ISSN 0012-9593

quatrième série - tome 44 fascicule 1 janvier-février 2011

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Peng SHAN

Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4$^{\text{e}}$ série, t. 44, 2011, p. 147 à 182

CRYSTALS OF FOCK SPACES AND CYCLOTOMIC RATIONAL DOUBLE AFFINE HECKE ALGEBRAS

BY PENG SHAN

ABSTRACT. – We define the i-restriction and i-induction functors on the category $\mathcal{O}$ of the cyclotomic rational double affine Hecke algebras. This yields a crystal on the set of isomorphism classes of simple modules, which is isomorphic to the crystal of a Fock space.

RÉSUMÉ. – On définit les foncteurs de i-restriction et i-induction sur la catégorie $\mathcal{O}$ des algèbres de Hecke doublement affines rationnelles cyclotomiques. Ceci donne lieu à un cristal sur l'ensemble des classes d'isomorphismes de modules simples, qui est isomorphe au cristal d'un espace de Fock.

Introduction

In [1], S. Ariki defined the i-restriction and i-induction functors for cyclotomic Hecke algebras. He showed that the Grothendieck group of the category of finitely generated projective modules of these algebras admits a module structure over the affine Lie algebra of type $A^{(1)}$, with the action of Chevalley generators given by the i-restriction and i-induction functors.

The restriction and induction functors for rational DAHA's (= double affine Hecke algebras) were recently defined by R. Bezrukavnikov and P. Etingof. With these functors, we give an analogue of Ariki's construction for the category $\mathcal{O}$ of cyclotomic rational DAHA's: we show that as a module over the type $A^{(1)}$ affine Lie algebra, the Grothendieck group of this category is isomorphic to a Fock space. We also construct a crystal on the set of isomorphism classes of simple modules in the category $\mathcal{O}$. It is isomorphic to the crystal of the Fock space. Recall that this Fock space also enters in some conjectural description of the decomposition numbers for the category $\mathcal{O}$ considered here. See [16], [17], [14] for related works.

Notation

For A an algebra, we will write A-mod for the category of finitely generated A-modules. For $f : A \to B$ an algebra homomorphism from A to another algebra B such that B is finitely generated over A, we will write

$$f_* : B\text{-mod} \to A\text{-mod}$$

for the restriction functor and we write

$$f^* : A\text{-mod} \to B\text{-mod}, \quad M \mapsto B \otimes_A M.$$

A $\mathbb{C}$-linear category $\mathcal{A}$ is called artinian if the Hom sets are finite dimensional $\mathbb{C}$-vector spaces and every object has a finite length. Given an object M in $\mathcal{A}$, we denote by $\mathrm{soc}(M)$ (resp. $\mathrm{head}(M)$) the socle (resp. the head) of M, which is the largest semi-simple subobject (quotient) of M.

Let $\mathcal{C}$ be an abelian category. The Grothendieck group of $\mathcal{C}$ is the quotient of the free abelian group generated by objects in $\mathcal{C}$ modulo the relations $M = M' + M''$ for all objects M, M', M'' in $\mathcal{C}$ such that there is an exact sequence $0 \to M' \to M \to M'' \to 0$. Let $K(\mathcal{C})$ denote the complexified Grothendieck group, a $\mathbb{C}$-vector space. For each object M in $\mathcal{C}$, let $[M]$ be its class in $K(\mathcal{C})$. Any exact functor $F : \mathcal{C} \to \mathcal{C}'$ between two abelian categories induces a vector space homomorphism $K(\mathcal{C}) \to K(\mathcal{C}')$, which we will denote by F again. Given an algebra A we will abbreviate $K(A) = K(A\text{-mod})$.

Denote by $\mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ the category of functors from a category $\mathcal{C}$ to a category $\mathcal{C}'$. For $F \in \mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ write $\mathrm{End}(F)$ for the ring of endomorphisms of the functor F. We denote by $1_F : F \to F$ the identity element in $\mathrm{End}(F)$. Let $G \in \mathrm{Fct}(\mathcal{C}', \mathcal{C}'')$ be a functor from $\mathcal{C}'$ to another category $\mathcal{C}''$. For any $X \in \mathrm{End}(F)$ and any $X' \in \mathrm{End}(G)$ we write $X'X : G \circ F \to G \circ F$ for the morphism of functors given by $X'X(M) = X'(F(M)) \circ G(X(M))$ for any $M \in \mathcal{C}$.

