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FANO MANIFOLDS OF DEGREE TEN AND EPW SEXTICS

BY Atanas ILIEV AND Laurent MANIVEL

Abstract. – O'Grady showed that certain special sextics in $\mathbb{P}^5$ called EPW sextics admit smooth double covers with a holomorphic symplectic structure. We propose another perspective on these symplectic manifolds, by showing that they can be constructed from the Hilbert schemes of conics on Fano fourfolds of degree ten. As applications, we construct families of Lagrangian surfaces in these symplectic fourfolds, and related integrable systems whose fibers are intermediate Jacobians.

Résumé. – O'Grady a démontré que certaines sextiques spéciales dans $\mathbb{P}^5$, les sextiques EPW, admettent pour revêtements doubles des variétés symplectiques holomorphes lisses. Nous proposons une nouvelle approche de ces variétés symplectiques, en montrant qu'elles se construisent à partir des schémas de Hilbert de coniques sur des variétés de Fano de dimension quatre et de degré dix. En guise d'application, nous construisons des familles de surfaces lagrangiennes dans ces variétés symplectiques, puis des systèmes intégrables dont les fibres sont des jacobiennes intermédiaires.

1. Introduction

EPW sextics (named after their discoverers, Eisenbud, Popescu and Walter) are some special hypersurfaces of degree six in $\mathbb{P}^5$, first introduced in [7] as examples of Lagrangian degeneracy loci. These hypersurfaces are singular in codimension two, but O'Grady realized in [17, 18, 20] that they admit smooth double covers which are irreducible holomorphic symplectic fourfolds. In fact, the first examples of such double covers were discovered by Mukai in [16], who constructed them as moduli spaces of stable rank two vector bundles on a polarized K3 surface of genus six. From this point of view, the symplectic structure is induced from the K3 surface. It carries over to double covers of EPW sextics by a deformation argument.

The main goal of this paper is to provide another point of view on this symplectic structure. Our starting point will be smooth Fano fourfolds Z of index two, obtained by cutting the six dimensional Grassmannian $G(2,5)$, considered in its Plücker embedding, by a hyperplane and a quadric. Our main observation is that the Hodge number $h^{3,1}(Z)$ equals one

(Lemma 4.1). By the results of e.g. [11], a generator of $H^{3,1}(Z)$ induces a closed holomorphic two-form on the smooth part of any Hilbert scheme of curves on Z. We focus on the case of conics. The most technical part of the paper consists in proving that for Z general, the Hilbert scheme $F_g(Z)$ of conics in Z is smooth (Theorem 3.2). It is thus endowed with a canonical (up to scalar) global holomorphic two-form.

Since $F_g(Z)$ has dimension five, it can certainly not be a symplectic variety. However, it admits a natural map to a sextic hypersurface $Y_Z^\vee$ in $\mathbb{P}^5$. We consider the Stein factorization

$$F_g(Z) \to \tilde{Y}_Z^\vee \to Y_Z^\vee.$$

It turns out that $\tilde{Y}_Z^\vee$ is a smooth fourfold, over which $F_g(Z)$ is essentially a smooth fibration in projective lines. Thus the two-form on $F_g(Z)$ descends to $\tilde{Y}_Z^\vee$. We show that this makes of $\tilde{Y}_Z^\vee$ a holomorphic symplectic fourfold (Theorem 4.13). Moreover the map $\tilde{Y}_Z^\vee \to Y_Z^\vee$ is a double cover, such that the associated involution of $\tilde{Y}_Z^\vee$ is anti-symplectic. This implies that $Y_Z^\vee$ is an EPW sextic (Proposition 4.17), and that $\tilde{Y}_Z^\vee$ does indeed coincide with the double cover constructed by O'Grady (Proposition 4.18).

Apart from making O'Grady's construction more transparent, at least from our point of view, our approach has several interesting consequences.

First, it shows that double covers of EPW sextics are very close to another classical example of symplectic fourfolds, namely the Fano varieties of lines on cubic fourfolds. Indeed, a smooth cubic fourfold Z also has $h^{3,1}(Z) = 1$, and the symplectic form on its Fano scheme of lines $F(Z)$ can be seen as induced from a generator of $H^{3,1}(Z)$, exactly as above. Note that a similar line of ideas has been used to explain the existence of a non degenerate two-form on the symplectic fourfolds in $G(6, 10)$ recently discovered in [5].

