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Derived invariance of the number of holomorphic 1*-forms and vector fields*

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DERIVED INVARIANCE OF THE NUMBER OF HOLOMORPHIC 1-FORMS AND VECTOR FIELDS

BY MIHNEA POPA AND CHRISTIAN SCHNELL

ABSTRACT. – We prove that smooth projective varieties with equivalent derived categories have isogenous Picard varieties. In particular their irregularity and number of independent vector fields are the same. This implies that all Hodge numbers are the same for arbitrary derived equivalent threefolds, as well as other consequences of derived equivalence based on numerical criteria.

RÉSUMÉ. – Nous montrons que deux variétés projectives lisses dont les catégories dérivées sont équivalentes, ont des variétés de Picard isogènes. En particulier, elles ont la même irrégularité et le même nombre de champs de vecteurs indépendants. On en déduit l'invariance des nombres de Hodge par l'équivalence dérivée pour les variétés de dimension trois, ainsi que quelques autres conséquences numériques.

1. Introduction

Given a smooth projective variety X, we denote by $\mathbf{D}(X)$ the bounded derived category of coherent sheaves $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}(X))$. All varieties we consider below are over the complex numbers. A result of Rouquier, [17] Théorème 4.18, asserts that if X and Y are smooth projective varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ (as linear triangulated categories), then there is an isomorphism of algebraic groups

$$\mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \simeq \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y).$$

We refine this by showing that each of the two factors is almost invariant under derived equivalence. According to Chevalley's theorem $\mathrm{Aut}^0(X)$, the connected component of the identity in $\mathrm{Aut}(X)$, has a unique maximal connected affine subgroup $\mathrm{Aff}(\mathrm{Aut}^0(X))$, and the quotient $\mathrm{Alb}(\mathrm{Aut}^0(X))$ by this subgroup is an abelian variety, the Albanese variety of $\mathrm{Aut}^0(X)$. The affine parts $\mathrm{Aff}(\mathrm{Aut}^0(X))$ and $\mathrm{Aff}(\mathrm{Aut}^0(Y))$, being also the affine parts of the two sides in the isomorphism above, are isomorphic. The main result of the paper is

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THEOREM A. – *Let X and Y be smooth projective varieties such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then*
(1) $\mathrm{Pic}^0(X)$ *and* $\mathrm{Pic}^0(Y)$ *are isogenous; equivalently,* $\mathrm{Alb}(\mathrm{Aut}^0(X))$ *and* $\mathrm{Alb}(\mathrm{Aut}^0(Y))$ *are isogenous.*
(2) $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(Y)$ *unless X and Y are étale locally trivial fibrations over isogenous positive dimensional abelian varieties (hence $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 0$).*

The key content is part (1), while (2) simply says that $\mathrm{Aut}^0(X)$ and $\mathrm{Aut}^0(Y)$ are affine unless the geometric condition stated there holds (hence the presence of abelian varieties is essentially the only reason for the failure of the derived invariance of the Picard variety).

COROLLARY B. – *If $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then*

$$h^0(X, \Omega_X^1) = h^0(Y, \Omega_Y^1) \quad \textit{and} \quad h^0(X, T_X) = h^0(Y, T_Y).$$

The Hodge number $h^{1,0}(X) = h^0(X, \Omega_X^1)$ is also called the *irregularity* $q(X)$, the dimension of the Picard and Albanese varieties of X. The invariance of the sum $h^0(X, \Omega_X^1) + h^0(X, T_X)$ was already known, and is a special case of the derived invariance of the Hochschild *co*homology of X ([15], [7]; cf. also [9] §6.1). Alternatively, it follows from Rouquier's result above. Corollary B, together with the derived invariance of Hochschild homology (cf. *loc. cit.*), implies the invariance of all Hodge numbers for all derived equivalent threefolds. This was expected to hold as suggested by work of Kontsevich [12] (cf. also [1]).

COROLLARY C. – *Let X and Y be smooth projective threefolds with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then*

$$h^{p,q}(X) = h^{p,q}(Y)$$

for all p and q.

