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RABINOWITZ FLOER HOMOLOGY AND SYMPLECTIC HOMOLOGY

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Abstract. – The first two authors have recently defined Rabinowitz Floer homology groups $RFH_*(M, W)$ associated to a separating exact embedding of a contact manifold (M, ξ) into a symplectic manifold (W, ω). These depend only on the bounded component V of $W \setminus M$. We construct a long exact sequence in which symplectic cohomology of V maps to symplectic homology of V, which in turn maps to Rabinowitz Floer homology $RFH_*(M, W)$, which then maps to symplectic cohomology of V. We compute $RFH_*(ST^*L, T^*L)$, where ST^*L is the unit cosphere bundle of a closed manifold L. As an application, we prove that the image of a separating exact contact embedding of ST^*L cannot be displaced away from itself by a Hamiltonian isotopy, provided dim $L \geq 4$ and the embedding induces an injection on π_1.

Résumé. – Étant donné un plongement exact et séparant d'une variété de contact (M, ξ) dans une variété symplectique (W, ω), les deux premiers auteurs ont défini des groupes d'homologie dits de Rabinowitz Floer $RFH_*(M, W)$. Ceux-ci dépendent uniquement de la composante bornée V de $W \setminus M$. Nous construisons une suite exacte longue dans laquelle la cohomologie symplectique de V est envoyée vers l'homologie symplectique de V, qui à son tour est envoyée vers l'homologie de Rabinowitz Floer $RFH_*(M, W)$, qui finalement est envoyée vers la cohomologie symplectique de V. Nous calculons $RFH_*(ST^*L, T^*L)$ pour le fibré cotangent unitaire ST^*L d'une variété compacte sans bord L. Nous démontrons que l'image d'un plongement exact et séparant de ST^*L ne peut pas être disjointe d'elle-même par une isotopie hamiltonienne, à condition que le plongement induise une injection sur le groupe fondamental et dim $L \geq 4$.

1. Introduction

Let (W, λ) be a complete convex exact symplectic manifold, with symplectic form $\omega = d\lambda$ (see Section 3 for the precise definition). An embedding $\iota : M \hookrightarrow W$ of a contact manifold (M, ξ) is called an *exact contact embedding* if there exists a 1-form α on M such that ker $\alpha = \xi$

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and $\alpha - \lambda|_M$ is exact. We identify M with its image $\iota(M)$. We assume throughout the paper that $W \setminus M$ consists of two connected components and denote the bounded component of $W \setminus M$ by V. One can classically [29] associate to such an exact contact embedding the *symplectic (co)homology groups* $SH_*(V)$ and $SH^*(V)$. We refer to Section 2 for the definition and basic properties, and to [25] for a recent survey.

The first two authors have recently defined for such an exact contact embedding Floer homology groups $RFH_*(M, W)$ for the Rabinowitz action functional [10]. We refer to Section 3 for a recap of the definition and of some useful properties. We will show in particular that these groups do not depend on W, but only on V (the same holds for $SH_*(V)$ and $SH^*(V)$). We shall use in this paper the notation $RFH_*(V)$ and call them *Rabinowitz Floer homology* groups.

REMARK 1.1. – All (co)homology groups are taken with field coefficients. Without any further hypotheses on the first Chern class $c_1(V)$ of the tangent bundle, the symplectic (co)homology and Rabinowitz Floer homology groups are $\mathbb{Z}_2$-graded. If $c_1(V) = 0$ they are $\mathbb{Z}$-graded, and if $c_1(V)$ vanishes on $\pi_2(V)$ the part constructed from contractible loops is $\mathbb{Z}$-graded. This $\mathbb{Z}$-grading on Rabinowitz Floer homology differs from the one in [10] (which takes values in $\frac{1}{2} + \mathbb{Z}$) by a shift of $1/2$ (see Remark 3.2).

Our purpose is to relate these two constructions. The relevant object is a new version of symplectic homology, denoted by $\check{SH}_*(V)$, associated to "$\bigvee$-shaped" Hamiltonians like the one in Figure 1 on page 976 below. This version of symplectic homology is related to the usual ones via the long exact sequence in the next theorem.

