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FOLIATED STRUCTURE OF THE KURANISHI SPACE AND ISOMORPHISMS OF DEFORMATION FAMILIES OF COMPACT COMPLEX MANIFOLDS

BY Laurent MEERSSEMAN

Abstract. – Consider the following uniformization problem. Take two holomorphic (parametrized by some analytic set defined on a neighborhood of 0 in $\mathbb{C}^p$, for some $p > 0$) or differentiable (parametrized by an open neighborhood of 0 in $\mathbb{R}^p$, for some $p > 0$) deformation families of compact complex manifolds. Assume they are pointwise isomorphic, that is for each point t of the parameter space, the fiber over t of the first family is biholomorphic to the fiber over t of the second family. Then, under which conditions are the two families locally isomorphic at 0? In this article, we give a sufficient condition in the case of holomorphic families. We show then that, surprisingly, this condition is not sufficient in the case of differentiable families. We also describe different types of counterexamples and give some elements of classification of the counterexamples. These results rely on a geometric study of the Kuranishi space of a compact complex manifold.

Résumé. – Considérons le problème d'uniformisation suivant. Prenons deux familles de déformation holomorphes (paramétrées par un ensemble analytique défini dans un voisinage de 0 dans $\mathbb{C}^p$ pour $p > 0$) ou différentiables (paramétrées par un voisinage de 0 dans $\mathbb{R}^p$ pour $p > 0$) de variétés compactes complexes. Supposons-les ponctuellement isomorphes, c'est-à-dire que, pour tout point t de l'espace des paramètres, la fibre en t de la première famille est biholomorphe à la fibre en t de la deuxième famille. Sous quelle(s) condition(s) les deux familles sont-elles localement isomorphes en 0? Dans cet article, nous donnons une condition suffisante dans le cas de familles holomorphes. Nous montrons ensuite que, de façon surprenante, la condition n'est pas suffisante dans le cas des familles différentiables. Nous décrivons également plusieurs types de contre-exemples et donnons quelques éléments de classifications de ces contre-exemples. Ces résultats reposent sur une étude géométrique de l'espace de Kuranishi d'une variété compacte complexe.

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Introduction

This article deals with the problem of giving a useful criterion to ensure that two holomorphic (respectively differentiable) deformation families are isomorphic as families. This takes the form of the following uniformization problem. For $i = 1, 2$, let

$$\pi_i : \mathcal{X}_i \to U \quad \text{respectively} \quad \pi_i : \mathcal{X}_i \to V$$

be two holomorphic (respectively differentiable) families of compact complex manifolds parametrized by some analytic set U defined on a neighborhood of 0 in $\mathbb{C}^p$, for some $p > 0$ (respectively an open neighborhood V of 0 in $\mathbb{R}^p$, for some $p > 0$). Assume that they are *pointwise isomorphic*, that is, for all $t \in U$ (respectively $t \in V$), the fiber $X_1(t) = \pi_1^{-1}(\{t\})$ is biholomorphic to the fiber $X_2(t) = \pi_2^{-1}(\{t\})$. Then the question is

Question 1. – *Under which hypotheses are the families $\mathcal{X}_1$ and $\mathcal{X}_2$ locally isomorphic at 0?*

By *locally isomorphic*, we mean that there exist an open neighborhood W of $0 \in U$ (respectively in V), and a biholomorphism Φ (respectively a CR-isomorphism) between $\mathcal{X}_1(W) = \pi_1^{-1}(W)$ and $\mathcal{X}_2(W) = \pi_2^{-1}(W)$ such that the following diagram is commutative

$$\begin{tikzcd} \mathcal{X}_1(W) \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & \mathcal{X}_2(W) \arrow[d, "\pi_2"] \\ W \arrow[r, "\text{Identity}"'] & W. \end{tikzcd}$$

We are also interested in the following broader problem.

Question 2. – *Under which hypotheses are the families $\mathcal{X}_1$ and $\mathcal{X}_2$ locally equivalent at 0?*

By *locally equivalent*, we mean that there exist open neighborhoods W_1 and W_2 of $0 \in U$ (respectively in V), a biholomorphism ϕ between W_1 and W_2 (respectively a diffeomorphism) and a biholomorphism Φ (respectively a CR-isomorphism) between $\mathcal{X}_1(W_1) = \pi_1^{-1}(W_1)$ and $\mathcal{X}_2(W_2) = \pi_2^{-1}(W_2)$ such that the following diagram is commutative

$$\begin{tikzcd} \mathcal{X}_1(W_1) \arrow[r, "\Phi"] \arrow[d, "\pi_1"'] & \mathcal{X}_2(W_2) \arrow[d, "\pi_2"] \\ W_1 \arrow[r, "\phi"'] & W_2. \end{tikzcd}$$

In other words, $\mathcal{X}_1$ and $\mathcal{X}_2$ are locally equivalent at 0 if $\phi^*\mathcal{X}_2$ and $\mathcal{X}_2$ are locally isomorphic for some local biholomorphism ϕ of U (respectively V) fixing 0.

Fix a family $\mathcal{X}_1$. In this paper, we shall say that this family has the *local isomorphism property* (at 0), respectively has the *local equivalence property* (at 0) if every other family $\mathcal{X}_2$ which is pointwise isomorphic to it is locally isomorphic to it (at 0), respectively locally equivalent (at 0).

It is known since Kodaira-Spencer (see [8], [15] and Section 5.1 of this article) that there exist pointwise isomorphic families of primary Hopf surfaces which are not locally isomorphic, both in the differentiable and the holomorphic cases.