Proof. – The fact that the Hochschild homology of X and Y is the same gives

$$\sum_{p-q=i} h^{p,q}(X) = \sum_{p-q=i} h^{p,q}(Y) \tag{1.1}$$

for all i. A straightforward calculation shows that this implies the invariance of all Hodge numbers except for $h^{1,0}$ and $h^{2,1}$, about which we only get that $h^{1,0} + h^{2,1}$ is invariant. We then apply Corollary B. □

Corollary C is already known (in arbitrary dimension) for varieties of general type: for these derived equivalence implies K-equivalence by a result of Kawamata [11], while K-equivalent varieties have the same Hodge numbers according to Batyrev [2] and Kontsevich, Denef-Loeser [8]. It is also well known for Calabi-Yau threefolds; more generally it follows easily for threefolds with numerically trivial canonical bundle (condition which is preserved by derived equivalence, see [11] Theorem 1.4). Indeed, since for threefolds Hirzebruch-Riemann-Roch gives $\chi(\omega_X) = \frac{1}{24} c_1(X) c_2(X)$, in this case $\chi(\omega_X) = 0$, hence $h^{1,0}(X)$ can be expressed in terms of Hodge numbers that are known to be derived invariant as above. Finally, in general the invariance of $h^{1,0}$ would follow automatically if X and Y were birational, but derived equivalence does not necessarily imply birationality.

The proof of Theorem A in §3 uses a number of standard facts in the study of derived equivalences: invariance results and techniques due to Orlov and Rouquier, Mukai's description of semi-homogeneous vector bundles, and Orlov's fundamental characterization of derived equivalences. The main new ingredients are results of Nishi-Matsumura and Brion on actions of non-affine algebraic groups (see §2). Further numerical applications of Corollary B to fourfolds or abelian varieties are provided in Remark 3.3.

Finally, the case of abelian varieties shows the existence of Fourier-Mukai partners with non-isomorphic Picard varieties. We expect however the following stronger form of Theorem A(1).

Conjecture. – *If* $\mathbf{D}(X) \simeq \mathbf{D}(Y)$*, then* $\mathbf{D}(\mathrm{Pic}^0(X)) \simeq \mathbf{D}(\mathrm{Pic}^0(Y))$.

Derived equivalent curves must be isomorphic (see e.g. [9], Corollary 5.46), while in the case of surfaces the conjecture is checked in the upcoming thesis of Pham [16] using the present methods and the classification of Fourier-Mukai equivalences in [3] and [11].

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2. Actions of non-affine algebraic groups

Most of the results in this section can be found in Brion [5], [4], or are at least implicit there. Let G be a connected algebraic group. According to Chevalley's theorem (see e.g. [5] p. 1), G has a unique maximal connected affine subgroup $\mathrm{Aff}(G)$, and the quotient $G/\mathrm{Aff}(G)$ is an abelian variety. We denote this abelian variety by $\mathrm{Alb}(G)$, since the map $G \to \mathrm{Alb}(G)$ is the Albanese map of G, i.e. the universal morphism to an abelian variety (see [19]). Thus $G \to \mathrm{Alb}(G)$ is a homogeneous fiber bundle with fiber $\mathrm{Aff}(G)$.

Lemma 2.1 ([4], Lemma 2.2). – *The map* $G \to \mathrm{Alb}(G)$ *is locally trivial in the Zariski topology.*

Now let X be a smooth projective variety. We abbreviate $G_X := \mathrm{Aut}^0(X)$, and let $a(X)$ be the dimension of the abelian variety $\mathrm{Alb}(G_X)$. The group G_X naturally acts on the Albanese variety $\mathrm{Alb}(X)$ as well (see [5] §3).

Lemma 2.2. – *The action of* G_X *on* $\mathrm{Alb}(X)$ *induces a map of abelian varieties*

$$\mathrm{Alb}(G_X) \to \mathrm{Alb}(X),$$

whose image is contained in the Albanese image $\mathrm{alb}_X(X)$*. More precisely, the composition* $G_X \to \mathrm{Alb}(X)$ *is given by the formula* $g \mapsto \mathrm{alb}_X(gx_0 - x_0)$*, where* $x_0 \in X$ *is an arbitrary point.*

Proof. – From $G_X \times X \to X$, we obtain a map of abelian varieties

$$\mathrm{Alb}(G_X) \times \mathrm{Alb}(X) \simeq \mathrm{Alb}(G_X \times X) \to \mathrm{Alb}(X).$$

It is clearly the identity on $\mathrm{Alb}(X)$, and therefore given by a map of abelian varieties $\mathrm{Alb}(G_X) \to \mathrm{Alb}(X)$. To see what it is, fix a base-point $x_0 \in X$, and write the Albanese map of X in the form $X \to \mathrm{Alb}(X)$, $x \mapsto \mathrm{alb}_X(x - x_0)$. Let $g \in G_X$ be an automorphism of X.

