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EIGENVALUES AND SIMPLICITY OF INTERVAL EXCHANGE TRANSFORMATIONS

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Abstract. – For a class of d-interval exchange transformations, which we call the symmetric class, we define a new self-dual induction process in which the system is successively induced on a union of sub-intervals. This algorithm gives rise to an underlying graph structure which reflects the dynamical behavior of the system, through the Rokhlin towers of the induced maps. We apply it to build a wide assortment of explicit examples on four intervals having different dynamical properties: these include the first nontrivial examples with eigenvalues (rational or irrational), the first ever example of an exchange on more than three intervals satisfying Veech's simplicity (though this weakening of the notion of minimal self-joinings was designed in 1982 to be satisfied by interval exchange transformations), and an unexpected example which is non uniquely ergodic, weakly mixing for one invariant ergodic measure but has rational eigenvalues for the other invariant ergodic measure.

Résumé. – Pour une classe d'échanges de d intervalles, que nous appelons la classe symétrique, nous définissons un nouveau processus d'induction autoduale, où le système est induit successivement sur des unions de sous-intervalles. Cet algorithme crée une structure de graphes qui reflète le comportement dynamique du système grâce aux tours de Rokhlin des applications induites. Nous l'utilisons pour construire un large choix d'exemples explicites sur quatre intervalles, avec différentes propriétés dynamiques : on y trouve entre autres les premiers exemples non triviaux possédant des valeurs propres (rationnelles ou irrationnelles), le premier exemple d'un échange de plus de trois intervalles qui soit simple au sens de Veech (alors que cette notion, affaiblissant celle d'autocouplages minimaux, a été introduite en 1982 avec les échanges d'intervalles en vue), et un exemple inattendu qui est non uniquement ergodique, faiblement mélangeant pour une des mesures ergodiques invariantes, mais a des valeurs propres rationnelles pour l'autre mesure ergodique invariante.

1. Preliminaries

Interval exchange transformations have been introduced by Oseledec [32], following an idea of Arnold [1]; an exchange of d intervals is defined by a probability vector of d lengths and a permutation on d letters; the unit interval is then partitioned according to the vector of lengths, and T exchanges the intervals according to the permutation, see Sections 1.1 and 1.2 below for all definitions. Katok and Stepin [24] used these transformations to exhibit a class of systems with simple continuous spectrum. Then Keane [25] defined a condition called i.d.o.c. ensuring minimality, and was the first to use the idea of induction, which was later formalized by Rauzy [34], as a generalization of the continued fraction algorithm. These tools formed the basis for future studies of various ergodic and spectral properties for these dynamical systems. For general properties of interval exchange transformations, the reader can consult the courses of Viana [41] and Yoccoz [42] [43].

In this paper we study d-interval exchange transformations T, defined by a vector $(\alpha_1, \dots \alpha_d)$ of lengths and the *symmetric* permutation $\pi i = d+1-i$, $1 \leq i \leq d$; we call $\mathscr{I}$ the set of $(\lambda_1, \dots, \lambda_d)$ in $\mathbb{R}^{+d}$ for which T, defined by the vector $(\frac{\lambda_1}{\lambda_1+\dots\lambda_d}, \dots, \frac{\lambda_d}{\lambda_1+\dots\lambda_d})$, satisfies the i.d.o.c. condition; henceforth we shall consider only transformations satisfying this condition: let $\mathscr{U}$, resp. $\mathscr{M}'$, $\mathscr{M}$, $\mathscr{N}$, $\mathscr{S}$, be the subset of $\mathscr{I}$ for which T is uniquely ergodic, resp. topologically weakly mixing, resp. weakly mixing for at least one invariant measure, resp. not weakly mixing for at least one invariant measure, resp. simple for at least one invariant measure. A great part of the history of this area is made by the difficult results about these sets. After Keane proved $m(\mathbb{R}^{d+} \setminus \mathscr{I}) = 0$ for the Lebesgue measure m on $\mathbb{R}^d$ and the surprising result that (for $d = 4$) $\mathscr{U}^c$ (for $X \in \{\mathscr{U}, \mathscr{M}', \mathscr{M}, \mathscr{N}, \mathscr{S}\}$ we call X^c its complement in $\mathscr{I}$) is not empty [26], he conjectured that $m(\mathscr{U}^c) = 0$. This conjecture was proved by Masur [29] and Veech [39], see also Boshernitzan [6] for a combinatorial proof closer to the spirit of the present paper. Then Veech [40] proved that $m(\mathscr{M}^c) = 0$ for some permutations, not including the symmetric one for $d = 4$; it took quite a long time to have, for all permutations outside the rotation class, first $m(\mathscr{M}'^c) = 0$ (Nogueira-Rudolph [30]), then at last $m(\mathscr{M}^c) = m(\mathscr{N}) = 0$ (Avila-Forni [4]); whether $m(\mathscr{S}^c) = 0$ is still an open question asked by Veech [38]; note that the result on weak mixing in [4] is valid both for one invariant measure and all invariant measures because $m(\mathscr{U}^c) = 0$.

