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EXCELLENT CONNECTIONS IN THE MOTIVES OF QUADRICS

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ABSTRACT. – In this article we prove the conjecture claiming that the motive of a real quadric is the "most decomposable" among anisotropic quadrics of given dimension over *all* fields. This imposes severe restrictions on the motive of arbitrary anisotropic quadric. As a corollary we estimate from below the rank of indecomposable direct summand in the motive of a quadric in terms of its dimension. This generalizes the well-known Binary Motive Theorem. Moreover, we have the description of the Tate motives involved. This, in turn, gives another proof of Karpenko's Theorem on the value of the first higher Witt index. But also other new relations among higher Witt indices follow.

RÉSUMÉ. – Dans cet article, nous prouvons la conjecture qui dit que le motif d'une quadrique réelle est le « plus décomposable » parmi ceux des quadriques de la même dimension sur *n'importe quel* corps. Cela restreint sûrement les motifs possibles pour une quadrique anisotrope quelconque. Nous en tirons en corollaire une minoration du rang d'un facteur direct indécomposable du motif d'une quadrique en fonction de sa dimension, ce qui généralise le théorème bien connu du motif binaire. De plus, nous obtenons une description des motifs de Tate qui apparaissent, ce qui implique alors une nouvelle preuve du théorème de Karpenko sur les valeurs du premier indice de Witt. D'autres relations entre les indices de Witt supérieurs s'en suivent également.

1. Introduction

Let Q be a smooth projective quadric of dimension n over a field k of characteristic not 2, and $M(Q)$ be its *motive* in the category $Chow(k)$ of Chow motives over k (see [12], or Chapter XII of [2]). Over the algebraic closure $\overline{k}$, our quadric becomes completely split, and so, cellular. This implies that $M(Q|_{\overline{k}})$ becomes isomorphic to a direct sum of Tate motives:

$$M(Q|_{\overline{k}}) \cong \oplus_{\lambda\in\Lambda(Q)}\mathbf{Z}(\lambda)[2\lambda],$$

where $\Lambda(Q)=\Lambda(n)$ is $\{i|\,0\leqslant i\leqslant[n/2]\}\bigsqcup\{n-i|\,0\leqslant i\leqslant[n/2]\}$. But over the ground field k our motive could be much less decomposable. The *Motivic Decomposition Type* invariant $MDT(Q)$ measures what kind of decomposition we have in $M(Q)$. Any direct summand N of $M(Q)$ also splits over $\overline{k}$, and $N|_{\overline{k}}\cong\sum_{\lambda\in\Lambda(N)}\mathbf{Z}(\lambda)[2\lambda]$, where $\Lambda(N)\subset\Lambda(Q)$ (see [12] for details). We say that $\lambda, \mu \in \Lambda(Q)$ are *connected*, if for any direct summand N of $M(Q)$, either both λ and μ are in $\Lambda(N)$, or both are out. This is an equivalence relation, and it splits $\Lambda(Q) = \Lambda(n)$ into a disjoint union of *connected components*. This decomposition is called the *Motivic Decomposition Type*. It interacts in a nontrivial way with the *Splitting pattern*, and using this interaction one proves many results about both invariants. The (absolute) *Splitting pattern* $\mathbf{j}(q)$ of the form q is defined as an increasing sequence $\{\mathfrak{j}_0, \mathfrak{j}_1, \dots, \mathfrak{j}_h\}$ of all possible Witt indices of $q|_E$ over all possible field extensions E/k. We will also use the (relative) *Splitting pattern* $\mathbf{i}(q)$ defined as $\{\mathfrak{i}_0, \dots, \mathfrak{i}_h\} := \{\mathfrak{j}_0, \mathfrak{j}_1 - \mathfrak{j}_0, \mathfrak{j}_2 - \mathfrak{j}_1, \dots, \mathfrak{j}_h - \mathfrak{j}_{h-1}\}$.

Let us denote the elements $\{\lambda | 0 \leqslant \lambda \leqslant [n/2]\}$ of $\Lambda(n)$ as λ_{lo}, and the elements $\{n - \lambda | 0 \leqslant \lambda \leqslant [n/2]\}$ as λ^{up}. See the appendix for the detailed explanation. The principal result relating the splitting pattern and the motivic decomposition type claims that all elements of $\Lambda(Q)$ come in pairs whose structure depends on the splitting pattern.

Proposition 1.1 ([12, Proposition 4.10], cf. [2, Theorem 73.26])

Let λ and μ be such that $\mathfrak{j}_{r-1} \leqslant \lambda, \mu < \mathfrak{j}_r$, where $1 \leqslant r \leqslant h$, and $\lambda + \mu = \mathfrak{j}_{r-1} + \mathfrak{j}_r - 1$. Then λ_{lo} is connected to μ^{up}.