By the universal property of $\mathrm{Alb}(X)$, it induces an automorphism $\tilde{g} \in \mathrm{Aut}^0(\mathrm{Alb}(X))$, making the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ \mathrm{Alb}(X) \arrow[r, "\tilde{g}"] & \mathrm{Alb}(X) \end{tikzcd}$$

commute; in other words, $\tilde{g}\big(\mathrm{alb}_X(x - x_0)\big) = \mathrm{alb}_X(gx - x_0)$. Any such automorphism is translation by an element of $\mathrm{Alb}(X)$, and the formula shows that this element has to be $\mathrm{alb}_X(gx_0 - x_0)$. It follows that the map $G_X \to \mathrm{Alb}(X)$ is given by $g \mapsto \mathrm{alb}_X(gx_0 - x_0)$. By Chevalley's theorem, it factors through $\mathrm{Alb}(G_X)$. □

A crucial fact is the following theorem of Nishi and Matsumura (cf. also [5]).

THEOREM 2.3 ([13], Theorem 2). – *The map $\mathrm{Alb}(G_X) \to \mathrm{Alb}(X)$ has finite kernel. More generally, any connected algebraic group G of automorphisms of X acts on $\mathrm{Alb}(X)$ by translations, and the kernel of the induced homomorphism $G \to \mathrm{Alb}(X)$ is affine.*

Consequently, the image of $\mathrm{Alb}(G_X)$ is an abelian subvariety of $\mathrm{Alb}(X)$ of dimension $a(X)$. This implies the inequality $a(X) \le q(X)$. Brion observed that X can always be fibered over an abelian variety which is a quotient of $\mathrm{Alb}(G_X)$ of the same dimension $a(X)$; the following proof is taken from [5], p. 2 and §3, and is included for later use of its ingredients.

LEMMA 2.4. – *There is an affine subgroup $\mathrm{Aff}(G_X) \subseteq H \subseteq G_X$ with $H/\mathrm{Aff}(G_X)$ finite, such that X admits a G_X-equivariant map $\psi\colon X \to G_X/H$. Consequently, X is isomorphic to the equivariant fiber bundle $G_X \times^H Z$ with fiber $Z = \psi^{-1}(0)$.*

Proof. – By the Poincaré complete reducibility theorem, the map $\mathrm{Alb}(G_X) \to \mathrm{Alb}(X)$ splits up to isogeny. This means that we can find a subgroup H containing $\mathrm{Aff}(G_X)$, such that there is a surjective map $\mathrm{Alb}(X) \to G_X/H$ with $\mathrm{Alb}(G_X) \to G_X/H$ an isogeny. It follows that $H/\mathrm{Aff}(G_X)$ is finite, and hence that H is an affine subgroup of G_X whose identity component is $\mathrm{Aff}(G_X)$. Let $\psi\colon X \to G_X/H$ be the resulting map; it is equivariant by construction. Since G_X acts transitively on G_X/H, we conclude that ψ is an equivariant fiber bundle over G_X/H with fiber $Z = \psi^{-1}(0)$, and therefore isomorphic to

$$G_X \times^H Z = (G_X \times Z)/H,$$

where H acts on the product by $(g, z) \cdot h = (g \cdot h, h^{-1} \cdot z)$. □

Note that the group H naturally acts on Z; the proof shows that we obtain X from the principal H-bundle $G_X \to G_X/H$ by replacing the fiber H by Z (see [18], §3.2). While $X \to G_X/H$ is not necessarily locally trivial, it is so in the étale topology.

LEMMA 2.5. – *Both $G_X \to G_X/H$ and $X \to G_X/H$ are étale locally trivial.*