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GEOMETRIC THETA-LIFTING FOR THE DUAL PAIR $\mathbb{SO}_{2m}, \mathbb{S}\mathrm{p}_{2n}$

BY Sergey LYSENKO

Abstract. – Let X be a smooth projective curve over an algebraically closed field of characteristic > 2. Consider the dual pair $H = \mathrm{SO}_{2m}, G = \mathrm{Sp}_{2n}$ over X with H split. Write Bun_G and Bun_H for the stacks of G-torsors and H-torsors on X. The theta-kernel $\mathrm{Aut}_{G,H}$ on $\mathrm{Bun}_G \times \mathrm{Bun}_H$ yields theta-lifting functors $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ and $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ between the corresponding derived categories. We describe the relation of these functors with Hecke operators.

In two particular cases these functors realize the geometric Langlands functoriality for the above pair (in the non ramified case). Namely, we show that for $n = m$ the functor $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ commutes with Hecke operators with respect to the inclusion of the Langlands dual groups $\check{H} \widetilde{\to} \mathrm{SO}_{2n} \hookrightarrow \mathrm{SO}_{2n+1} \widetilde{\to} \check{G}$. For $m = n+1$ we show that the functor $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ commutes with Hecke operators with respect to the inclusion of the Langlands dual groups $\check{G} \widetilde{\to} \mathrm{SO}_{2n+1} \hookrightarrow \mathrm{SO}_{2n+2} \widetilde{\to} \check{H}$.

In other cases the relation is more complicated and involves the SL_2 of Arthur. As a step of the proof, we establish the geometric theta-lifting for the dual pair $\mathrm{GL}_m, \mathrm{GL}_n$. Our global results are derived from the corresponding local ones, which provide a geometric analog of a theorem of Rallis.

Résumé. – Soit X une courbe projective lisse sur un corps algébriquement clos de caractéristique $>$ 2. On considère la paire duale $H = \mathrm{SO}_{2m}$, $G = \mathrm{Sp}_{2n}$ sur X où H est déployé. Notons Bun_G et Bun_H les champs de modules des G-torseurs et des H-torseurs sur X. Le faisceau thêta $\mathrm{Aut}_{G,H}$ sur $\mathrm{Bun}_G \times \mathrm{Bun}_H$ donne lieu aux foncteurs de thêta-lifting $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ et $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ entre les catégories dérivées correspondantes. On décrit la relation entre ces foncteurs et les opérateurs de Hecke.

Dans deux cas particuliers cela devient la fonctorialité de Langlands géométrique pour cette paire (cas partout non ramifié). À savoir, on montre que pour $n = m$ le foncteur $F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$ commute avec les opérateurs de Hecke par rapport à l'inclusion des groupes duaux de Langlands $\check{H} \widetilde{\to} \mathrm{SO}_{2n} \hookrightarrow \mathrm{SO}_{2n+1} \widetilde{\to} \check{G}$. Pour $m = n+1$ on montre que le foncteur $F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$ commute avec les opérateurs de Hecke par rapport à l'inclusion des groupes duaux de Langlands $\check{G} \widetilde{\to} \mathrm{SO}_{2n+1} \hookrightarrow \mathrm{SO}_{2n+2} \widetilde{\to} \check{H}$.

Dans d'autres cas la relation est plus complexe et fait intervenir le SL_2 d'Arthur. Comme une étape de la preuve, on établit le thêta-lifting géométrique pour la paire duale $\mathrm{GL}_m, \mathrm{GL}_n$. Nos résultats globaux sont déduits de résultats locaux correspondants, qui géométrisent un théorème de Rallis.

1. Introduction

1.1. – The Howe correspondence for dual reductive pairs is known to realize the Langlands functoriality in some particular cases (cf. [23], [1], [15]). In this paper, which is a continuation of [16], we develop a similar geometric theory for the dual reductive pairs $(\mathbb{S}\mathrm{p}_{2n}, \mathbb{S}\mathbb{O}_{2m})$ and $(\mathrm{GL}_n, \mathrm{GL}_m)$. We consider only the everywhere unramified case.

