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OVERCONVERGENT DE RHAM-WITT COHOMOLOGY

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Abstract. – The goal of this work is to construct, for a smooth variety X over a perfect field k of finite characteristic $p > 0$, an overconvergent de Rham-Witt complex $W^{\dagger}\Omega_{X/k}$ as a suitable subcomplex of the de Rham-Witt complex of Deligne-Illusie. This complex, which is functorial in X, is a complex of étale sheaves and a differential graded algebra over the ring $W^{\dagger}(\mathcal{O}_X)$ of overconvergent Witt-vectors. If X is affine one proves that there is an isomorphism between Monsky-Washnitzer cohomology and (rational) overconvergent de Rham-Witt cohomology. Finally we define for a quasiprojective X an isomorphism between the rational overconvergent de Rham-Witt cohomology and the rigid cohomology.

Résumé. – Le but de ce travail est de construire, pour X une variété lisse sur un corps parfait k de caractéristique finie, un complexe de de Rham-Witt surconvergent $W^{\dagger}\Omega_{X/k}$ comme un sous-complexe convenable du complexe de de Rham-Witt de Deligne-Illusie. Ce complexe qui est fonctoriel en X est un complexe de faisceaux étales et une algèbre différentielle graduée sur l'anneau $W^{\dagger}(\mathcal{O}_X)$ des vecteurs de Witt surconvergents. Lorsque X est affine, on démontre qu'il existe un isomorphisme canonique entre la cohomologie de Monsky-Washnitzer et la cohomologie (rationnelle) de de Rham-Witt surconvergente. Finalement on définit pour X quasi-projectif un isomorphisme entre la cohomologie rigide de X et la cohomologie de de Rham-Witt surconvergente rationnelle.

Introduction

Let X be a smooth variety over a perfect field k of finite characteristic. The purpose of this work is to define an overconvergent de Rham-Witt complex $W^{\dagger}\Omega^{\cdot}_{X/k}$ of sheaves on X. This complex is a differential graded algebra contained in the de Rham-Witt complex $W\Omega^{\cdot}_{X/k}$ of Illusie and Deligne.

If X is quasiprojective we define a canonical isomorphism from rigid cohomology of X in the sense of Berthelot:

$$H^i_{\mathrm{rig}}(X/W(k) \otimes \mathbb{Q}) \to \mathbb{H}^i(X, W^{\dagger}\Omega^{\cdot}_{X/k}) \otimes \mathbb{Q}.$$

In particular these are finite dimensional vector spaces over $W(k)\otimes\mathbb{Q}$ by [2]. We conjecture that the image of the morphism

$$\mathbb{H}^i(X, W^\dagger\Omega^{\cdot}_{X/k}) \to \mathbb{H}^i(X, W^\dagger\Omega^{\cdot}_{X/k})\otimes\mathbb{Q}$$

is a finitely generated $W(k)$-module. If X is projective we expect that the image of $\mathbb{H}^i(X, W^\dagger\Omega^{\cdot}_{X/k})$ under the comparison isomorphism between rigid cohomology and crystalline cohomology coincides with the image of crystalline cohomology.

In the case where $X = \operatorname{Spec} A$ is affine we obtain more precise results. The cohomology groups of the individual sheaves $W^\dagger\Omega^j_{X/k}$ are zero for $i > 0$. The complex $H^0(X, W^\dagger\Omega^{\cdot}_{X/k})$ will be denoted by $W^\dagger\Omega^{\cdot}_{A/k}$. Let $\tilde{A}$ be a lifting of A to a smooth algebra $\tilde{A}$ over $W(k)$. We denote by $\tilde{A}^\dagger$ the weak completion of $\tilde{A}$ in the sense of Monsky-Washnitzer. The absolute Frobenius endomorphism on A lifts (non canonically) to $\tilde{A}^\dagger$. This defines a homomorphism $\tilde{A}^\dagger \to W(A)$. We show that the image of this map lies in $W^\dagger(A)$. This defines morphisms

$$(1) \quad H^i(\Omega^{\cdot}_{\tilde{A}^\dagger/W(k)}) \to H^i(W^\dagger\Omega^{\cdot}_{A/k}), \quad \text{for } i \geq 0.$$

We show that the kernel and cokernel of this map is annihilated by $p^{2\kappa}$, where $\kappa = \lfloor \log_p \dim A \rfloor$. If we tensor the morphism (1) by $\mathbb{Q}$ it becomes independent of the lift of the absolute Frobenius chosen.

We note that Lubkin [12] used another growth condition on Witt vectors. His bounded Witt vectors are different from our overconvergent Witt vectors.