Let $e \geqslant 2$ be an integer and z be a formal parameter. Denote by $\mathfrak{sl}_e$ the Lie algebra of traceless $e \times e$ complex matrices. The type $A^{(1)}$ affine Lie algebra is

$$\widetilde{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}\partial$$

equipped with the Lie bracket

$$[\xi \otimes z^m + ac + b\partial, \xi' \otimes z^n + a'c + b'\partial] = [\xi, \xi'] \otimes z^{m+n} + m\delta_{m,-n}\mathrm{tr}(\xi\xi')c + nb\xi' \otimes z^n - mb'\xi \otimes z^m,$$

for $\xi, \xi' \in \mathfrak{sl}_e$, $a, a', b, b' \in \mathbb{C}$. Here $\mathrm{tr} : \mathfrak{sl}_e \to \mathbb{C}$ is the trace map. Let

$$\widehat{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c.$$

It is the Lie subalgebra of $\widetilde{\mathfrak{sl}}_e$ generated by the Chevalley generators

$$\begin{aligned} e_i &= E_{i,i+1} \otimes 1, \quad & f_i &= E_{i+1,i} \otimes 1, \quad 1 \leqslant i \leqslant e-1 \\ e_0 &= E_{e1} \otimes z, \quad & f_0 &= E_{1e} \otimes z^{-1}. \end{aligned}$$

Here E_{ij} is the elementary matrix with 1 in the position (i, j) and 0 elsewhere. Let $h_i = [e_i, f_i]$ for $0 \leqslant i \leqslant e-1$. We consider the Cartan subalgebra

$$\mathfrak{t} = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} \mathbb{C}h_i \oplus \mathbb{C}\partial,$$

and its dual $\mathfrak{t}^*$. For $i \in \mathbb{Z}/e\mathbb{Z}$ let $\alpha_i \in \mathfrak{t}^*$ (resp. $\alpha_i^\vee \in \mathfrak{t}$) be the simple root (resp. coroot) corresponding to e_i. The fundamental weights are $\{\Lambda_i \in \mathfrak{t}^* : i \in \mathbb{Z}/e\mathbb{Z}\}$ such that $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$ and $\Lambda_i(\partial) = 0$ for any $i, j \in \mathbb{Z}/e\mathbb{Z}$. Let $\delta \in \mathfrak{t}^*$ be the element given by $\delta(h_i) = 0$ for all i and $\delta(\partial) = 1$. We will write P for the weight lattice of $\widetilde{\mathfrak{sl}}_e$. It is the free abelian group generated by the fundamental weights and δ.

1. Reminders on Hecke algebras, rational DAHA's and restriction functors

1.1. Hecke algebras

Let $\mathfrak{h}$ be a finite dimensional vector space over $\mathbb{C}$. Recall that a pseudo-reflection is a non trivial element s of $GL(\mathfrak{h})$ which acts trivially on a hyperplane, called the reflecting hyperplane of s. Let $W \subset GL(\mathfrak{h})$ be a finite subgroup generated by pseudo-reflections. Let $\mathscr{S}$ be the set of pseudo-reflections in W and $\mathscr{A}$ be the set of reflecting hyperplanes. We set $\mathfrak{h}_{\mathrm{reg}} = \mathfrak{h} - \bigcup_{H \in \mathscr{A}} H$, it is stable under the action of W. Fix $x_0 \in \mathfrak{h}_{\mathrm{reg}}$ and identify it with its image in $\mathfrak{h}_{\mathrm{reg}}/W$. By definition the braid group attached to $(W, \mathfrak{h})$, denoted by $B(W, \mathfrak{h})$, is the fundamental group $\pi_1(\mathfrak{h}_{\mathrm{reg}}/W, x_0)$.