Second, it sheds some light on the intriguing interplay between the varieties of type $Z = G(2, 5) \cap Q \cap L$ of different dimensions N, where L denotes a linear space of dimension $N + 4$. For $N = 4$ we have seen how to construct an EPW sextic from the family of conics on Z. If $N = 2$, one gets for Z the genus six K3 surfaces which were, thanks to Mukai's observations, at the beginning of this story, but whose associated sextics form only a codimension one family in the moduli space of all EPW sextics (see [19]). If $N = 5$, it is very easy to see that there is an EPW sextic attached to Z; we explain this in Proposition 2.1, as a way to introduce these special sextics. Finally the case $N = 3$ was the main theme of investigations of [12] and [4]; in these studies the surface of conics on Z played a crucial role; it is very closely related to the singular locus of the EPW sextic attached to Z. We will prove that for any $N = 3, 4$ or 5, a general EPW sextic is attached to a general Z, in fact a certain family of such sextics. For sure there is more to understand about this, see Section 4.5 for a tentative discussion.

Third, from the fourfold Z we obtain a rather concrete description of the symplectic form on $\tilde{Y}_Z^\vee$ (while in [18] its existence was only guaranteed by a deformation argument). This allows us to exhibit certain Lagrangian surfaces in $\tilde{Y}_Z^\vee$, that we construct either from threefolds that are hyperplane sections of Z (Proposition 5.2), or fivefolds that contain Z as a hyperplane section (Proposition 5.6). Moreover, we are able to construct, over the moduli stacks parametrizing these families of threefolds (respectively, fivefolds), two integrable systems whose Liouville tori are the corresponding intermediate Jacobians (Theorems 5.3 and 5.7). Again, this is strikingly similar to the constructions of [10], of two integrable systems over the moduli stacks parametrizing cubic threefolds (respectively, fivefolds) contained in (respectively, containing) a given cubic fourfold.

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Notation

V_5 is a five-dimensional complex vector space. The Grassmannian $G = G(2,5) = G(2,V_5)$ parametrizes two-dimensional vector spaces in V_5.

$Z = G \cap Q \cap H$ is the intersection of G, considered in its Plücker embedding, with a quadric Q and a hyperplane $H = \mathbb{P}V_9$, where $V_9 \subset \wedge^2 V_5$.

$I_2(Z) = H^0(\mathcal{I}_Z(2))$ denotes the space of quadrics containing Z. The hyperplane of quadrics containing G, called Pfaffian quadrics, is $I_2(G) = H^0(\mathcal{I}_G(2)) \simeq V_5$. The hyperplane of Pfaffian quadrics in the projectivization $I = \mathbb{P}(I_2(Z)) \simeq \mathbb{P}^5$ is denoted H_P. In the dual projective space $I^\vee$, it defines a point h_P called the Plücker point.

$Y_Z \subset I$ denotes the closure of the locus of singular non Pfaffian quadrics. The projectively dual hypersurface is $Y_Z^\vee \subset I^\vee$. The variety $\hat{Y}_Z^\vee$ parametrizes pairs $(h, V_4) \in I^\vee \times \mathbb{P}V_5^\vee$ such that a quadric in h cuts $\mathbb{P}(\wedge^2 V_4) \cap H$ along a singular quadric.

$F_g(G)$ is the Hilbert scheme of conics in G, $F(G)$ is the nested Hilbert scheme of pairs $(c, V_4) \in F_g(G) \times \mathbb{P}V_5^\vee$ such that $c \subset G(2, V_4)$.

$F_g(Z)$ is the Hilbert scheme of conics in Z, $F(Z)$ its preimage in $F(G)$.

For c a generic conic in Z, there is a unique V_4 such that $(c, V_4) \in F(Z)$, and we set $G_c = G(2, V_4)$, $P_c = G(2, V_4) \cap H$ and $S_c = P_c \cap Q$. In the pencil of quadrics containing S_c, the unique quadric containing the plane $\langle c \rangle$ spanned by c is denoted Q_{c,V_4}.

In 4.4 we construct maps $F(Z) \to \hat{Y}_Z^\vee$ and $F_g(Z) \to Y_Z^\vee$. The varieties $\bar{Y}_Z^\vee$ and $\tilde{Y}_Z^\vee$ are then defined by the Stein factorizations $F(Z) \to \bar{Y}_Z^\vee \to \hat{Y}_Z^\vee$ and $F_g(Z) \to \tilde{Y}_Z^\vee \to Y_Z^\vee$.

2. EPW sextics in duality

2.1. Quadratic sections of $G(2,5)$

Let V_5 be a five dimensional complex vector space. Denote by $G = G(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)$ the Grassmannian of planes in V_5, considered in the Plücker embedding. Let $X = G \cap Q$ be a general quadric section: this is a Fano fivefold of index three and degree ten. In the sequel, when we will talk about a Fano manifold of degree ten, this will always mean a variety of this type, or possibly a linear section.