While all these extremely powerful articles establish generic results for general interval exchange transformations, here we aim to provide a detailed analysis of the dynamical behaviour of specific families of interval exchanges; more precisely, we want to address problems concerning relations between the sets defined above, nothing of which was known until recently for $d > 3$, except obvious relations as $\mathscr{M}' \subset \mathscr{M}$, $\mathscr{U} \cap \mathscr{N} \cap \mathscr{M} = \varnothing$ and $(\mathscr{U} \cap \mathscr{N}) \cup (\mathscr{U} \cap \mathscr{M}) = \mathscr{U}$. It was not known whether $\mathscr{N}$ is nonempty or even that $\mathscr{S}$, which is likely to have full measure (indeed, the whole notion of simplicity has been devised for that, and Veech's question has been much investigated), is nonempty; we can also ask about the non-emptiness of some intersections such as $\mathscr{U}^c \cap \mathscr{M}$ or (more difficult as these are two small sets) $\mathscr{U}^c \cap \mathscr{N}$. Another problem is to find explicit examples (in the sense that maybe the vector of lengths is not given, but it can be computed by an explicit algorithm), and not only existence theorems; very few of them were known: for $d = 4$, explicit elements of $\mathscr{U}^c$ are given by Keane [26] while explicit elements of $\mathscr{U}$ can be deduced from the same paper, or built from substitutions, or pseudo-Anosov maps, by a classical construction; but none were known in other sets, even in the bigger ones, until, during the preparation of the present paper, Sinai and Ulcigrai [35] found explicit elements of $\mathscr{M}$, while Yoccoz [42] built explicit elements of $\mathscr{U}^c$ for every d; other related results [20][8] were derived after preliminary versions of the present paper were circulated, see the discussion in Section 6 below.

Similar questions have been addressed for the (by unanimous consent much easier) case $d = 3$, by Veech [36], del Junco [21], and the present authors plus Holton [13][14][15][16]; the methods of these papers have had to be considerably upgraded to tackle the next case, $d = 4$. Thus we have introduced a new notion of induction, beside the classical ones due to Rauzy [34], Zorich [44], and more recently Yoccoz ([28] where a good survey of all these notions can also be found). This *self-dual* induction, studied in more details in [19], is a variant of the less well-known induction of da Rocha [27] [11], and for $d = 3$ its measure-theoretic properties and self-duality are studied in [18]. We present it in Section 2 below, and use it in Sections 3 and 4 to build families of explicit examples of four-interval exchanges; each example is described by four families of Rokhlin towers, depending on partial quotients of our induction algorithm. After a good choice of these partial quotients, our transformation will have the required properties through a measure-theoretic isomorphism with a rank one system. Whether and why this new induction was necessary to answer the questions we addressed will be discussed at the end of Section 6 below.

What we obtain in the end is some groups of examples for $d = 4$: two in $\mathcal{U} \cap \mathcal{M}' \cap \mathcal{M}^c$, one having rational eigenvalues and the other being measure-theoretically isomorphic to an irrational rotation, one in $\mathcal{U} \cap \mathcal{M}'^c \cap \mathcal{M} \cap \mathcal{S}$, and one in $\mathcal{U}^c \cap \mathcal{M}' \cap \mathcal{M} \cap \mathcal{N}$. We find also elements of $\mathcal{U} \cap \mathcal{M}$ which are measure-theoretically isomorphic to some of the so-called Arnoux-Rauzy systems. All the examples we produce come from expansions having (very) unbounded partial quotients in our induction algorithm. That makes our elements of $\mathcal{M}$ a priori different from Sinai-Ulcigrai's ones, these being obtained from periodic examples relative to a different induction algorithm; in particular, our examples are all rigid, and completely new; their existence was not unexpected, but the existence of an example with irrational eigenvalues for the simpler case $d = 3$ was the object of a question of Veech (1984) which was solved only in [15] (2004); our examples prove also that Avila-Forni's result is strictly stronger than Nogueira-Rudolph's. The first example of an exchange on more than three intervals which is simple is not surprising, but this resisted the efforts of specialists during 25 years, and constitutes a first step towards Veech's open question. As for our last example, which is weakly mixing for one of the two invariant ergodic measures but has rational eigenvalues for the other, it came as a surprise even for the authors.