On the other hand, the classical Fischer-Grauert Theorem [2], can be restated as follows. Let X be a compact complex manifold and U be an open neighborhood of 0 in some $\mathbb{C}^p$. Then every trivial family $X \times U$ has the local isomorphism property. This works also for differentiable families. Indeed the proof given in [2] for holomorphic families is easily adapted to the differentiable case, the core of the proof being Theorem 6.2 of [7] which is valid both for differentiable and holomorphic families.

Moreover, J. Wehler proved in [15] that, over a smooth base, holomorphic families of compact complex tori (in any dimension) as well as holomorphic families of compact manifolds with negatively curved holomorphic curvature (this implies that they are Kobayashi hyperbolic) have the local isomorphism property. This time, the proofs do not adapt to the differentiable case.

Observe that in the two previous cases, the function $h^0(t)$, that is the dimension of the cohomology group $H^0(X_t, \Theta_t)$ (where Θ_t is the sheaf of holomorphic vector fields along X_t) is constant for all $t \in U$. It is equal to n in the case of n-dimensional tori, and to 0 in the case of negatively curved manifolds. Wehler asks in the introduction of [15] if this condition is sufficient to have the local isomorphism property.

In this paper, we prove that this is the case, even over a singular base. Namely,

Theorem 3. – *If U is reduced and if the function h^0 is constant for all the fibers of a holomorphic deformation family $\pi : \mathcal{X} \to U$, then $\mathcal{X}$ has the local isomorphism property.*

We then give examples (both in the differentiable and holomorphic setting) of families not having the local equivalence property, as well as of locally equivalent but not locally isomorphic families. We classify these counterexamples into two types, and we give in Theorem 4 a complete classification of 1-dimensional families of type II not having the local equivalence property.

Coming back to the search for a criterion, we prove that, surprisingly, things are completely different in the differentiable case.

Theorem 5. – *There exist differentiable families of 2-dimensional compact complex tori parametrized by an interval that are pointwise isomorphic but not locally isomorphic at a given point.*

To solve the uniformization problems stated above, we first study the geometry of the Kuranishi space K of a compact complex manifold X. We show in Theorem 1 that it has a natural holomorphic foliated structure: two points belonging to the same leaf correspond to biholomorphic complex structures. More precisely, K admits an analytic stratification such that each piece of the induced decomposition (see Section III for more details) is foliated. The leaves are complex manifolds, but the transverse structure of the foliation may be singular (this happens when the Kuranishi space is singular).

The foliation may be of dimension or of codimension zero. In Theorem 2, we prove that there exists leaves of positive dimension (that is the foliation has positive dimension on some piece of the decomposition) if and only if the function h^0 is not constant in the neighbourhood of 0 in K (0 representing the central point X). In particular, in many examples, the foliation is a foliation by points.

Although Theorems 3, 4 and 5 on the uniformization problems are not strictly speaking a consequence of Theorems 1 and 2 on the foliated structure of K, the geometric picture of K they bring played an essential role in the understanding and resolution of the problem. The key ingredients to prove the theorems are some trivial remarks on diffeomorphisms of the Kuranishi space (see Section II), the Fischer-Grauert Theorem [2], a result of Namba [12] and a fundamental proposition proved by Kuranishi in [10].

We end the article with a discussion of the relationship between the uniformization problem and the universality of the Kuranishi space.

1. Notations and background

Let X be a compact complex manifold. We denote by X^{diff} the underlying smooth manifold and by J the corresponding complex operator.

A *(holomorphic) deformation family* of X is a proper and flat projection π from a $\mathbb{C}$-analytic space $\mathcal{X}$ (possibly non-reduced) over an analytic set U defined on an open neighborhood of 0 in some $\mathbb{C}^p$. A *differentiable deformation family* (see [7]) is a smooth submersion π from a smooth manifold $\mathcal{X}$ endowed with a Levi-flat integrable almost CR-structure over an open neighborhood V of 0 in some $\mathbb{R}^p$, whose level sets are tangent to the CR-structure. If the almost CR-structure on $\mathcal{X}$ is not supposed to be integrable, one has a *differentiable deformation family of almost-complex structures* of X.

In the three cases, the central fiber $X_0 = \pi^{-1}(\{0\})$ is assumed to be biholomorphic to X. Sometimes, we consider *marked* deformation families of X, that is we fix a precise holomorphic identification $i : X \to X_0$.

Let us recall some features of the construction of the Kuranishi space following [9]. The set of almost complex structures close to J is identified with a neighbourhood A of 0 in the space A^1 of $(0,1)$-forms on X with values in $T^{1,0}$. In particular, 0 represents the complex structure J we started with (here and in the sequel, the topology used on spaces of sections of a vector bundle over X is induced by some Sobolev norms, see [9] for more details).

Put a hermitian metric h on X. Then we have a $\bar{\partial}$-operator on A^p, the space of $(0,p)$-forms with values in $T^{1,0}$, a formal adjoint operator δ with respect to the induced hermitian product on A^p and a Laplace-like operator $\square$. Let SH^1 denote the set of δ-closed forms in A^1. Kuranishi proves in [9]

Proposition K1. – *For A small enough, there exist a neighborhood B of 0 in SH^1 and an application Ξ from A to B mapping an almost complex structure α onto a δ-closed representant $\Xi(\alpha)$. Moreover, if $\alpha(t)$ is a smooth family of almost complex structures, then so is $\Xi(\alpha(t))$.*

By representant, we mean that $\Xi(\alpha)$ and α induce isomorphic almost complex structures on X^{diff}.

Then Kuranishi defines a holomorphic map G from A^1 to A^1 and proves that it is a biholomorphism between a special subset of A^1 (containing in particular all integrable