Let $I = |I_X(2)|$ denote the linear system of quadrics containing X. Then $I \simeq \mathbb{P}^5$ is generated by Q and the hyperplane $H_P = |I_G(2)|$ of Pfaffian quadrics. Note that once we have chosen an isomorphism $\wedge^5 V_5 \simeq \mathbb{C}$, there is a natural isomorphism

$$V_5 \simeq I_G(2), \qquad v \mapsto P_v(x) = v \wedge x \wedge x.$$

To be more precise, $I_2(G) \simeq \wedge^4 V_5^\vee \simeq V_5 \otimes \det(V_5^\vee)$.

The Pfaffian quadrics P_v are all of rank six. Therefore, the divisor D_X of degree ten parametrizing singular quadrics in I decomposes as

$$D_X = 4H_P + Y_X,$$

for some sextic hypersurface $Y_X \subset I$.

On the other hand, consider a hyperplane V_4 of V_5. Then the Plücker quadrics cut $\mathbb{P}(\wedge^2 V_4) \subset \mathbb{P}(\wedge^2 V_5)$ along the same quadric, namely the Grassmannian $G(2, V_4)$. Therefore the quadrics in $|I_X(2)|$ cut out a pencil of quadrics in $\mathbb{P}(\wedge^2 V_4)$. If V_4 is general, the generic quadric in this pencil is smooth, and there is a finite number of hyperplanes in $|I_X(2)|$ (six, to be precise) restricting to singular quadrics in $\mathbb{P}(\wedge^2 V_4)$. This condition defines a hypersurface $Y'_X \subset I^\vee$. The following statement is essentially contained in [18] (see in particular Propositions 7.1 and 3.1).

Proposition 2.1. – *The two hypersurfaces $Y_X \subset I$ and $Y'_X \subset I^\vee$ are projectively dual EPW sextics.*

First we need to recall briefly the definition of an EPW sextic (for more details see [7, 18]; the version we give here follows [19], Section 3.2). One starts with a six-dimensional vector space U_6. Then $\wedge^3 U_6$ is twenty-dimensional and admits a natural non degenerate skew-symmetric form (once we have fixed a generator of $\wedge^3 U_6 \simeq \mathbb{C}$). Let then $A \subset \wedge^3 U_6$ be a ten-dimensional Lagrangian subspace. There are two associated EPW sextics $Y_A \subset \mathbb{P}(U_6)$ and $Y_A^\vee \subset \mathbb{P}(U_6^\vee)$, one being the projective dual to the other. They are defined as

$$Y_A = \{\ell \subset U_6, \quad \ell \wedge (\wedge^2 U_6) \cap A \neq 0\} \subset \mathbb{P}(U_6),$$
$$Y_A^\vee = \{H \subset U_6, \quad \wedge^3 H \cap A \neq 0\} \subset \mathbb{P}(U_6^\vee)$$

(ℓ denotes a line and H a hyperplane in U_6). We will be mostly interested in $Y_A^\vee$. If A is general enough, then $Y_A^\vee$ is singular exactly along

$$S_A = \{H \subset U_6, \quad \dim(\wedge^3 H \cap A) \geq 2\},$$

which is a smooth surface.

Proof. – The quadric Q in $G(2, V_5)$ is defined by a tensor in $S^2(\wedge^2 V_5)^\vee$ modded out by the space of Pfaffian quadrics. We choose a representative Q_0 in $S^2(\wedge^2 V_5)^\vee$. In particular, the choice of Q_0 induces a decomposition $I_2(X) = I_2(G) \oplus \mathbb{C}Q_0$, hence a decomposition

$$\wedge^3 I_2(X) \simeq \wedge^3 I_2(G) \oplus \wedge^2 I_2(G) \otimes Q_0.$$

Observe that if we let $D = \det V_5^\vee$, then $I_2(G) \simeq V_5 \otimes D$, hence $\wedge^2 I_2(G) \simeq \wedge^2 V_5 \otimes D^2$ and $\wedge^3 I_2(G) \simeq \wedge^3 V_5 \otimes D^3 \simeq \wedge^2 V_5^\vee \otimes D^2$. We can therefore attach to Q_0 the subspace $A(Q_0)$ of $\wedge^3 I_2(X)$ defined as

$$A(Q_0) := \left\{(Q_0(x, \bullet) \otimes d^2,\ x \otimes d^2 \otimes Q_0), \quad x \in \wedge^2 V_5\right\},$$

where d is any generator of D. Then $A(Q_0)$ is a Lagrangian subspace of $\wedge^3 I_2(X)$ (this follows from the symmetry of Q_0), canonically attached to the point defined by Q_0 in $I - H_P \simeq \mathbb{C}^5$. Consider the EPW sextic $Y_{A(Q_0)}^\vee \subset I^\vee$. Note that for Q_0 generic, $A(Q_0)$ is a generic Lagrangian subspace of $\wedge^3 I_2(X)$, so $Y_{A(Q_0)}^\vee$ is a generic EPW sextic.

Lemma 2.2. – *We have $Y_{A(Q_0)}^\vee \simeq Y'_X$.*