THEOREM 1.2. – *There is a long exact sequence*

$$\cdots \longrightarrow SH^{-*}(V) \longrightarrow SH_*(V) \longrightarrow \check{SH}_*(V) \longrightarrow SH^{-*+1}(V) \longrightarrow \cdots \tag{1}$$

One can think of the long exact sequence (1) in two complementary ways: either as measuring the defect from being an isomorphism for the canonical map $SH^{-*}(V) \to SH_*(V)$, defined in Section 2.7, or, in view of Theorem 1.5 below, as being an obstruction against having an isomorphism $RFH_*(V) \simeq SH_*(V) \oplus SH^{-*+1}(V)$. An interesting fact is that we have a very precise description of the map $SH^{-*}(V) \to SH_*(V)$. To state it, let us recall that there are canonical morphisms induced by truncation of the range of the action $H_{*+n}(V, \partial V) \xrightarrow{c_*} SH_*(V)$ and $SH^*(V) \xrightarrow{c^*} H^{*+n}(V, \partial V)$ (see [29] or Lemma 2.1 below).

PROPOSITION 1.3. – *The map $SH^{-*}(V) \to SH_*(V)$ fits into a commutative diagram*

$$\begin{tikzcd} SH^{-*}(V) \arrow[r] \arrow[d, "c^*"'] & SH_*(V) \\ H^{-*+n}(V, \partial V) \arrow[r] & H_{*+n}(V, \partial V) \arrow[u, "c_*"'] \end{tikzcd} \tag{2}$$

in which the bottom arrow is the composition of the map induced by the inclusion $V \hookrightarrow (V, \partial V)$ with the Poincaré duality isomorphism

$$H^{-*+n}(V, \partial V) \xrightarrow{PD} H_{*+n}(V) \xrightarrow{\mathrm{incl}_*} H_{*+n}(V, \partial V).$$

We also define in Section 2.7 truncated versions $\check{SH}_*^{\geq 0}(V)$ and $\check{SH}_*^{\leq 0}(V)$ of the symplectic homology groups $\check{SH}_*(V)$.

Proposition 1.4. – *There are commuting diagrams of long exact sequences as below, where PD denotes Poincaré duality and the top exact sequence is the (co)homological long exact sequence of the pair $(V, M = \partial V)$:*

$$\begin{tikzcd}
\cdots \arrow[r] & H_{*+n}(V) \arrow[r] \arrow[d, equal, "PD"] & H_{*+n}(V,M) \arrow[r] \arrow[d] & H_{*+n-1}(M) \arrow[r] \arrow[d] & H_{*+n-1}(V) \arrow[r] \arrow[d, equal, "PD"] & \cdots \\
\cdots \arrow[r] & H^{-*+n}(V,M) \arrow[r] & SH_*(V) \arrow[r] & \check{SH}_*^{\geq 0}(V) \arrow[r] & H^{-*+1+n}(V,M) \arrow[r] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r] & H^{-*+n}(V,M) \arrow[r] & H^{-*+n}(V) \arrow[r] & H^{-*+n}(M) \arrow[r] & H^{-*+n+1}(V) \arrow[r] & \cdots \\
\cdots \arrow[r] & SH^{-*}(V) \arrow[r] \arrow[u] & H_{*+n}(V,M) \arrow[r] \arrow[u, equal, "PD"] & \check{SH}_*^{\leq 0}(V) \arrow[r] \arrow[u] & SH^{-*+1}(V) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

The main result of this paper is the following.

Theorem 1.5. – *We have an isomorphism*

$$RFH_*(V) \simeq \check{SH}_*(V).$$

Theorem 1.5 is proved in Section 6. It follows that the Rabinowitz Floer homology groups fit into a long exact sequence

$$\cdots \longrightarrow SH^{-*}(V) \longrightarrow SH_*(V) \longrightarrow RFH_*(V) \longrightarrow SH^{-*+1}(V) \longrightarrow \cdots \tag{3}$$

We also recall the following vanishing result for Rabinowitz Floer homology from [10].

Theorem 1.6 ([10, Theorem 1.2]). – *If $M = \partial V$ is Hamiltonianly displaceable in W, then*

$$RFH_*(V) = 0.$$

To state the next corollary, we recall that the symplectic (co)homology and Rabinowitz Floer homology groups decompose as direct sums

$$SH_*(V) = \oplus_c SH_*^c(V), \quad SH^*(V) = \oplus_c SH_c^*(V), \quad RFH_*(V) = \oplus_c RFH_*^c(V)$$

indexed over free homotopy classes of loops *in* V. We denote the free homotopy class of the constant loops by $c = 0$.

