ISSN 0012-9593

*quatrième série - tome 44 fascicule 4 juillet-août 2011*

# ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Chloé PERIN

*Elementary embeddings in torsion-free hyperbolic groups*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

*Ann. Scient. Éc. Norm. Sup.*
4e série, t. 44, 2011, p. 631 à 681

# ELEMENTARY EMBEDDINGS IN TORSION-FREE HYPERBOLIC GROUPS

BY CHLOÉ PERIN

ABSTRACT. – We describe first-order logic elementary embeddings in a torsion-free hyperbolic group in terms of Sela's hyperbolic towers. Thus, if $H$ embeds elementarily in a torsion free hyperbolic group $\Gamma$, we show that the group $\Gamma$ can be obtained by successive amalgamations of groups of surfaces with boundary to a free product of $H$ with some free group and groups of closed surfaces. This gives as a corollary that an elementary subgroup of a finitely generated free group is a free factor. We also consider the special case where $\Gamma$ is the fundamental groups of a closed hyperbolic surface. The techniques used to obtain this description are mostly geometric, as for example actions on real or simplicial trees, or the existence of JSJ splittings. We also rely on the existence of factor sets, a result used in the construction of Makanin-Razborov diagrams for torsion-free hyperbolic groups.

RÉSUMÉ. – On obtient une description des plongements élémentaires (au sens de la logique du premier ordre) dans un groupe hyperbolique sans torsion, en termes de tours hyperboliques de Sela. Ainsi, si $H$ est plongé élémentairement dans un groupe hyperbolique sans torsion $\Gamma$, on peut obtenir $\Gamma$ en amalgamant successivement des groupes de surfaces à bord à un produit libre de $H$ avec des groupes libres et des groupes de surfaces fermées. Ceci permet en corollaire de montrer qu'un sous-groupe plongé élémentairement dans un groupe libre de type fini est un facteur libre. On considère également le cas où $\Gamma$ est le groupe fondamental d'une surface hyperbolique fermée. Les techniques utilisées pour obtenir cette description sont essentiellement géométriques : actions sur des arbres réels ou simpliciaux, décompositions JSJ. On s'appuie également sur des résultats d'existence d'ensembles de factorisation utilisés dans la construction de diagrammes de Makanin-Razborov pour un groupe hyperbolique sans torsion.

## 1. Introduction

Tarski's problem asks whether any two finitely generated non abelian free groups are elementary equivalent, namely whether they satisfy the same closed first-order formulas over the language of groups. In a series of articles starting with [26] and culminating in [31], Sela answered this question positively (see also the work of Kharlampovich and Myasnikov [15]).

Sela's approach is very geometric, and thus enables him in [33] to tackle problems on the first-order theory of torsion-free hyperbolic groups as well.

Another notion of interest in first-order theory is that of an elementary subgroup, or elementary embedding. Informally, a subgroup $H$ of a group $G$ is elementary if any tuple of elements of $H$ satisfies the same first-order properties in $H$ and $G$ (see Section 2 for a definition).

Denote by $\mathbb{F}_n$ the free group on $n$ generators. To prove that finitely generated free groups of rank at least 2 all have the same elementary theory, Sela shows in fact the following stronger result:

Theorem 1.1 ([31, Theorem 4]). – *Suppose* $2 \leq k \leq n$. *The standard embedding* $\mathbb{F}_k \hookrightarrow \mathbb{F}_n$ *is elementary.*

In this paper, we use some of Sela's techniques to give a description of elementary subgroups of torsion-free hyperbolic groups. Our main result is

Theorem 1.2. – *Let $G$ be a torsion-free hyperbolic group. Let $H \hookrightarrow G$ be an elementary embedding. Then $G$ is a hyperbolic tower based on $H$.*

Hyperbolic towers are groups built by successive addition of hyperbolic floors, which can be described as follows. A group $G$ has a hyperbolic floor structure over a subgroup $G'$ if it is the fundamental group of a complex $X$ built by gluing some surfaces $\Sigma_1, \dots, \Sigma_m$ along their boundary to the disjoint union of complexes $X'_1, \dots, X'_l$, such that $G'$ is the fundamental group of a subcomplex $X'$ which contains the subcomplexes $X'_i$, and whose intersection with each surface $\Sigma_j$ is contractible (in particular $G'$ is isomorphic to the free product of the groups $\pi_1(X'_i)$). We require moreover the existence of a retraction $r : G \to G'$ which sends the fundamental groups $\pi_1(\Sigma_j)$ to non abelian images.