For any $H \in \mathscr{A}$, let W_H be the pointwise stabilizer of H. This is a cyclic group. Write e_H for the order of W_H. Let s_H be the unique element in W_H whose determinant is $\exp(\frac{2\pi\sqrt{-1}}{e_H})$. Let q be a map from $\mathscr{S}$ to $\mathbb{C}^*$ that is constant on the W-conjugacy classes. Following [6, Definition 4.21] the Hecke algebra $\mathscr{H}_q(W, \mathfrak{h})$ attached to $(W, \mathfrak{h})$ with parameter q is the quotient of the group algebra $\mathbb{C}B(W, \mathfrak{h})$ by the relations:

$$(T_{s_H} - 1) \prod_{t \in W_H \cap \mathscr{S}} (T_{s_H} - q(t)) = 0, \quad H \in \mathscr{A}. \tag{1.1}$$

Here T_{s_H} is a generator of the monodromy around H in $\mathfrak{h}_{\mathrm{reg}}/W$ such that the lift of T_{s_H} in $\pi_1(W, \mathfrak{h}_{\mathrm{reg}})$ via the map $\mathfrak{h}_{\mathrm{reg}} \to \mathfrak{h}_{\mathrm{reg}}/W$ is represented by a path from x_0 to $s_H(x_0)$. See [6, Section 2B] for a precise definition. When the subspace $\mathfrak{h}^W$ of fixed points of W in $\mathfrak{h}$ is trivial, we abbreviate

$$B_W = B(W, \mathfrak{h}), \quad \mathscr{H}_q(W) = \mathscr{H}_q(W, \mathfrak{h}).$$

1.2. Parabolic restriction and induction for Hecke algebras

In this section we will assume that $\mathfrak{h}^W = 1$. A parabolic subgroup W' of W is by definition the stabilizer of a point $b \in \mathfrak{h}$. By a theorem of Steinberg, the group W' is also generated by pseudo-reflections. Let q' be the restriction of q to $\mathscr{S}' = W' \cap \mathscr{S}$. There is an explicit inclusion $\imath_q : \mathscr{H}_{q'}(W') \hookrightarrow \mathscr{H}_q(W)$ given by [6, Section 2D]. The restriction functor

$$^{\mathscr{H}}\mathrm{Res}_{W'}^{W} : \mathscr{H}_q(W)\text{-mod} \to \mathscr{H}_{q'}(W')\text{-mod}$$

is the functor $(\imath_q)_*$. The induction functor

$$^{\mathscr{H}}\mathrm{Ind}_{W'}^{W} = \mathscr{H}_q(W) \otimes_{\mathscr{H}_{q'}(W')} -$$

is left adjoint to $^{\mathscr{H}}\mathrm{Res}_{W'}^{W}$. The coinduction functor

$$^{\mathscr{H}}\mathrm{coInd}_{W'}^{W} = \mathrm{Hom}_{\mathscr{H}_{q'}(W')}(\mathscr{H}_q(W), -)$$

is right adjoint to $^{\mathscr{H}}\mathrm{Res}_{W'}^{W}$. The three functors above are all exact.

Let us recall the definition of $\imath_q$. It is induced from an inclusion $\imath : B_{W'} \hookrightarrow B_W$, which is in turn the composition of three morphisms ℓ, κ, $\jmath$ defined as follows. First, let $\mathcal{A}' \subset \mathcal{A}$ be the set of reflecting hyperplanes of W'. Write

$$\overline{\mathfrak{h}} = \mathfrak{h}/\mathfrak{h}^{W'}, \quad \overline{\mathcal{A}} = \{\overline{H} = H/\mathfrak{h}^{W'} : H \in \mathcal{A}'\}, \quad \overline{\mathfrak{h}}_{\mathrm{reg}} = \overline{\mathfrak{h}} - \bigcup_{\overline{H}\in\overline{\mathcal{A}}} \overline{H}, \quad \mathfrak{h}'_{\mathrm{reg}} = \mathfrak{h} - \bigcup_{H\in\mathcal{A}'} H.$$

The canonical epimorphism $p : \mathfrak{h} \to \overline{\mathfrak{h}}$ induces a trivial W'-equivariant fibration $p : \mathfrak{h}'_{\mathrm{reg}} \to \overline{\mathfrak{h}}_{\mathrm{reg}}$, which yields an isomorphism

$$\ell : B_{W'} = \pi_1(\overline{\mathfrak{h}}_{\mathrm{reg}}/W', p(x_0)) \xrightarrow{\sim} \pi_1(\mathfrak{h}'_{\mathrm{reg}}/W', x_0). \tag{1.2}$$