Let $A = k[T_1, \ldots, T_d]$ be the polynomial ring. For each real $\epsilon > 0$ we defined ([5]) the Gauss norm γ_ϵ on $W(A)$. We extend them to the de Rham-Witt complex $W\Omega^{\cdot}_{A/k}$. A Witt differential from $W\Omega^{\cdot}_{A/k}$ is called overconvergent if its Gauss norm is finite for some $\epsilon > 0$. We denote the subcomplex of all overconvergent Witt differentials by $W^\dagger\Omega^{\cdot}_{A/k}$. Following the description in [10], $W\Omega^{\cdot}_{A/k}$ decomposes canonically into an integral part and an acyclic fractional part and this decomposition continues to hold for the complex of overconvergent Witt differentials. The integral part is easily identified with the de Rham complex associated to the weak completion of the polynomial algebra $W(k)[T_1, \ldots, T_d]$ in the sense of Monsky and Washnitzer. This explains the terminology "overconvergent" for Witt differentials. For an arbitrary smooth k-algebra B we choose a presentation $A \to B$. We define the complex of overconvergent Witt differentials $W^\dagger\Omega^{\cdot}_{B/k}$ as the image of $W^\dagger\Omega^{\cdot}_{A/k}$. This is independent of the presentation. It is a central result that the functor which associates to a smooth affine scheme $\operatorname{Spec} B$ the group $W^\dagger\Omega^m_{B/k}$ is a sheaf for the étale topology, and that $H^i_{\mathrm{Zar}}(\operatorname{Spec} B, W^\dagger\Omega^m_{B/k}) = 0$ for $i \geq 1$. For this we generalize ideas of Meredith [13]. One also uses that the ring of overconvergent Witt vectors is weakly complete in the sense of Monsky-Washnitzer [5] and the complex of overconvergent Witt differentials satisfies a similar property of weak completeness. The étale sheaf property depends on an explicit description - for a finite étale extension C/B - of $W^\dagger\Omega^{\cdot}_{C/k}$ in terms of $W^\dagger\Omega^{\cdot}_{B/k}$. The result is as nice as one can hope for. By a result of Kedlaya [9] any smooth variety can be covered by affines which are finite étale over a localized polynomial algebra. It then remains to show a localization property of overconvergence; namely a Witt differential of a localized polynomial algebra which becomes overconvergent after further localization is already overconvergent. This requires a detailed study of suitable Gauss norms (that are all equivalent) on the truncated de Rham-Witt complex of a localized polynomial algebra.

In the final section we globalize the comparison with rigid cohomology from the affine case. In our approach it is essential to use Grosse-Klönne's dagger spaces [6]. Let Z be an affine smooth scheme over k. Let $Z \to F$ be a closed embedding in a smooth affine scheme over $W(k)$. We call (Z, F) a special frame. To a special frame we associate canonically a dagger space $]Z[^{\dagger}_{\hat{F}}$. Its de Rham cohomology coincides with the rigid cohomology of Z:

$$\mathbf{R}\Gamma(]Z[^{\dagger}_{\hat{F}}, \Omega^{\cdot}_{]Z[^{\dagger}_{\hat{F}}}) = R\Gamma_{\mathrm{rig}}(Z).$$

If $F \times_{\operatorname{Spec} W(k)} \operatorname{Spec} k = Z$ the dagger space $]Z[^{\dagger}_{\hat{F}}$ is affinoid. Therefore the hypercohomology is not needed

$$\Gamma(]Z[^{\dagger}_{\hat{F}}, \Omega^{\cdot}_{]Z[^{\dagger}_{\hat{F}}}) = \mathbf{R}\Gamma(]Z[^{\dagger}_{\hat{F}}, \Omega^{\cdot}_{]Z[^{\dagger}_{\hat{F}}}).$$

We show that the latter is true for a big enough class of special frames. Then simplicial methods allow a globalization to the quasiprojective case.

0. Definition of the overconvergent de Rham-Witt complex

Let R be an $\mathbb{F}_p$-algebra which is an integral domain. We consider the polynomial algebra $A = R[T_1, \ldots, T_d]$. Before we recall the de Rham-Witt complex, we review a few properties of the de Rham complex $\Omega_{A/R}$.

There is a natural morphism of graded rings

$$F : \Omega_{A/R} \to \Omega_{A/R},$$

which is the absolute Frobenius on $\Omega^0_{A/R}$ and such that ${}^F dT_i = T_i^{p-1} dT_i$. As shown in [10], $\Omega_{A/R}$ has an R-basis of so called basic differentials. Their definition depends on certain choices which we will fix now in a more special way than in loc. cit.

We consider functions $k : [1, d] \to \mathbb{Z}_{\geq 0}$ called weights. On the support $\operatorname{Supp} k = \{i_1, \ldots, i_r\}$ we fix an order $i_1, \ldots, i_r$ with the following properties:

(i) $\operatorname{ord}_p k_{i_1} \leq \operatorname{ord}_p k_{i_2} \leq \cdots \leq \operatorname{ord}_p k_{i_r}$.

(ii) If $\operatorname{ord}_p k_{i_n} = \operatorname{ord}_p k_{i_{n+1}}$, then $i_n \leq i_{n+1}$.