For generalizations (to other permutations and values of d), see Section 6 below.

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1.1. The main definitions

DEFINITION 1.1. – *A* symmetric d-interval exchange transformation *is a d-interval exchange transformation T with probability vector $(\alpha_1, \dots, \alpha_d)$, and permutation $\pi i = d+1-i$, $1 \le i \le d$, defined by*

$$Tx = x + \sum_{\pi^{-1}j < \pi^{-1}i} \alpha_j - \sum_{j<i} \alpha_j.$$

when x is in the interval

$$\Delta_i = \left[\sum_{j<i}\alpha_j, \sum_{j\leq i}\alpha_j\right[.$$

We denote by β_i, $1 \leq i \leq d-1$, the i-th discontinuity of T^{-1}, namely $\beta_i = \sum_{j=d+1-i}^{d}\alpha_j$, while γ_i is the i-th discontinuity of T, namely $\gamma_i = \sum_{j=1}^{i}\alpha_j = 1-\beta_{d-j}$. Then $\Delta_1 = [0,\gamma_1[$, $\Delta_i = [\gamma_{i-1},\gamma_i[$, $2 \leq i \leq d-1$ and $\Delta_d = [\gamma_{d-1},1[$.

Definition 1.2. – *T satisfies the* infinite distinct orbit condition *(or i.d.o.c. for short) of Keane* [25] *if the $d-1$ negative trajectories $\{T^{-n}(\gamma_i)\}_{n\geq 0}$, $1 \leq i \leq d-1$, of the discontinuities of T are infinite disjoint sets.*

The i.d.o.c. condition for T is (strictly) weaker than the *total irrationality* condition on the lengths, where the only rational relation between α_i, $1 \leq i \leq d$, is $\sum_{i=1}^{d}\alpha_i = 1$. As here π is primitive, the i.d.o.c. condition implies that T is *minimal* (every orbit is dense) [25].

1.2. A few notions from ergodic theory

A general reference for this section is [10].

Definition 1.3. – *A system (X,T) is* uniquely ergodic *if it admits only one invariant probability measure.*

Definition 1.4. – *Let (X,T,μ) be a finite measure-preserving dynamical system.*

A real number $0 \leq \gamma < 1$ is an eigenvalue *of T (denoted additively) if there exists a non-constant f in $\mathcal{L}^2(X,\mathbb{R}/\mathbb{Z})$ such that $f \circ T = f + \gamma$ in $\mathcal{L}^2(X,\mathbb{R}/\mathbb{Z})$; f is then an* eigenfunction *for the eigenvalue γ. As, following* [10]*, we consider only non-constant eigenfunctions, $\gamma = 0$ is not an eigenvalue if T is ergodic. T is* weakly mixing *if it has no eigenvalue.*

Definition 1.5. – *(X,T) is* topologically weakly mixing *if it has no continuous (non-constant) eigenfunction.*

In the particular case of interval-exchange transformations, the topology we use here is the standard one (induced by the Lebesgue measure) on the interval $[0,1[$ (though T itself is not continuous), but the proofs in the present paper work in the same way if we view T as the shift on the symbolic trajectories, equipped with the product topology on $\{1,\dots,d\}^{\mathbb{N}}$; the two topologies are not equivalent, and it does not seem to be known whether a continuous eigenfunction for one has to be continuous for the other.

Definition 1.6. – *(X,T,μ) is* rigid *if there exists a sequence $s_n \to \infty$ such that, for any measurable set A, $\mu(T^{s_n}A\Delta A) \to 0$.*

Definition 1.7. – *In (X,T), a (Rokhlin)* tower *of* base *F is a collection of disjoint measurable sets called* levels *F, TF, ..., $T^{h-1}F$. If X is equipped with a partition P such that each level T^rF is contained in one atom $P_{w(r)}$, the* name *of the tower is the word $w(0)\dots w(h-1)$.*