Endow $\mathfrak{h}$ with a W-invariant hermitian scalar product. Let $||\cdot||$ be the associated norm. Set

$$\Omega = \{x \in \mathfrak{h} : ||x - b|| < \varepsilon\}, \tag{1.3}$$

where ε is a positive real number such that the closure of Ω does not intersect any hyperplane that is in the complement of $\mathcal{A}'$ in $\mathcal{A}$. Let $\gamma : [0,1] \to \mathfrak{h}$ be a path such that $\gamma(0) = x_0$, $\gamma(1) = b$ and $\gamma(t) \in \mathfrak{h}_{\mathrm{reg}}$ for $0 < t < 1$. Let $u \in [0,1[$ such that $x_1 = \gamma(u)$ belongs to Ω, write γ_u for the restriction of γ to $[0,u]$. Consider the homomorphism

$$\sigma : \pi_1(\Omega \cap \mathfrak{h}_{\mathrm{reg}}, x_1) \to \pi_1(\mathfrak{h}_{\mathrm{reg}}, x_0), \quad \lambda \mapsto \gamma_u^{-1} \cdot \lambda \cdot \gamma_u.$$

The canonical inclusion $\mathfrak{h}_{\mathrm{reg}} \hookrightarrow \mathfrak{h}'_{\mathrm{reg}}$ induces a homomorphism $\pi_1(\mathfrak{h}_{\mathrm{reg}}, x_0) \to \pi_1(\mathfrak{h}'_{\mathrm{reg}}, x_0)$. Composing it with σ gives an invertible homomorphism

$$\pi_1(\Omega \cap \mathfrak{h}_{\mathrm{reg}}, x_1) \to \pi_1(\mathfrak{h}'_{\mathrm{reg}}, x_0).$$

Since Ω is W'-invariant, its inverse gives an isomorphism

$$\kappa : \pi_1(\mathfrak{h}'_{\mathrm{reg}}/W', x_0) \xrightarrow{\sim} \pi_1((\Omega \cap \mathfrak{h}_{\mathrm{reg}})/W', x_1). \tag{1.4}$$

Finally, we see from above that σ is injective. So it induces an inclusion

$$\pi_1((\Omega \cap \mathfrak{h}_{\mathrm{reg}})/W', x_1) \hookrightarrow \pi_1(\mathfrak{h}_{\mathrm{reg}}/W', x_0).$$

Composing it with the canonical inclusion $\pi_1(\mathfrak{h}_{\mathrm{reg}}/W', x_0) \hookrightarrow \pi_1(\mathfrak{h}_{\mathrm{reg}}/W, x_0)$ gives an injective homomorphism

$$\jmath : \pi_1((\Omega \cap \mathfrak{h}_{\mathrm{reg}})/W', x_1) \hookrightarrow \pi_1(\mathfrak{h}_{\mathrm{reg}}/W, x_0) = B_W. \tag{1.5}$$

By composing ℓ, κ, $\jmath$ we get the inclusion

$$\imath = \jmath \circ \kappa \circ \ell : B_{W'} \hookrightarrow B_W. \tag{1.6}$$

It is proved in [6, Section 4C] that $\imath$ preserves the relations in (1.1). So it induces an inclusion of Hecke algebras which is the desired inclusion

$$\imath_q : \mathcal{H}_{q'}(W') \hookrightarrow \mathcal{H}_q(W).$$

For $\imath$, $\imath'$: $B_{W'} \hookrightarrow B_W$ two inclusions defined as above via different choices of the path γ, there exists an element $\rho \in P_W = \pi_1(\mathfrak{h}_{\mathrm{reg}}, x_0)$ such that for any $a \in B_{W'}$ we have $\imath(a) = \rho\imath'(a)\rho^{-1}$. In particular, the functors $\imath_*$ and $(\imath')_*$ from B_W-mod to $B_{W'}$-mod are isomorphic. Also, we have $(\imath_q)_* \cong (\imath'_q)_*$. So there is a unique restriction functor ${}^{\mathcal{H}}\mathrm{Res}_{W'}^{W}$ up to isomorphisms.