Consequently, any direct summand in the motive of anisotropic quadric consists of even number of Tate motives when restricted to $\overline{k}$, in particular, of at least two Tate motives. If it consists of just two Tate motives we will call it *binary*. It can happen that $M(Q)$ splits into *binary motives*. As was proven by M.Rost ([10, Proposition 4]), this is the case for *excellent* quadrics, and, hypothetically, it should be the only such case. The *excellent* quadratic forms introduced by M. Knebusch ([9]) are sort of substitutes for the Pfister forms in dimensions which are not powers of two. Namely, if one wants to construct such a form of dimension, say, m, one needs first to present m in the form $2^{r_1} - 2^{r_2} + \dots + (-1)^{s-1}2^{r_s}$, where $r_1 > r_2 > \dots > r_{s-1} > r_s + 1 \geqslant 1$ (it is easy to see that such presentation is unique), and then choose pure symbols $\alpha_i \in \mathrm{K}^M_{r_i}(k)/2$ such that $\alpha_1 \vdots \alpha_2 \vdots \dots \vdots \alpha_s$. Then the respective excellent form is (any scalar multiple of) an m-dimensional form $(\langle\langle \alpha_1 \rangle\rangle - \langle\langle \alpha_2 \rangle\rangle + \dots + (-1)^{s-1}\langle\langle \alpha_s \rangle\rangle)_{\mathrm{an}}$. In particular, if $m = 2^r$ one gets an r-fold Pfister form. Anisotropic quadrics over $\mathbb{R}$ give us examples of excellent quadrics in all dimensions. It follows from the mentioned result of M. Rost that the only connections in the motives of excellent quadrics are binary ones coming from Proposition 1.1. At the same time, the experimental data suggested that in the motive of anisotropic quadric Q of dimension n we should have not only connections coming from the splitting pattern $\mathbf{i}(Q)$ of Q but also ones coming from the excellent splitting pattern:

Conjecture 1.2 ([12, Conjecture 4.22]). – *Let Q and P be anisotropic quadrics of dimension n with P-excellent. Then we can identify $\Lambda(Q) = \Lambda(n) = \Lambda(P)$, and for $\lambda, \mu \in \Lambda(n)$,*

$$\lambda, \mu \ \textit{connected in} \ \Lambda(P) \implies \lambda, \mu \ \textit{connected in} \ \Lambda(Q).$$

Remark. – Connections in $\Lambda(P)$ depend only on the excellent splitting pattern, and thus, only on n. In particular, they are the same as for the real anisotropic quadric of dimension n.

Partial case of this conjecture, where λ and μ belong to the *outer excellent shell* (that is, $\lambda, \mu < \mathrm{j}_1(P)$), was proven earlier and presented by the author at the conference in Eilat, Feb. 2004. The proof used *Symmetric operations*, and the Grassmannian $G(1, Q)$ of projective lines on Q, and is a minor modification of the proof of [13, Theorem 4.4] (assuming $char(k) = 0$). Another proof using Steenrod operations and $Q^{\times 2}$ appears in [2, Corollary 80.13] (here $char(k) \neq 2$).

The principal aim of the current paper is to prove the whole conjecture for all field of characteristic different from 2.

THEOREM 1.3. – *Conjecture 1.2 is true.*

This theorem shows that the connections in the motive of an excellent quadric are *minimal* among anisotropic quadrics of a given dimension (one can put it also as follows: any decomposition which one can find in the motive of anisotropic quadric over any field is also present in the motive of the real anisotropic quadric of the same dimension). Moreover, for a given anisotropic quadric Q we get not just one set of such connections, but $h(Q)$ sets, where $h(Q)$ is a *height* of Q, since we can apply the theorem not just to q but to $q_i := (q|_{k_i})_{\mathrm{an}}$ for all fields $k_i, 0 \leqslant i < h$, from the *generic splitting tower of Knebusch* (see [8]). And the more splitting pattern of Q differs from the excellent splitting pattern, the more nontrivial conditions we get, and the more indecomposable $M(Q)$ will be.

As an application of this philosophy, we get a result bounding from below the rank of an indecomposable direct summand in the motive of a quadric in terms of its dimension—see Theorem 2.1. This is a generalization of the Binary Motive Theorem ([6, Theorem 6.1], see also other proofs in [13, Theorem 4.4] and [2, Corollary 80.11]) which claims that the dimension of a binary direct summand in the motive of a quadric is equal to $2^r - 1$, for some r, and which has many applications in the quadratic form theory. Moreover, we can describe which particular Tate motives must be present in $N|_{\overline{k}}$ depending on the dimension on N. An immediate corollary of this is another proof of the theorem of Karpenko (formerly known as the conjecture of Hoffmann) describing possible values of the first higher Witt index of q in terms of $\dim(q)$. This approach to the Hoffmann's conjecture based on the Conjecture 1.2 is, actually, the original one introduced by the author in 2001, and it is pleasant to see it working, after all. But, aside from the value of the first Witt index, the Theorem 2.1 gives many other relations on higher Witt indices.