A hyperbolic tower over $H$ is built by successively adding hyperbolic floors to a "ground floor" which is the free product of $H$, closed surface groups and a free group (see Figure 1). For a precise definition, see Definition 5.4.

Hyperbolic towers are defined by Sela in [26], and enable him to give in [31] a description of finitely generated groups which are elementary equivalent to free groups. This structure is also used in [33] to give a classification of elementary equivalence classes of torsion-free hyperbolic groups. In fact, Proposition 7.6 of [33] shows that some particular subgroups of a torsion-free hyperbolic group $\Gamma$ (its "elementary cores"), over which $\Gamma$ has a structure of hyperbolic tower, are elementarily embedded in $\Gamma$. According to Sela, the specific properties of these subgroups (apart from the structure of hyperbolic tower $\Gamma$ admits over them) are not used in the proof, which in fact shows that the converse of Theorem 1.2 holds [32].

In the particular case where $G$ is a free group, we show that Theorem 1.2 implies the converse of Theorem 1.1, so that we have

Theorem 1.3. – *Let $H$ be a proper subgroup of $\mathbb{F}_n$. The embedding of $H$ in $\mathbb{F}_n$ is elementary if and only if $H$ is a non abelian free factor of $\mathbb{F}_n$.*

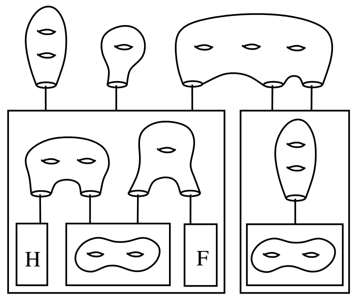


FIGURE 1. A hyperbolic tower over $H$.

Let us consider the case where $G$ is the fundamental group of a closed hyperbolic surface. In the example represented on Figure 2, the element corresponding to $\gamma$ can be written as a product of two commutators in $\pi_1(\Sigma)$, though not in $\pi_1(\Sigma_1)$. This is a property which can be expressed by a first-order formula, and that $\gamma$ satisfies on $\pi_1(\Sigma)$, though not in $\pi_1(\Sigma_1)$: the embedding of $\pi_1(\Sigma_1)$ in $\pi_1(\Sigma)$ is not elementary.

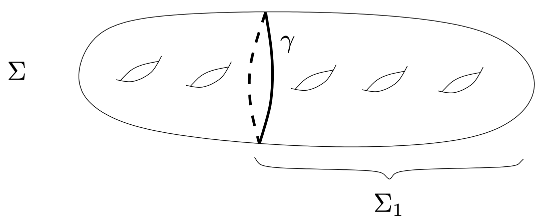


FIGURE 2. $\pi_1(\Sigma_1)$ is not elementarily embedded in $\pi_1(\Sigma)$.

This example seems to suggest that an elementary subgroup of the fundamental group of a hyperbolic surface cannot be too big. In fact, we show that applying Theorem 1.2 gives

THEOREM 1.4. – *Let $S$ be the fundamental group of a closed hyperbolic surface $\Sigma$. Suppose $H$ is a proper subgroup of $S$ whose embedding in $S$ is elementary.*

*Then $H$ is a non abelian free factor of the fundamental group of a connected subsurface $\Sigma_0$ of $\Sigma$ whose complement in $\Sigma$ is connected, and which satisfies $|\chi(\Sigma_0)| \leq |\chi(\Sigma)|/2$ (with equality if and only if $\Sigma$ is the double of $\Sigma_0$).*

To prove Theorem 1.2, we need to uncover a decreasing sequence $G = G_0 \geq G_1 \geq G_2 \geq \cdots$ of subgroups of $G$ which contain $H$, each of the subgroups $G_i$ forming a floor of a hyperbolic tower above the next subgroup $G_{i+1}$.