Let $\mathscr{P} = \{I_0, I_1, \ldots, I_l\}$ be a partition of $\operatorname{Supp} k$ as in [10]. A basic differential is a differential of the form:

$$\mathfrak{e}(k, \mathscr{P}) = T^{k_{I_0}} \left(\frac{dT^{k_{I_1}}}{p^{\operatorname{ord}_p k_{I_1}}} \right) \cdots \left(\frac{dT^{k_{I_l}}}{p^{\operatorname{ord}_p k_{I_l}}} \right). \tag{0.1}$$

It is shown in [10] Proposition 2.1 that the elements (0.1) form a basis of the de Rham complex $\Omega_{A/R}$ as an R-module. The de Rham-Witt complex $W\Omega_{A/R}$ has a similar description, but now fractional weight functions are involved. More precisely, an element $\omega \in W\Omega^r_{A/R}$ has a unique decomposition as a sum of basic Witt differentials [10]

$$\omega = \sum_{k, \mathscr{P}} e\left(\xi_{k,\mathscr{P}}, k, \mathscr{P}\right), \tag{0.2}$$

where $k : [1, d] \to \mathbb{Z}_{\geq 0}[\frac{1}{p}]$ is any weight ([10], 2.2) and $\mathscr{P} = \{I_0, I_1, \ldots, I_r\}$ runs through all partitions of $\operatorname{Supp} k$. Moreover, the coefficients $\xi_{k,\mathscr{P}} \in W(R)$ satisfy a certain convergence condition ([10], Theorem 2.8).

For each real number $\varepsilon > 0$ we define the Gauss norm of ω:

$$\gamma_\varepsilon(\omega) = \inf_{k,\mathcal{P}}\{\mathrm{ord}_V \xi_{k,\mathcal{P}} - \varepsilon|k|\}. \tag{0.3}$$

We will also use the truncated Gauss norms for a natural number $n \geq 0$:

$$\gamma_\varepsilon[n](\omega) = \inf_{k,\mathcal{P}}\{\mathrm{ord}_V \xi_{k,\mathcal{P}} - \varepsilon|k| \mid \mathrm{ord}_V \xi_{k,\mathcal{P}} \leq n\}.$$

The truncated Gauss norms factor over $W_{n+1}\Omega_{A/R}$. We note that in the truncated case the inf is over a finite set.

If $\gamma_\varepsilon(\omega) > -\infty$, we say that ω has radius of convergence ε.

We call ω overconvergent, if there is an $\varepsilon > 0$ such that ω has radius of convergence ε. It follows from the definitions that

$$\gamma_\varepsilon(\omega_1 + \omega_2) \geq \min\left(\gamma_\varepsilon(\omega_1), \gamma_\varepsilon(\omega_2)\right). \tag{0.4}$$

This inequality shows that the overconvergent Witt differentials form a subgroup of $W\Omega_{A/R}$ which is denoted by $W^\dagger\Omega_{A/R}$. We have $W^\dagger\Omega_{A/R} = \bigcup_\varepsilon W^\varepsilon\Omega_{A/R}$ where $W^\varepsilon\Omega_{A/R}$ are the overconvergent Witt differentials with radius of convergence ε.

If $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$, then an $\overline{\mathbb{R}}$-valued function c on an abelian group M which satisfies (0.4), so that $c(a+b) \geq \min\{c(a), c(b)\}$, is called an order function.

Definition 0.5. – *We say that ω is homogeneous of weight k if in the sum $\omega = \sum e\left(\xi_{k,\mathcal{P}}, k, \mathcal{P}\right)$ the weight k is fixed. We write weight$(\omega) = k$.*

If $g \in \mathbb{Q}$, then we can consider sums which are homogeneous of degree g, i.e.

$$\omega = \sum_{|k|=g,\mathcal{P}} e\left(\xi_{k,\mathcal{P}}, k, \mathcal{P}\right).$$

Then we define $\deg(\omega) = g$. *If ω is homogeneous of a fixed degree, we define*

$$\mathrm{ord}_V\,\omega = \min \mathrm{ord}_V\,\xi_{k,\mathcal{P}}.$$

It is easy to see that $\gamma_\varepsilon(\omega) > -\infty$ if and only if there are real constants C_1, C_2, with $C_1 > 0$ such that for all weights k occurring in ω we have

$$|k| \leq C_1 \mathrm{ord}_V\,\xi_{k,\mathcal{P}} + C_2. \tag{0.6}$$

One can take $C_1 = \frac{1}{\varepsilon}$.

Using this equivalent definition one can show that the product of two overconvergent Witt differentials is again overconvergent, as follows: For two homogeneous forms ω_1, ω_2 one has $\mathrm{ord}_V\,(\omega_1 \wedge \omega_2) \geq \max\left(\mathrm{ord}_V\,\omega_1, \mathrm{ord}_V\,\omega_2\right)$. This follows from a (rather tedious) case by case calculation with basic Witt differentials.

We have $\deg(\omega_1 \wedge \omega_2) = \deg\omega_1 + \deg\omega_2$.

Assume now that

$$\deg\omega \leq C_1 \mathrm{ord}_V\,\omega + C_2$$

and

$$\deg\omega' \leq C_1' \mathrm{ord}_V\,\omega' + C_2'$$

for two homogeneous forms ω, ω' of fixed degrees. Then

$$\deg(\omega \wedge \omega') = \deg\omega + \deg\omega' \leq (C_1 + C_1')\,\mathrm{ord}_V\,(\omega_1 \wedge \omega_2) + C_2 + C_2'.$$