Recall the classical construction of the theta-lifting operators. Let X be a smooth projective geometrically connected curve over $k = \mathbb{F}_q$. Let $F = k(X)$, $\mathbb{A}$ be the adèles ring of X, $\mathcal{O}$ be the integer adèles. Let G, H be split connected reductive groups over $\mathbb{F}_q$ that form a dual pair inside some symplectic group $\mathbb{S}\mathrm{p}_{2r}$. Assume that the metaplectic covering $\widetilde{\mathbb{S}\mathrm{p}}_{2r}(\mathbb{A}) \to \mathbb{S}\mathrm{p}_{2r}(\mathbb{A})$ splits over $G(\mathbb{A}) \times H(\mathbb{A})$. Let S be the corresponding Weil representation of $G(\mathbb{A}) \times H(\mathbb{A})$. A choice of a theta-functional $\theta : S \to \bar{\mathbb{Q}}_\ell$ yields a morphism of modules over the global non ramified Hecke algebras $\mathcal{H}_G \otimes \mathcal{H}_H$

$$S^{(G\times H)(\mathcal{O})} \to \mathrm{Funct}((G\times H)(F)\backslash(G\times H)(\mathbb{A})/(G\times H)(\mathcal{O}))$$

sending ϕ to the function $(g,h) \mapsto \theta((g,h)\phi)$. The space $S^{(G\times H)(\mathcal{O})}$ has a distinguished non ramified vector, its image ϕ_0 under the above map is the classical theta-function. Viewing ϕ_0 as a kernel of integral operators, one gets the classical theta-lifting operators

$$F_G : \mathrm{Funct}(H(F)\backslash H(\mathbb{A})/H(\mathcal{O})) \to \mathrm{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O}))$$

and

$$F_H : \mathrm{Funct}(G(F)\backslash G(\mathbb{A})/G(\mathcal{O})) \to \mathrm{Funct}(H(F)\backslash H(\mathbb{A})/H(\mathcal{O})).$$

For the dual pairs $(\mathbb{S}\mathrm{p}_{2n}, \mathbb{S}\mathbb{O}_{2m})$ and $(\mathrm{GL}_n, \mathrm{GL}_m)$ these operators realize the Langlands functoriality between the corresponding automorphic representations (as we will see below, its precise formulation involves the SL_2 of Arthur). We establish a geometric analog of this phenomenon.

Recall that $S \widetilde{\to} \otimes'_{x\in X} S_x$ is the restricted tensor product of local Weil representations of $G(F_x) \times H(F_x)$. Here F_x denotes the completion of F at $x \in X$. The above functoriality in the classical case is a consequence of a local result describing the space of invariants $S_x^{G(\mathcal{O}_x)\times H(\mathcal{O}_x)}$ as a module over the tensor product ${}_x\mathcal{H}_G \otimes {}_x\mathcal{H}_H$ of local (non ramified) Hecke algebras. In the geomeric setting the main step is also to prove a local analog of this and then derive the global functoriality. The proof of this local result due to Rallis ([23]) does not geometrise in an obvious way, as it makes essential use of functions with infinite-dimensional support. Their geometric counterparts should be perverse sheaves, however the notion of a perverse sheaf with infinite-dimensional support is not known. We get around this difficulty using inductive systems of perverse sheaves rather then perverse sheaves themselves.