Let us give an idea of the proof in the special case where $G$ is freely indecomposable relative to $H$. For each decomposition of $G$ as an amalgamated product or an HNN extension above a cyclic group for which $H$ lies in one of the factors, we consider the Dehn twists which fix the factor containing $H$. We then define the modular group $\mathrm{Mod}_H(G)$ as the group of automorphisms of $G$ generated by all such Dehn twists.

The shortening argument of Rips and Sela gives the following result (this is a special case of one of the two key results for the construction of restricted Makanin-Razborov diagrams in [33]):

THEOREM 1.5. – *Let $G$ be a hyperbolic group which is freely indecomposable with respect to a non abelian subgroup $H$. There exists a finite set of proper quotients of $G$, such that for any non injective morphism $h : G \to G$ which fixes $H$, there is an element $\sigma$ of $\mathrm{Mod}_H(G)$ such that $h \circ \sigma$ factors through one of the corresponding quotient maps.*

The key idea in the proof of Theorem 1.2 is to try and express this factorization result by a first-order formula that $H$ satisfies.

Suppose now that $G$ does not admit any non trivial splitting over a cyclic subgroup in which $H$ is elliptic, so that the modular group $\mathrm{Mod}_H(G)$ is trivial. Theorem 1.5 then implies that any non injective morphism $G \to G$ fixing $H$ factors through one of a finite set of quotients. Note that if $H$ is a proper subgroup of $G$, a morphism $G \to H$ fixing $H$ is in particular a non injective morphism $G \to G$ fixing $H$. Let $U$ be a finite set containing one non trivial element in the kernel of each one of the quotient maps given by Theorem 1.5. If $H$ has a finite generating set $\{h_1, \dots, h_n\}$, we can express by a first-order formula $\phi(h_1, \dots, h_n)$ satisfied by $H$ the following statement: "any morphism $G \to H$ fixing the elements $h_i$ sends one of the elements of $U$ to a trivial image". Since $H$ is elementary in $G$, the formula $\phi(h_1, \dots, h_n)$ is satisfied by $G$, and its interpretation on $G$ is the following statement: "any morphism $G \to G$ fixing each of the elements $h_i$ sends one of the elements of $U$ to a trivial image". This is obviously not true of the identity map $G \to G$, so we get a contradiction. Thus if $H$ is elementary in $G$, there is at least one non trivial splitting of $G$ over a cyclic group in which $H$ is elliptic.

In general, we do not know a priori that $H$ is finitely generated (though this is a consequence of Theorem 1.2), and thus we cannot express the fact that a morphism fixes $H$ in a first-order formula. We can in fact generalize Theorem 1.5 to show that any non injective morphism $G \to G$ which fixes a large enough finitely generated subgroup $H_0$ of $H$ factors after precomposition by a modular automorphism through one of finitely many quotients. We also need to show that a morphism $G \to H$ which fixes a large enough finitely generated subgroup $H_0$ of $H$ cannot be injective. For this, we prove that if $G$ is freely indecomposable with respect to a subgroup $H$, then it is freely indecomposable with respect to a finitely generated subgroup $H_0$ of $H$, and we combine this with the following result:

THEOREM 1.6. – *Let $G$ be a torsion-free hyperbolic group. Let $H_0$ be a non cyclic subgroup of $G$ relative to which $G$ is freely indecomposable. Then $G$ is co-Hopf relative to $H_0$, that is, if a morphism $\phi : G \to G$ is injective and fixes $H_0$ then it is an isomorphism.*

A more difficult problem is that of expressing precomposition by a modular automorphism even when the modular group is not trivial. To overcome this, we consider the cyclic JSJ decomposition of $G$ relative to $H$: it is a decomposition of $G$ as a graph of group with cyclic edge stabilizers in which $H$ is elliptic, and which is maximal in some sense among all such decompositions of this type that $G$ admits. Some of the vertex groups of this decomposition are fundamental groups of surfaces with boundary. The point is that the modular group preserves some of the structure of the JSJ. The JSJ decomposition gives some combinatorial