Corollary 1.7. – *Assume $M = \partial V$ is Hamiltonianly displaceable in W.*

– *For $c \neq 0$ we have*

$$SH_*^c(V) = 0, \quad SH_c^*(V) = 0.$$

– *Suppose that $c_1(W)|_{\pi_2(W)} = 0$. Then for $c = 0$ we have*

$$SH_*^{c=0}(V) = 0, \quad SH^*_{c=0}(V) = 0 \tag{4}$$

if $* \geq n$ or $* \leq -n$. Moreover, if V is Stein then (4) holds for $* \neq 0$, and if V is Stein subcritical then (4) holds for all $* \in \mathbb{Z}$.

Proof. – The long exact sequence (3) splits into a direct sum of long exact sequences, indexed over free homotopy classes of loops in V. The assumption that M is Hamiltonianly displaceable implies $RFH_*(V) = 0$, hence the map $SH_c^{-*}(V) \to SH_*^c(V)$ is an isomorphism for any c.

We now use the commutative diagram in Proposition 1.3 and the fact that the canonical map $c_* : H_{*+n}(V, \partial V) \to SH_*(V)$ takes values into the direct summand $SH_*^{c=0}(V)$, and similarly, the map $c^* : SH^*(V) \to H^{*+n}(V, \partial V)$ factors through $SH_{c=0}^*(V)$ (see Lemma 2.1 and Lemma 2.4 below).

Let us assume $c \neq 0$. Then the above discussion shows that the map $SH_c^{-*}(V) \to SH_*^c(V)$ is at the same time an isomorphism and vanishes. This implies the conclusion.

Let us now assume $c = 0$ and $c_1(W)|_{\pi_2(W)} = 0$, so that all homology groups are $\mathbb{Z}$-graded. By Proposition 1.3, the map $SH_{c=0}^{-*}(V) \to SH_*^{c=0}(V)$ is the composition

$$SH_{c=0}^{-*}(V) \to H^{-*+n}(V, \partial V) \simeq H_{*+n}(V) \to H_{*+n}(V, \partial V) \to SH_*^{c=0}(V),$$

and therefore vanishes if $H_{*+n}(V) = 0$ or $H_{*+n}(V, \partial V) \cong H^{n-*}(V) = 0$. This is always the case if $* \geq n$ or $* \leq -n$. If V is Stein, this holds if $* \neq 0$, and if V is Stein subcritical, this holds for all $* \in \mathbb{Z}$. The conclusion follows. □

Corollary 1.8 ([8]). – *If V is Stein subcritical and $c_1(V)|_{\pi_2(V)} = 0$, then $SH_*(V) = 0$.*

Proof. – Any compact set in a subcritical Stein manifold is Hamiltonianly displaceable [3]. Thus V is displaceable in its symplectic completion (see Section 2 for the definition), and therefore $SH_*(V) = 0$ by Corollary 1.7. □

Remark. – The original proof of Corollary 1.8 in [8] uses a handle decomposition for W. The proof given above only uses the fact that the subcritical skeleton can be displaced from itself [3]. On the other hand, the proof given above uses the grading in an essential way and hence only works under the hypothesis $c_1(V)|_{\pi_2(V)} = 0$, whereas the original proof does not need this assumption.

Remark. – Ritter proved in [18, Theorem 96] that Rabinowitz Floer homology vanishes if and only if symplectic homology vanishes, if and only if symplectic cohomology vanishes. This is essentially a consequence of the unital ring structure of symplectic homology, with the unit living in degree $* = n$.

Corollary 1.9 (Weinstein conjecture in displaceable manifolds)

Assume that V is Hamiltonianly displaceable in W and $c_1(W)|_{\pi_2(W)} = 0$. Then any hypersurface of contact type $\Sigma \subset V$ carries a closed characteristic, i.e. a closed integral curve of the line distribution $\ker \omega|_\Sigma$.

Proof. – This follows from the fact that $SH_{c=0}^n(V) = 0$, as proved in Corollary 1.7 above. In particular the canonical map $SH_{c=0}^n(V) \to H^{2n}(V, \partial V)$ vanishes, and thus V satisfies the Strong Algebraic Weinstein Conjecture in the sense of Viterbo [29]. The conclusion is then a consequence of the Main Theorem in [29] (see also [16, Theorem 4.10] for details). □