Let us underline the following phenomenon in the proof that we find striking. Let $G = \mathbb{S}\mathrm{p}_{2n}$, $H = \mathbb{S}\mathbb{O}_{2m}$. The Langlands dual groups are $\check{G} \widetilde{\to} \mathbb{S}\mathbb{O}_{2n+1}$ and $\check{H} \widetilde{\to} \mathbb{S}\mathbb{O}_{2m}$ over $\bar{\mathbb{Q}}_\ell$. Write $\mathrm{Rep}(\check{G})$ for the category of finite-dimensional representations of $\check{G}$ over $\bar{\mathbb{Q}}_\ell$, and similarly for $\check{H}$. There will be ind-schemes Y_H, Y_G over k and fully faithful functors $f_H : \mathrm{Rep}(\check{H}) \to \mathrm{P}(Y_H)$ and $f_G : \mathrm{Rep}(\check{G}) \to P(Y_G)$ taking values in the categories of perverse sheaves (pure of weight zero) on Y_H (resp., Y_G) with the following properties. Extend f_H to a functor

$$f_H : \mathrm{Rep}(\check{H} \times \mathbb{G}_m) \to \oplus_{i\in\mathbb{Z}} P(Y_H)[i] \subset \mathrm{D}(Y_H)$$

as follows. If V is a representation of $\check{H}$ and I is the standard representation of $\mathbb{G}_m$ then $f_H(V \boxtimes (I^{\otimes i})) \widetilde{\to} f_H(V)[i]$ is placed in perverse cohomological degree $-i$. For $n \geq m$ there will be an ind-proper map $\pi : Y_G \to Y_H$ such that the following diagram is 2-commutative

$$\begin{array}{ccc} \mathrm{Rep}(\check{G}) & \xrightarrow{f_G} & P(Y_G) \\ \downarrow \mathrm{Res}^\kappa & & \downarrow \pi_! \\ \mathrm{Rep}(\check{H} \times \mathbb{G}_m) & \xrightarrow{f_H} & \oplus_{i\in\mathbb{Z}} P(Y_H)[i] \end{array}$$

for some homomorphism $\kappa : \check{H} \times \mathbb{G}_m \to \check{G}$. For $n = m$ the restriction of κ to $\mathbb{G}_m$ is trivial, so $\pi_! f_G$ takes values in the category of perverse sheaves in this case. Both f_G and f_H send an irreducible representation to an irreducible perverse sheaf. So, for $V \in \mathrm{Rep}(\check{G})$ the decomposition of $\mathrm{Res}^\kappa(V)$ into irreducible ones can be seen via the decomposition theorem of Beilinson, Bernstein and Deligne ([2]). Actually here $\pi : \Pi(K) \times \mathrm{Gr}_G \to \Pi(K)$ is the projection, where $K = k((t))$, Π is a finite-dimensional k-vector space, and Gr_G is the affine grassmanian for G. There will also be an analog of the above result for $n < m$ (and also for the dual pair $\mathrm{GL}_n, \mathrm{GL}_m$).

The above phenomenon is a part of our main local results (Proposition 4 in Section 5.1, Theorem 7 in Section 6.2). They provide a geometric analog of the local theta correspondence for these dual pairs. The key technical tools in the proof are *the weak geometric analogs of the Jacquet functors* (cf. Section 4.7).

1.2. – In the global setting let Ω denote the canonical line bundle on X. Let G be the group scheme over X of automorphisms of $\mathcal{O}_X^n \oplus \Omega^n$ preserving the natural symplectic form $\wedge^2(\mathcal{O}_X^n \oplus \Omega^n) \to \Omega$. Let $H = \mathbb{SO}_{2m}$. Write Bun_H for the stack of H-torsors on X, similarly for G. Using the construction from [19], we introduce a geometric analog $\mathrm{Aut}_{G,H}$ of the above function ϕ_0, this is an object of the derived category of ℓ-adic sheaves on $\mathrm{Bun}_G \times \mathrm{Bun}_H$. It yields the theta-lifting functors

$$F_G : \mathrm{D}(\mathrm{Bun}_H) \to \mathrm{D}(\mathrm{Bun}_G)$$

and

$$F_H : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_H)$$

between the corresponding derived categories. Our main global results for the pair (G, H) are Theorems 3 and 4 describing the relation between the theta-lifting functors and the Hecke functors on Bun_G and Bun_H. They agree with the conjectures of Adams ([1]). One of the advantages of the geometric setting compared to the classical one is that the SL_2 of Arthur appears naturally.

