $(0, p)$ of Shimura varieties of Hodge type (or more generally, for quasi Shimura p-varieties of Hodge type), there exists a level m stratification that parametrizes BT_m's with additional structures (see Subsection 6.2). The proof of Theorem 1.2 can be adapted to show that all level m stratifications are pure (see Theorem 6.3), provided they are either in characteristic $p \geq 5$ or are in characteristic $p \in \{2, 3\}$ and an additional condition holds.

In this introduction, we will only state the Siegel modular varieties variant of Theorem 1.2 (see Example 6.5). Let $N \geq 3$ be an integer prime to p. Let $\mathscr{A}_{d,1,N}$ be the Mumford moduli scheme that parameterizes principally polarized abelian schemes over $\mathbb{F}_p$-schemes of relative dimension d and equipped with a symplectic similitude level N structure (cf. [18], Theorems 7.9 and 7.10). Let $(\mathscr{U}, \Lambda)$ be the principally quasi-polarized p-divisible group of the universal principally polarized abelian scheme over $\mathscr{A}_{d,1,N}$. If k is algebraically closed and if (D, λ) is a principally quasi-polarized p-divisible group of height $2d$ over k, let $\mathfrak{s}_{D,\lambda}(m)$ be the unique reduced locally closed subscheme of $\mathscr{A}_{d,1,N,k}$ that satisfies the following identity of sets

$$\mathfrak{s}_{D,\lambda}(m)(k) = \{\, y \in \mathscr{A}_{d,1,N}(k) \mid y^*(\mathscr{U}, \Lambda)[p^m] \cong (D, \lambda)[p^m] \,\}.$$

Then $\mathfrak{s}_{D,\lambda}(m)$ is regular and equidimensional (see [25], Corollary 4.3 and Example 4.5; Subsection 2.3 below can be easily adapted to prove the existence and the smoothness of the k-scheme $\mathfrak{s}_{D,\lambda}(m)$). Moreover we have:

Theorem 1.6. – *If either $p = 3$ and $d \leq 6$ or $p \geq 5$, then the locally closed subscheme $\mathfrak{s}_{D,\lambda}(m)$ is pure in $\mathscr{A}_{d,1,N,k}$.*

We remark that for $m = 1$, Theorem 1.6 neither implies nor is implied by Oort's result ([20], Theorem 1.2) which asserts (for all primes p) that the scheme $\mathfrak{s}_{D,\lambda}(1)$ is quasi-affine.

Finally, we investigate briefly the following stronger notion of purity.

DEFINITION 1.7. – *Let $T \to S$ be a quasi-compact immersion and let $\bar{T}$ be the scheme-theoretic closure of T in S. Then T is called* Zariski locally principally pure in S *if locally for the Zariski topology of $\bar{T}$, there exists a function $f \in \Gamma(\bar{T}, \mathscr{O}_{\bar{T}})$ such that we have $T = \bar{T}_f$, where $\bar{T}_f$ is the largest open subscheme of $\bar{T}$ over which f is invertible.*

We obtain variants of this notion by replacing the Zariski topology by another Grothendieck topology $\mathscr{T}$ of S. If $\mathscr{T}$ is coarser than the fpqc topology (e.g., the Zariski or the étale topology), each $\mathscr{T}$ locally principally pure subscheme is pure (as affineness for morphisms is a local property for the fpqc topology). Principal purity for p-rank strata corresponds to the existence of generalized Hasse–Witt invariants. They have been investigated by T. Itō for certain unitary Shimura varieties (see [10]) and by E. Z. Goren for Hilbert modular varieties (see [4]).

In Section 7, we will show that this stronger notion of purity does not hold in general. In fact, we have:

PROPOSITION 1.8. – *Let $c, d \geq 2$ and $s \in \{1, \dots, c-1\}$. Then the strata of p-rank equal to s associated to BT_1's over $\mathbb{F}_p$-schemes of codimension c and dimension d, are not étale locally principally pure in general.*

We now give an overview of the structure of the paper. In Section 2, we define the level m strata $S_{D_m}(X_m)$ and we prove some basic properties of them. Then we make a dévissage to the following situation.

ESSENTIAL SITUATION 1.9. – Let k be an algebraically closed field of characteristic $p > 0$ and let D_m be a BT_m over k which satisfies condition (C). Let D be a p-divisible group over k such that $D[p^m] = D_m$. Let $S = \mathscr{A}$ be a smooth k-scheme of finite type which is equidimensional of dimension cd and for which the following two properties hold:

(a) There exists a p-divisible group $\mathscr{E}$ of codimension c and dimension d over $\mathscr{A}$ which is a versal deformation at each k-valued point of $\mathscr{A}$.
(b) There exists a point $y_D \in \mathscr{A}(k)$ such that $y_D^*(\mathscr{E})$ is isomorphic to D.

In this case we simply write $\mathfrak{s}_D(m)$ instead of $\mathscr{A}_D(\mathscr{E}[p^m])$. In Subsection 2.3, we will prove that $\mathfrak{s}_D(m)$ is smooth over k (by [25], Theorem 1.2 (a) and (b) and Remark 3.1.2 we know already that the reduced scheme of $\mathfrak{s}_D(m)$ is a smooth equidimensional k-scheme, although this fact is not used in the proof below). Then we show that Theorem 1.2 follows if $\mathfrak{s}_D(m)$ is pure in $\mathscr{A}$.

We remark that for $m \geq n_D$ (where n_D is the integer defined above before Corollary 1.5) the fact that $\mathfrak{s}_D(m)$ is pure in $\mathscr{A}$ is proved in [24], Theorem 5.3.1 (c). This result of [24] and thus Corollary 1.5 also, hold even if condition (C) does not hold for $D[p]$.

The proof of Theorem 1.2 is presented in Section 5. There we show that purity follows from the affineness of a certain orbit $\mathscr{O}_m$ of a group action

$$\mathbb{T}_m \colon \mathscr{H}_m \times \mathscr{D}_m \to \mathscr{D}_m,$$

which was introduced in [25]. The orbits of $\mathbb{T}_m$ parameterize isomorphism classes of BT_m's over perfect fields. In fact we show that $\mathscr{O}_m$ is affine for all m provided $\mathscr{O}_1$ is affine.

The definition of the action $\mathbb{T}_m$ is recalled in Section 3, and in Section 4 the main properties of the action $\mathbb{T}_1$ we need are presented. There the condition (C) is introduced and used.