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2. Applications of the Main Theorem

For the direct summand N of $M(Q)$ let us denote as $\mathrm{rank}(N)$ the cardinality of $\Lambda(N)$ (that is, the number of Tate motives in $N|_{\overline{k}}$), as $a(N)$ and $b(N)$ the minimal and maximal element in $\Lambda(N)$, respectively, and as $\dim(N)$ the difference $b(N)-a(N)$. Unless otherwise stated, n will always be the dimension of a quadric.

THEOREM 2.1. – *Let N be an indecomposable direct summand in the motive of anisotropic quadric with* $\dim(N)+1=2^{r_1}-2^{r_2}+\cdots+(-1)^{s-1}2^{r_s}$, *where* $r_1>r_2>\cdots>r_{s-1}>r_s+1\geqslant 1$. *Then:*

(1) $\operatorname{rank}(N)\geqslant 2s$;
(2) *for* $1\leqslant k\leqslant s$, *let* $d_k=\sum_{i=1}^{k-1}(-1)^{i-1}2^{r_i-1}+\varepsilon(k)\cdot\sum_{j=k}^{s}(-1)^{j-1}2^{r_j}$, *where* $\varepsilon(k)=1$, *if k is even, and* $\varepsilon(k)=0$, *if k is odd. Then*

$$(a(N)+d_k)_{\mathrm{lo}}\in\Lambda(N),\qquad \textit{and}\qquad (n-b(N)+d_k)^{\mathrm{up}}\in\Lambda(N).$$

REMARK. – In particular, we get: if $\operatorname{rank}(N)=2$, that is, N is *binary*, then $\dim(N)=2^r-1$, for some r—the Binary Motive Theorem.

Proof. – First, we reduce to a special case:

LEMMA 2.2. – *It is sufficient to prove Theorem 2.1 in the case* $\mathfrak{i}_1(q)=1$, *and* $a(N)=0$, $b(N)=\dim(Q)$.

Proof. – It follows from [12, Corollary 4.14] that there exists $1\leqslant t\leqslant h(q)$ such that $\mathfrak{j}_{t-1}(q)\leqslant a(N),(n-b(N))<\mathfrak{j}_t(q)$, and

$$\dim(N)=n-\mathfrak{j}_{t-1}(q)-\mathfrak{j}_t(q)+1.$$

Then we can pass to the field k_{t-1} from the generic splitting tower of Knebusch, and $N|_{k_{t-1}}$ shifted by $(-\mathfrak{j}_{t-1})[-2\mathfrak{j}_{t-1}]$ will be a direct summand in the motive of Q_{t-1}, where $q_{t-1}=(q|_{k_{t-1}})_{\mathrm{an}}$. It follows from Corollary 4.2, that under this transformation *lower motives* λ_{lo} are transformed into lower motives $(\lambda-\mathfrak{j}_{t-1})_{\mathrm{lo}}$, while *upper motives* λ^{up} are transformed into the upper ones $(\lambda-\mathfrak{j}_{t-1})^{\mathrm{up}}$.

It can happen that $N|_{k_{t-1}}(-\mathfrak{j}_{t-1})[-2\mathfrak{j}_{t-1}]$ is decomposable, but it follows from [12, Corollary 4.14] that it should contain indecomposable submotive N' of the same dimension. Since we estimate the rank of N from below, it is sufficient to prove the statement for N' and $q'=q_{t-1}$. Thus, everything is reduced to the case $t=1$. Considering the subform q'' of q' of codimension $\mathfrak{i}_1(q')-1$, we get from [12, Theorem 4.15] that $M(Q'')$ contains a direct summand isomorphic to $N'(-a(N'))[-2a(N')]$, while $\mathfrak{i}_1(q'')=1$ by [12, Corollary 4.9(3)]. Again, Corollary 4.2 shows that separation into upper and lower motives is preserved under these manipulations. Hence, we reduced everything to the case: $\mathfrak{i}_1(q)=1$, and $a(N)=0$, $b(N)=\dim(Q)$. □

We will use the following observation relating the Motivic Decomposition Types of a form and of its anisotropic kernel.

OBSERVATION 2.3. – *Let ρ be some quadratic form over k, E/k be some field extension,* $m=i_W(\rho|_E)$, *and* $\rho'=(\rho|_E)_{\mathrm{an}}$. *Then $\Lambda(\rho')$ is naturally embedded into $\Lambda(\rho)$ by the rule:* $\lambda_{\mathrm{lo}}\mapsto(\lambda+m)_{\mathrm{lo}}$, $\lambda^{\mathrm{up}}\mapsto(\lambda+m)^{\mathrm{up}}$, *and connections in $\Lambda(\rho')$ imply ones in $\Lambda(\rho)$.*