An essential difficulty in the proof was the fact that the complex $\mathrm{Aut}_{G,H}$ is not perverse (it has infinitely many perverse cohomologies), it is not even a direct sum of its perverse cohomologies (cf. Section 8.3).

We also establish the global theta-lifting for the dual pair $(\mathrm{GL}_n, \mathrm{GL}_m)$ (cf. Theorem 5).

1.3. – Let us briefly discuss how the paper is organized. Our main results are collected in Section 2. In Section 3 we remind some constructions at the level of functions, which we have in mind for geometrization.

In Section 4 we develop a geometric theory for the following classical objects. Let $K = \mathbb{F}_q((t))$ and $\mathcal{O} = \mathbb{F}_q[[t]]$. Given a reductive group G over $\mathbb{F}_q$ and a finite dimensional representation M, the space of invariants in the Schwarz space $\mathcal{S}(M(K))^{G(\mathcal{O})}$ is a module over the non ramified Hecke algebra $\mathcal{H}_G$. We introduce the geometric analogs of the Fourier transform on this space and (some weak analogs) of the Jacquet functors. A way to relate this with the global case is proposed in Section 4.6.

In Section 5 we develop the local theta correspondence for the dual pair $(\mathrm{GL}_n, \mathrm{GL}_m)$. The key ingredients here are decomposition theorem from [2], the dimension estimates from [21] and hyperbolic localization results from [4].

In Section 6 we develop the local theta correspondence for the dual pair $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2m})$. In addition to the above tools, we use the classical result (Proposition 2) in the proof of our Proposition 7.

In Section 7 we derive the global theta-lifting results for the dual pair $(\mathrm{GL}_n, \mathrm{GL}_m)$.

In Section 8 we prove our main global results (Theorems 3 and 4) about theta-lifting for the dual pair $(\mathbb{S}\mathrm{p}_{2n}, \mathrm{S}\mathbb{O}_{2m})$. The relation between the local theory and the theta-kernel $\mathrm{Aut}_{G,H}$ comes from the results of [16]. In that paper we have introduced a scheme $\mathcal{L}_d(M(F_x))$ of discrete lagrangian lattices in a symplectic Tate space $M(F_x)$ and a certain μ_2-gerb $\widetilde{\mathcal{L}}_d(M(F_x))$ over it. The complex $\mathrm{Aut}_{G,H}$ on $\mathrm{Bun}_{G,H}$ comes from the stack $\widetilde{\mathcal{L}}_d(M(F_x))$ simply as the inverse image. The key observation is that it is much easier to prove the Hecke property of $\mathrm{Aut}_{G,H}$ on $\widetilde{\mathcal{L}}_d(M(F_x))$, because over the latter stack it is perverse.

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2. Main results

2.1. Notation

Let k be an algebraically closed field of characteristic $p > 2$ (except in Section 3, where $k = \mathbb{F}_q$). All the schemes (or stacks) we consider are defined over k.

Let X be a smooth projective connected curve. Set $F = k(X)$. For a closed point $x \in X$ write F_x for the completion of F at x, let $\mathcal{O}_x \subset F_x$ be the ring of integers. Let $D_x = \mathrm{Spec}\, \mathcal{O}_x$ denote the disc around x. Write Ω for the canonical line bundle on X.

Fix a prime $\ell \neq p$. For a k-stack S locally of finite type write $\mathrm{D}(S)$ for the category introduced in ([17], Remark 3.21) and denoted by $\mathrm{D}_c(S, \bar{\mathbb{Q}}_\ell)$ in *loc. cit*. It should be thought of as the unbounded derived category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on S. For $* = +, -, b$ we