ISSN 0012-9593

quatrième série - tome 44 fascicule 4 juillet-août 2011

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4e série, t. 44, 2011, p. 683 à 721

HARMONIC MEASURES VERSUS QUASICONFORMAL MEASURES FOR HYPERBOLIC GROUPS

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Abstract. – We establish a dimension formula for the harmonic measure of a finitely supported and symmetric random walk on a hyperbolic group. We also characterize random walks for which this dimension is maximal. Our approach is based on the Green metric, a metric which provides a geometric point of view on random walks and, in particular, which allows us to interpret harmonic measures as quasiconformal measures on the boundary of the group.

Résumé. – On établit une formule de la dimension de la mesure harmonique d'une marche aléatoire de loi de support fini et symétrique sur un groupe hyperbolique. On caractérise aussi les lois pour lesquelles la dimension est maximale. Notre approche repose sur la distance de Green, une distance qui permet de développer un point de vue géométrique sur les marches aléatoires et, en particulier, d'interpréter les mesures harmoniques comme des mesures quasiconformes.

1. Introduction

Let Γ be a non-elementary word hyperbolic group. There are two main constructions of measures on the boundary $\partial\Gamma$ of Γ: quasiconformal measures and harmonic measures. Let us recall these constructions.

Given a cocompact properly discontinuous action of Γ by isometries on a pointed proper geodesic metric space (X, w, d), the Patterson-Sullivan procedure consists in taking weak limits of

$$\frac{1}{\sum_{\gamma\in\Gamma} e^{-sd(w,\gamma(w))}} \sum_{\gamma\in\Gamma} e^{-sd(w,\gamma(w))}\delta_{\gamma(w)}$$

as s decreases to the logarithmic volume growth

$$v \stackrel{\text{def}}{=} \limsup_{R\to\infty} \frac{1}{R} \log |B(w,R)\cap\Gamma(w)|\,.$$

Patterson-Sullivan measures are quasiconformal measures and Hausdorff measures of ∂X when endowed with a visual metric.

Given a probability measure μ on Γ, the random walk $(Z_n)_n$ starting from the neutral element e associated with μ is defined by

$$Z_0 = e\,;\; Z_{n+1} = Z_n \cdot X_{n+1}\,,$$

where (X_n) is a sequence of independent and identically distributed random variables of law μ. Under some mild assumptions on μ, the walk $(Z_n)_n$ almost surely converges to a point $Z_\infty \in \partial\Gamma$. The law of Z_∞ is by definition the harmonic measure ν.

The purpose of this work is to investigate the interplay between these two classes of measures and take advantage of this interplay to derive information on the geometry of harmonic measures.

We show that, for a general hyperbolic group, the Hausdorff dimension of the harmonic measure satisfies a 'dimension-entropy-rate of escape' formula and we characterize those harmonic measures of maximal dimension. These results are intimately connected with a sharp control of the deviation of sample paths of the random walk with respect to geodesics.

Not surprisingly, the starting point is A. Ancona's theorem stating that the Green function is almost multiplicative along geodesics [2, Thm 6.1].

The usual tool for studying selfsimilar measures is to replace the action of the group by a linear-in-time action of a dynamical system and then to apply the thermodynamic formalism to it. Indeed it is tempting to interpret Ancona's estimate as a kind of Gibbs property of the harmonic measure. This strategy was successfully used in special cases: for free groups and Fuchsian groups, a Markov-map F_Γ has been introduced on the boundary which is orbit-equivalent to Γ [12, 35]. For discrete subgroups of isometries of a Cartan-Hadamard manifold, one may work with the geodesic flow [24, 26, 33, 34]. Both these methods seem difficult to implement for general hyperbolic groups. On the one hand, it is not obvious how to associate a Markov map with a general hyperbolic group; in particular, not enough is known on the automatic structure of a hyperbolic group to derive a coding of its boundary. On the other hand, the construction of the geodesic flow for general hyperbolic spaces is delicate and its mixing properties do not seem strong enough to apply the thermodynamic formalism.

Our approach is completely different: it directly combines geometric and probabilistic arguments without any reference to the thermodynamic formalism. We make a heavy use of the so-called Green metric associated with the random walk. Ancona's theorem implies that the Green metric is hyperbolic in the sense of Gromov. It is then a simple but crucial observation that the harmonic measure is actually a Patterson-Sullivan measure. Hence the Green metric provides us with a genuine geometric framework for random walks. The computation of the dimension of the harmonic measure now follows from a straightforward argument. The explicit expression of the Green metric in terms of the hitting probability of the random walk makes it possible to directly take advantage of the independence of the increments of the walk. The combination of both facts —hyperbolicity and independence— yields very precise estimates on how random paths deviate from geodesics. This is used for analyzing the harmonic measures of maximal dimension. We also get an alternative and rather straightforward proof of the fact that the harmonic measure of a random walk on a Fuchsian group with cusps is singular, a result previously established in [19] and [15] by completely different methods.

The Green metric is not geodesic in general, so that we are led to developing a new framework for a special class of non-geodesic hyperbolic spaces that we call *quasiruled spaces*. Thus, we extend classical results such as the approximation of finite configurations by trees and the theory of quasiconformal measures.

We believe that our approach is also significant from a purely geometric point of view. For instance, it raises the question whether—in general—Patterson-Sullivan measures of maximal Hausdorff dimension can be similarly characterized. To our knowledge, this problem is completely open; compare with [13].

The rest of this introduction is devoted to a more detailed description of our results.

1.1. Geometric setting

Given a hyperbolic group Γ, we let $\mathcal{D}(\Gamma)$ denote the collection of hyperbolic left-invariant metrics on Γ which are quasi-isometric to a word metric induced by a finite generating set of Γ. In general these metrics do not come from proper geodesic metric spaces as we will see (cf. Theorem 1.1 for instance). In the sequel, we will distinguish the group as a space and as acting on a space: we keep the notation Γ for the group, and we denote by X the group as a metric space endowed with a metric $d \in \mathcal{D}(\Gamma)$. We may equivalently write $(X, d) \in \mathcal{D}(\Gamma)$. We will often require a base point which we will denote by $w \in X$. We recall that the boundary can be supplied with a family of so-called *visual distances* d_ε which essentially depend on a *visual parameter* $\varepsilon > 0$ (cf. § 2).

This setting enables us to capture in particular the following two situations.

- Assume that Γ admits a cocompact properly discontinuous action by isometries on a proper geodesic space (Y, d). Pick $w \in Y$ such that $\gamma \in \Gamma \mapsto \gamma(w)$ is a bijection, and consider $X = \Gamma(w)$ with the restriction of d.
- We may choose $(X, d) = (\Gamma, d_G)$ where d_G is the Green metric associated with a random walk (see Theorem 1.1).

Let μ be a symmetric probability measure the support of which generates Γ. Even if the support of μ may be infinite, we will require some compatibility with the geometry of the quasi-isometry class of $\mathcal{D}(\Gamma)$. Thus, we will often assume one of the following two assumptions. Given a metric $(X, d) \in \mathcal{D}(\Gamma)$, we say that the random walk has *finite first moment* if

$$\sum_{\gamma \in \Gamma} d(w, \gamma(w))\mu(\gamma) < \infty\,.$$

We say that the random walk has an *exponential moment* if there exists $\lambda > 0$ such that

$$\sum_{\gamma \in \Gamma} e^{\lambda d(w, \gamma(w))}\mu(\gamma) < \infty\,.$$

Note that both these conditions only depend on the quasi-isometry class of the metric.

1.2. The Green metric

The analogy between both families of measures—quasiconformal and harmonic—has already been pointed out in the literature e.g. [13, 26, 36]. Our first task is to make this empirical fact a theorem, i.e., we prove that harmonic measures are indeed quasiconformal measures for a well-chosen metric: given a symmetric law μ on Γ such that its support generates Γ, let $F(x, y)$ be the probability that the random walk started at x ever hits y, i.e., the probability there is some n such that $xZ_n = y$. Let

$$d_G(x, y) \stackrel{\text{def}}{=} -\log F(x, y)\,.$$

This function d_G is known to be a left-invariant metric on Γ. It was first introduced by S. Blachère and S. Brofferio in [8] and further studied in [9]. It is non-degenerate as soon as the walk is transient, i.e., eventually leaves any finite set. This is the case as soon as Γ is a non-elementary hyperbolic group.

The Green function is defined by

$$G(x, y) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \mathbb{P}[xZ_n = y] = \sum_{n=0}^{\infty} \mu^n(x^{-1}y)\,,$$

where, for each $n \geq 1$, μ^n is the law of Z_n, i.e., the nth convolution power of the measure μ. The Markov property and the invariance by the group imply

$$G(x, y) = F(x, y)G(y, y) = F(x, y)G(e, e)$$

so that

$$d_G(x, y) = \log G(e, e) - \log G(x, y)\,.$$

Non-elementary hyperbolic groups are non-amenable and for such groups and finitely supported laws μ, it was proved in [8] that the Green and word metrics are quasi-isometric. Nevertheless it does not follow from this simple fact that d_G is hyperbolic.

We first prove the following:

THEOREM 1.1. – *Let Γ be a non-elementary hyperbolic group, μ a symmetric probability measure on Γ the support of which generates Γ.*

(i) *Assume that μ has an exponential moment, then $d_G \in \mathcal{D}(\Gamma)$ if and only if for any r there exists a constant $C(r)$ such that*

$$F(x, y) \leq C(r)F(x, v)F(v, y) \tag{1}$$

whenever x, y and v are points in a locally finite Cayley graph of Γ and v is at distance at most r from a geodesic segment between x and y.

(ii) *If $d_G \in \mathcal{D}(\Gamma)$ then the harmonic measure is Ahlfors regular of dimension $1/\varepsilon$, when $\partial\Gamma$ is endowed with a visual metric d_ε^G of parameter $\varepsilon > 0$ induced by d_G.*

A. Ancona proved that (1) holds for finitely supported laws μ [2, Thm 6.1] and he used this estimate as the key ingredient in proving that the Martin boundary coincides with the geometric (hyperbolic) boundary; we may track down this idea to [1], see also [26, Thm 3.1]. The relationships between the Green metric and the Martin boundary are further discussed in § 1.5 and § 3.2.

Theorem 1.1 in particular yields